GRACE-LIKE POLYNOMIALS.

by David Ruelle*.

Abstract. *Results of somewhat mysterious nature are known on the location of zeros of certain polynomials associated with statistical mechanics (Lee-Yang circle theorem) and also with graph counting. In an attempt at clarifying the situation we introduce and discuss here a natural class of polynomials. Let* $P(z_1, \ldots, z_m, w_1, \ldots, w_n)$ *be separately of degree 1 in each of its* $m + n$ *arguments. We say that* P *is a Grace-like polynomial if* $P(z_1, \ldots, w_n) \neq 0$ *whenever there is a circle in* $\mathbf{C}$ *separating* $z_1, \ldots, z_m$ *from* $w_1, \ldots, w_n$*. A number of properties and characterizations of these polynomials are obtained.*

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I had the luck to meet Steve Smale early in my scientific career, and I have read his 1967 article in the Bulletin of the AMS more times than any other scientific paper. It took me a while to realize that Steve had worked successively on a variety of subjects, of which "differentiable dynamical systems" was only one. Progressively also I came to appreciate his independence of mind, expressed in such revolutionary notions as that the beaches of Copacabana are a good place to do mathematics. Turning away from scientific nostalgy, I shall now discuss a problem which is not very close to Steve's work, but has relations to his interests in recent years: finding where zeros of polynomials are located in the complex plane.

0 Introduction.

One form of the Lee-Yang circle theorem [3] states that if $|a_{ij}| \leq 1$ for $i, j = 1, \ldots, n$, and $a_{ij} = a_{ji}^*$, the polynomial

$$\sum_{X \subset \{1,\ldots,n\}} z^{\mathrm{card} X} \prod_{i \in X} \prod_{j \notin X} a_{ij}$$

has all its zeros on the unit circle $\{z : |z| = 1\}$.

Let now Γ be a finite graph. We denote by Γ' the set of dimer subgraphs γ (at most one edge of γ meets any vertex of Γ), and by Γ'' the set of unbranched subgraphs γ (no more than two edges of γ meet any vertex of Γ). Writing $|\gamma|$ for the number of edges in γ, on proves that

$$\sum_{\gamma \in \Gamma'} z^{|\gamma|}$$

has all its zeros on the negative real axis (Heilmann-Lieb [2]) and

$$\sum_{\gamma \in \Gamma''} z^{|\gamma|}$$

has all its zeros in the left-hand half plane $\{z : \mathrm{Im} z < 0\}$ (Ruelle [6]).

The above results can all be obtained in a uniform manner by studying the zeros of polynomials

$$P(z_1, \ldots, z_n)$$

which are *multiaffine* (separately of degree 1 in their n variables), and then taking $z_1 = \ldots = z_n = z$. The multiaffine polynomials corresponding to the three examples above are obtained by multiplying factors for which the location of zeros is known and performing *Asano contractions*:

$$Auv + Bu + Cv + D \qquad \to \qquad Az + D$$

The *key lemma* (see [5]) is that if K, L are closed subsets of $\mathbf{C} \backslash \{0\}$ and if

$$u \notin K, v \notin L \qquad \Rightarrow \qquad Auv + Bu + Cv + D \neq 0$$

then

$$z \notin -K * L \qquad \Rightarrow \qquad Az + D \neq 0$$

where we have written $K * L = \{uv : u \in K, v \in L\}$.

To get started, let $P(z_1, \ldots, z_n)$ be a multiaffine *symmetric* polynomial. If $W_1, \ldots, W_n$ are the roots of $P(z, \ldots, z) = 0$, we have

$$P(z_1, \ldots, z_n) = \text{const.} \sum_{\pi} \prod_{j=1}^{n} (z_j - W_{\pi(j)})$$

where the sum is over all permutations π of n objects. *Grace's theorem* asserts that if $Z_1, \ldots, Z_n$ are separated from $W_1, \ldots, W_n$ by a circle of the Riemann sphere, then $P(Z_1 \ldots, Z_n) \neq 0$. For example, if a is real and $-1 \leq a \leq 1$, the roots of $z^2 + 2az + 1$ are on the unit circle, and therefore

$$uv + au + av + 1$$

cannot vanish when $|u| < 1$, $|v| < 1$; from this one can get the Lee-Yang theorem.

In view of the above, it is natural to consider multiaffine polynomials

$$P(z_1, \ldots, z_m, w_1, \ldots, w_n)$$

which cannot vanish when $z_1, \ldots, z_m$ are separated from $w_1, \ldots, w_n$ by a circle. We call these polynomials Grace-like, and the purpose of this note is to study and characterize them.

I. General theory.

We say that a complex polynomial $P(z_1, z_2, \ldots)$ in the variables z_1, $z_2, \ldots$ is a Multi-Affine Polynomial (*MA-nomial* for short) if it is separately of degree 1 in z_1, $z_2, \ldots$. We say that a circle $\Gamma \subset \mathbf{C}$ *separates* the sets A', $A'' \subset \mathbf{C}$ if $\mathbf{C} \backslash \Gamma = C' \cup C''$, where C', C'' are open, $C' \cap C'' = \emptyset$ and $A' \subset C'$, $A'' \subset C''$. We say that the MA-nomial $P(z_1, \ldots, z_m, w_1, \ldots, w_n)$ is Grace-like (or a G-nomial for short) if it satisfies the following condition

(G) *Whenever there is a circle* Γ *separating* $\{Z_1, \ldots, Z_m\}$, $\{W_1, \ldots, W_n\}$, *then*

$$P(Z_1, \ldots, W_n) \neq 0$$

[Note that we call circle either a straight line $\Gamma \subset \mathbf{R}$ or a *proper circle* $\Gamma = \{z \in \mathbf{C} : |z - a| = R\}$ with $a \in \mathbf{C}$, $0 < R < \infty$].

1 Lemma (homogeneity).

The G-nomial P is homogeneous of degree $k \leq \min(m, n)$.

If there is a circle Γ separating $\{z_1, \ldots, z_m\}$, $\{w_1, \ldots, w_n\}$, then the polynomial $\lambda \mapsto P(\lambda z_1, \ldots, \lambda w_n)$ does not vanish when $\lambda \neq 0$, hence it is of the form $C\lambda^k$, where $C = P(z_1, \ldots, w_n)$. Thus

$$P(\lambda z_1, \ldots, \lambda w_n) = \lambda^k P(z_1, \ldots, w_n)$$

on an open set of $\mathbf{C}^{m+n}$, hence identically, *i.e.*, P is homogeneous of degree k.

Assuming $k > n$, each monomial in P would have a factor z_i, hence

$$P(0, \ldots, 0, 1, \ldots, 1) = 0$$

in contradiction with the fact that $\{0, \ldots, 0\}$, $\{1, \ldots, 1\}$ are separated by a circle. Thus $k \leq n$, and similarly $k \leq m$. $\square$

2 Lemma (degree).

If all the variables $z_1, \ldots, w_n$ effectively occur in the G-nomial P, then $m = n$ and P has degree $k = n$.

By assumption

$$(\prod_{i=1}^{m} z_i)(\prod_{j=1}^{n} w_j) P(z_1^{-1}, \ldots, w_n^{-1})$$

is a homogeneous MA-nomial $\tilde{P}(z_1, \ldots, w_n)$ of degree $m + n - k$. If $Z_1, \ldots, W_n$ are all $\neq 0$ and $\{Z_1, \ldots, Z_m\}$, $\{W_1, \ldots, W_n\}$ are separated by a circle Γ, we may assume that Γ does not pass through 0. Then $\{Z_1^{-1}, \ldots, Z_m^{-1}\}$, $\{W_1^{-1}, \ldots, W_n^{-1}\}$ are separated by Γ^{-1}, hence $\tilde{P}(Z_1, \ldots, W_n) \neq 0$. Let $\mathcal{V}$ be the variety of zeros of $\tilde{P}$ and

$$\mathcal{Z}_i = \{(z_1, \ldots, w_n) : z_i = 0\} \qquad , \qquad \mathcal{W}_j = \{(z_1, \ldots, w_n) : w_j = 0\}$$

Then

$$\mathcal{V} \subset (\mathcal{V} \backslash \cup_{i,j} (\mathcal{Z}_i \cup \mathcal{W}_j)) \cup \cup_{i,j} (\mathcal{Z}_i \cup \mathcal{W}_j)$$

Since all the variables $z_1, \ldots, w_n$ effectively occur in $P(z_1, \ldots, w_n)$, none of the hyperplanes $\mathcal{Z}_i$, $\mathcal{W}_j$ is contained in $\mathcal{V}$, and therefore

$$\mathcal{V} \subset \text{closure}(\mathcal{V} \backslash \cup_{i,j} (\mathcal{Z}_i \cup \mathcal{W}_j))$$

We have seen that the points $(Z_1, \ldots, W_n)$ in $\mathcal{V} \backslash \cup_{i,j} (\mathcal{Z}_i \cup \mathcal{W}_j)$ are such that $\{Z_1, \ldots, Z_m\}$, $\{W_1, \ldots, W_m\}$ cannot be separated by a circle Γ, and the same applies to their limits. Therefore $\tilde{P}$ again satisfies (G). Applying Lemma 1 to P and $\tilde{P}$ we see that $k \leq \min(m, n)$, $m + n - k \leq \min(m, n)$. Therefore $m + n \leq 2 \min(m, n)$, thus $m = n$, and also $k = n$. $\square$

3 Proposition (reduced G-nomials).

If $P(z_1, \ldots, z_m, w_1, \ldots, w_n)$ is a G-nomial, then P depends effectively on a subset of variables which may be relabelled $z_1, \ldots, z_k, w_1, \ldots, w_k$ so that

$$P(z_1, \ldots, z_m, w_1, \ldots, w_n) = \alpha R(z_1, \ldots, z_k, w_1, \ldots, w_k)$$

where $\alpha \neq 0$, the G-nomial R is homogeneous of degree k, and the coefficient of $z_1 \cdots z_k$ in R is 1.

This follows directly from Lemma 2. ∎

We call a G-nomial R as above a *reduced* G-nomial.

4 Lemma (translation invariance).

If $P(z_1, \ldots, w_n)$ is a G-nomial, then

$$P(z_1 + s, \ldots, w_n + s) = P(z_1, \ldots, w_n)$$

i.e., P is translation invariant.

If there is a circle Γ separating $\{z_1, \ldots, z_m\}$, $\{w_1, \ldots, w_n\}$, then the polynomial

$$p(s) = P(z_1 + s, \ldots, w_n + s)$$

satisfies $p(s) \neq 0$ for all $s \in \mathbf{C}$. This implies that $p(s)$ is constant, or $dp/ds = 0$, for $(z_1, \ldots, w_n)$ in a nonempty open subset of $\mathbf{C}^{2n}$. Therefore $dp/ds = 0$ identically, and p depends only on $(z_1, \ldots, w_n)$. From this the lemma follows. ∎

5 Proposition (properties of reduced G-nomials).

If $P(z_1, \ldots, w_n)$ is a reduced G-nomial, the following properties hold:

(reduced form) *there are constants C_π such that P has the reduced form*

$$P(z_1, \ldots, w_n) = \sum_{\pi} C_\pi \prod_{j=1}^{n} (z_j - w_{\pi(j)})$$

where the sum is over all permutations π of $(1, \ldots, n)$

(conformal invariance) *if $ad - bc \neq 0$, then*

$$P(\frac{az_1 + b}{cz_1 + d}, \ldots, \frac{aw_n + b}{cw_n + d}) = P(z_1, \ldots, w_n) \prod_{j=1}^{n} \frac{ad - bc}{(cz_j + d)(cw_j + d)}$$

in particular we have the identity

$$(\prod_{i=1}^{k} z_i)(\prod_{j=1}^{k} w_j) R(z_1^{-1}, \ldots, w_k^{-1}) = (-1)^k R(z_1, \ldots, w_k)$$

(roots) *the polynomial*

$$\hat{P}(z) = P(z, \ldots, z, w_1, \ldots, w_n)$$

is equal to $\prod_{k=1}^{n} (z - w_k)$, so that its roots are the w_k (repeated according to multiplicity).

Using Proposition 3 and Lemma 4, the above properties follow from Proposition A2 and Corollary A3 in Appendix A. $\square$

6 Proposition (compactness).

The space of MA-nomials in $2n$ *variables which are homogeneous of degree* n *may be identified with* $\mathbf{C}^{\binom{2n}{n}}$. *The set* G_n *of reduced G-nomials of degree* n *is then a compact subset of* $\mathbf{C}^{\binom{2n}{n}}$. *We shall see later (Corollary 15) that* G_n *is also contractible.*

Let $P_k \in G_n$ and $P_k \to P_\infty$. In particular P_∞ is homogeneous of degree n, and the monomial $z_1 \cdots z_n$ occurs with coefficient 1. Suppose now that

$$P_\infty(Z_1, \dots, Z_m, W_1, \dots, W_n) = 0$$

with $\{Z_1, \dots, Z_m\}$, $\{W_1, \dots, W_n\}$ separated by a circle Γ. One can then choose discs D_1, $\dots$, D_{2n} containing $\{Z_1, \dots, W_n\}$ and not intersecting Γ. Lemma A1 in Appendix A would then imply that P_∞ vanishes identically in contradiction with the fact that P_∞ contains the term $z_1 \cdots z_n$. Therefore $P_\infty \in G_n$, *i.e.*, G_n is closed.

Suppose now that G_n were unbounded. There would then be P_k such that the largest coefficient (in modulus) c_k in P_k tends to ∞. Going to a subsequence we may assume that

$$c_k^{-1} P_k \to P_\infty$$

where P_∞ is a homogeneous MA-nomial of degree n, and does not vanish identically. The same argument as above shows that P_∞ is a G-nomial, hence (by Proposition 3) the coefficient α of $z_1 \cdots z_n$ does not vanish, but since $\alpha = \lim c_k^{-1}$, c_k cannot tend to infinity as supposed. G_n is thus bounded, hence compact. $\square$

7 Proposition (the cases $n = 1$, 2).

The reduced G-nomials with $n = 1, 2$ *are as follows:*

For $n = 1$: $P = z_1 - w_1$.

For $n = 2$: $P = (1-\theta)(z_1 - w_1)(z_2 - w_2) + \theta(z_1 - w_2)(z_2 - w_1)$ *with real* $\theta \in [0, 1]$.

We use Proposition 5 to write P in reduced form.

In the case $n = 1$, we have $P = C(z_1 - w_1)$, and $C = 1$ by normalization.

In the case $n = 2$, we have

$$P = C'(z_1 - w_1)(z_2 - w_2) + C''(z_1 - w_2)(z_2 - w_1)$$

In view of (G), C', C'' are not both 0. Assume $C' \neq 0$, then (G) says that

$$\frac{z_1 - w_1}{z_1 - w_2} : \frac{z_2 - w_1}{z_2 - w_2} + \frac{C''}{C'} \neq 0 \tag{1}$$

when $\{z_1, z_2\}$, $\{w_1, w_2\}$ are separated by a circle. If C''/C' were not real, we could find z_1, z_2, w_1, w_2 such that

$$\frac{z_1 - w_1}{z_1 - w_2} : \frac{z_2 - w_1}{z_2 - w_2} = -\frac{C''}{C'} \tag{2}$$

but the fact that the cross-ratio in the left hand side of (2) is not real means that z_1, z_2, w_1, w_2 are not on the same circle, and this implies that there is a circle separating $\{z_1, z_2\}$, $\{w_1, w_2\}$. Therefore (1) and (2) both hold, which is impossible. We must therefore assume C''/C' real, and it suffices to check (1) for z_1, z_2, w_1, w_2 on a circle. The condition that $\{z_1, z_2\}$, $\{w_1, w_2\}$ are separated by a circle is now equivalent to the cross-ratio being > 0, and therefore (G) is equivalent to $C''/C' \geq 0$. If we assume $C'' \neq 0$, the argument is similar and gives $C'/C'' \geq 0$. The normalization condition yields then $C' = 1 - \theta$, $C'' = \theta$ with $\theta \in [0, 1]$ $\square$

8 Proposition (determinants).

Let Δ_z be the diagonal $n \times n$ matrix where the j-th diagonal element is z_j and similarly for Δ_w. Also let U be a unitary $n \times n$ matrix ($U\Delta_w U^{-1}$ is thus an arbitrary normal matrix with eigenvalues $w_1, \ldots, w_n$). Then

$$P(z_1, \ldots, z_n, w_1, \ldots, w_n) = \det(\Delta_z - U\Delta_w U^{-1})$$

is a reduced G-nomial. We may assume that $\det U = 1$ *and write*

$$\det(\Delta_z - U\Delta_w U^{-1}) = \det((U_{ij}(z_i - w_j)))$$

Let $\{z_1, \ldots, z_n\}$, $\{w_1, \ldots, w_n\}$ be separated by a circle Γ. We may assume that Γ is a proper circle. Suppose first that the z_j are inside the circle Γ and the w_j outside. We want to prove that $\det(\Delta_z - U\Delta_w U^{-1}) \neq 0$. By translation we may assume that Γ is centered at the origin, say $\Gamma = \{z : |z| = R\}$; then, by assumption, using the operator norm,

$$||\Delta_z|| < R \qquad , \qquad ||\Delta_w^{-1}|| < R^{-1}$$

Therefore

$$||\Delta_z (U\Delta_w U^{-1})^{-1}|| < 1$$

so that

$$\det(\Delta_z - U\Delta_w U^{-1}) = \det(-U\Delta_w U^{-1}) \det(1 - \Delta_z (U\Delta_w U^{-1})^{-1}) \neq 0$$

as announced. The case where the w_j are inside Γ and the z_j outside is similar (consider $\det(\Delta_w - U^{-1}\Delta_z U)$). $\square$

9 Proposition (Grace's theorem).

The polynomial

$$P_\Sigma(z_1, \ldots, z_n, w_1, \ldots, w_n) = \frac{1}{n!} \sum_\pi \prod_{j=1}^n (z_j - w_{\pi(j)}) \tag{3}$$

where the sum is over all permutations of $(1,\ldots,n)$ *is a reduced G-nomial.*

See Polya and Szegö [4] Exercise V 145. ∎

This result will also follow from our proof of Corollary 15 below.

10 Proposition (permanence properties).

(Permutations) *If* $P(z_1,\ldots,z_n,w_1,\ldots,w_n)$ *is a reduced G-nomial, permutation of the* z_i*, or the* w_j*, or interchange of* $(z_1,\ldots,z_n)$ *and* $(w_1,\ldots,w_n)$ *and multiplication by* $(-1)^n$ *produces again a reduced G-nomial.*

(Products) *If* $P'(z'_1,\ldots,w'_{n'})$, $P''(z''_1,\ldots,w''_{n''})$ *are reduced G-nomials, then their product* $P'\otimes P''(z'_1,\ldots,z''_{n''},w'_1,\ldots,w''_{n''})$ *is a reduced G-nomial.*

(Symmetrization) *Let* $P(z_1,\ldots,z_n,w_1,\ldots,w_n)$ *be a reduced G-nomial, and*

$$P_S(z_1,\ldots,z_n,w_1,\ldots,w_n)$$

be obtained by symmetrization with respect to a subset S *of the variables* $z_1,\ldots,z_n$*, then* P_S *is again a reduced G-nomial. Symmetrization with respect to all variables* $z_1,\ldots,z_n$ *produces the polynomial* P_Σ *given by (3).*

The part of the proposition relative to permutations and products follows readily from the definitions. To prove the symmetrization property we may relabel variables so that S consists of $z_1,\ldots,z_s$. We denote by $\hat{P}(z)$ the polynomial obtained by replacing $z_1,\ldots,z_s$ by z in P (the dependence on $z_{s+1},\ldots,w_n$ is not made explicit). With this notation P_S is the only MA-nomial symmetric with respect to $z_1,\ldots,z_n$ and such that $\hat{P}_S(z)=\hat{P}(z)$. We may write

$$\hat{P}(z)=\alpha(z-a_1)\cdots(z-a_s) \tag{4}$$

where $\alpha,a_1,\ldots,a_s$ may depend on $z_{s+1},\ldots,w_n$. If Γ is a circle separating the regions C', C'', and $z_{s+1},\ldots,z_n\in C'$, $w_1,\ldots,w_n\in C''$, (G) implies that $\alpha\neq 0$ and $a_1,\ldots,a_s\notin C'$. Grace's theorem implies that P_S does not vanish when $z_1,\ldots,z_s$ are separated by a circle from $a_1,\ldots,a_s$. Therefore P_S does not vanish when $z_1,\ldots,z_s\in C'$, hence P_S is a G-nomial, which is easily seen to be reduced. If $s=n$, (4) becomes

$$\hat{P}(z)=(z-w_1)\cdots(z-w_n)$$

in view of Proposition 5, hence symmetrisation of P gives P_Σ.

II. Further results.

We define now G_0-nomials as a class of MA-nomials satisfying a new condition (G_0) weaker than (G). It will turn out later that G_0-nomials and G-nomials are in fact the same (Proposition 12). The new condition is

(G_0) *If there are two proper circles, or a proper circle and a straight line* Γ', $\Gamma''\subset\mathbf{C}$ *such that* $z_1,\ldots,z_m\in\Gamma'$, $w_1,\ldots,w_n\in\Gamma''$*, and* $\Gamma'\cap\Gamma''=\emptyset$*, then*

$$P(z_1,\ldots,z_m,w_1,\ldots,w_n)\neq 0$$

Remember that a proper circle is of the form $\{z : |z - a| = R\}$, with $0 < R < \infty$. For the purposes of (G_0) we may allow $R = 0$ (because a circle Γ' or Γ'' reduced to a point a' or a'' can be replaced by a small circle through a' or a'').

11 Lemma.

Let $P(z_1, \ldots, w_n)$ be a G_0-nomial, and define

$$\tilde{P}(z_1, \ldots, w_n) = (\prod_{i=1}^{m} z_i)(\prod_{j=1}^{n} w_j) P(z_1^{-1}, \ldots, w_n^{-1}) \tag{5}$$

(a) P is translation invariant.

(b) If P depends effectively on $z_1, \ldots, w_n$, then $\tilde{P}$ is translation invariant, and therefore a G_0-nomial.

The polynomial $a \mapsto p(a) = P(z_1 + a, \ldots, w_n + a)$ does not vanish, and is therefore constant if $z_1, \ldots, z_m \in \Gamma'$, $w_1, \ldots, w_n \in \Gamma''$, and $\Gamma' \cap \Gamma'' = \emptyset$. But this means $dp/da = 0$ under the same conditions, and therefore dp/da vanishes identically. This proves (a).

From the fact that P depends effectively on $z_1, \ldots, w_n$, we obtain that none of the $m + n$ polynomials

$$\tilde{P}(0, z_2 - z_1, \ldots, w_n - z_1)$$

$$\ldots$$

$$\tilde{P}(z_1 - w_n, \ldots, w_{n-1}, 0)$$

vanishes identically. The union $\mathcal{Z}$ of their zeros has thus a dense complement in $\mathbf{C}^{m+n}$. Let now Γ', Γ'' be disjoint proper circles in $\mathbf{C}$. If $z_1, \ldots, z_m \in \Gamma'$, $w_1, \ldots, w_n \in \Gamma''$, the polynomial

$$a \mapsto \tilde{p}(a) = \tilde{P}(z_1 + a, \ldots, w_n + a)$$

can vanish only if $a \in \{-z_1, \ldots, -w_n\}$. [This follows from ($G_0$) and the fact that $(a+\Gamma')^{-1}$, $(a + \Gamma'')^{-1}$ are disjoint and are proper circles or a proper circle and a straight line]. To summarize, $\tilde{p}(a)$ can vanish only if

$$a \in \{-z_1, \ldots, -w_n\} \qquad \text{and} \qquad (z_1, \ldots, w_n) \in \mathcal{Z}$$

Since a polynomial vanishing on a nonempty open set of $\Gamma'^m \times \Gamma''^n$ must vanish identically on $\mathbf{C}^{m+n}$, we have

$$(\mathbf{C}^{m+n} \backslash \mathcal{Z}) \cap (\Gamma'^m \times \Gamma''^n) \neq \emptyset$$

There is thus a nonempty open set $U \subset (\Gamma'^m \times \Gamma''^n) \backslash \mathcal{Z}$. For $(z_1, \ldots, w_n) \in U$, $\tilde{p}(\cdot)$ never vanishes, and is thus constant, *i.e.*, $d\tilde{p}(a)/da = 0$. Therefore $d\tilde{p}(a)/da = 0$ for all $(z_1, \ldots, w_n) \in \mathbf{C}^{m+n}$. In conclusion, $\tilde{P}$ is translation invariant. This implies immediately that $\tilde{P}$ is a G_0-nomial. $\square$

12 Proposition.

If the MA-nomial $P(z_1, \ldots, z_m, w_1, \ldots, w_n)$ satisfies (G_0), it also satisfies (G).

If the sets $\{z_1, \ldots, z_m\}$ and $\{w_1, \ldots, w_n\}$ are separated by a circle Γ, we can find two disjoint proper circles Γ' and Γ'' close to Γ and separating them. By a transformation $\Phi : z \mapsto (z+a)^{-1}$, we may assume that $\Phi z_1, \ldots, \Phi z_m$ are *inside* of the circle $\Phi\Gamma'$, and $\Phi w_1, \ldots, \Phi w_n$ *inside* of the circle $\Phi\Gamma''$.

We may write

$$\Phi\Gamma' = \{z : |z-u| = r'\} \qquad , \qquad \Phi\Gamma'' = \{w : |w-v| = r''\}$$

The assumption that P is a G_0-nomial and Lemma 11 imply that $\tilde{P}$ (defined by (5)) satisfies $\tilde{P}(z_1, \ldots, w_n) \neq 0$ if

$$z_1, \ldots, z_m \in \{z : |z-u| = \rho'\}, \qquad , \qquad w_1, \ldots, w_n \in \{w : |w-v| = \rho''\}$$

whenever $0 \leq \rho' \leq r'$ and $0 \leq \rho'' \leq r''$. Considered as a function of the $\xi_i = \log(z_i - u)$ and $\eta_j = \log(w_j - v)$, $\tilde{P}$ has no zero, and $1/\tilde{P}$ is thus analytic in a region

$$\{\mathrm{Re}\,\xi_i < c \text{ for } i = 1, \ldots, m \text{ and } \mathrm{Re}\,\eta_j < c \text{ for } j = 1, \ldots, n\}$$

$$\cup\{\mathrm{Re}\,\xi_1 = \ldots = \mathrm{Re}\,\xi_m < \log r' \text{ and } \mathrm{Re}\,\eta_1 = \ldots = \mathrm{Re}\,\eta_n < \log r''\}$$

for suitable (large negative) c. This is a tube and by the Tube Theorem* $1/\tilde{P}$ is analytic in

$$\{\mathrm{Re}\,\xi_i < \log r' \text{ for } i = 1, \ldots, m \text{ and } \mathrm{Re}\,\eta_j < \log r'' \text{ for } j = 1, \ldots, n\}$$

and therefore $\tilde{P}$ does not vanish when $z_1, \ldots, z_m$ are inside of $\Phi\Gamma'$ and $w_1, \ldots, w_n$ inside $\Phi\Gamma''$. Going back to the polynomial P, we see that it cannot vanish when $\{z_1, \ldots, z_m\}$ and $\{w_1, \ldots, w_n\}$ are separated by Γ' and Γ''. $\square$

13 Proposition.

Suppose that $P(z_1, \ldots, z_n, w_1, \ldots, w_n)$ satisfies the conditions of Proposition A2 and that

$$P(z_1, \ldots, z_n, w_1, \ldots, w_n) \neq 0$$

when $|z_1| = \ldots = |z_n| = a$, $|w_1| = \ldots = |w_n| = b$ and $0 < a \neq b$. Then P is a G-nomial.

Taking $z_1, \ldots, z_n = 3/2$, $w_1, \ldots, w_n = 1/2$, we have $0 \neq P(3/2, \ldots, 1/2) = P(1, \ldots, 0)$ $= \alpha$, *i.e.*, the coefficient α of the monomial $z_1 \ldots z_n$ in P is different from 0 . Therefore we have

$$P(z_1, \ldots, z_n, w_1, \ldots, w_n) \neq 0 \tag{6}$$

if $|z_1| = \ldots = |z_n| = a$, $|w_1| = \ldots = |w_n| = b$ and $0 \leq a < b$; (6) also holds if $|w_1| = \ldots = |w_n| = b$ provided $|z_1|, \ldots, |z_n| < e^c$ for suitable (large negative) c. Applying the Tube Theorem as in the proof of Proposition 12 we find thus that (6) holds when

$$|z_1|, \ldots, |z_n| < b \qquad , \qquad |w_1| = \ldots = |w_n| = b$$

* For the standard Tube Theorem see for instance [7] Theorem 2.5.10. Here we need a variant, the Flattened Tube Theorem, for which see Epstein [1]

In particular, $P(z_1,\ldots,w_n) \neq 0$ if $z_1,\ldots,z_n \in \Gamma'$, $w_1,\ldots,w_n \in \Gamma''$ where Γ', Γ'' are proper circles such that Γ' is entirely inside Γ'' and Γ'' is centered at 0. But by conformal invariance (Corollary A3) we can replace these conditions by $\Gamma' \cap \Gamma'' = \emptyset$. Proposition 12 then implies that P is a G-nomial. $\square$

14 Proposition.

Suppose that $P_0(z_1,\ldots,w_n)$ and $P_1(z_1,\ldots,w_n)$ are reduced G-nomials which become equal when $z_1 = z_2$:

$$P_0(z,z,z_3,\ldots,w_n) = P_1(z,z,z_3,\ldots,w_n)$$

Then, for $0 \leq \alpha \leq 1$

$$P_\alpha(z_1,\ldots,w_n) = (1-\alpha)P_0(z_1,\ldots,w_n) + \alpha P_1(z_1,\ldots,w_n)$$

is again a reduced G-nomial.

[Note that instead of the pair (z_1,z_2) one could take any pair (z_i,z_j)].

We have to prove that if the proper circle Γ separates $\{z_1,\ldots,z_n\}$, $\{w_1,\ldots,w_n\}$, then $P_\alpha(z_1,\ldots,w_n) \neq 0$.

Let $p_\alpha(z_1,z_2)$ be obtained from $P_\alpha(z_1,\ldots,w_n)$ by fixing $z_3,\ldots,z_n$ on one side of Γ and $w_1,\ldots,w_n$ on the other side. By assumption

$$p_\alpha(z_1,z_2) = az_1z_2 + b_\alpha z_1 + c_\alpha z_2 + d \tag{7}$$

where $b_\alpha = (1-\alpha)b_0 + \alpha b_1$, $c_\alpha = (1-\alpha)c_0 + \alpha c_1$, and $b_0 + c_0 = b_1 + c_1$. We have to prove:

(A) *If $z_1, z_2 \in \Delta$ where Δ is the region bounded by Γ and not containing $w_1,\ldots,w_n$, then $p_\alpha(z_1,z_2) \neq 0$.*

We remark now that, as functions of $z_3,\ldots,w_n$, the expressions

$$a \qquad , \qquad -\frac{(b_0+c_0)^2}{4a} + d$$

cannot vanish identically. For a this is because the coefficient of $z_1\cdots z_n$ in (7) is 1. Note now that if we decompose a in prime factors, these cannot occur with an exponent > 1 because a is of degree ≤ 1 in each variable $z_3,\ldots,w_n$. Therefore if $-(b_0+c_0)^2/4a + d = 0$, *i.e.*, if a divides $(b_0+c_0)^2$, then a divides (b_0+c_0) and the quotient is homogeneous of degree 1. But then $(b_0+c_0)^2/4a$ contains some variables with an exponent 2, in contradiction with the fact that in d all variables occur with an exponent ≤ 1. In conclusion $-(b_0+c_0)^2/4a+d$ cannot vanish identically.

By a small change of $z_3,\ldots,w_n$ we can thus assume that

$$a \neq 0 \qquad , \qquad -\frac{(b_0+c_0)^2}{4a} + d \neq 0 \tag{8}$$

We shall first consider this case and later use a limit argument to prove (A) when (8) does not hold. By the change of coordinates

$$z_1 = u_1 - \frac{b_0 + c_0}{2a} \quad , \quad z_2 = u_2 - \frac{b_0 + c_0}{2a}$$

(linear in z_1, z_2) we obtain

$$p_\alpha = a u_1 u_2 + \frac{1}{2}(b_\alpha - c_\alpha)(u_1 - u_2) - \frac{(b_0 + c_0)^2}{4a} + d$$

(Note that $b_\alpha + c_\alpha = b_0 + c_0$). Write

$$A = (b_0 + c_0)^2 - 4ad \quad , \quad \beta = \frac{\sqrt{A}}{2a} \quad , \quad \lambda(\alpha) = \frac{b_\alpha - c_\alpha}{\sqrt{A}}$$

for some choice of the square root of A, and

$$u_1 = \beta v_1 \quad , \quad u_2 = \beta v_2$$

then

$$p_\alpha = \frac{A}{4a}(v_1 v_2 + \lambda(\alpha)(v_1 - v_2) - 1)$$

If we write $v_1 = (\zeta_1 + 1)/(\zeta_1 - 1)$, $v_2 = (\zeta_2 + 1)/(\zeta_2 - 1)$, the condition $p_\alpha \neq 0$ becomes

$$\zeta_1(1 - \lambda(\alpha)) + \zeta_2(1 + \lambda(\alpha)) \neq 0 \tag{9}$$

Note that $\lambda(\alpha) = \pm 1$ means $(b_\alpha - c_\alpha)^2 - A = 0$, *i.e.*, $ad - b_\alpha c_\alpha = 0$ and

$$p_\alpha = a(z_1 - S_\alpha)(z_2 - T_\alpha)$$

By assumption $p_0(z, z) \neq 0$ when $z \in \Delta$. Therefore, the image Δ_v of Δ in the v-variable does not contain $+1$, -1, and the image Δ_ζ in the ζ-variable does not contain 0, ∞. In particular Δ_ζ is a circular disc or a half-plane, and thus *convex.*

If $\lambda(\alpha)$ is real and $-1 \leq \lambda(\alpha) \leq 1$, (9) holds when $\zeta_1, \zeta_2 \in \Delta_\zeta$. [This is because Δ_ζ is convex and $\Delta_\zeta \not\ni 0$]. Therefore in that case (A) holds.

We may thus exclude the values of α such that $-1 \leq \lambda(\alpha) \leq 1$, and reduce the proof of the proposition to the case when at most one of $\lambda(0)$, $\lambda(1)$ is in $[-1, 1]$, and the other $\lambda(\alpha) \notin [-1, 1]$. Exchanging possibly P_0, P_1, we may assume that all $\lambda(\alpha) \notin [-1, 1]$ except $\lambda(1)$. Exchanging possibly z_1, z_2, (*i.e.*, replacing λ by $-\lambda$) we may assume that $\lambda(1) \neq 1$. We may finally assume that

$$|\lambda(0) + 1| + |\lambda(0) - 1| \geq |\lambda(1) + 1| + |\lambda(1) - 1| \tag{10}$$

where the left hand side is > 2 while the right hand side is $=2$ if $\lambda(1) \in [-1, 1]$.

For $\alpha \in [0,1]$ we define the map

$$f_\alpha : \zeta \mapsto \frac{\lambda(\alpha)+1}{\lambda(\alpha)-1}\zeta$$

When $\alpha = 0, 1$ the inequality (9) holds by assumption for $\zeta_1, \zeta_2 \in \Delta_\zeta$. [Note that the point $v = \infty$, *i.e.*, $\zeta = 1$ does not make a problem: if $\lambda \neq \pm 1$ this is seen by continuity; if $\lambda = \pm 1$ this follows from $\Delta_\zeta \not\ni 0$]. Therefore

$$\Delta_\zeta \cap f_0 \Delta_\zeta = \emptyset \qquad , \qquad \Delta_\zeta \cap f_1 \Delta_\zeta = \emptyset$$

We want to show that $\Delta_\zeta \cap f_\alpha \Delta_\zeta = \emptyset$ for $0 < \alpha < 1$. In fact, it suffices to prove

$$\Delta'_\zeta \cap f_\alpha \Delta'_\zeta = \emptyset$$

for slightly smaller $\Delta' \subset \Delta_\zeta$, *viz*, the inside of a proper circle Γ' such that 0 is outside of Γ'. Since we may replace Δ' by any $c\Delta'$ where $c \in \mathbf{C}\backslash\{0\}$, we assume that Δ' is the interior of a circle centered at $\lambda(0) - 1$ and with radius $r_-^0 < |\lambda(0) - 1|$. Then $f_0\Delta'$ is the interior of a circle centered at $\lambda(0) + 1$ and with radius r_+^0. The above two circles are disjoint, but we may increase r_-^0 until they touch, obtaining

$$r_-^0 + r_+^0 = 2 \qquad , \qquad r_+^0 = |\frac{\lambda(0)+1}{\lambda(0)-1}| r_-^0$$

i.e.,

$$r_-^0 = \frac{2|\lambda(0)-1|}{|\lambda(0)+1| + |\lambda(0)-1|} \qquad , \qquad r_+^0 = \frac{2|\lambda(0)+1|}{|\lambda(0)+1| + |\lambda(0)-1|}$$

We define r_-^α and r_+^α similarly, with $\lambda(0)$ replaced by $\lambda(\alpha)$ for $\alpha \in [0,1]$. To prove that $\Delta' \cap f_\alpha \Delta' = \emptyset$ for $0 < \alpha < 1$, we may replace Δ' by $\frac{\lambda(\alpha)-1}{\lambda(0)-1}\Delta'$ (which is a disc centered at $\lambda(\alpha) - 1$) and it suffices to prove that the radius $|\frac{\lambda(\alpha)-1}{\lambda(0)-1}| r_-^0$ of this disc is $\leq r_-^\alpha$, *i.e.*,

$$\frac{2|\lambda(\alpha)-1|}{|\lambda(0)+1| + |\lambda(0)-1|} \leq \frac{2|\lambda(\alpha)-1|}{|\lambda(\alpha)+1| + |\lambda(\alpha)-1|}$$

or

$$|\lambda(0)+1| + |\lambda(0)-1| \geq |\lambda(\alpha)+1| + |\lambda(\alpha)-1| \tag{11}$$

Note now that $\{\lambda \in \mathbf{C} : |\lambda + 1| + |\lambda - 1| = \mathit{const.}\}$ is an ellipse with foci ± 1, and since $\lambda(\alpha)$ is affine in α, the maximum value of $|\lambda(\alpha) + 1| + |\lambda(\alpha) - 1|$ for $\alpha \in [0,1]$ is reached at 0 or 1, and in fact at 0 by (10). This proves (11).

This concludes the proof of (A) under the assumption (8). Consider now a limiting case when (8) fails and suppose that (A) does not hold. Then, by Lemma A1, p_α vanishes identically. In particular this would imply $p_\alpha(z,z) = 0$, in contradiction with the assumption that P_0 is a G-nomial.

We have thus shown that P_α is a G-nomial, and since it is homogeneous of degree n in the $2n$ variables $z_1, \ldots, w_n$, and contains $z_1 \cdots z_n$ with coefficient 1, P_α is a reduced G-nomial. $\square$

15 Corollary (contractibility).

The set G_n of reduced G-nomials is contractible.

In the linear space of MA-nomials $P(z_1, \ldots, w_n)$ satisfying the conditions of Proposition A2 we define a flow by

$$\frac{dP}{dt} = -P + \binom{n}{2}^{-1} \sum{}^* \pi P \tag{12}$$

where Σ^* is the sum over the $\binom{n}{2}$ transpositions π, *i.e.* interchanges of two of the variables $z_1, \ldots, z_n$ of P. In view of Proposition 14, the positive semiflow defined by (12) preserves the set G_n of reduced G-nomials. Condition (b)$_n$ of Proposition A2 shows that the only fixed point of (12) is, up to a normalizing factor, Grace's polynomial P_Σ. We have thus a contraction of G_n to $\{P_\Sigma\}$, and G_n is therefore contractible. $\square$

A. Appendix.

A1 Lemma (limits).

Let $D_1, \ldots, D_r$ be open discs, and assume that the MA-nomials $P_k(z_1, \ldots, z_r)$ do not vanish when $z_1 \in D_1, \ldots, z_r \in D_r$. If the P_k have a limit P_∞ when $k \to \infty$, and if $P_\infty(\hat{z}_1, \ldots, \hat{z}_r) = 0$ for some $\hat{z}_1 \in D_1, \ldots, \hat{z}_r \in D_r$, then $P_\infty = 0$ identically.

There is no loss of generality in assuming that $\hat{z}_1 = \ldots = \hat{z}_r = 0$. We prove the lemma by induction on r. For $r = 1$, if the affine function P_∞ vanishes at 0 but not identically, the implicit function theorem shows that P_k vanishes for large k at some point close to 0, contrary to assumption. For $r > 1$, the induction assumption implies that putting any one of the variables $z_1, \ldots, z_r$ equal to 0 in P_∞ gives the zero polynomial. Therefore $P_\infty(z_1, \ldots, z_r) = \alpha z_1 \cdots z_r$. Fix now $z_j = a_i \in D_i \backslash \{0\}$ for $j = 1, \ldots, r-1$. Then $P_k(a_1, \ldots, a_{r-1}, z_r) \neq 0$ for $z_r \in D_r$, but the limit $P_\infty(a_1, \ldots, a_{r-1}, z_r) = \alpha a_1 \cdots a_{r-1} z_r$ vanishes at $z_r = 0$ and therefore identically, *i.e.*, $\alpha = 0$, which proves the lemma. $\square$

A2 Proposition (reduced forms).

For $n \geq 1$, the following conditions on a MA-nomial $P(z_1, \ldots, z_n, w_1, \ldots, w_n)$ not identically zero are equivalent:

(a)$_n$ P satisfies

$$P(z_1 + \xi, \ldots, w_n + \xi) = P(z_1, \ldots, w_n) \qquad \text{(translation invariance)}$$

$$P(\lambda z_1, \ldots, \lambda w_n) = \lambda^n P(z_1, \ldots, w_n) \qquad \text{(homogeneity of degree } n)$$

(b)$_n$ There are constants C_π such that

$$P(z_1, \ldots, w_n) = \sum_\pi C_\pi \prod_{j=1}^n (z_j - w_{\pi(j)})$$

where the sum is over all permutations π *of* $(1,\ldots,n)$.

We say that $(\mathrm{b})_n$ gives a *reduced form* of P (it need not be unique).

Clearly $(\mathrm{b})_n \Rightarrow(\mathrm{a})_n$. We shall prove $(\mathrm{a})_n \Rightarrow(\mathrm{b})_n$ by induction on n, and obtain at the same time a bound $\sum|C_\pi| \leq k_n.||P||$ for some norm $||P||$ (the space of P's is finite dimensional, so all norms are equivalent). Clearly, $(\mathrm{a})_1$ implies that $P(z_1,w_1)=C(z_1-w_1)$, so that $(\mathrm{b})_1$ holds. Let us now assume that P satisfies $(\mathrm{a})_n$ for some $n>1$.

If X is an n-element subset of $\{z_1,\ldots,w_n\}$, let $A(X)$ denote the coefficient of the corresponding monomial in P. We have

$$\sum_X A(X) = P(1,\ldots,1) = P(0,\ldots,0) = 0$$

In particular

$$\max_{X',X''} |A(X') - A(X'')| \geq \max_X A(X)$$

Note also that one can go from X' to X'' in a bounded number of steps exchanging a z_j and a w_k. Therefore one can choose z_j, w_k, Z containing z_j and not w_k, and W obtained by replacing z_j by w_k in Z so that

$$|A(Z) - A(W)| \geq \alpha(\sum_X |A(X)|^2)^{1/2}$$

where α depends only on n.

Write now

$$P = az_jw_k + bz_j + cw_k + d$$

where the polynomials a, b, c, d do not contain z_j, w_k. We have thus

$$P = P_1 + \frac{1}{2}(b-c)(z_j - w_k)$$

where

$$P_1 = az_jw_k + \frac{1}{2}(b+c)(z_j+w_k) + d$$

Let $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$ be obtained by adding ξ to all the arguments of a, b, c, d. By translation invariance we have thus

$$az_jw_k + bz_j + cw_k + d = \tilde{a}(z_j+\xi)(w_k+\xi) + \tilde{b}(z_j+\xi) + \tilde{c}(w_k+\xi) + \tilde{d}$$

$$= \tilde{a}z_jw_k + (\tilde{a}\xi+\tilde{b})z_j + (\tilde{a}\xi+\tilde{c})w_k + \tilde{a}\xi^2 + (\tilde{b}+\tilde{c})\xi + \tilde{d}$$

hence $\tilde{b}-\tilde{c} = b-c$. Therefore $b-c$ satisfies $(\mathrm{a})_{n-1}$ and, using the induction assumption we see that

$$\frac{1}{2}(b-c)(z_j-w_k)$$

has the form given by $(\mathrm{b})_n$. In particular P_1 again satisfies $(\mathrm{a})_n$.

We compare now the coefficients $A_1(X)$ for P_1 and $A(X)$ for P:

$$\sum_X |A(X)|^2 - \sum_X |A_1(X)|^2 \geq |A(Z)|^2 + |A(W)|^2 - \frac{1}{2}|A(Z) + A(W)|^2$$
$$= \frac{1}{2}|A(Z) - A(W)|^2 \geq \frac{\alpha^2}{2}\sum_X |A(X)|^2$$

so that

$$\sum |A_1(X)|^2 \leq (1 - \frac{\alpha^2}{2})\sum_X |A(X)|^2$$

We have thus a geometrically convergent approximation of P by expressions satisfying $(\mathrm{b})_n$, and an estimate of $\sum |C_\pi|$ as desired. □

A3 Corollary

If the MA-nomial $P(z_1, \ldots, w_n)$ satisfies the conditions of Proposition A2, the following properties hold:

(conformal invariance) *if $ad - bc \neq 0$, then*

$$P(\frac{az_1 + b}{cz_1 + d}, \ldots, \frac{aw_n + b}{cw_n + d}) = P(z_1, \ldots, w_n) \prod_{j=1}^{n} \frac{ad - bc}{(cz_j + d)(cw_j + d)}$$

(roots) *the polynomial*

$$\hat{P}(z) = P(z, \ldots, z, w_1, \ldots, w_n)$$

has exactly the roots $w_1, \ldots, w_n$ (repeated according to multiplicity).

These properties follow directly if one writes P in reduced form. □

References.

[1] H. Epstein. "Some analytic properties of scattering amplitudes in quantum field theory." in M. Chretien and S. Deser eds. *Axiomatic Field Theory.* Gordon and Breach, New York, 1966.

[2] O.J. Heilmann and E.H. Lieb. "Theory of monomer-dimer systems." Commun. Math. Phys. **25**,190-232(1972); **27**,166(1972).

[3] T. D. Lee and C. N. Yang. "Statistical theory of equations of state and phase relations. II. Lattice gas and Ising model." Phys. Rev. **87**,410-419(1952).

[4] G. Polya and G. Szegö. *Problems and theorems in analysis II.* Springer, Berlin, 1976.

[5] D. Ruelle. "Extension of the Lee-Yang circle theorem." Phys. Rev. Letters **26**,303-304(1971).

[6] D. Ruelle. "Counting unbranched subgraphs." J. Algebraic Combinatorics **9**,157-160(1999); "Zeros of graph-counting polynomials." Commun. Math. Phys. **200**,43-56(1999).

[7] L. Hörmander. *An introduction to complex analysis in several variables.* D. Van Nostrand, Princeton, 1966.