NONEQUILIBRIUM STATISTICAL MECHANICS NEAR EQUILIBRIUM:

COMPUTING HIGHER ORDER TERMS.

by David Ruelle*

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Abstract. Using SRB measures to describe nonequilibrium steady states, one can in principle compute the coefficients of expansions around equilibrium. We discuss here how this can be done in practice, and how the results correspond to the zero noise limit when there is a stochastic perturbation. The approach used is formal rather than rigorous.

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0. Generalities.

This article follows the lines of [12] in discussing nonequilibrium steady states near equilibrium. Let (f^t) describe the microscopic time evolution of a system submitted to external forces and the action of a thermostat. We assume that (f^t) acts on a compact phase space M (this is the situation obtained with a so-called *Gaussian thermostat* [13]). In general (*i.e.*, outside of equilibrium), (f^t) has no invariant measure absolutely continuous with respect to a Riemannian volume element dx on M. Let us assume that the time evolution (f^t) is strongly chaotic (for instance *uniformly hyperbolic*), and let ρ be an *SRB measure* (*i.e.*, a limit when $t \to \infty$ of the image by f^t of a measure absolutely continuous with respect to dx). We assume that a nonequilibrium steady state is described by the SRB measure ρ, and study the dependence of ρ on (f^t) near equilibrium.

We call *chaotic principle* (after Gallavotti and Cohen [11]) the above assumption that physical time evolution satisfies strong chaoticity (or hyperbolicity) conditions, and that the nonequilibrium steady state is an SRB measure. The present paper is part of the rediscussion of fundamental issues of nonequilibrium statistical mechanics currently taking place on the basis of such ideas. The idea of using SRB measures to describe nonequilibrium states is not new (see in particular [20]) but has only recently become productive of specific results when Gallavotti and Cohen [11] proved their fluctuation theorem for entropy production. There is some question as to what chaoticity assumption one wants to make*: the maximal requirement that (f^t) is an exponentially mixing Anosov flow is ideal for proving theorems, but physically unrealistic. At the other extreme one may make only the minimal request that the SRB measures considered have nonzero Lyapunov exponents, and that certain time correlation functions tend to zero sufficiently fast at infinity to give convergent integrals. These assumptions reflect the idea that the microscopic time evolution is very chaotic. Note that in physical applications a thermodynamic limit (many degrees of freedom) is often implied, and this may help in obtaining time correlation functions which tend to zero fast at infinity. Here we shall leave the explicit assumptions at a minimum, and lay no claim to mathematical rigor. Even with this mathematical haziness, the chaotic principle yields more definite results than the usual physical approaches. Specifically, we shall see that it provides explicit expressions for higher order terms in the expansion of nonequilibrium quantities around equilibrium. These results can probably be made rigorous for uniformly hyperbolic systems, but should remain exact in more general (and physically interesting) circumstances.

For the purposes of the present paper the chaotic principle will mostly be invoqued at equilibrium. We assume that the *equilibrium time evolution* (f_0^t) preserves the volume element dx associated with some Riemann metric. (In coordinates $dx = \sqrt{g}\prod dx^i$, where g is the determinant of the metric tensor (g_{ij})). Therefore, at equilibrium, dx is an SRB measure, and we write $\rho_0(dx) = dx$. Note that a measure $\phi(x)dx$ with smooth density $\phi > 0$ may be rewritten as the Riemann volume for a modified Riemann metric. To justify our choice of equilibrium SRB measure, notice that for a Hamiltonian time evolution (f^t)

* In his *chaotic hypothesis* Gallavotti usually also assumes (microscopic) reversibility, which will not be used here.

the *microcanonical ensemble*, *i.e.*, the Liouville volume element restricted to an energy surface M, can be written in the form $\rho_0(dx) = dx$.

We refer the reader to the literature for the theory of SRB measures*. Here we recall a couple of basic facts. For any ergodic measure ρ one can define Lyapunov exponents λ_i (as many as the dimension $\dim M$ of M) and, at ρ-almost every $x \in M$, stable and unstable subspaces V_x^s, $V_x^u \subset T_x M$ (the tangent space to M at x). If $\xi \in V_x^u$ we have $(1/t)\log\|Tf^t\xi\| \to$ some $\lambda_i > 0$ for $t \to -\infty$, and if $\xi \in V_x^s$ we have $(1/t)\log\|Tf^t\xi\| \to$ some $\lambda_i < 0$ for $t \to +\infty$. We assume that only one of the λ_i vanishes (corresponding to the direction $F(x) = df^t(x)/dt|_{t=0}$ of the flow, so that $T_x M = \mathbf{R}.F(x) \oplus V_x^s \oplus V_x^u$. Through ρ-almost all points, there is an *unstable manifold* $\mathcal{V}^u \subset M$. This is a smooth manifold, with dimension equal to the number of positive Lyapunov exponents, tangent to V_x^u at each $x \in \mathcal{V}^u$. There is a natural volume element $\sigma^u(dx)$ on each $\mathcal{V}^u$ (defined up to a multiplicative constant) such that $(f^t)^*(\sigma^u|\mathcal{V})$ is (up to multiplication by a constant) $\sigma^u|f^t\mathcal{V}^u$. One can always write a *disintegration* $\rho = \int d\alpha\, \rho_\alpha$ where each ρ_α is carrried by an unstable manifold. The SRB measures ρ are those for which each ρ_α is absolutely continuous with respect to the Riemann volume on the unstable manifold carrying it, *i.e.*, ρ_α is proportional to σ^u on a piece of unstable manifold.

We shall use below the *divergence* $\mathrm{div}X$ of a vector field X on M defined with respect to the volume element dx. (In coordinates, $\mathrm{div}X = (1/\sqrt{g})\sum_i \partial(\sqrt{g}\,X^i)/\partial x^i$, where g is the determinant of the metric tensor (g_{ij})). In particular, if ρ is an SRB measure for the flow (f^t) generated by the vector field F, the corresponding rate of *entropy production*** is

$$e = \rho(-\mathrm{div}F)$$

We shall also need the *unstable divergence* $\mathrm{div}^u X = \mathrm{div}^u X^u$. To define this we first take the component X_x^u of X along the unstable subspace V_x^u at x by projecting along $\mathbf{R}.F(x) \oplus V_x^s$ (where V_x^s is the stable subspace, and $F(x) = df^t(x)/dt|_{t=0}$ as above). By definition, $\mathrm{div}^u X^u$ is the divergence of X^u along an unstable manifold $\mathcal{V}^u$, taken with respect to the volume element $\sigma^u(dx)$. In the uniformly hyperbolic case, $\mathrm{div}^u X^u$ is a honest Hölder continuous function on the support of ρ, otherwise little can be said in general.

The dependence of the SRB measure ρ on (f^t) was studied in [23]; in the linear approximation near equilibrium this yields a justification of the Onsager relations (see [12], following the earlier [9], [10]). In the present paper we want to go beyond the linear approximation, and obtain power series expansions for the state ρ, the rate of entropy production e, and the thermodynamic fluxes $\mathcal{J}$ around equilibrium. In Section 1, we discuss the simplest case of a continuous time deterministic evolution or *flow* (f^t). In order to obtain more explicit formulae, we deal in Section 2 with the special example of the *isokinetic model.* In Section 3, we examine the complications that arise when we have

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* See Sinai [24], Ruelle [19], Bowen and Ruelle [4] for the uniformly hyperbolic case, Ledrappier and Strelcyn [17], and Ledrappier and Young [18] for the general theory, and Eckmann and Ruelle [7] for a physical introduction.

** This formula appears for instance in Andrey [1]; for a general discussion of entropy production, see Ruelle [21].

random forces, and look in particular at the zero noise limit for a stochastic perturbation of a deterministic flow. Finally, in Section 4 we consider *discrete time* dynamics.

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1. Continuous time deterministic evolution.

Let (f^t_λ) be the flow associated with the vector field $F+\lambda X$ on M. (The assumed affine dependence of the vector field on the parameter λ simplifies formulae; the isokinetic example discussed below satisfies this condition). We define operators P, Q^t acting on smooth functions $M\to\mathbf{R}$ by

$$P\Psi=\sum_i X^i\partial_i\Psi$$

$$Q^t\Psi=\Psi\circ f^t_0$$

We have used local coordinates x^i to define $\partial_i=\partial/\partial x^i$ and the components X^i of X.

1.1. SRB states.

If $\Phi:M\to\mathbf{R}$ is smooth and independent of λ we have*

$$\frac{\delta^r}{\delta\lambda^r}(\Phi\circ f^T_\lambda)\big|_{\lambda=0}$$

$$=r!\int\cdots\int_{0<t_1<\cdots<t_r<T}dt_1\ldots dt_r\,Q^{T-t_r}PQ^{t_r-t_{r-1}}P\ldots PQ^{t_2-t_1}PQ^{t_1}\Phi$$

From this we can compute formally the derivatives with respect to λ of the SRB measure ρ_λ for the flow (f^t_λ). We assume that $(f^T_\lambda)^*\mu\to\rho_\lambda$ when $T\to\infty$, for a probability measure μ with smooth density**. Thus (formally) at $\lambda=0$

$$(\frac{\delta^r}{\delta\lambda^r}\rho_\lambda|_{\lambda=0})(\Phi)=\lim_{T\to\infty}\mu(\frac{\delta^r}{\delta\lambda^r}(\Phi\circ f^T))$$

$$=r!\int_0^\infty d\tau_1\cdots\int_0^\infty d\tau_r\,\rho_0(PQ^{\tau_r}P\ldots PQ^{\tau_1}\Phi)$$

so that

$$\rho_\lambda(\Phi)=\sum_{r=0}^\infty\frac{\lambda^r}{r!}(\frac{\delta^r}{\delta\lambda^r}\rho|_{\lambda=0})(\Phi)$$

$$=\sum_{r=0}^\infty\lambda^r\int_0^\infty d\tau_1\cdots\int_0^\infty d\tau_r\,\rho_0(PQ^{\tau_r}P\ldots PQ^{\tau_1}\Phi)\tag{1.1}$$

This formula deserves a couple of comments.

(a) We may write (1.1) formally as

$$\rho_\lambda(\Phi)=\rho_0((1-\lambda\int_0^\infty d\tau PQ^\tau)^{-1}\Phi)$$

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* δ is used to denote differentiation with respect to λ.

** In the uniformly hyperbolic Axiom A case, one would assume that ρ_f has its support on an attractor, and that μ has support close to this attractor.

Since the orthogonal projection π_0 on constants in $L^2(\rho_0)$ is given by

$$\pi_0\Phi = \int \rho_0(dx)\Phi(x)$$

we have

$$P\pi_0 = 0 \qquad , \qquad \pi_0 Q^\tau = Q^\tau \pi_0$$

hence

$$PQ^\tau = P(1-\pi_0)Q^\tau = PQ^\tau(1-\pi_0)$$

Therefore (1.1) may also be rewritten as

$$\rho_\lambda(\Phi) = \rho_0((1-\lambda\int_0^\infty d\tau PQ^\tau(1-\pi_0))^{-1}\Phi)$$

(b) The simplicity of (1.1) is deceptive: it is not clear why the integrals should converge. The following calculation will clarify this point partially. We may take $X = \phi F + X^s + X^u$, where $F(x) = df_0^t(x)/dt|_{t=0}$, and X^s, X^u are the components of X in the (strong) stable and unstable directions with respect to (f_0^t). Then, if div^u denotes the unstable divergence defined in Section 0, we may write (using as in [23] the fact that ρ_0 is SRB to perform integrations by part)

$$\int_0^\infty d\sigma \int_0^\infty d\tau\, \rho_0(\Psi.\, Q^\sigma P Q^\tau \Phi)$$

$$= \int_0^\infty d\sigma \int_0^\infty d\tau\, \rho_0\big((\Psi\circ f_0^{-\sigma}).[(\phi F + X^s + X^u)\cdot \mathrm{grad}(\Phi\circ f_0^\tau)]\big)$$

$$= \int_0^\infty d\sigma \int_0^\infty d\tau\, \rho_0\big([(-X^u\cdot \mathrm{grad})(\Psi\circ f_0^{-\sigma}].(\Phi\circ f_0^\tau) + (\Psi\circ f_0^{-\sigma}).(-\mathrm{div}^u X^u).(\Phi\circ f_0^\tau)$$
$$+(\Psi\circ f_0^{-\sigma}).\phi.[(F\cdot \mathrm{grad}(\Phi\circ f_0^\tau)] + (\Psi\circ f_0^{-\sigma}).[(X^s\cdot \mathrm{grad})(\Phi\circ f_0^\tau)]\big)$$

$$= \int_0^\infty d\sigma \int_0^\infty d\tau\, \rho_0\big(-[(Tf_0^{-\sigma})X^u)\cdot((\mathrm{grad}\Psi)\circ f_0^{-\sigma})].(\Phi\circ f_0^\tau) - (\Psi\circ f_0^{-\sigma}).(\mathrm{div}^u X^u).(\Phi\circ f_0^\tau)$$
$$+(\Psi\circ f_0^{-\sigma}).\phi.((F\cdot \mathrm{grad}\Phi)\circ f_0^\tau) + (\Psi\circ f_0^{-\sigma}).[((Tf_0^\tau)X^s)\cdot((\mathrm{grad}\Phi)\circ f_0^\tau)]\big) \qquad (1.2)$$

If we assume that Φ, Ψ are smooth and satisfy $\rho_0(\Phi) = \rho_0(\Psi) = 0$, the chaotic principle implies that the integrand in the right-hand side tends to 0 rapidly when σ, $\tau \to \infty$, so that the double integral is convergent and the above expressions are well defined.

Note that the above formulae do not assume that we have equilibrium at $\lambda = 0$.

1.2. Entropy production and thermodynamic fluxes.

Let us replace the vector field $F + \lambda X$ by $F + \sum_\alpha \lambda_\alpha X_\alpha$, and define P_α by

$$P_\alpha \Psi = \sum_i X_\alpha^i \partial_i \Psi$$

We write $\lambda = (\lambda_\alpha)$ and denote by ρ_λ the SRB measure corresponding to $F + \sum_\alpha \lambda_\alpha X_\alpha$, obtaining formally

$$\rho_\lambda(\Phi) = \rho_0((1 - \sum_\alpha \lambda_\alpha \int_0^\infty d\tau P_\alpha Q^\tau)^{-1}\Phi)$$

Let us now assume that we have equilibrium at $\lambda = 0$, *i.e.*, $\rho_0(dx) = dx$, hence $\mathrm{div}F = 0$. The entropy production is then

$$e_\lambda = \rho_0((1 - \sum_\alpha \lambda_\alpha \int_0^\infty d\tau P_\alpha Q^\tau)^{-1}(-\sum_\alpha \lambda_\alpha \mathrm{div}X_\alpha))$$

and can be expanded in powers of the λ_α. Omitting the index λ, we define the *fluxes* (see [9], [10], [12])

$$\mathcal{J}_\alpha = \rho_\lambda(\frac{\partial}{\partial\lambda_\alpha}(-\mathrm{div}(F + \sum_\alpha \lambda_\alpha X_\alpha))) = -\rho_\lambda(\mathrm{div}X_\alpha)$$

so that

$$e_\lambda = \sum_\alpha \lambda_\alpha \mathcal{J}_\alpha$$

The $\mathcal{J}_\alpha$ are nonlinear functions of λ vanishing for $\lambda = 0$ (because $\rho_0(\mathrm{divergence}) = 0$). The positivity of the entropy production (see [21]) implies that

$$\sum_\alpha \lambda_\alpha \mathcal{J}_\alpha \geq 0$$

identically. We may write

$$\mathcal{J}_\alpha = \sum_\beta L^{(2)}_{\alpha\beta}\lambda_\beta + \sum_{\beta\gamma} L^{(3)}_{\alpha\beta\gamma}\lambda_\beta\lambda_\gamma + \ldots \tag{1.3}$$

where the matrix $(L^{(2)}_{\alpha\beta})$ of *transport coefficients* is the object of the Onsager reciprocity relations $L^{(2)}_{\alpha\beta} = \pm L^{(2)}_{\beta\alpha}$ as discussed in [9], [10], [12] and an expression for $L^{(2)}_{\alpha\beta}$, $L^{(3)}_{\alpha\beta\gamma}$, is given below.

1.3. Coefficients for the expansion of e_λ, $\mathcal{J}_\alpha$.

In the simple case of a vector field $F + \lambda X$ (with $\rho_0(dx) = dx$) we may compute the entropy production as

$$e_\lambda = -\lambda\rho_\lambda(\mathrm{div}X) = -\sum_{r=1}^\infty \lambda^{r+1} \int_0^\infty d\tau_1 \cdots \int_0^\infty d\tau_r\, \rho_0(PQ^{\tau_r}P \ldots PQ^{\tau_1}\mathrm{div}X)$$

$$= \sum_{r=1}^\infty \lambda^{r+1} \int_0^\infty d\tau_1 \cdots \int_0^\infty d\tau_r\, \rho_0(\mathrm{div}X \cdot Q^{\tau_r}P \ldots PQ^{\tau_1}\mathrm{div}X) = \sum_{r=1}^\infty L^{(r+1)}\lambda^{r+1}$$

where

$$L^{(2)} = \int_0^\infty d\tau_1 \rho_0(\mathrm{div}X.Q^{\tau_1}\mathrm{div}X) = \frac{1}{2}\int_{-\infty}^\infty d\tau_1\, \rho_0(\mathrm{div}X.\,(\mathrm{div}X)\circ f^{\tau_1})$$

$$L^{(3)} = \int_0^\infty d\tau_1 \int_0^\infty d\tau_2\, \rho_0(\mathrm{div}X.Q^{\tau_2}PQ^{\tau_1}\mathrm{div}X)$$

$$= \int_0^\infty d\tau_1 \int_0^\infty d\tau_2\, \rho_0\big(((\mathrm{div}X)\circ f^{-\tau_2}).P((\mathrm{div}X)\circ f^{\tau_1})\big)$$

Restoring the vector field $F+\sum_\alpha \lambda_\alpha X_\alpha$ we may similarly write $\mathcal{J}_\alpha$ in the form (1.3), hence

$$e = \sum_{\alpha\beta} L^{(2)}_{\alpha\beta}\lambda_\alpha\lambda_\beta + \sum_{\alpha\beta\gamma} L^{(3)}_{\alpha\beta\gamma}\lambda_\alpha\lambda_\beta\lambda_\gamma + \ldots$$

where

$$L^{(2)}_{\alpha\beta} = \int_0^\infty d\tau_1\, \rho_0(\mathrm{div}X_\beta.(\mathrm{div}X_\alpha)\circ f^{\tau_1}) \tag{1.4}$$

$$L^{(3)}_{\alpha\beta\gamma} = \int_0^\infty d\tau_1 \int_0^\infty d\tau_2\, \rho_0\big(((\mathrm{div}X_\beta)\circ f^{-\tau_2}).P_\gamma((\mathrm{div}X_\alpha)\circ f^{\tau_1})\big) \tag{1.5}$$

Similar formulae are easily obtained for higher orders. Note that the above expressions for $L^{(3)}$ and $L^{(3)}_{\alpha\beta\gamma}$ can be rewritten following (1.2) yielding, hopefully, convergent integrals. In practice, however, this rewriting cannot be done explicitly, because of the difficulty of determining the stable and unstable directions. The point of invoquing (1.2) is to give evidence that the integrals for $L^{(3)}$, $L^{(3)}_{\alpha\beta\gamma}$ do in fact converge.

2. Application to the isokinetic model.

Consider the $2N-1$-dimensional manifold $S\times M$, where $S=\{\mathbf{p}\in\mathbf{R}^N:\mathbf{p}\cdot\mathbf{p}/2m=K\}$ and M is a bounded open subset of $\mathbf{R}^N$ or of the torus $\mathbf{T}^N$, with piecewise smooth boundary ∂M. A time evolution (f^t) is defined on (a large subset of) $S\times\overline{M}$ by

$$\frac{d}{dt}\begin{pmatrix}\mathbf{p}\\ \mathbf{q}\end{pmatrix}=\begin{pmatrix}\lambda(\xi-\alpha\mathbf{p})\\ \mathbf{p}/m\end{pmatrix} \tag{2.1}$$

when $(\mathbf{p},\mathbf{q})$ is in $S\times M$, and by elastic reflection at $S\times\partial M$. In (2.1), ξ is a smooth vector field and $\alpha=\mathbf{p}\cdot\xi/\mathbf{p}\cdot\mathbf{p}=\mathbf{p}\cdot\xi/2mK$. This *isokinetic* time evolution is of particular interest when ξ is locally a gradient, in view of the *pairing property* which then holds for Lyapunov exponents*. When $\lambda=0$, the time evolution is that of a billiard (or of a hard sphere system). For $\lambda\neq 0$ it includes the model studied in [5].

We have here

$$X=\begin{pmatrix}\xi-\alpha\mathbf{p}\\ 0\end{pmatrix}$$

so that (without the assumption that ξ is locally a gradient):

$$\mathrm{div}X=-(N-1)\alpha \tag{2.2}$$

Define $N\times N$ matrices $L_{pp}(t)$, $L_{pq}(t)$, $L_{pq}(t)$, $L_{qq}(t)$, depending on $\mathbf{p}=\mathbf{p}(0)$, $\mathbf{q}=\mathbf{q}(0)$ and t, so that

$$\begin{pmatrix}d\mathbf{p}(t)\\ d\mathbf{q}(t)\end{pmatrix}=\begin{pmatrix}L_{pp}(t) & L_{pq}(t)\\ L_{qp}(t) & L_{qq}(t)\end{pmatrix}\begin{pmatrix}d\mathbf{p}(0)\\ d\mathbf{q}(0)\end{pmatrix}$$

[This is obtained by solving the "linearized equation" corresponding to (2.1) with the initial conditions $L_{pp}(0)=1$, $L_{pq}(0)=0$, $L_{pq}(0)=0$, $L_{qq}(0)=1$]. The gradient of ξ is a $N\times N$ matrix $M(\mathbf{q})$ such that

$$M(\mathbf{q})\mathbf{u}=\mathrm{grad}_{\mathbf{q}}(\xi(\mathbf{q})\cdot\mathbf{u})$$

With this notation we have

$$P((\mathrm{div}X)\circ f^t)=-\frac{(N-1)}{2mK}P(\mathbf{p}(t)\cdot\xi(\mathbf{q}(t)))$$

$$=-\frac{(N-1)}{2mK}(\xi(\mathbf{q})-\alpha\mathbf{p})\cdot\mathrm{grad}_{\mathbf{p}}(\mathbf{p}(t)\cdot\xi(\mathbf{q}(t)))$$

$$=-\frac{(N-1)}{2mK}[\xi(\mathbf{q}(t))\cdot L_{pp}(t)(\xi(\mathbf{q})-\alpha\mathbf{p})+(M(\mathbf{q}(t))\mathbf{p}(t))\cdot L_{qp}(t)(\xi(\mathbf{q})-\alpha\mathbf{p})] \tag{2.3}$$

Using (2.2) and (2.3) one can compute the coefficients $L^{(2)}$ and $L^{(3)}$ in the λ-expansion of the entropy production e_λ, and similarly for $L^{(2)}_{\alpha\beta}$, $L^{(3)}_{\alpha\beta\gamma}$.

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* The pairing, observed empirically by Evans, Cohen and Morriss [8], was proved by Dettmann and Morriss [6], and also Wojtkowski and Liverani [25].

3. Continuous time random dynamical systems.

Here we start with a probability space $(\Omega, \mathbf{P})$ and a continuous one-parameter group (θ^t) of transformations of Ω with respect to which $\mathbf{P}$ is ergodic. A one-parameter group $(\mathbf{f}^t)$ of transformations of $\Omega \times M$ is given such that $\mathbf{f}^t(\omega, x) = (\theta^t\omega, f^t_\omega x)$. One can again define SRB states under suitable hyperbolicity conditions (the discrete time case is discussed in [3]).

3.1. λ-expansions.

In what follows we shall forget about hyperbolicity, and discuss the Markov case, which is realized by stochastic differential equations, (see for instance Ikeda and Watanabe [14], Arnold [2]). This situation gives simple formulae, which will again be derived only formally (in particular we sail clear off the subtleties of stochastic integration). Let thus $\mathbf{f}^t_\lambda = (\theta^t, f^t_{\omega\lambda})$ where $(f^t_{\omega\lambda})$ is a family of diffeomorphisms obtained by integrating the stochastic differential equation (*Langevin equation*) $dx(t)/dt = F(\theta^t\omega, x) + \lambda X(x)$. Define

$$P\Psi = \sum_i X_i \partial_i X$$

$$Q^t_\omega \Psi = \Psi \circ f^t_{\omega 0}$$

Then

$$\frac{\delta^r}{\delta\lambda^r}(\Phi \circ f^T_{\omega\lambda})|_{\lambda=0} = r! \int \cdots \int_{0<t_1<\cdots<t_r<T} dt_1 \ldots dt_r Q^{T-t_r}_{\theta^{t_r}\omega} P Q^{t_r - t_{r-1}}_{\theta^{t_{r-1}}\omega} P \ldots P Q^{t_1}_\omega \Phi$$

Let μ denote a probability measure with smooth density on M and let an "SRB stationary measure" ρ_λ on M be given by the weak limit $\int \mathbf{P}(d\omega)(f^T_{\lambda\omega})^*\mu \to \rho_\lambda$ when $T \to \infty$. Using the Markov property we have then (formally), at $\lambda = 0$,

$$(\frac{\delta^r}{\delta\lambda^r}\rho_\lambda|_{\lambda=0})(\Phi) = \lim_{T\to\infty} \int \mathbf{P}(d\omega)\mu(\frac{\delta^r}{\delta\lambda^r}(\Phi \circ f^T_{\omega\lambda}))|_{\lambda=0}$$

$$= r! \int \mathbf{P}(d\omega_1) \int_0^\infty d\tau_1 \ldots \int \mathbf{P}(d\omega_r) \int_0^\infty d\tau_r \rho_0(PQ^{\tau_r}_{\omega_r} P \ldots PQ^{\tau_1}_{\omega_1}\Phi)$$

and therefore

$$\rho_\lambda(\Phi) = \rho_0((1 - \lambda \int \mathbf{P}(d\omega) \int_0^\infty d\tau P Q^\tau_\omega)^{-1}\Phi) \tag{3.1}$$

The diffusion process associated with the Langevin equation $dx/dt = F(\theta^t\omega, x)$ is defined by the operators p^τ (for $\tau > 0$) such that

$$(\int \mathbf{P}(d\omega) Q^\tau_\omega \Phi)(x) = (p^\tau \Phi)(x) = \int p^\tau(x, y)\Phi(y)dy$$

We have

$$p^{0+} = 1$$

$$\frac{d}{d\tau}p^{\tau} = p^{\tau}A = Ap^{\tau} \tag{3.2}$$

where the second order elliptic differential operator A (infinitesimal generator of the diffusion process) is given in coordinates by

$$A\Phi = \frac{1}{\sqrt{a}}\sum_{ij}\partial_i(\sqrt{a}\,a^{ij}\partial_j)\Phi + \sum_i b^i\partial_i\Phi$$

Here the a^{ij}, b^i are functions on M, (a^{ij}) is a positive definite matrix, and a is the inverse of the determinant of (a^{ij}). The second term $(b\cdot\text{grad})$ in the expression for A corresponds to a *drift*. The absence of term of order 0 means that

$$\frac{d}{d\tau}\int\sqrt{a(y)}\prod dy^i p^{\tau}(x,y) = 0$$

because, writing $\Phi(\cdot) = p^{\tau}(x,\cdot)$ we have $dp^{\tau}(x,\cdot)/d\tau = A^*\Phi$ where, in coordinates,

$$A^*\Phi = \frac{1}{\sqrt{a}}\sum_{ij}\partial_i(\sqrt{a}\,a^{ij}\partial_j)\Phi - \frac{1}{\sqrt{a}}\sum_i\partial_i(\sqrt{a}b^i\Phi)$$

We choose the metric on M such that (g_{ij}) is proportional to the inverse of (a^{ij}), obtaining

$$A\Phi = \epsilon\triangle\Phi + b\cdot\text{grad}\Phi$$

$$A^*\Phi = \epsilon\triangle\Phi - \text{div}(\Phi b)$$

where $\triangle$ is the *Laplace-Beltrami operator*, ϵ a constant, and div the divergence with respect to the volume element $dx = \sqrt{g}\prod dx^i$.

We assume now that at $\lambda = 0$ we have equilibrium, *i.e.*, the measure dx is stationary for the diffusion process (p^t), or $A^*1 = 0$, or $\text{div}b = 0$, so that

$$A^*\Phi = \epsilon\triangle\Phi - b\cdot\text{grad}\Phi$$

In the present situation, we have $\rho_0(dx) = dx$; furthermore the operator A vanishes on constants and maps the space $\{\Phi : \rho_0(\Phi) = 0\}$ (orthogonal to constants with respect to ρ_0) to itself. We have in $L^2(dx)$

$$\frac{d}{d\tau}\|p^{\tau}\Phi\|^2 = \rho_0((p^{\tau}\Phi)^*(A+A^*)p^{\tau}\Phi) = 2\epsilon((p^{\tau}\Phi)^*\triangle p^{\tau}\Phi)$$

Assuming M to be a connected compact manifold, we have $-\triangle \geq c > 0$ on $\{\Phi : \rho_0(\Phi) = 0\}$, so that

$$\frac{d}{d\tau}\|p^{\tau}\Phi\|^2 \leq -2\epsilon c\|p^{\tau}\Phi\|^2$$

hence

$$\|p^{\tau}\Phi\| \leq e^{-\epsilon c\tau}\|\Phi\|$$

If π_0 denotes the orthogonal projection on constants in $L^2(dx)$, we have thus $\|p^\tau(1-\pi_0)\| \leq e^{-\epsilon c\tau}$ in the operator norm on $L^2(dx)$ and we can thus define

$$R = \int_0^\infty d\tau \int \mathbf{P}(d\omega) Q_\omega^\tau (1-\pi_0) = \int d\tau p^\tau (1-\pi_0)$$

so that R has operator norm $\leq 1/c\epsilon$ on $L^2(dx)$. From (3.2) we see that

$$1 + RA = 1 + AR = \pi_0$$

hence

$$P(1 + RA) = 0 \tag{3.3}$$

Since

$$P \int_0^\infty d\tau \int \mathbf{P}(d\omega) Q_\omega^\tau = PR$$

(3.1) yields

$$\rho_\lambda(\Phi) = \rho_0((1-\lambda PR)^{-1}\Phi)$$

[It is satisfactory to check that ρ_λ is invariant under the diffusion process (p_λ^τ) associated with the elliptic operator

$$A_\lambda = \epsilon\triangle + (b + \lambda X)\cdot \mathrm{grad} = A + \lambda P$$

We have indeed

$$\frac{d}{dt}\rho_\lambda(p_\lambda^t \Phi) = \rho_0((1-\lambda PR)^{-1}(A + \lambda X\cdot \mathrm{grad})p_\lambda^t\Phi) = 0$$

because, using (3.3),

$$(1-\lambda PR)^{-1}(A+\lambda P) = [1 + (1-\lambda PR)^{-1}\lambda PR](A + \lambda P)$$

$$= A + \lambda P + (1-\lambda PR)^{-1}(-\lambda P + \lambda PR\lambda P) = A + \lambda P - (1-\lambda PR)^{-1}(1-\lambda PR)\lambda P = A$$

and $\rho_0(A\Phi) = 0$].

The formula expected for the entropy production is

$$e_\lambda = \int \mathbf{P}(d\omega)\rho_\lambda(-\mathrm{div}F_\omega(t) - \lambda \mathrm{div}X)$$

(the discrete time version of this formula is discussed in [22]). Since the measure dx is stationary for the process associated with F_ω we write $\int \mathbf{P}(d\omega)\mathrm{div}F_\omega = 0$ so that

$$e_\lambda = -\lambda\rho_\lambda(\mathrm{div}X) = -\lambda\rho_0((1-\lambda PR)^{-1}\mathrm{div}X)$$

$$= -\sum_{r=1}^\infty \lambda^{r+1}\rho_0((PR)^r\mathrm{div}X) = \sum_{r=1}^\infty \lambda^{r+1}\rho_0(\mathrm{div}X.RP\dots PR\,\mathrm{div}X)$$

$$= \sum_{r=1}^{\infty} \tilde{L}^{(r+1)} \lambda^{r+1}$$

Let us now replace λX by $\sum_\alpha \lambda_\alpha X_\alpha$ and define the fluxes

$$\mathcal{J}_\alpha = -\rho_\lambda(\mathrm{div} X_\alpha) = \sum_\beta \tilde{L}^{(2)}_{\alpha\beta}\lambda_\beta + \sum_{\beta\gamma} \tilde{L}^{(3)}_{\alpha\beta\gamma}\lambda_\beta\lambda_\gamma + \dots$$

We have then

$$e_\lambda = \sum_\alpha \lambda_\alpha \mathcal{J}_\alpha$$

The coefficients of $\mathcal{J}_\alpha$ are given by

$$\tilde{L}_{\alpha\beta} = \rho_0(\mathrm{div} X_\beta . R \mathrm{div} X_\alpha) \tag{3.4}$$

$$\tilde{L}^{(3)}_{\alpha\beta\gamma} = \rho_0(\mathrm{div} X_\beta . R P_\gamma R \mathrm{div} X_\alpha) \tag{3.5}$$

and similarly for higher order.

3.2. Small random perturbations.

Various results are known showing that the SRB states for uniformly hyperbolic systems are stable under small random perturbations (see [15], [16]). Comparison of the formulae in Sections 1.3 and 3.1 yield a result of this type for the coefficients L, $\tilde{L}$ of the λ-expansions of e_λ and the $\mathcal{J}_\alpha$. In fact, inspection of (1.4), (1.5) and (3.4), (3.5) shows that the coefficients L corresponding to a deterministic evolution are obtained from the coefficients $\tilde{L}$ corresponding to a random evolution by the replacement

$$R = \int_0^\infty d\tau \int \mathbf{P}(d\omega) Q^\tau_\omega (1 - \pi_0) \qquad \longrightarrow \qquad \int_0^\infty d\tau Q^\tau (1 - \pi_0)$$

This means that the limit $\epsilon \to 0$ (zero noise limit) of the coefficients $\tilde{L}$ reproduces the coefficients L. At least this holds formally, and assuming that the integrals defining the coefficients L converge, as discussed in Section 1. The introduction of the term $\epsilon\triangle$ in A is however a singular perturbation, and the analysis of the ϵ-dependence of the coefficients $\tilde{L}$ near $\epsilon = 0$ would require more definite mathematical assumption than we have chosen to make.

4. Diffeomorphism case.

In this Section we derive the formulae which correspond for diffeomorphisms to those established in Section 1 for flows. While it was natural in Section 1 to assume an affine dependence on the parameter λ, this is no longer possible in the discrete time situation studied here. The formulae obtained will therefore be somewhat more complicated.

Let the diffeomorphism $f : M \to M$ depend on the real parameter λ, and let X_λ be the vector field on M such that $\delta f \circ f^{-1} = X_\lambda . \delta\lambda$. We write $X_{\lambda k} = \delta^{k-1} X_\lambda / \delta\lambda^{k-1}$, and define operators $P_{\lambda k}$, Q_λ acting on smooth functions $M \to \mathbf{R}$ by

$$P_{\lambda k}\Psi = \sum_i X^i_{\lambda k} \partial_i \Psi$$

$$Q_\lambda \Psi = \Psi \circ f$$

[We have used local coordinates x_i to define $\partial_i = \partial/\partial x_i$ and the components $X^i_{\lambda k}$ of $X_{\lambda k}$; also δ denotes differentiation with respect to λ]. By induction on k, we define operators $R_{\lambda k}$ such that $R_{\lambda 1} = P_{\lambda 1}$ and

$$(k+1) R_{\lambda(k+1)} = P_{\lambda 1} . R_{\lambda k} + [\frac{\delta}{\delta\lambda}, R_{\lambda k}]$$

Since $[\frac{\delta}{\delta\lambda}, P_{\lambda k}] = P_{\lambda(k+1)}$, we see that the $R_{\lambda k}$ are polynomials in the $P_{\lambda k}$.

We claim that, if $\Phi : M \to \mathbf{R}$ is smooth and independent of λ,

$$\frac{\delta^r}{\delta\lambda^r}(\Phi \circ f^N)$$

$$= r! \sum_{s=1}^{r} \sum_{k_1,\ldots,k_s \geq 1}^{*} \sum_{n_1,\ldots,n_s \geq 1}^{**} Q^{n_s}_\lambda R_{\lambda k_s} Q^{n_{s-1}}_\lambda \ldots Q^{n_1}_\lambda R_{\lambda k_1} Q^{N - \sum n_i}_\lambda \Phi \tag{4.1}$$

where $\sum^*$ is subjected to the condition $\sum k_i = r$, while $\sum^{**}$ is subjected to the condition $\sum n_i \leq N$. We have indeed

$$\frac{\delta}{\delta\lambda}(\Phi \circ f^N) = \sum_{n=0}^{N-1} [X^i_\lambda \partial_i (\Phi \circ f^n)](f^{N-n} x)$$

$$= \sum_{n=0}^{N-1} Q^{N-n}_\lambda P_{\lambda 1} Q^n_\lambda = \sum_{n=1}^{N} Q^n_\lambda P_{\lambda 1} Q^{N-n}_\lambda$$

which proves (4.1) for $r = 1$. Note the identities

$$[\frac{\delta}{\delta\lambda}, Q^n_\lambda] = \sum_{n'=1}^{n-1} Q^{n-n'}_\lambda R_{\lambda 1} Q^{n'}_\lambda + Q^n_\lambda R_{\lambda 1}$$

$$[\frac{\delta}{\delta\lambda}, R_{\lambda k}] = (k+1)R_{\lambda(k+1)} - R_{\lambda 1}R_{\lambda k}$$

and therefore also

$$[\frac{\delta}{\delta\lambda}, Q_\lambda^n R_{\lambda k}] = \sum_{n'=1}^{n-1} Q_\lambda^{n-n'} R_{\lambda 1} Q_\lambda^{n'} R_{\lambda k} + (k+1) Q_\lambda^n R_{\lambda(k+1)}$$

The proof of (4.1) for higher r is then readily obtained by induction.

Note that, to third order, we have

$$R_{\lambda 1} = P_{\lambda 1} \quad , \quad R_{\lambda 2} = \frac{1}{2}(P_{\lambda 1}P_{\lambda 1} + P_{\lambda 2}) \tag{4.2}$$

$$R_{\lambda 3} = \frac{1}{6}(P_{\lambda 1}P_{\lambda 1}P_{\lambda 1} + 2P_{\lambda 1}P_{\lambda 2} + P_{\lambda 2}P_{\lambda 1} + P_{\lambda 3})$$

We shall now compute formally the derivatives of the SRB measure ρ_λ associated with f. We assume that $f^{*N}\mu \to \rho_\lambda$ when $N \to \infty$, for a probability measure μ with smooth density*. Thus (formally)

$$(\frac{\delta^r}{\delta\lambda^r}\rho_\lambda)(\Phi) = \lim_{N\to\infty} \mu(\frac{\delta^r}{\delta\lambda^r}(\Phi \circ f^N))$$

and, using (4.1),

$$\rho_\lambda(\Phi) = \sum_{r=0}^{\infty} \frac{\lambda^r}{r!}[(\frac{\delta^r}{\delta\lambda^r}\rho_\lambda|_{\lambda=0})(\Phi)]$$

$$= \sum_{s=0}^{\infty} \sum_{k_1=1}^{\infty} \lambda^{k_1} \ldots \sum_{k_s=1}^{\infty} \lambda^{k_s} \sum_{n_0=0}^{\infty} \sum_{n_1=1}^{\infty} \cdots \sum_{n_{s-1}=1}^{\infty} \rho_0(R_{0k_s} Q_0^{n_{s-1}} \cdots Q_0^{n_1} R_{0k_1} Q_0^{n_0} \Phi)$$

$$= \rho_0((1 - \sum_{k=1}^{\infty} \lambda^k Q_0 R_{0k} \sum_{n=0}^{\infty} Q_0^n)^{-1} \Phi) \tag{4.3}$$

As in Section 1, one could further rewrite the above formula by separating the stable and unstable directions. Let us write $X_{\lambda k} = X_k^{(s)} + X_k^{(u)}$ where $X_k^{(s)}$ and $X_k^{(u)}$ are respectively in the stable and unstable subbundles of TM. Then $P_{\lambda k} = P_k^{(s)} + P_k^{(u)}$ where $P_k^{(s)} = X_k^{(s)} \cdot \operatorname{grad} = X_k^{(s)} \cdot \operatorname{grad}^s$, $P_k^{(u)} = X_k^{(u)} \cdot \operatorname{grad} = X_k^{(u)} \cdot \operatorname{grad}^u$, and grad^s, grad^u are the gradients in the stable and unstable directions. We have for example

$$P_1^{(s)} Q_\lambda^n \Phi = X^{(s)} \cdot \operatorname{grad}^s(\Phi \circ f^n) = (Tf^n X^{(s)}) \cdot (\operatorname{grad}^s \Phi) \circ f^n$$

* In the uniformly hyperbolic Axiom A case, one would assume that ρ_λ has its support on an attractor, and that μ has support close to this attractor. For this case, a rigorous proof of differentiability with respect to λ has been given in [23].

Using the fact that ρ_λ is SRB we also have

$$\rho_\lambda(\Psi Q^n_\lambda P^{(u)}_1 \ldots) = \rho_\lambda((\Psi\circ f^{-n})X^{(u)}\cdot \mathrm{grad}^u \ldots) = -\rho_\lambda(\mathrm{div}^u((\Psi\circ f^{-n})\cdot X^{(u)})\ldots)$$

$$= -\rho_\lambda((\Psi\circ f^{-n})\mathrm{div}^u X^{(u)}\ldots) - \rho_\lambda(((\mathrm{grad}^u\Psi)\circ f^{-n})\cdot(Tf^{-n}X^{(u)})\ldots)$$

We evaluate now the derivatives of $\log J_\lambda(x)$, where $J_\lambda(x)$ is the Jacobian of f. We have

$$J_\lambda(x) = |\det(A^i_j)|$$

where $A^i_j(x) = \partial_j f^i(x)$, hence

$$\frac{\delta}{\delta\lambda}\log J_\lambda(x) = \frac{\delta}{\delta\lambda}\mathrm{tr}\,\log(\partial_j f^i(x)) = \sum_{i,j}(A^{-1})^i_j\frac{\delta}{\delta\lambda}(\partial_i f^j(x))$$

$$= \sum_{i,j}(A^{-1})^i_j\partial_i X^k_\lambda(f(x))\partial_k f^j(x) = \sum_i(\partial_i X^i_\lambda)(f(x)) = (Q_\lambda \mathrm{div} X_\lambda)(x)$$

We have (using (1) with $N=1$)

$$\frac{\delta^k}{\delta\lambda^k}\log J_\lambda = \frac{\delta^{k-1}}{\delta\lambda^{k-1}}Q_\lambda\mathrm{div}X_\lambda = Q_\lambda\mathrm{div}X_{\lambda k} + \sum_{r=1}^{k-1}\binom{k-1}{r}r!QR_{\lambda r}\frac{\delta^{k-r-1}}{\delta\lambda^{k-r-1}}\mathrm{div}X_\lambda$$

$$= Q_\lambda\mathrm{div}X_{\lambda k} + \sum_{r=1}^{k-1}\frac{(k-1)!}{(k-r-1)!}Q_\lambda R_{\lambda r}\mathrm{div}X_{\lambda,k-r}$$

so that

$$\frac{\delta}{\delta\lambda}\log J_\lambda = \sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!}(Q_0\mathrm{div}X_{0k} + \sum_{r=1}^{k-1}\frac{(k-1)!}{(k-r-1)!}Q_0R_{0r}\mathrm{div}X_{0,k-r})$$

$$= \sum_{s=0}^{\infty}\frac{\lambda^s}{s!}Q_0\mathrm{div}X_{0,s+1} + \sum_{r=1}^{\infty}\sum_{s=0}^{\infty}\frac{\lambda^{r+s}}{s!}Q_0R_{0r}\mathrm{div}X_{0,s+1}$$

$$= (Q_0 + \sum_{r=1}^{\infty}\lambda^r Q_0R_{0r})\sum_{s=0}^{\infty}\frac{\lambda^s}{s!}\mathrm{div}X_{0,s+1} \tag{4.4}$$

We shall now for simplicity drop the indices 0, so that Q, X, X_k, P_k are taken at $\lambda = 0$ in accordance with the notation of Section 1. We shall however write ρ_0, f_0, J_λ for ρ_λ, f_λ, J_λ at $\lambda=0$. To second order (4.4) gives then, using (4.2),

$$\frac{\delta}{\delta\lambda}\log J_\lambda$$

$$= Q\mathrm{div}X + \lambda Q(P_1\mathrm{div}X + \mathrm{div}X_2) + \frac{1}{2}\lambda^2 Q((P_1P_1 + P_2)\mathrm{div}X + 2P_1\mathrm{div}X_2 + \mathrm{div}X_3)$$

and therefore to third order

$$\log J_\lambda = \log J_0 + \lambda Q_0\mathrm{div}X + \frac{\lambda^2}{2}Q(P_1\mathrm{div}X + \mathrm{div}X_2)$$
$$+\frac{\lambda^3}{6}Q((P_1P_1 + P_2)\mathrm{div}X + 2P_1\mathrm{div}X_2 + \mathrm{div}X_3) \tag{4.5}$$

We assume now that $\rho_0(dx) = dx$, so that $J_0 = 1$, and $\rho_0(\text{divergence}) = 0$. To third order (4.3) and (4.5) give thus

$$\rho_\lambda(\log J_\lambda) = \lambda^2[\frac{1}{2}\rho_0(P_1\mathrm{div}X) + \sum_{n=1}^{\infty}\rho_0(P_1Q^n\mathrm{div}X)]$$
$$+\lambda^3[\frac{1}{6}\rho_0((P_1P_1 + P_2)\mathrm{div}X + 2P_1\mathrm{div}X_2) + \sum_{n=1}^{\infty}\rho_0(P_1Q^n\frac{1}{2}(P_1\mathrm{div}X + \mathrm{div}X_2))$$
$$+\sum_{n=1}^{\infty}\rho_0(\frac{1}{2}(P_1P_1 + P_2)Q^n\mathrm{div}X) + \sum_{n=1}^{\infty}\sum_{n'=1}^{\infty}\rho_0(P_1Q^nP_1Q^{n'}\mathrm{div}X)]$$

Therefore the entropy production, to third order, is

$$e_\lambda = \rho_\lambda(-\log J_\lambda) = \lambda^2[\frac{1}{2}\rho_0(\mathrm{div}X.\mathrm{div}X) + \sum_{n=1}^{\infty}\rho_0(\mathrm{div}X.Q^n\mathrm{div}X)]$$
$$+\lambda^3[\frac{1}{6}\rho_0(\mathrm{div}X.P_1\mathrm{div}X + \mathrm{div}X_2.\mathrm{div}X + 2\mathrm{div}X.\mathrm{div}X_2)$$
$$+\sum_{n=1}^{\infty}\frac{1}{2}\rho_0(\mathrm{div}X.Q^nP_1\mathrm{div}X + \mathrm{div}X.Q^n\mathrm{div}X_2) + \mathrm{div}X.P_1Q^n\mathrm{div}X + \mathrm{div}X_2.Q^n\mathrm{div}X$$
$$+\sum_{n=1}^{\infty}\sum_{n'=1}^{\infty}\rho_0(\mathrm{div}X.Q^nP_1Q^{n'}\mathrm{div}X)]$$

This can be rewritten as

$$e_\lambda = L^{(2)}\lambda^2 + L^{(3)}\lambda^3 + \ldots$$

where

$$L^{(2)} = \frac{1}{2}\sum_{n=-\infty}^{\infty}\rho_0\big(\mathrm{div}X.((\mathrm{div}X)\circ f^n)\big)$$

$$L^{(3)} = [\frac{1}{6}\rho_0\big(\mathrm{div}X.P_1\mathrm{div}X\big) + \frac{1}{2}\sum_{n=1}^{\infty}\rho_0\big(((\mathrm{div}X)\circ f^{-n}).P_1\mathrm{div}X\big)$$
$$+\frac{1}{2}\sum_{n=1}^{\infty}\rho_0\big(\mathrm{div}X.P_1((\mathrm{div}X)\circ f^n)\big) + \sum_{n=1}^{\infty}\sum_{n'=1}^{\infty}\rho_0\big(((\mathrm{div}X)\circ f^{-n}).P_1((\mathrm{div}X)\circ f^{n'})\big)$$
$$+\frac{1}{2}\sum_{n=-\infty}^{\infty}\rho_0\big((\mathrm{div}X_2).((\mathrm{div}X)\circ f^n)\big)$$

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