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Is There Screening in Turbulence?

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The statistical mechanics of some electric models predicts exponential decay of space correlations (screening). This suggests that one look also for screening in 2- and 3-dimensional hydrodynamic turbulence.

KEY WORDS: Turbulence; screening; vortices; XY model; Debye–Hückel theory.

It is well known that the kinetic energy of a 2-dimensional incompressible fluid can be interpreted as the electrostatic energy of a charge distribution equal to the distribution of (scalar) vorticity ω of the fluid. Similarly, the kinetic energy of a 3-dimensional incompressible fluid can be interpreted as the magnetic energy of a distribution of electric current equal to the distribution of (vector) vorticity ω of the fluid. These facts suggest that one may try to understand statistical properties of fluids (turbulence in particular) in terms of the statistical mechanics of a system of electric charges ($d=2$) or currents ($d=3$). In fact, Onsager[17] has proposed that, for $d=2$ and point vortices, the statistical mechanics would correspond to negative temperature, leading to coalescence of vortices of the same sign. (It was shown later[9] that negative-temperature states, however, do not exist for 2-dimensional electrostatic interactions.)

It should be noted that, in the above electric analogies, the time evolution of the fluid has no direct relation with the time evolution of the charges or currents. In fact, the *total* (i.e., kinetic) energy of the fluid is equal to the *potential* energy of the charges or currents. One should therefore be prudent in ascribing a statistical mechanics to the fluid (or its vorticity) even if the statistical mechanics of the corresponding charge or current systems makes good sense.

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For $d=2$, Kraichnan[15] has predicted the existence of an inverse energy cascade on the basis of a statistical equilibrium argument, using energy and entropy conservation. This suggests that equilibrium statistical mechanics might apply to vortices with mutual distances $\geqslant$ the injection wavelength. (The vortices would correspond to the small-$|k|$ modes, with a suitable admixture of the large-$|k|$ modes.) For $d=3$, there is no clear argument in favor of equidistribution of energy, but one might hope that something like it applies to large-size vortex structures.

For the statistical mechanics of a neutral system of point vortices, with vorticity ± 1, in 2 dimensions, the existence of the large-volume limit of thermodynamic functions has been proved (see ref. 9 for a review), and also the existence of the large-volume limit for the correlation functions in the canonical ensemble.[18,8] In particular, let ρ be the density and $\rho^{+-}(\xi, \eta)$, $\rho^{++}(\xi, \eta)$ be the correlation functions for a pair of vortices of signs $+, -$ or $+, +$, then the expectation value of the product of the vorticity at ξ and the vorticity at η is

$$\langle \omega(\xi)\, \omega(\eta) \rangle = \rho\delta(\xi - \eta) - 2[\rho^{+-}(\xi, \eta) - \rho^{++}(\xi, \eta)]$$

It is known that $\rho^{+-} - \rho^{++}$ is a positive convex function of $|\xi - \eta|$ which tends to 0 as $|\xi - \eta| \to \infty$.[11,6,7] It has been proved (at least in a lattice approximation) by Brydges and Federbush[3,4] that, at high temperature, there is a *screening phase* as suggested by Debye–Hückel theory. In particular, $\langle \omega(\xi)\, \omega(\eta) \rangle$ tends to zero exponentially fast with $|\xi - \eta|$. There is also a low-temperature phase with formation of dipoles and no screening.[11]

For $d=3$, the statistical mechanics of a discretized version of the current model also exhibits screening at high temperature, but no screening at low temperature (see refs. 10 and 12 and the review in ref. 13 of the XY model). Similar results hold for a related model[1,5]: there is a high-temperature phase with long vorticity lines, and screening.[1] Another phase with small loops of vorticity (or current) behaves like a gas of dipoles and does not exhibit screening.[5] Screening means that the vorticity in a bounded region A and that in the translated region $A + x$ have correlations which decay exponentially when $|x| \to \infty$. (In particular, $\langle \omega(\xi)\, \omega(\eta) \rangle$, and $\langle \omega(\xi)^2\, \omega(\eta)^2 \rangle - \langle \omega^2 \rangle^2$ should tend to zero exponentially fast when $|\xi - \eta| \to \infty$.)

Numerical studies[14,19–21] have indicated that vorticity in 3-dimensional turbulence is organized in long, thin tubes. This suggests that screening might indeed be present in 3-dimensional turbulence. This could be simply checked by studying numerically the decay of correlations such as $\langle \omega(\xi)\, \omega(\eta) \rangle$ and $\langle \omega(\xi)^2\, \omega(\eta)^2 \rangle - \langle \omega^2 \rangle^2$ for large $|\xi - \eta|$.

Similarly, it would be interesting to study numerically the vorticity correlations in 2 dimensions. In particular, one would like to check if $\langle \omega(\xi)\, \omega(\eta) \rangle < 0$ when $|\xi - \eta|$ is larger than the diameter of vortices, and to see if $\langle \omega(\xi)\, \omega(\eta) \rangle$ tends to zero exponentially when $|\xi - \eta| \to \infty$. (For the numerical observation of discrete vortices in 2 dimensions see refs. 2 and 16.)

Note that screening (for $d = 2$ or $d = 3$) is an exponential decoupling of vorticity variables, *not* of velocities.

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