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Deterministic chaos: the science and the fiction

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Under the name of chaos the mathematical ideas of differentiable dynamics have had a profound impact on physics. Some success stories are discussed. Attention is also directed to the excessive optimism of some currently attempted applications. It is shown how and why they should fail.

1. INTRODUCTION

The study of deterministic time evolutions has been of central interest to science since the days of Isaac Newton. Some of these time evolutions are notoriously complicated, as shown by the example of hydrodynamic turbulence. In recent years it has become clear that complicated deterministic evolutions are amenable to analysis, and to a certain extent, prediction. A field of studies has thus developed which we now call *chaos*. The success of chaos has made it a fashionable topic, and therefore subject to the imperatives of contemporary science, calling for spectacular results right away. Several recent examples (outside chaos) show that spectacular does not necessarily mean correct, and that what is fashionable one day often becomes disreputable the next day.

This puts me in a somewhat uncomfortable position. On one hand I believe that the ideas of chaos are among the more original ones in modern science, and will have many useful applications. On the other hand, I think that some of the notions that have had greatest success in the media are either very questionable or demonstrably wrong. I shall have to come back to this question later, but for the moment I shall concentrate on the more positive task of saying what chaos is about.

There is a huge literature of rather uneven quality on chaos. An introduction to the subject is afforded by the reprint collections of Cvitanović (1984) and Hao Bai-Lin (1984). See also Bergé *et al.* (1984), Eckmann & Ruelle (1985) for reviews, and the Royal Society Discussion Meeting edited by Berry *et al.* (1987).

2. SENSITIVE DEPENDENCE ON INITIAL CONDITIONS

We shall concern ourselves with deterministic time evolutions of the form

$$\mathrm{d}x(t)/\mathrm{d}t = X(x(t)) \quad \text{(continuous time } t\text{)},$$

or

$$x(t+1) = f(x(t)) \quad \text{(discrete time } t\text{)}.$$

Here $x(t)$ is a vector in finite or infinite dimension. We assume that the evolution is autonomous (i.e. X or f has no explicit t dependence). We also assume that X and

f depend differentiably on their argument. One then says that one has differentiable dynamics, or a (differentiable) dynamical system.

If an infinitesimal change $\delta x(0)$ is made to the initial condition, there will be at time t a corresponding change $\delta x(t)$. We say that we have sensitive dependence on initial conditions if $\delta x(t)$ grows exponentially with t:

$$|\delta x(t)| \sim |\delta x(0)|\, e^{\lambda t},$$

where $\lambda > 0$ (λ is called a Liapunov exponent).

It is not hard to produce examples of sensitive dependence on initial conditions where $x(0)$ either is an unstable fixed point or is on an unstable periodic orbit. This fits well with the commonplace observation that small causes may have large effects. What is not so obvious is that in many cases, for all (or almost all) initial conditions $x(0)$, and almost all $\delta x(0)$, we have exponential growth of $\delta x(t)$. This is what is now called chaos. Chaos is thus the prevalence of sensitive dependence on initial conditions, whatever the initial condition is. This situation was first observed by J. Hadamard (1898) at the end of the nineteenth century in a rather special system called the geodesic flow on a manifold of constant negative curvature. Interestingly, the philosophical importance of this discovery was soon realized by such people as P. Duhem (1906) and H. Poincaré (1908). They understood that for a system such as that studied by Hadamard one could not make useful long-term predictions. With remarkable insight, Poincaré goes on to discuss the problem of weather predictability, and of the evolution of a gas of hard spheres (these are prime examples to which the ideas of chaos have been applied in recent times).

After Duhem and Poincaré, the ideas that they had promoted seem to have been forgotten, at least by physicists. When they were rediscovered, the scientific landscape had changed: mathematics and experimental physics had progressed, and computers were available.

Mathematicians, following the ideas of Hadamard, have now proved the existence of chaos in a number of dynamical systems (see Smale 1967). Furthermore, computer studies have shown that chaos is present in many situations not yet analysed mathematically. Roughly speaking, one might say that if a system is sufficiently complicated, its time evolution is likely to be chaotic.

An early example of a computer study exhibiting chaos is the analysis of the system.

$$\begin{aligned} \dot{x}_1 &= -10x_1 + 10x_2, \\ \dot{x}_2 &= -x_1 x_3 + 28x_1 - x_2, \\ \dot{x}_3 &= x_1 x_2 - \tfrac{8}{3}x_3, \end{aligned}$$

by E. Lorenz (1963). The time evolution provided by this system yields a caricature of the time evolution of a convecting fluid layer. Because the atmosphere is a convecting fluid layer, Lorenz could argue that the dynamics of the atmosphere is chaotic, and this therefore explains the difficulty in predicting the weather long in advance. Pursuing this idea, Lorenz and others (see Ghil *et al.* 1985) have shown that one cannot expect to make reasonable forecasts more than one or two weeks ahead. As can be seen, Lorenz rediscovered ideas that Poincaré had published more than half a century earlier, but the time was then ripe for an implementation of these ideas. Poincaré's writings were at the level of scientific philosophy, while the work of Lorenz and other meteorologists reached the level of quantitative science.

3. Strange attractors and turbulence

The turbulent flows of fluids offer rather obvious examples of complicated time evolutions. If one is awake to the ideas of sensitive dependence on initial conditions, and of chaos, it is rather natural to propose that turbulence is chaotic. F. Takens and myself wrote a paper to make just this proposal (Ruelle & Takens 1971), and it is interesting that it was received at first rather coldly. We concentrated on the onset of turbulence, for which the theory accepted at the time was that of Landau (1944) and Hopf (1948). According to this theory, a weakly turbulent fluid contains only a finite number of frequencies, or modes, and the fluid becomes more and more turbulent as the number of modes increases. In this 'modes' picture, there can be no sensitive dependence on initial conditions. We argued that much more general time evolutions were possible, asymptotic to what we called strange attractors. Strange attractors are finite dimensional, and correspond in some sense to exciting only a finite number of degrees of freedom. Yet they have an infinite number of basic frequencies. This situation is incompatible with the 'modes' paradigm, and this may explain the scepticism with which our proposal was received. The proposal, however, could be tested experimentally, and was found in accord with experiments by Gollub & Swiney (1975), while the Landau–Hopf theory did not fit (see also the earlier work of McLaughlin & Martin (1974) and by Ahlers (1974)).

Following this success, chaos was looked for and found in a variety of systems (for instance, chemical reactions (see Ruelle 1973; Vidal & Pacault 1981)). Furthermore, detailed 'roads to turbulence' or to chaos were investigated, namely scenarios in which a system with simple time evolution (periodic, say) becomes chaotic when some parameters are changed. See Eckmann (1981) for a discussion of such scenarios, of which the most spectacular is M. Feigenbaum's period-doubling cascade (Feigenbaum 1978, 1979).

I summarize very briefly a vast amount of experimental and theoretical work. For certain well-controlled systems (hydrodynamical, chemical, electromechanical, etc.) one can demonstrate precisely the presence of chaos and study in detail scenarios by which this is reached. We are here in the domain of quantitative science. Furthermore, for some less well-controlled systems, one may confidently infer that chaos and unpredictability are present, even when a precise demonstration of that fact is not practically possible. For instance, knowing that the time evolution of a fluid at the onset of turbulence is chaotic, it is hard to escape the conclusion that it is also chaotic for fully developed turbulence, even if this cannot be verified experimentally. An important consequence is that theories of turbulence based on a large but finite number of basic frequencies (modes) are necessarily wrong. Other examples are provided by various complicated time evolutions that are observed in biology, climatology, ecology and economics. We don't have here really convincing models of deterministic time evolution, but it is somehow reasonable to think that we have sensitive dependence on initial conditions. Notice, however, that we are back at the level of scientific philosophy. Attempts have been made to bring such problems into the realm of quantitative science, and those will be discussed in a moment.

4. Information and correlation dimensions

Let $B_x(r)$ be the ball (i.e. full sphere) of radius r centered at x, and $\rho(B_x(r))$ its Lebesgue volume. Then

$$\rho(B_x(r)) = C_d\, r^d,$$

where d is the dimension and C_d a constant. If ρ is not Lebesgue, but some singular measure, one may still be able to define a dimension by

$$\lim_{r\to 0} \frac{\log_{10} \rho(B_x(r))}{\log_{10} r}.$$

If we have a time evolution for which time averages are defined, it is interesting to take for ρ the measure that gives the fraction of time spent by the system in various regions of phase space. The above limit is then called the *information dimension* (we assume that the limit exists, it is then almost everywhere independent of x if ρ is ergodic, which is a natural assumption). If we know that the information dimension of a system is 3.2, say, this is physically interesting and means something like the fact that the system has 3.2 degrees of freedom. Furthermore, information dimension ties in with other mathematical concepts, for which a general theory exists. (For a review, see Eckmann & Ruelle 1985.)

The information dimension can be extracted from experimental data, but a closely related quantity, the *correlation dimension*, can be obtained more readily. We shall now describe the algorithm devised by Grassberger & Procaccia (1983) to compute this correlation dimension. Let me immediately point out that this algorithm has played a very important role in allowing us to say something about systems that otherwise defied analysis.

We start from a time series (u_i), with $i = 1, \ldots, N$. The u_i are measured values of a scalar quantity at regular intervals of time Δt. Choosing an integer $m > 0$, called the *embedding dimension*, we reconstruct a trajectory (x_i), with $i = 1, \ldots, N-m+1$, in m dimensions by putting

$$x_i = (u_i, u_{i+1}, \ldots, u_{i+m-1}).$$

(This reconstruction, which I had advocated verbally, first appeared in print in Packard *et al.* (1980), see the footnote 8 of that paper.) Given $r > 0$, let now $N(r)$ be the number of pairs $\{i,j\}$ such that the distance of x_i and x_j is $\leqslant r$. If the graph of $\log_{10} N(r)$ against $\log_{10} r$ has nearly constant slope for small r and sufficiently large embedding dimension m, that slope is the correlation dimension. There are magnificent examples where the correlation dimension has been computed convincingly from long time series of high quality. See the volume edited by Bergé

(1988) for reviews of this topic, in particular the article by Atten & Malraison (1988).

There are, however, limitations to the Grassberger–Procaccia algorithm. The range $[r_{\min}, r_{\max}]$ of values of r is limited ($r_{\min}$ and $r_{\max}$ are the smallest and largest values of the distance of pairs $\{x_i, x_j\}$, in particular $r_{\max}$ is the diameter of the reconstructed attractor). In fact only part of this range is usable; the lower part of the range is spoilt by statistical fluctuations, and the higher part by nonlinearities. One can try to cheat, and increase the number of points in the time series by interpolation. This lowers $r_{\min}$, but produces a spurious slope 1 at small $\log_{10} r$ (see Eckmann & Ruelle 1985; Atten & Malraison 1988). Let me now show that if the slope in the Grassberger–Procaccia algorithm is measured over at least one decade one finds necessarily

$$\text{correlation dimension} \leqslant 2\log_{10} N$$

(with a base 10 logarithm). Indeed, the slope of interest is

$$(\log_{10} N(r'') - \log_{10} N(r'))/(\log_{10} r'' - \log_{10} r'),$$

where

$$r_{\min} \leqslant r' < r'' \leqslant r_{\max}$$

and therefore

$$\begin{aligned} &N(r') \geqslant 1, \\ &N(r'') \leqslant \tfrac{1}{2}N(N-1) < N^2, \\ &\log_{10} N(r'') - \log_{10} N(r') \leqslant \log_{10} N^2. \end{aligned}$$

But because $r'' \geqslant 10r'$ we have also

$$\log_{10} r'' - \log_{10} r' \geqslant \log_{10} 10$$

and therefore

$$\text{slope} \leqslant 2\log_{10} N$$

as announced.

The above argument is due to Eckmann and myself (J.-P. Eckmann & D. Ruelle 1988, unpublished work). We can of course increase N by interpolation but, as we have seen above, this kind of cheating does not help. Atten & Malraison (1988) arrive at similar conclusions (they allow half a decade for the measure of the slope and half a decade for the unusable part of the range of r). One can of course replace 10 by some other number a, and obtain

$$\text{correlation dimension} \leqslant 2\log_a N,$$

but it seems unreasonable to measure a slope over much less than a decade.

In conclusion, one should not believe dimension estimates that are not well below $2\log_{10} N$. Let us note that, in their article, Atten & Malraison do not bother to discuss the published studies where this condition is violated. They are, however, numerous, and we shall now look at some of them.

5. Finance

It is an important scientific activity to verify accepted laws of nature. It is even more important to discover new laws. Consider time series obtained from electroencephalograms (EEGS), climate, or financial data. Is it possible that these correspond to deterministic time evolutions of relatively low dimension, at least at some level of accuracy? The possibility is well worth checking, even if one is *a priori* sceptical. In fact, it appears that the EEGS of some very sick people (with the Creutzfeldt–Jacob disease) correspond to low-dimensional dynamics (A. Babloyantz, personal communication); for other EEGS this is not the case.

The Grassberger–Procaccia algorithm provides a most natural test as to whether or not a given time series corresponds to a dynamical system of low dimensionality. One has, however, to proceed cautiously. Suppose that we start from a time series (of length N) that does not correspond to a deterministic time evolution of low dimension. The Grassberger–Procaccia algorithm will then presumably try to indicate a high dimension by saturating (or nearly saturating) the bound $2\log_{10} N$.

Let us take an example. The economists Scheinkman & Le Baron (1989), starting from a good-quality financial time series, of length $N = 1100$, computed a dimension of around 6. They carefully noted that by scrambling the time series (u_i) one increases to 11 the dimension obtained by the Grassberger–Procaccia algorithm. (Scrambling is making a random permutation of the time series. The value 11 violates the bound $2\log_{10} N$, which means that the slope is computed over less than a decade.)

Scheinkman was very prudent in suggesting that a low-dimensional time evolution may be underlying the data. In fact my present opinion is that this is not the case. Probably, there is some smoothness in the original time series, which explains that scrambling increases the computed dimension. Note that there are quantities other than the correlation dimension that one may try to compute, and that may be better behaved (this may be the case of Liapunov exponents). I think, however, that attempts to predict the future of the time series (see Anderson *et al.* 1988) by methods of dynamical systems are unlikely to be successful.

6. Controversies in Nature

It is noteworthy that the inventors of the Grassberger–Procaccia algorithm have been themselves rather prudent regarding its use. This is reflected in some controversies concerning work of Nicolis & Nicolis (see Nicolis & Nicolis 1984, 1987; Grassberger 1986, 1987) and Tsonis & Elsner (see Tsonis & Elsner 1988; Procaccia 1988). Nicolis & Nicolis claim to find a dimension 3.1 for a 'climatic attractor' with $N = 500$. Grassberger, analysing the same problem, but with $N = 230$, does not see a small-dimensional attractor. He remarks that in fact 'both analyses are based on the same set of 184 data points' interpolated and smoothened differently (the basic data are oxygen isotope records from a deep sea core). We see that we are dangerously close to the $2\log_{10} N$ bound.

Tsonis & Elsner analyse a series of 3960 values of '10-second averages of the vertical wind velocity recorded 10 metres above the ground over an 11-hour period' and find a 'dimension' equal to 7.3. The comments of Procaccia are rather sarcastic. Indeed, what we know of the dynamics of the atmosphere precludes the existence of a low-dimensional attractor. Note that the dimension proposed is about equal to $2 \log_{10} N$.

I think that I have made my point, and that it is unnecessary to quote more examples that could easily be found in the recent literature.

7. Conclusions

After a brief general review of the ideas of chaos, I have concentrated our attention on a special problem. The problem is to find out if various time evolutions observed in Nature correspond to low-dimensional deterministic dynamics. Various authors, using the Grassberger–Procaccia algorithm have claimed that this was the case. A careful examination of the method indicates, however, that the 'dimensions' of the order 6 that are obtained are very close to the upper bound $2 \log_{10} N$ permitted by the Grassberger–Procaccia algorithm (N is the length of the time series used, of the order of 10^3). The 'proof' that one has low-dimensional dynamics is therefore inconclusive, and the suspicion is that the time evolutions under discussion do not correspond to low-dimensional dynamics. It is possible that interesting information can nevertheless be extracted from the time series examined, but this would probably require new ideas. In the meantime, prudence is in order, and claims that one can predict the stock market – for instance – using the ideas of dynamical systems appear somewhat unrealistic.

Along the lines of the 'science and fiction' title of this talk, let me conclude on a lighter note. Readers of '*The hitchhiker's guide to the galaxy*', that masterpiece of British literature by D. Adams, know that a huge supercomputer has answered 'the great problem of life, the universe, and everything'. The answer obtained after many years of computation is 42. Unfortunately, one does not know to what precise question this is the answer, and what to make of it. It think that what happened is this. The supercomputer took a very long time series describing all it knew about 'life, the universe, and everything' and proceeded to compute the correlation dimension of the corresponding dynamics, using the Grassberger–Procaccia algorithm. This time series had a length N somewhat larger than 10^{21}. And you can imagine what happened. After many years of computation the answer came: dimension is approximately $2 \log_{10} N \approx 42$.

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