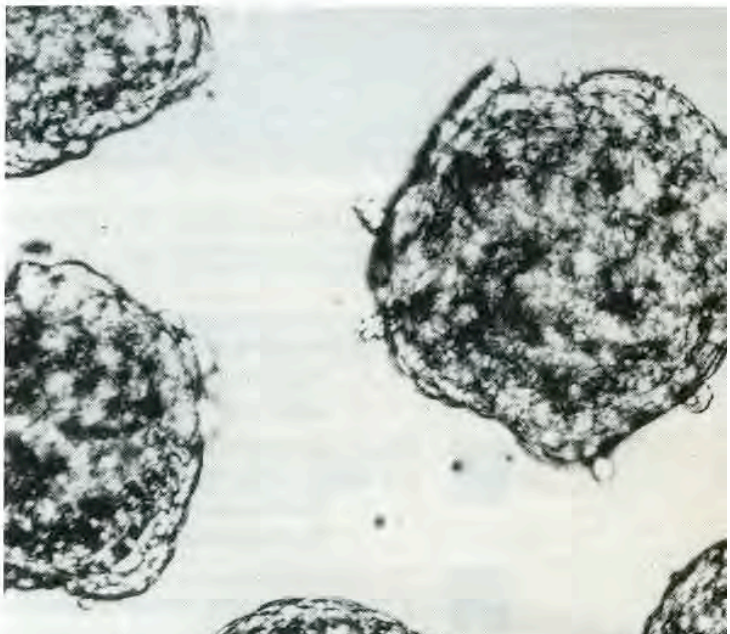
WHERE CAN ONE HOPE TO PROFITABLY APPLY THE IDEAS OF CHAOS?



The success of the ideas of chaos has led to attempts to apply them to a great variety of situations. This is in principle a good strategy, but the results are not always up to expectations. In some cases the results are predictably of little interest. Suppose you have concocted a mathematical model in biology or economics; you put this model on your computer and you discover a Feigenbaum period-doubling cascade, which is often a sign that chaos is present. Is this result interesting? Well, probably not. One reason is that the detailed dynamical properties of your model may not have anything to do with the properties of the real-life system. Another reason why your discovery may be without interest is that the occurrence of a Feigenbaum cascade need not have any particular biological or economic significance: You still have to address the problem of the relevance of your finding for biology or economics.

There is thus a problem of where and when one can hope to profitably apply the ideas of chaos to a real system. I will discuss some of the issues involved in this problem by considering a variety of examples. Readers wanting to know more about the mathematics of chaos may look at the review article by Jean-Pierre Eckmann and myself.[1] Very useful reprint collections of papers on chaos have been published by Pedrag Cvitanović[2] and Hao Bai-Lin.[3]

Differentiable dynamics, bifurcations and chaos

I will consider dynamical systems occurring in astronomy, meteorology, climate, chemical kinetics, biology and possibly economics. Figure 1 shows an excellent example of a real biological system that exhibits chaos as well as other dynamical behaviors. We model such systems mathematically with deterministic time evolutions that have either a discrete time,

$$x(t+1) = f(x(t))$$

or continuous time,

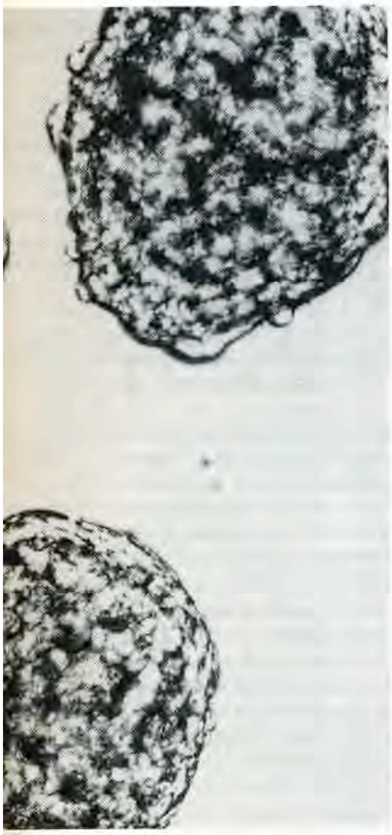
$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = X(x(t))$$

At each time t, $x(t)$ is a point in some manifold M (the phase space of the system) and the map f or the vector field X is differentiable (usually infinitely differentiable). This situation is known as differentiable dynamics, and the pair (M, f) or (M, X) is a differentiable dynamical system.

We focus our attention on time evolutions that have a property known as recurrence. This means that the system of interest, after some transients, establishes itself in an asymptotic regime where it comes back close to the same points of M again and again. The set of points in phase space to which it returns is called an attractor of the dynamical system. (See figure 2.) Furthermore, an attractor carries a natural measure $\rho(x)\,\mathrm{d}x$ describing time averages. That is, the integral $\int\rho(x)\,\mathrm{d}x$ over a region of phase space tells us how much time the system spends

A number of theoretical and practical issues must be considered when attempting to carry out meaningful analyses of real systems such as planetary orbits, heartbeats and economics in terms of chaos theory.

David Ruelle



Chicken embryo heart cell aggregates beating spontaneously. When stimulated by a periodic electric signal, such aggregates produce a variety of types of dynamical behavior, including chaos. This is one of the rare cases of a biological system with well-understood nontrivial dynamics. (Courtesy of Alvin Shrier, Leon Glass and Michael R. Guevara, McGill University. See also ref. 7.) **Figure 1**

in that region in the asymptotic regime. For frictionless mechanical systems, the natural measure is the Liouville measure ($\Pi_i \, dp_i \, dq_i$, where the p's and q's are momenta and coordinates) restricted to the energy shell.

Real systems often have parameters that can have different values. In an experimental system, for example, the experimenter may choose the temperature or the electric power supplied to have one value or another. We call such parameters bifurcation parameters, and they stay constant in a given experiment but may be changed from one experiment to the next.

We are interested mostly in the long-time, or asymptotic, behavior of dynamical systems, and this depends in general on the values of the bifurcation parameters. The attractor of a system may change qualitatively as the bifurcation parameters are smoothly varied. (See figure 3.) For instance, suppose we are considering a horizontal layer of water heated from below. For small rates of heating we have a conducting regime, and the water remains motionless. For larger rates of heating the fluid starts moving: We are in a convecting regime. In this situation the intensity of heating is a bifurcation parameter, and the transition from the conducting to the convecting regime is an example of bifurcation. An important part of the study of differentiable dynamical systems is the analysis of their bifurcations.

I shall define chaos in terms of sensitivity to initial conditions. If $\delta x(0)$ is an infinitesimal change of the initial condition for our dynamical system, and $\delta x(t)$ is the corresponding change at time $t > 0$, then in general

$$|\delta x(t)| \approx e^{\lambda t} |\delta x(0)|$$

More precisely, if $\| \partial x(t) / \partial x(0) \|$ denotes the norm of the Jacobian matrix, the limit

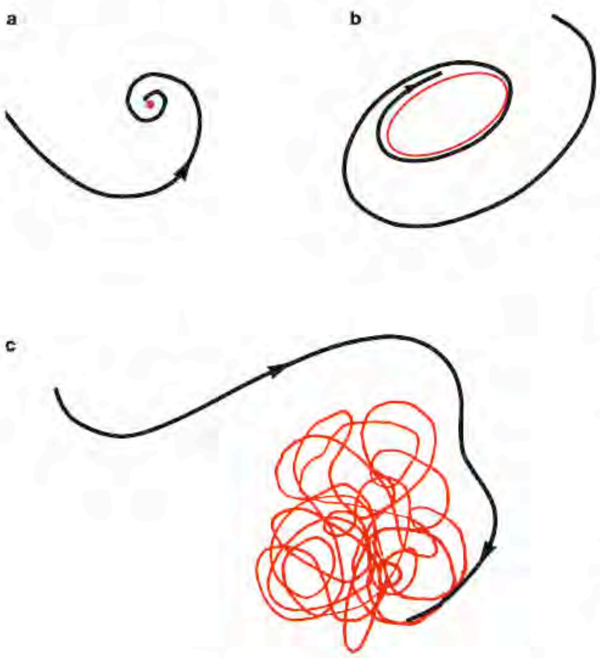
$$\lambda = \lim_{t \to \infty} \frac{1}{t} \log \left\| \frac{\partial x(t)}{\partial x(0)} \right\|$$

exists and is called the largest Lyapunov exponent. (See figure 4.) Other Lyapunov exponents arise for special choices of $\delta x(0)$. The above formula expresses that λ is the rate of separation (per unit time) of orbits passing near $x(0)$. If $\lambda > 0$, the orbits diverge exponentially, and we have sensitive dependence on initial conditions: Any microscopic uncertainty in the initial conditions or any errors in our computation render the long-term motion macroscopically unpredictable. In the case of recurrent motion on an attractor, λ is the same almost everywhere on the attractor. (Here, "almost everywhere" is with respect to the natural measure $\rho(x)dx$ mentioned earlier;

David Ruelle is a professor of theoretical physics at the Institut des Hautes Etudes Scientifiques in Bures-sur-Yvette, France.

stated another way, λ varies only on a set of points that the system visits on a negligible number of occasions.) If the largest Lyapunov exponent λ thus associated with the asymptotic motion on the attractor is greater than zero, we say that we have chaos.

A chaotic time evolution necessarily has irregular oscillations, because periodic or quasiperiodic behavior implies $\lambda = 0$. (Quasiperiodic motion involves two or more characteristic frequencies.) I have recounted the above definition of chaos to make one important point: To show the existence of chaos, it is not sufficient to prove sensitive dependence on initial conditions; this has to be done in the asymptotic regime (that is, for the measure $\rho(x)\mathrm{d}x$). One finds claims in the literature that chaos has been proved because a so-called homoclinic point has been exhibited. (A homoclinic point for a dynamical system (M,f) is a point x such that the limits $\lim_{k\to+\infty} f^k x$ and $\lim_{k\to-\infty} f^k x$ exist and are equal.) Starting with work by Henri Poincaré, it has long been understood that the existence of a homoclinic point implies very complicated dynamics. But a homoclinic point is not necessarily in an attractor, it may not have anything to do with the long-time behavior of the system, and its presence is not enough to prove chaos. In fact it is rare that one can prove the existence of chaos analytically. The standard way to show that a dynamical system is chaotic is by computer study and numerical estimation of λ. Finding that $\lambda > 0$ numerically does not prove mathematically that the behavior of the dynamical system is chaotic, but it is better evidence than the existence of a homoclinic point.



Attractors of dynamical systems take many forms. The time evolution of a system is described by the trajectory of a representative point in phase space. The asymptotic motion of this point (after transients have died out) is a set called an attractor. **a:** A trajectory spiraling to a point attractor. **b:** A trajectory approaching a periodic orbit (a limit cycle). **c:** A trajectory approaching a more complicated attractor. **Figure 2**

Real systems

Many published papers give the superficial impression that they deal with real physical, biological or economic systems, while in reality they present only computer studies of models. By "real system" I mean a system in, say, astronomy,[4,5] mechanics, physics, geophysics,[6] chemistry, biology[7] or economics[8] with a time evolution that one wants to investigate. Computer study of a model is an important method of investigation, but the results can only be as good as the model. (We shall come back to this problem later.)

The demonstration that chaos occurs in a real system is an important finding: It explains irregular oscillations of the system and limits the predictability of its future. Indeed, a chaotic system is one for which a small uncertainty in the initial conditions leads to an exponentially growing uncertainty in the predictions of its future. Notice, however, that this exponential growth exists only for an infinitesimal change in the initial conditions and may saturate to small values for certain systems. That is, a small but finite change might lead to only a small change in the asymptotic behavior. In other words there are chaotic systems that have mostly predictable time evolutions with a little bit of superimposed chaotic noise. Saying that a system is chaotic is thus not the end of the story. In fact we should not be obsessed by chaos: The whole of the dynamics of a real system is interesting, and not just the occurrence of chaos.

To show that we should be open-minded in analyzing the dynamics of real systems, let me discuss the problem of heartbeats. The beating of the heart is roughly periodic, but notoriously not quite regular. We might guess that the normally periodic cardiac rhythm would, in some diseases, bifurcate to a chaotic rhythm, with possibly fatal consequences. On the other hand Ary Goldberger and Arnold Mandell have noted that some very sick people have extremely regular heartbeats shortly before dying.[9] Goldberger and Mandell suggest that a chaotic cardiac regime is physiologically normal, while a regular periodic regime would be pathological. Functionally, however, one does not see why this should be so. Recently, Joseph Zbilut and, independently, Leon Glass have proposed an

Bifurcation occurs when an attractor changes qualitatively in response to a small change in one of the system's parameters. In the example shown here, a limit cycle of period T is succeeded by a limit cycle with a period of approximately $2T$ (period doubling), which is in turn succeeded by one with a period of approximately $4T$. The infinite repetition of this phenomenon is the Feigenbaum period-doubling cascade. **Figure 3**

explanation that I find very interesting. They note that the cardiac rhythm must constantly adjust to the activity of the organism (respiration, mental activity and so on) in response to internal or external demands. This adjustment produces an irregular rhythm that need not be the low-dimensional deterministic chaos of physicists. In certain diseases the heart loses the ability to adjust to the activity of the organism and thus exhibits an extremely regular periodic rhythm before ultimate failure. To gain a reasonable understanding of the cardiac rhythm will require much more work, along with a combination of dynamical and functional insight.

We come now to a difficult but unavoidable question: To what extent can a real system be described as a deterministic differentiable dynamical system?

Real systems can in general be described as deterministic systems with some added noise. This description is sufficiently vague that it apears to cover everything. In economics, for instance, such a description is familiar and the noise is called "shocks." A first remark concerning the above picture is that the separation between noise and the deterministic part of the evolution is ambiguous, because one can always interpret "noise" as a deterministic time evolution in infinite dimension. In the discrete-time case one does this by replacing the noisy signal $x(t)$ with a vector $\mathbf{x}(t) = (x(t+i))_{i \in \mathbb{Z}}$, where i runs over the integers ($\mathbb{Z}$); we have then a deterministic time evolution $\mathbf{x}(t) = (x(t+i))_{i \in \mathbb{Z}} \mapsto \mathbf{x}(t+1) = (x(t+i+1))_{i \in \mathbb{Z}}$. To describe a signal as "deterministic but with noise" is thus meaningless unless we put conditions on both the deterministic and the noisy contributions to the signal:

▷ The deterministic part of the time evolution (in the absence of noise) must be low dimensional. This means that either it takes place in a low-dimensional manifold or it corresponds to a low-dimensional attractor in infinite-dimensional space. Low dimensionality is important if one is to be able to reconstruct the dynamics from experimental data. (See figure 5.)

▷ The noisy part must be of small amplitude. It is also very useful to know if the noise is uncorrelated and how the noise comes into the signal—as an additive contribution or by changing some parameters in the deterministic dynamical system that produces the signal.

An additive noisy contribution to a low-dimensional deterministic signal is easiest to remove (even if its amplitude is not small). There is a vast literature on the problem of noisy dynamics, and there are computer studies where noise is introduced in a well-defined manner and it is shown how to remove it. In realistic situations such as economics time series, however, the interpretation of data as "deterministic plus noise" is extremely difficult.

A standard way in which one tries to remove noise is by filtering the experimental signal. The idea is to keep the components of the signal that correspond to a certain frequency band and to discard other components—getting rid, for instance, of high-frequency noise. In fact, however, the filter is a dynamical system driven by the incoming signal. If this signal is sinusoidal, the filter will change its amplitude in a well-defined manner. If the incoming signal is complicated, the intrinsic dynamics of the filter may make it more complicated and more difficult to analyze. And since experimental signals are often routinely filtered, this is really something to worry about. For instance, if you try to do something with an electroencephalogram handed to you by your physiologist friend, you may find out after a while (as happened to me) that this EEG has been narrowly filtered around 10 hertz (because your friend happened to be interested in alpha waves). The signal may also have been massaged by various smoothing and interpolation processes. The analysis of such a signal is very problematic. In general a theorist analyzing an "experimental" time series would be well advised to look how much the raw data have already been massaged before getting to him or her.

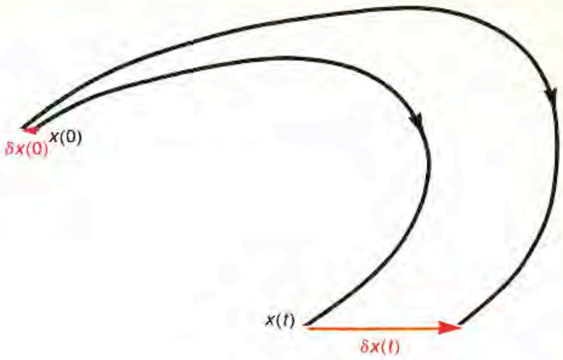
After all the above warnings, let me hasten to say that the ideas of nonlinear dynamics have led to quite a bit of improvement over the more traditional, linear methods of analyzing time series.[10,11]

The methods of differentiable dynamics are numerous and varied, making possible the analysis of rather different types of real systems. To get oriented we shall now consider various questions that can be asked about real systems.

How well do we know the basic equations?

A particularly favorable case is that of the astronomy of the solar system: There are finitely many degrees of freedom, and the equations of motion are known with great precision. This is why eclipses can be accurately predicted a fairly long time in advance—one thousand years, say. Beyond that, the precision drops, but because of dissipation (tides) rather than because of chaos. Over longer times, however, computer studies by Jack Wisdom, Jacques Laskar and others have shown the presence of chaos in the solar system.[4,5] The characteristic time of this chaos for the motion of the Earth is on the order of 5 million years, so that one cannot make accurate predictions for more than a few tens of millions of years.

The basic equations are also fairly well known for meteorology. Here we have an infinite number of degrees of freedom, but the simplifying feature that the Earth's atmosphere is basically two dimensional. By introducing a number of *ad hoc* parameters (obtained from observation) one gets a realistic and reliable model of the atmos-



The Lyapunov exponent is a measure of chaos. A small change $\delta x(0)$ is made in the initial point $x(0)$ of a trajectory. The corresponding change of $x(t)$ is $\delta x(t) \approx e^{\lambda t}\delta x(0)$, where λ is a Lyapunov exponent. If the motion on an attractor has $\lambda > 0$, there is sensitive dependence on initial conditions, and the motion is by definition chaotic (and the attractor is a "strange attractor"). **Figure 4**

phere, which is seen to be chaotic with a prediction time on the order of the week. That is, meteorologists cannot predict the weather more than about a week in advance.

The problem of climate seems close to that of meteorology, but the time scale is different and parameters that can be considered constant in the case of meteorology vary in the case of the climate. This makes the study of the climate difficult and explains, for instance, the uncertainty that remains about the warming of the atmosphere due to increased CO_2 concentration.

An early success of the theory of chaos was the demonstration that chaos is present in hydrodynamic turbulence. There has since been detailed study of the onset of turbulence and of weak turbulence, in which the motion of the fluid is described by a low-dimensional attractor. But hydrodynamicists are also interested in fully developed turbulence, a notoriously difficult topic. The fundamental equation of time evolution here is the Navier–Stokes equation, which is reasonably well satisfied by real fluids. The problem is that turbulence is a complicated three-dimensional motion that is at the limit of what present-day computers can handle. In fact, recent progress in the understanding of fully developed turbulence is due to the progress in computing, which now allows numerical simulation of truly turbulent flows (with the possibility of visualizing quantities that are not accessible in laboratory experiments with real fluids). (See the March 1993 special issue of PHYSICS TODAY on high-performance computing and physics, in particular the article by George Karniadakis and Steven Orszag on page 34.)

Chemical kinetics yields a number of interesting time evolutions. There are indeed oscillating chemical reactions, such as the Belousov–Zhabotinsky reaction, that behave chaotically under certain circumstances. (See figure 6.) Here there is no doubt that one has well-defined, low-dimensional, deterministic time evolutions, but the evolution equations cannot usefully be derived from first principles.

Finance, economics and social science phenomena yield time evolutions of great interest but perplexing difficulty. One has the impression that while there is an element of deterministic low-dimensional dynamics, a useful model should also include noise (shocks) and perhaps drift of the deterministic dynamics (that is, some parameters of the deterministic part of the dynamics change with time). Here, basically, one has not been able to obtain quantitatively useful models. One tentative conclusion of studies in this domain is that many time evolutions in finance or economics are chaotic in the sense that a small change in the initial conditions would have important consequences for later evolution. This means that the complicated arrangements elaborated by political decision makers in search of a subtle optimal equilibrium may be doomed to failure. Accepting the idea that the future is chaotic and not foreseeable in detail, one should set up regulations and arrangements that are robust in the presence of unforeseen events.

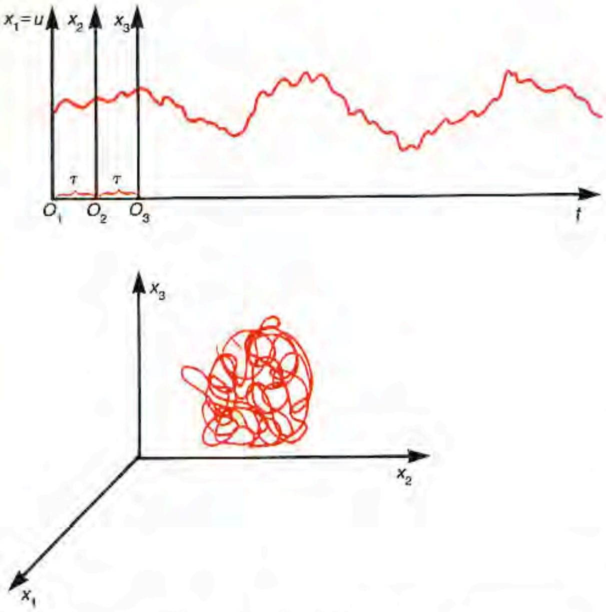
How complicated is the system?

One way to define the complication of a real system is by the dimension of the attractor describing its asymptotic motion. In the case of a low-dimensional attractor, the time evolution will repeatedly bring the system close to any point on the attractor. One can use this recurrence to predict the time evolution from initial condition x_0: Find an x_{-T} observed in the past that is close to x_0 and predict that $x_t = x_{-T+t}$. Variants of this idea using several x_{-T} values and linear or polynomial interpolation have been described. These methods are not applicable when the dimension of the attractor is large, because the recurrence time is then too large and the system is not observed twice near the same configuration.

Take the case of meteorology: The "dimension of the attractor" depends on the details of the modeling, but it certainly is large. One could think of predicting the weather by finding a global state of the atmosphere in the past (time $-T$) similar to that observed today (time 0) and inferring the weather tomorrow (time 1) from the weather observed at time $-T+1$. The method fails because one cannot find a time $-T$ with a global state of the atmosphere close to that at time 0. So instead one uses both theoretical principles and observational data to obtain realistic evolution equations and integrates these on a computer to make weather predictions. The results are valid for a few days, and then sensitive dependence on initial conditions ruins them.

Consider now the case of oscillating chemical reactions. While the basic equations of evolution escape us, the low dimension of the attractor permits a complete reconstruction of the dynamics. In one well-studied case of the Belousov–Zhabotinsky reaction, the dimension is approximately 2.1, and the attractor is therefore a fractal set (a "strange attractor"), as is typical when chaos is present.

For systems that are not too complicated, that is, those for which sufficient recurrence is observed, one can



Time-delay method allows reconstruction of an attractor from a signal. Given a scalar signal $u(t)$ (top), one can construct a vector signal $\mathbf{x}(t) = (x_1(t), x_2(t), x_3(t), \ldots)$ by writing $x_1(t) = u(t)$, $x_2(t) = u(t+\tau)$, $x_3(t) = u(t+2\tau), \ldots$ for some time delay τ. The vector signal gives an n-dimensional representation of the trajectory and of the corresponding attractor (bottom). **Figure 5**

also use the Grassberger–Procaccia algorithm to obtain the "correlation dimension" of the time evolution. There have been beautiful applications of this algorithm, but also a number of incorrect applications, in which it was not checked that the system had sufficient recurrence. Given a measure ρ (think of the natural measure on an attractor), we denote the mass inside a sphere of radius r centered at x by $\rho(x, r)$. If $\rho(x, r) \approx cr^\alpha$ for some constants c and α, we may interpret α as a dimension. The Grassberger–Procaccia algorithm uses a variation of this idea and the time-delay reconstruction (see figure 5) to compute a correlation dimension associated with a time evolution. More details can be found in references 1–3, and reference 12 tells a cautionary tale about the use of the Grassberger–Procaccia algorithm.

Is the modeling quantitative or qualitative?

A standard way to analyze a real system is to make a mathematical model, which can then be studied on a computer (or by rigorous mathematics). As we have seen, there are cases for which the mathematical model is quantitatively satisfactory and its study very useful. In many published studies, however, the relation between real system and model remains obscure, and the results of a computer study of the model are therefore worthless.

But we should not be too pessimistic. In some cases, interesting qualitative conclusions are possible even in the absence of a quantitative model. We have already mentioned the case of economics and finance, where sensitive dependence on initial conditions is probably present (further complicated by shocks), so that these systems are unpredictable in a strong sense. There are also interesting qualitative studies in biology concerning the dynamics of locomotion. The problem there is to find a qualitatively satisfactory model of a centipede or a fish that will actually emulate crawling or swimming.

What does one want to know?

There have been rather different *good* reasons to study real dynamical systems. (Just writing another research paper is not considered a good reason!) Let us examine some specific cases.

In meteorology, once it is shown that sensitivity to initial conditions is present and sets limits on predictability, the interest has been to make a quantitative estimate of those limits. In particular, it is known that certain situations known as blocking (a high-pressure area at high latitude) lead to increased predictability.

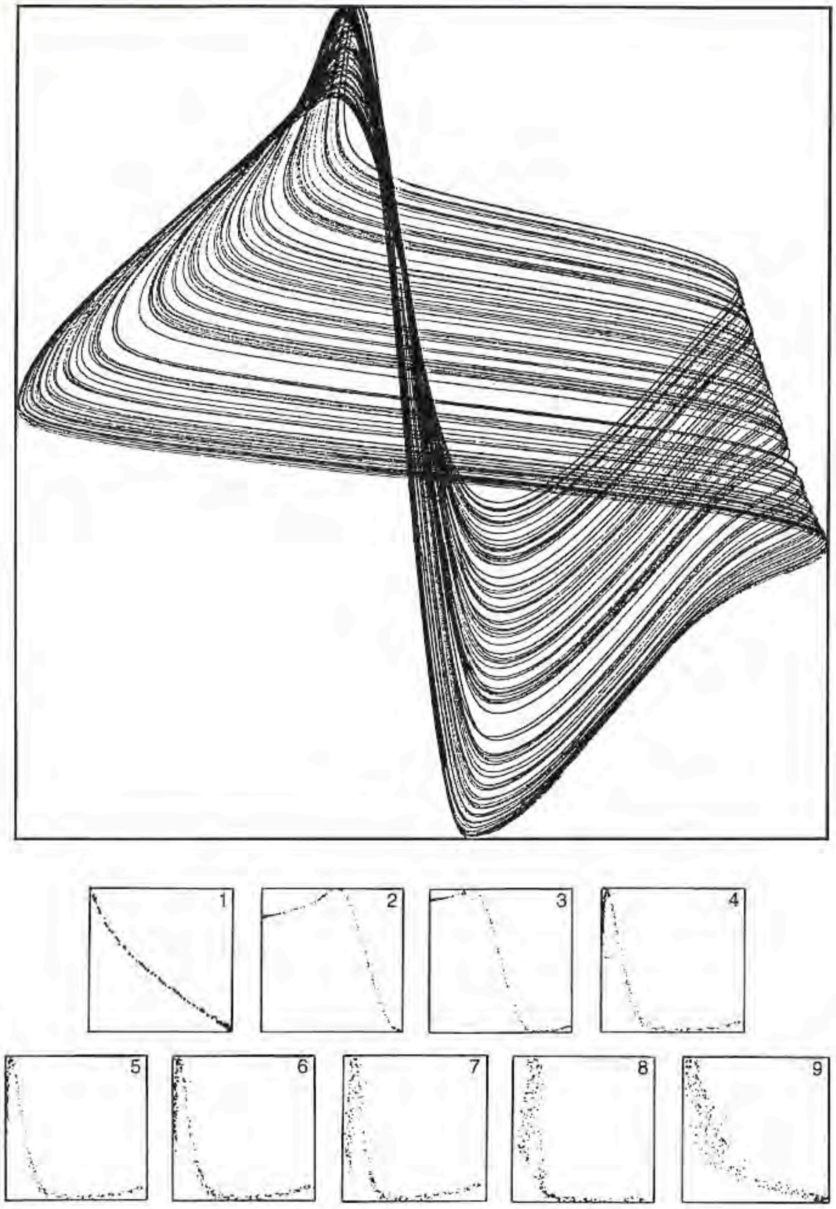
In the theory of hydrodynamic turbulence, the original question was to determine if Lev Landau's "modes" paradigm was valid or if it had to be replaced by a "chaotic" paradigm. In the modes paradigm the time evolution is a superposition of a finite number of periodic oscillations (modes); in the chaotic model there is sensitive dependence on initial conditions, and a description in terms of a finite number of modes is impossible. A number of studies (largely experimental) have shown that chaos is indeed present. Actually, it was found that a weakly excited fluid shows the typical behavior of a low-dimensional dynamical system (Hopf bifurcations, Feigenbaum bifurcations, quasiperiodicity, chaos and so on). The fact that the system is a fluid (and thus has a phase space of infinite dimension) is not visible in the qualitative features of its dynamics.

The existence of chaotically oscillating chemical reactions was a surprise to some chemists. The detailed theory of these oscillations has shown that it is sometimes possible to gain a good dynamical understanding of systems even when we know little about the basic kinetic mechanisms.

Studies of chaos in the astronomy of the solar system have been particularly fruitful. Wisdom solved the long-outstanding question of the origin of the Kirkwood gaps in the asteroid belts between Mars and Jupiter by showing that these gaps correspond to asteroid orbits that have chaotic excursions: The excursions lead the asteroids to collide with Mars and thereby be removed from orbit.[4] Laskar gave evidence for chaotic motion of the inner planets, overturning the accepted views on the stability of the solar system.[5]

In the applied sciences, a good understanding of the dynamics of oscillating systems allows one to stabilize periodic oscillations (avoiding chaos by feedback mechanisms, for instance).[13]

Sometimes a negative result can also be interesting. For instance, one might have hoped that some of the complicated time evolutions that are observed in the time series of economics and finance correspond to low-dimen-



The Belousov–Zhabotinsky chemical reaction (oxidation of malonate by bromate in the presence of metallic ions) is known to proceed in an oscillating manner. Using a continuously stirred tank reactor, a group at the University of Texas at Austin studied such a reaction, monitoring one of the concentrations as a function of time. From this function they reconstructed an attractor in three dimensions by the time-delay method. The upper part of the figure shows a two-dimensional projection of this three-dimensional attractor. The lower part shows several sections through the attractor. Notice that in one circuit around the attractor (1, 2, 3, . . . , 9, 1) the cross section is stretched and folded, as is typical of strange attractors. Stretching is in fact what causes sensitive dependence on initial conditions and thus chaos. (Adapted from J.-C. Roux, R. H. Simoyi, H. L. Swinney, Physica D **8**, 257, 1983.) **Figure 6**

sional chaos. The evidence so far is that economics and finance evade description by low-dimensional chaos, as they have evaded other descriptions.

Relevance and usefulness

The theory of dynamical systems has made important contributions to our understanding of the "real world," and in particular the role played in it by "chaos." One can expect further valuable contributions, especially in the difficult domain of biology. But a necessary condition for progress is that the relation between models and the real world be properly assessed. More generally, the problems of relevance and usefulness should not be obliterated by the routine of "contributing to the scientific literature."

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This article is based on lectures given at the University of Cape Town, South Africa.

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