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Repellers for real analytic maps

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Abstract. The purpose of this note is to prove a conjecture of D. Sullivan† that when the Julia set J of a rational function f is hyperbolic, the Hausdorff dimension of J depends real analytically on f. We shall obtain this as corollary of a general result on repellers of real analytic maps (see corollary 5).

Let M be a real analytic manifold of finite dimension N, J a compact subset of M, and V an open neighbourhood of J in M. We say that J is a (mixing) *repeller* for the real analytic map $f: V \to M$ if the following conditions are satisfied

(*a*) *there exist* $C > 0$, $\alpha > 1$ *such that*

$$\|(T_x f^n)u\| \geq C\alpha^n \|u\| \tag{1}$$

for all $x \in J$, $u \in T_x M$, $n \geq 1$ (*and some Riemann metric on TM*),

(*b*) $J = \{x \in V : f^n x \in V \text{ for all } n > 0\}$,

(*c*) *f is topologically mixing on J, i.e. for every non-empty open set O intersecting J there is an $n > 0$ such that $f^n O \supset J$.*

From (*b*) and (*c*) it follows that $fJ = J$. Our results would extend easily to the case where J is topologically + transitive instead of topologically mixing (see [**12**]).

1. Proposition. *Let J be a mixing repeller for the real analytic map $f: V \mapsto M$, and let $\phi: V \mapsto \mathbb{R}$ be a real analytic function. Then the series*

$$\zeta(u) = \exp \sum_{n=1}^{\infty} \frac{u^n}{n} \sum_{x \in \text{Fix} f^n} \exp \sum_{k=0}^{n-1} \phi(f^k x)$$

has non-vanishing convergence radius and extends to a meromorphic function of u, again noted $\zeta(u)$. This function has a simple pole at $\exp P(\phi) > 0$, and every other zero or pole of ζ has modulus $> \exp P(\phi)$. The function $\phi \mapsto P(\phi)$ is convex. There is a unique Radon measure ρ on J such that

$$P(\phi + \psi) - P(\phi) \geq \rho(\psi) \tag{2}$$

for all ψ, and ρ is an f-invariant probability measure (*Gibbs measure*).

To see this, one observes that expanding maps have Markov partitions.‡ Markov partitions permit a study of the periodic points of f. Assuming only that ϕ is Hölder

† Formulated at the conference on dynamical systems in Rio de Janeiro, 1981, see [**15**].

‡ Markov partitions have been introduced by Sinai [**13**] for Anosov diffeomorphisms. Their existence for expanding maps is implicit in Bowen [**1**]. For an explicit discussion see Ruelle [**12**]. One may choose an 'adapted' metric on M such that $C = 1$ in (1). Characterizations of expanding maps as needed for the existence of Markov partitions are analysed in [**5**].

continuous one shows, by methods of statistical mechanics, that ζ extends to a circle of radius $>\exp P$ in which it has no zero and only a simple pole at $\exp P$.† One obtains then ρ satisfying (2) for all Hölder continuous functions $\phi, \psi: J \to \mathbb{R}$.

The real analyticity of f and ϕ is needed to prove the meromorphy of ζ in $\mathbb{C}$. Using the Markov partition and complex extensions of f and ϕ, one expresses ζ in the terms of Fredholm determinants in the form

$$\zeta(u) = \prod_{k=0}^{N} [\det(1 - u\mathscr{L}_k)]^{(-1)^{k+1}}$$

where the $\mathscr{L}_k$ have continuous kernels on compact sets, depending analytically on f and ϕ (see Ruelle [**11**, theorem 1], the application considered here is much the same as that of theorem 2 of [**11**]; the Fredholm theory used is based on Grothendieck [**6**]). In particular, if f and ϕ depend analytically on parameters, then ζ will depend analytically on the same parameters.‡ We now formulate this result more precisely.

2. PROPOSITION. *With the notation of proposition* 1, *let f and ϕ (now noted f_λ, ϕ_λ) depend on a parameter $\lambda \in U \subset \mathbb{R}^m$ such that $(\lambda, x) \mapsto f_\lambda x$, $\phi(x)$ are analytic, and f_λ has a repeller J_λ depending continuously on λ. We may take U open by Ω stability. Under these conditions $\zeta = d_1/d_2$ where d_1, d_2 are entire holomorphic in u and real analytic in $\lambda \in U$.*

3. COROLLARY. *The function $\lambda \mapsto P$ is real analytic and $\lambda \to \rho$ is real analytic in the sense that $\lambda \mapsto \rho(\psi)$ is analytic for real analytic $\psi: V \mapsto \mathbb{R}$. If $\phi_\lambda < 0$ on J_λ the function $\lambda \mapsto t$ is analytic, where t is defined by $P(t\phi_\lambda) = 0$.*

The analyticity of $\lambda \mapsto e^P$ (and thus $\lambda \mapsto P$) results from the implicit function theorem applied to the function $(\lambda, u) \mapsto 1/\zeta$. We consider now two applications of the analyticity of $\lambda \mapsto P$, where λ is replaced by (t, λ), $t \in \mathbb{R}$.

If $\psi: V \to \mathbb{R}$ is real analytic, we see that $(t, \lambda) \mapsto P(\phi_\lambda + t\psi)$ is real analytic, and therefore also

$$\lambda \mapsto \frac{d}{dt} P(\phi_\lambda + t\psi)|_{t=0} = \rho(\psi).$$

This proves the real analyticity of $\lambda \mapsto \rho$ as announced.

Similarly $(t, \lambda) \mapsto P(t\phi_\lambda)$ is real analytic. We also have the variational principle††

$$P(t\phi_\lambda) = \max\{h(\sigma) + t\sigma(\phi_\lambda) : \sigma \text{ invariant probability measure}\}$$

where h is the measure-theoretic entropy. Therefore if $\phi_\lambda < 0$ on J_λ, the function $t \mapsto P(t\phi_\lambda)$ has derivative < 0 and goes from positive to negative values.‡‡ Its unique zero is a real analytic function of λ by the implicit function theorem.

† See Ruelle [**10**] or [**12**], Mayer [**8**]. For related ζ-functions see Chen & Manning [**4**].

‡ One could also deduce this from the fact that the periodic points of f depend analytically on the parameters, and that one has control over their positions when the parameters become complex (see lemma 1 in [**11**]). Therefore the coefficients of ζ depend holomorphically on the parameters, and the same is true of ζ.

†† In its general form, this is due to Walters [**16**], see also Misiurewicz [**9**], Bowen [**1**], Ruelle [**12**]

‡‡ The existence of the Markov partition gives an explicit upper bound on h.

4. PROPOSITION. *Let J be a repeller for a map $f: V \mapsto M$. We assume that f is conformal with respect to some continuous Riemann metric, and of class $C^{1+\varepsilon}$ ($\varepsilon > 0$). If we write*

$$\phi(x) = -\log \|Tf(x)\|$$

the Hausdorff dimension t of J is defined by Bowen's formula $P(t\phi) = 0$. Furthermore the t-Hausdorff measure ν on J is equivalent to the Gibbs measure ρ corresponding to $t\phi$.

In the formulation of this proposition we have allowed f to be $C^{1+\varepsilon}$ rather than real analytic as in our earlier definitions. Apart from this, the proposition is due to Bowen [**2**] (who worked with groups of fractional linear transformations of the Riemann sphere). For the convenience of the reader, appendix 1 reproduces a proof of proposition 4. See Sullivan [**15**] for an analogous determination of t. Actually the results of Bowen and Sullivan allow the map f to be discontinuous, as we shall indicate below.

5. COROLLARY. *Let J_λ be a repeller for a real analytic conformal map f_λ, depending real analytically on λ. (Thus $(\lambda, x) \mapsto f_\lambda x$ is real analytic $U \times V \mapsto M$ and the linear maps Df_λ are of the form : scalar $\times$ isometry.) Then the Hausdorff dimension of J_λ is a real analytic function of λ.*

This follows from proposition 4 and corollary 3.

6. COROLLARY. *If the Julia set J of a rational function f is hyperbolic, the Hausdorff dimension of J depends real analytically on f.*

We let $f = P/Q$ where P, Q are polynomials of fixed degrees, so that f can be parametrized by a family of coefficients varying over $\mathbb{R}^m$. Hyperbolicity means that condition (a) in the definition of a repeller is satisfied. Conditions (b) and (c) in the definition of a repeller are satisfied for general Julia sets (see Brolin [**3**, theorems 4.2 and 4.3]). It follows therefore that the Hausdorff dimension of J depends analytically on f.

The polynomial map $z \mapsto z^q$, with $q \geq 2$, has the unit circle

$$\{z \in \mathbb{C} : |z| = 1\}$$

as hyperbolic Julia set. Corollary 6 applies therefore to the maps

$$z \mapsto z^q + \lambda$$

for small complex λ. A formal calculation (see appendix 2) gives

$$t = 1 + \frac{|\lambda|^2}{4 \log q} + \text{higher order terms in } \lambda.$$

The case $q = 2$ has been particularly studied (see Brolin [**3**] and references quoted there, and Mandelbrot [**7**] which also contains beautiful pictures of the corresponding J_λ). A computer calculation of t as a function of λ (real) for $z \mapsto z^2 + \lambda$ was performed by L. Garnett (unpublished) and prompted Sullivan's conjecture that

$\lambda \mapsto t$ is analytic.† Sullivan [**15**] proved that $t > 1$ when $\lambda \neq 0$ (and $|\lambda|$ is sufficiently small).

7. *Generalization*

As mentioned above, Bowen originally established the formula $P(t\phi) = 0$ for the Hausdorff dimension of a repeller J in the context of groups of fractional linear transformations of the Riemann sphere. (The Hausdorff dimension results were extended by Sullivan to more general groups of conformal maps [**14**].) In Bowen's study, J is the quasi-circle associated with a quasi-Fuchsian group G, and there is a Markov partition $\{S_\alpha\}$ of J such that f is a different fractional linear transformation on each S_α, and thus discontinuous. Arguments similar to those given above show in this case that the Hausdorff dimension of the quasi-circle depends real analytically on G or, equivalently, on pairs of points in Teichmüller space.

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Appendix 1: *Proof of proposition* 4

The pressure (function P) and Gibbs state ρ occurring in proposition 4 translate to similar concepts for the symbolic dynamical system associated with a Markov partition of J. A Markov partition $\{S_\alpha\}$ is a finite collection of closed non-empty subsets of J such that $\bigcup S_\alpha = J$ and $\operatorname{int} S_\alpha$ is dense in S_α (int denotes the interior in J). Furthermore,

(i) $\operatorname{int} S_\alpha \cap \operatorname{int} S_\beta = \emptyset$ if $\alpha \neq \beta$,

(ii) each fS_α is a union of sets S_β.

For a study of symbolic dynamics, the reader must be referred to Bowen [**2**] or Ruelle [**12**].

Let $\{S_\alpha\}$ be a Markov partition of J into small subsets. We call K the maximum number of S_β which intersect any S_α:

$$K = \max_\alpha \operatorname{card} \{S_\beta : S_\alpha \cap S_\beta \neq \emptyset\}.$$

Let $\tilde{S}_\alpha$ be a small open neighbourhood of S_α in V, for each α, such that

$$\tilde{S}_\alpha \cap \tilde{S}_\beta = \emptyset \quad \text{whenever} \quad S_\alpha \cap S_\beta = \emptyset.$$

We assume that for all α the diameter of $\tilde{S}_\alpha$ is $<\Delta$, and that $\tilde{S}_\alpha$ contains the δ-neighbourhood of S_α $(0 < \delta < \Delta)$. If $\xi_0, \xi_1, \ldots, \xi_n$ is an admissible sequence of elements of the Markov partition, i.e. $f\xi_{j-1} \supset \xi_j$ for $j = 1, \ldots, n$, we define

$$E(\xi_0, \ldots, \xi_n) = \bigcap_{j=0}^{n} f^{-j}\xi_j,$$

$$\tilde{E}(\xi_0, \ldots, \xi_n) = \bigcap_{j=0}^{n} f^{-j}\tilde{\xi}_j.$$

† The results of the calculation suggest $t = 1 + C|\lambda|^2$ and are compatible with $t = 1 + |\lambda|^2/(4 \log 2)$.

The sets $\tilde{E}(\xi_0, \dots, \xi_n)$ which intersect a given $\tilde{E}(\xi_0^*, \dots, \xi_n^*)$ are determined successively as follows:

(a) choose ξ_n such that $\xi_n \cap \xi_n^* = \varnothing$,

(b) ξ_j is uniquely determined for $k = n-1, \dots, 1, 0$ by

$$\left[\bigcap_{j=k}^{n} f^{-(j-k)}\xi_j\right] \cap \left[\bigcap_{j=k}^{n} f^{-(j-k)}\xi_j^*\right] \neq \varnothing.$$

In particular the sets $\tilde{E}(\xi_0, \dots, \xi_n)$ which intersect $\tilde{E}(\xi_0^*, \dots, \xi_n^*)$ correspond precisely to the sets $E(\xi_0, \dots, \xi_n)$ which intersect $E(\xi_0^*, \dots, \xi_n^*)$, and there are at most K of those. We also see that, if Δ has been taken sufficiently small, there are $\beta \in (0, 1)$ and $G > 0$ (β and G independent of $n, \xi_0^*, \dots, \xi_n^*$) such that

$$\operatorname{dist}(\xi, \xi^*) \leq G\beta^n \quad \text{if } \tilde{\xi} \in \tilde{E}(\xi_0, \dots, \xi_n) \text{ and } \xi^* \in \tilde{E}(\xi_0^*, \dots, \xi_n^*) \tag{A.1}$$

(use part (a) of the definition of a repeller). In particular,

$$\operatorname{diam} \tilde{E}(\xi_0^*, \dots, \xi_n^*) \leq G\beta^n.$$

Let

$$F_{\xi_0, \dots, \xi_n} : \tilde{\xi}_n \mapsto \tilde{E}(\xi_0, \dots, \xi_n)$$

be the inverse of the restriction of f^n to $\tilde{E}(\xi_0, \dots, \xi_n)$. If $x \in \tilde{\xi}_n$ we have, since f is conformal,

$$\begin{aligned}\log \|F'_{\xi_0, \dots, \xi_n}(x)\| &= \sum_{k=0}^{n-1} \log \|(f^{-1})'(F_{\xi_{k+1}, \dots, \xi_n}(x))\| = -\sum_{k=0}^{n-1} \log \|f'(F_{\xi_k, \dots, \xi_n} x)\| \\ &= \sum_{k=0}^{n-1} \phi(F_{\xi_k, \dots, \xi_n} x)\end{aligned} \tag{A.2}$$

where we have denoted the tangent map by a dash. If

$$\tilde{E}(\xi_0, \dots, \xi_n) \cap \tilde{E}(\xi_0^*, \dots, \xi_n^*) \neq \varnothing \quad \text{and} \quad x \in \tilde{\xi}_n, x^* \in \tilde{\xi}_n^*$$

we have thus, using (A.1),

$$|\log \|F'_{\xi_0, \dots, \xi_n}(x)\| - \log \|F'_{\xi_0^*, \dots, \xi_n^*}(x^*)\|| \leq C_\varepsilon \sum_{k=0}^{n-1} (G\beta^{n-k})^\varepsilon < \frac{C_\varepsilon G^\varepsilon}{1-\beta^\varepsilon} = D \tag{A.3}$$

where C_ε is the ε-Hölder norm of ϕ. In particular, *if* $x^* \in \xi_n$, *the ball of radius*

$$e^{-D}\delta \|F'_{\xi_0^*, \dots, \xi_n^*}(x^*)\|$$

centred at

$$F_{\xi_0^*, \dots, \xi_n^*} x^*$$

is entirely contained in

$$\tilde{E}(\xi_0^*, \dots, \xi_n^*).\dagger$$

† We assume here for simplicity that $\phi < 0$.

The Gibbs measure ρ corresponding to $t\phi$ is determined (since $P(t\phi)=0$) by the fact that there is a constant γ such that†

$$|\log \rho(E(\xi_0,\ldots,\xi_n)) - \sum_{k=0}^{n-1} t\phi(F_{\xi_k,\ldots,\xi_n}x)| < \gamma \qquad \text{(A.4)}$$

where γ is independent of n, $E(\xi_0,\ldots,\xi_n)$, and $x \in \xi_n$. Using (A.2) and (A.4) we have, for each $E(\xi_0,\ldots,\xi_n)$, the following estimate of the t-Hausdorff measure ν:

$$\begin{aligned}\nu(E(\xi_0,\ldots,\xi_n)) &\le \lim_{p\to\infty} \sum_{\xi_{n+1}\cdots\xi_{n+p}} (\operatorname{diam} \tilde{E}(\xi_0,\ldots,\xi_{n+p}))^t \\ &\le \lim_{p\to\infty} \sum_{\xi_{n+1}\cdots\xi_{n+p}} (2\Delta e^D \|F'_{\xi_0,\ldots,\xi_{n+p}}(f^p x)\|)^t \\ &\le (2\Delta e^D)^t \lim_{p\to\infty} \sum_{\xi_{n+1}\cdots\xi_{n+p}} \exp \sum_{k=0}^{n+p-1} t\phi(F_{\xi_k,\ldots,\xi_{n+p}} f^p x) \\ &\le (2\Delta e^D)^t e^\gamma \rho(E(\xi_0,\ldots,\xi_n)).\end{aligned}$$

This shows that ν is absolutely continuous with respect to ρ.

On the other hand $\nu(E(\xi_0,\ldots,\xi_n))$ is the infimum of

$$\sum_{j=1}^{\infty} (\operatorname{diam} U_j)^t$$

for an open cover $\{U_j\}$ of $E(\xi_0,\ldots,\xi_n)$ when diam $U_j \to 0$. For each j take

$$y_j \in E(\xi_0,\ldots,\xi_n) \cap U_j,$$

and notice that $E(\xi_0,\ldots,\xi_n)$ is covered by the balls

$$B_{y_j}(\operatorname{diam} U_j).$$

For each j let n_j be the smallest integer such that if

$$y_j \in E(\xi_0^*,\ldots,\xi_{n_j+1}^*)$$

then

$$e^{-D}\delta \|F'_{\xi_0^*,\ldots,\xi_{n_j+1}^*}(f^{n_j+1}y_j)\| \le \operatorname{diam} U_j. \qquad \text{(A.5)}$$

(We may assume that diam U_j is small, and therefore

$$n_j > n, \qquad \xi_0^* = \xi_0, \ldots, \xi_n^* = \xi_n,$$

the further ξ_k depend on j.) By assumption

$$e^{-D}\delta \|F'_{\xi_0^*,\ldots,\xi_{n_j}^*}(f^{n_j}y_j)\| > \operatorname{diam} U_j.$$

Therefore, the set $E(\xi_0,\ldots,\xi_n)$ is covered by the $\tilde{E}(\xi_0^*,\ldots,\xi_{n_j}^*)$ and, using (A.5) and (A.2) we see that

$$\begin{aligned}\sum_{j=1}^{\infty} (\operatorname{diam} U_j)^t &\ge e^{-Dt}\delta^t \sum_{j=1}^{\infty} \exp t \sum_{k=0}^{n_j} \phi(F_{\xi_k^*,\ldots,\xi_{n_j+1}^*} f^{n_j+1} y_j) \\ &\ge e^{-Dt-Et}\delta^t \sum_{j=1}^{\infty} \exp t \sum_{k=0}^{n_j-1} \phi(F_{\xi_k^*,\ldots,\xi_{n_j}^*} f^{n_j} y_j)\end{aligned}$$

† See Bowen [2] or Ruelle [6].

where E is an upper bound to $|\phi(x)|$. We recall that each $\tilde{E}(\xi_0^*, \ldots, \xi_{n_i}^*)$ intersects at most K sets $E(\xi_0, \ldots, \xi_{n_i})$. Redistributing the contribution of the index j among those, and using (A.2) and (A.3) we find

$$\sum_{j=1}^{\infty} (\text{diam } U_j)^t \geq K^{-1} e^{-2Dt-Et}\delta^t \sum_\lambda \exp t \sum_{k=0}^{n_\lambda - 1} \phi(F_{\xi_k^\lambda, \ldots, \xi_{n_\lambda}^\lambda} x_\lambda)$$

where the $E(\xi_0^\lambda, \ldots, \xi_{n_\lambda}^\lambda)$ cover $E(\xi_0, \ldots, \xi_n)$. So, finally, using (A.4), we obtain

$$\nu(E(\xi_0, \ldots, \xi_n)) \geq K^{-1} e^{-2Dt-Et}\delta^t e^{-\gamma} \rho(E(\xi_0, \ldots, \xi_n)).$$

This shows that ρ is absolutely continuous with respect to ν, completing the proof of the proposition. □

Appendix 2: Hausdorff dimension of the Julia set J of the map $f: z \mapsto z^q - p$.
We shall formally show that the Hausdorff dimension of J is

$$t = 1 + \frac{|p|^2}{4 \log q} + \text{terms of order} > 2 \text{ in } p.$$

For small $|p|$, f has a fixed point α close to 1, so that

$$\alpha + p = \alpha^q \quad \text{and} \quad \alpha = 1 + \frac{p}{q-1} + \cdots.$$

Write $\gamma = \exp 2i\pi/q$. With $\varepsilon_i = 0, 1, \ldots, q-1$ we define

$$\begin{aligned}\zeta(\varepsilon_1, \ldots, \varepsilon_n) &= \gamma^{\varepsilon_n}(p + \gamma^{\varepsilon_{n-1}}(p + \cdots (p + \gamma^{\varepsilon_1}\alpha)^{1/q} \cdots)^{1/q})^{1/q} \\ &= \exp[Q(\varepsilon_1, \ldots, \varepsilon_n) 2i\pi + r(\varepsilon_1, \ldots, \varepsilon_{n-1})]\end{aligned}$$

where

$$Q(\varepsilon_1, \ldots, \varepsilon_n) = \frac{\varepsilon_n}{q} + \frac{\varepsilon_{n-1}}{q^2} + \cdots + \frac{\varepsilon_1}{q^n},$$

$$\begin{aligned}r(\varepsilon_1, \ldots, \varepsilon_n) &= \frac{1}{q} r(\varepsilon_1, \ldots, \varepsilon_{n-1}) + \frac{1}{q} \log(1 + p/\zeta(\varepsilon_1, \ldots, \varepsilon_n)) \\ &\approx \frac{1}{q} r(\varepsilon_1, \ldots, \varepsilon_{n-1}) + \frac{1}{q} p/\zeta(\varepsilon_1, \ldots, \varepsilon_n) \\ &\approx \frac{1}{q} r(\varepsilon_1, \ldots, \varepsilon_{n-1}) + \frac{1}{q} p \exp(-Q(\varepsilon_1, \ldots, \varepsilon_n) \cdot 2i\pi)\end{aligned}$$

to first order in p. Therefore, if $u = \exp(-Q(\varepsilon_1, \ldots, \varepsilon_n) \cdot 2i\pi)$,

$$\begin{aligned}r(\varepsilon_1, \ldots, \varepsilon_n) &\approx p\left[\frac{1}{q} u + \frac{1}{q^2} u^q + \frac{1}{q^3} u^{q^2} + \cdots + \frac{1}{q^n} u^{q^{n-1}} + \frac{1}{q^n} \cdot \frac{1}{q-1}\right] \\ &= \frac{p}{q} \sum_{k=0}^{\infty} \frac{1}{q^k} u^{q^k}.\end{aligned}$$

Writing

$$\phi(z) = -\log|f'(z)| = -\log q|z|^{q-1}$$

we have

$$\phi(\zeta(\varepsilon_1,\ldots,\varepsilon_n)) = -\log q - \mathrm{Re}\,(q-1)r(\varepsilon_1,\ldots,\varepsilon_{n-1}),$$

hence

$$\sum_{k=1}^{n}\phi(\zeta(\varepsilon_1,\ldots,\varepsilon_k)) = -n\log q - \mathrm{Re}\,(q-1)\sum_{k=1}^{n} r(\varepsilon_1,\ldots,\varepsilon_{k-1}).$$

We have, to first order in p,

$$\mathrm{Re}\,(q-1)\sum_{k=1}^{n} r(\varepsilon_1,\ldots,\varepsilon_{k-1}) \approx \mathrm{Re}\, p\Phi_n(u)$$

where

$$\Phi_n(u) = \left(1-\frac{1}{q}\right)u + \left(1-\frac{1}{q^2}\right)u^q + \cdots + \left(1-\frac{1}{q^n}\right)u^{q^{n-1}} + \frac{q-q^{-n}}{q-1}.$$

To second order in p we have, using the induction formula,

$$\sum_{\varepsilon_1,\ldots,\varepsilon_n} r(\varepsilon_1,\ldots,\varepsilon_n) \approx \frac{1}{q}\sum_{\varepsilon_1,\ldots,\varepsilon_n}[r(\varepsilon_1,\ldots,\varepsilon_{n-1})(1-pu)+pu-\tfrac{1}{2}p^2u^2]$$

$$\approx \sum_{\varepsilon_1,\ldots,\varepsilon_{n-1}} r(\varepsilon_1,\ldots,\varepsilon_{n-1})$$

so that, for large n,

$$\sum_{\varepsilon_1,\ldots,\varepsilon_n}\sum_{k=1}^{n} r(\varepsilon_1,\ldots,\varepsilon_{k-1}) \approx O(q^n).$$

The Hausdorff dimension $t = 1+\beta$ of the Julia set J of $z \mapsto z^q - p$ is determined by

$$\sum_{\varepsilon_1,\ldots,\varepsilon_n} \exp\,(1+\beta)\sum_{k=1}^{n}\phi(\zeta(\varepsilon_1,\ldots,\varepsilon_k)) = O(1)$$

for large n or, to second order in p,

$$O(1) \approx \sum_{\varepsilon_1,\ldots,\varepsilon_n} q^{-n(1+\beta)}\exp\left[-\mathrm{Re}\,(q-1)\sum_{k=1}^{n} r(\varepsilon_1,\ldots,\varepsilon_{k-1})\right]$$

$$\approx \sum_{\varepsilon_1,\ldots,\varepsilon_n} q^{-n(1+\beta)}\left[1-\mathrm{Re}\,(q-1)\sum_{k=1}^{n} r(\varepsilon_1,\ldots,\varepsilon_{k-1}) + \tfrac{1}{2}(\mathrm{Re}\, p\Phi_n(u))^2\right]$$

$$\approx q^{-n\beta} + O(q^{-\beta})|p| + O(q^{-n\beta})|p|^2$$

$$+q^{-n(1+\beta)}\frac{1}{2}\left[\frac{q^n}{2}|p|^2\left(\left(1-\frac{1}{q}\right)^2+\left(1-\frac{1}{q^2}\right)^2+\cdots+\left(1-\frac{1}{q^{n-1}}\right)^2\right)+|p|^2o(n)\right].$$

We have used

$$\sum_{\varepsilon_1,\ldots,\varepsilon_n}(\mathrm{Re}\, pu^{q^{r-1}})(\mathrm{Re}\, pu^{q^{s-1}}) = 0 \quad \text{if } 1 \le r < s \le n+1,$$

$$\sum_{\varepsilon_1,\ldots,\varepsilon_n}(\mathrm{Re}\, pu^{q^{r-1}})^2 = \sum_{\varepsilon_1,\ldots,\varepsilon_n}\frac{1}{2}(|p|^2+\mathrm{Re}\, p^2u^{2q^{r-1}})\begin{cases} =\tfrac{1}{2}q^n|p|^2 & \text{if } r<n,\\ \le q^n|p|^2 & \text{if } r=n \text{ or } n+1.\end{cases}$$

Thus, omitting negligible terms

$$O(1) \approx q^{-n\beta}\left(1+\frac{|p|^2}{4}n\right) \approx \exp n\left(\frac{|p|^2}{4} - \beta \log q\right)$$

giving

$$\beta = \frac{|p|^2}{4 \log q} + \cdots, \quad \text{or} \quad t = 1 + \frac{|p|^2}{4 \log q} + \cdots.$$

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