Towards a theory of Diophantine geometry

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Diophantine geometry is the study of maps

$$Y \longrightarrow X$$

between schemes of finite type over $\mathbb{Q}$ or $\mathbb{Z}$. The starting point of such an inquiry might be equations

$$f(x_1, x_2, \ldots, x_n) = 0$$

with $f \in \mathbb{Z}[x_1, x_2, \ldots, x_n]$ and the study of rational or integer solutions. But then one gradually and naturally enlarges the possibilities for both the coefficients and the solutions, starting with algebraic number fields. For example, we quickly run into

$$\mathbb{Q}[\zeta_p]$$

when studying

$$x^p + y^p = z^p,$$

both as the domain of solutions, but also in the coefficients, when we find it convenient to factorize

$$x^p + y^p = \prod_i (x + \zeta_p^i y),$$

($p \geq 3$). And then, once we start regarding the equation as defining a geometric object, we are inevitably led to consider maps from other geometric objects[1], such as might be defined by

$$g(x_1, x_2, \ldots, x_m) = 0$$

or, equivalently, solutions to the original equation in the finitely generated ring

$$\mathbb{Z}[x_1, x_2, \ldots, x_m]/(g(x_1, x_2, \ldots, x_m)).$$

Thence, it is hard not to find the general geometric formulation rather sensible.

Nevertheless, there is some resistance to thinking of all such investigations as falling under the same rubric. Consider the following small list:

(1) Galois theory, where one is given a map $\mathrm{Spec}(K) \longrightarrow \mathrm{Spec}(F)$ between spectra of number fields, and classifies intermediate maps that fit into a diagram:

$$\begin{tikzcd} & \mathrm{Spec}(L) \arrow[dr] & \\ \mathrm{Spec}(K) \arrow[ur] \arrow[rr] & & \mathrm{Spec}(F) \end{tikzcd}$$

[1] When changing variables, for example, in the most elementary instance.

(2) Class field theory, where one attempts to classify maps between rings of algebraic integers:

$$\mathrm{Spec}(\mathcal{O}_K) \longrightarrow \mathrm{Spec}(\mathcal{O}_F).$$

(3) Conjectures and theorems of Shafarevich type, where one proves the finiteness or non-existence of certain kinds of maps to rings of integers [1, 6, 7]:

–There are no abelian schemes

$$f : \mathcal{A} \longrightarrow \mathrm{Spec}(\mathbb{Z}).$$

(Fontaine-Abrashkin)

–There are at most finitely many smooth proper curves

$$f : \mathcal{X} \longrightarrow \mathrm{Spec}(\mathcal{O}_{F,S})$$

of genus $g \geq 2$. (Faltings)

(4) Hasse-Minkowski theory, the study of local-to-global principles exemplified by the exact sequence [21]:

$$0 \longrightarrow H^2(F, \mathbb{G}_m) \longrightarrow \oplus_v H^2(F_v, \mathbb{G}_m) \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

(5) The conjecture of Birch and Swinnerton Dyer [9, 17]:

$$E(\mathbb{Z}) \otimes \mathbb{Q}_p \simeq H^1_f(G_{\mathbb{Q}}, H_1(\bar{E}, \mathbb{Q}_p))$$

$$|E(\mathbb{Z})| < \infty \Longleftrightarrow L(E,1) \neq 0.$$

According to our original definition, all these are conjectures or theorems about Diophantine geometry in a very direct sense, in spite of the diversity of techniques that occur in their study. Should one indeed think of them in a uniform manner?

Indication that such unity may be possible comes from *linearized Diophantine geometry*. This is the study of maps between objects one obtains by linearizing the category of algebraic varieties to the category of *motives* [15].

The key step is to replace maps

$$Y \longrightarrow X$$

by *correspondences*:

$$\begin{array}{ccc} & Z & \\ \swarrow & & \searrow \\ Y & & X \end{array}$$

and formal linear combinations. This idea, suitably adapted, leads to the notion of a *linearized map* from Y to X as an algebraic cycle

$$Z = \sum_i n_i Z_i \subset Y \times X.$$

That is, one constructs a new category $\mathcal{M}_*$ and a functor

$$H : Var \longrightarrow \mathcal{M}_*$$

thought of as *universal homology* such that

$$\mathrm{Hom}_{\mathcal{M}_*}(H(Y), H(X)) = CH_*(Y \times X)_{\mathbb{Q}},$$

the $\mathbb{Q}$-vector space of algebraic cycles on $Y \times X$ modulo an appropriate equivalence relation.

This idea, refined suitably, is expected to lead to a very harmonious theory with good conjectural properties. For example, when X and Y are smooth projective varieties over F (possibly $\mathrm{Spec}(F)$ itself), Tate has conjectured that

$$\mathrm{Hom}_{\mathcal{M}_{hom}}(H(Y), H(X)) \otimes \mathbb{Q}_p \simeq \mathrm{Hom}_{G_F}[H^{et}(\bar{Y}, \mathbb{Q}_p), H^{et}(\bar{X}, \mathbb{Q}_p)],$$

where H^{et} refers to the dual of étale cohomology, considered inside the category of representations of the Galois group $G_F = \mathrm{Gal}(\bar{F}/F)$. Furthermore,

$$\mathrm{Ext}^1_{\mathcal{M}_{hom}}(H(Y), H(X)) \otimes \mathbb{Q}_p \simeq \mathrm{Ext}^1_{G_F, f}(H^{et}(\bar{Y}, \mathbb{Q}_p), H^{et}(\bar{X}, \mathbb{Q}_p)).$$

There is also supposed to be a *numerical criterion* for the existence of maps and extensions in terms of L-functions:

$$\dim\mathrm{Ext}^1_{\mathcal{M}_{hom}}(H(Y), H(X)) - \dim\mathrm{Hom}_{\mathcal{M}_{hom}}(H(Y), H(X))$$
$$= \mathrm{ord}_{s=1} L(H(X)^* \otimes H(Y), s)$$

One may well question the efficacy of this methodology, even if all the conjectures were to be true[2]. Nevertheless, it is striking that all maps in this theory are treated in a uniform manner, without regard, for example, to the relative dimensions of the objects[3].

The obvious problem is that in linearized Diophantine geometry, X will pick up many more 'virtual' Y-points, making it difficult to recover the original maps of interest. That is, the knowledge that

$$\mathrm{Hom}_{\mathcal{M}}(Y, X)$$

is non-zero may tell us nothing as all about existence of actual maps from Y to X (unless X has some special structure like that of an abelian variety.)

For example, when $Y = F$ itself, then given a K point $x \in X(K)$ for a finite field extension K of F, the formal linear combination

$$\sum \sigma(x),$$

where $\sigma(x)$ runs over the Galois conjugates of x, will define a map in $\mathcal{M}$ from $\mathrm{Spec}(F)$ to X. This obviously tells us nothing about the F points of X. One can attempt to remedy this in various ways, for example, by focusing on zero-cycles of degree one over F. This was, to a certain extent, the (unsuccessful) approach taken by Weil in his thesis towards a proof of the Mordell conjecture [23].

But the appoach that is our main concern here is that of replacing $H(X)$ by $\pi(X)$, a *homotopy type*, whose precise nature remains undetermined. Weil [24] proposed the use of $Bun_n(X)$, the moduli stack of vector bundles of rank n on X. At that time, a precise relation between this and the homotopy of X was not known, but was later established through the work of Narasimhan-Seshadri and Simpson [19, 20]. In the 1980's Grothendieck [8] proposed the profinite étale π_1 as $\pi(X)$, at least for a certain class of schemes including algebraic number fields and curves of genus ≥ 2, while Deligne and Ihara emphazised the importance of *rational homotopy types* that have a motivic nature [5]. More recently, Toen proposed *schematic homotopy types* as a suitably refined version of $\pi(X)$.

One might express a fantastic hope in this context to develop non-linear refinements of all aspects of the theory of motives, especially criteria for the existence and classification of morphisms from one

[2] Can one really compute the L-function?

[3] Here, we are starting with Y and X given, and ignoring somewhat the question of classifying all possible Y in a manner that generalizes class field theory. One might view conjectures of Shafarevich type as a preliminary step. To move systematically in this direction, one needs to supplement the usual theory of motives with Langland's reciprocity conjecture [18].

scheme of finite type to another. In any case, the motivic theory seems to provide a good starting point from which one can seek fruitful avenues of 'non-linearization/non-abelianization.'

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For the remainder of this paper, we will illustrate the general principle of non-abelian lifting with a rather concrete example, that is, an extension of the conjecture of Birch and Swinnerton-Dyer to the realm of hyperbolic curves.

Phenomena of the type

$[\mathcal{X}(\mathbb{Q}) < \infty] \sim$ [non-vanishing of L-values].

we have discussed elsewhere [14], even while a precise formulation remains elusive. But equally interesting and perhaps more important is the problem of extending the statement

$$E(\mathbb{Z}) \otimes \mathbb{Q}_p \simeq H^1_f(G_{\mathbb{Q}}, H_1(\bar{E}, \mathbb{Q}_p))$$

to the hyperbolic setting.

For this, let $\mathcal{X}/\mathbb{Z}$ denote one of the following:

–$\mathbb{P}^1 \setminus \mathcal{D}$ where $\mathcal{D}$ is a reduced horizonal divisor with at least three $\mathbb{C}$-points.

–The regular minimal model of a compact smooth curve of genus ≥ 2;

–The complement of a non-empty reduced horizontal divisor $\mathcal{D}$ inside a regular minimal model $\mathcal{X}'$ of a compact smooth curve of genus ≥ 1.

Denote by X the generic fiber of $\mathcal{X}$ and $\bar{X}$ its base-change to $\bar{\mathbb{Q}}$. Denote by b either

–an integral point[4] of $\mathcal{X}$;

–or an integral tangent vector to $\mathcal{X}'$ at an integral point in $\mathcal{D}$ ([5], section 15). Note that such a point must remain in the smooth locus of $\mathcal{X}'$.

For the rest of this paper, p will be an odd prime of good reduction and T a finite set of primes containing all primes of bad reduction for $\mathcal{X}$ and p. In the case of $\mathcal{X} = \mathcal{X}' \setminus \mathcal{D}$ as above, T also includes the primes where $\mathcal{D} \longrightarrow \mathrm{Spec}(\mathbb{Z})$ is not étale.

Clearly, the main case of genus zero to keep in mind is $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, while in genus one we should think about $\mathcal{E} \setminus O$, where $\mathcal{E}$ is the regular minimal model of an elliptic curve and O is its origin.

Recall [11] that the $\mathbb{Q}_p$-unipotent fundamental group U comes with a lower central series $U^1 = U$, $U^{n+1} = [U, U^n]$ and corresponding quotients $U_n = U/U^{n+1}$ that can then be viewed as a projective system

$$\cdots \longrightarrow U_4 \longrightarrow U_3 \longrightarrow U_2 \longrightarrow U_1$$

of algebraic groups.

For any other point $x \in \mathcal{X}(\mathbb{Z})$, we have

$$P(x) := \pi_1^{\mathbb{Q}_p}(\bar{X}; b, x) = [\pi_1^{et}(\bar{X}; b, x) \times U]/\pi_1^{et}(\bar{X}, b),$$

the homotopy classes of $\mathbb{Q}_p$-pro-unipotent étale paths from b to x. Here, $\gamma \in \pi_1^{et}(\bar{X}, b)$ acts on $\pi_1^{et}(\bar{X}; b, x) \times U$ by

$$(p, u) \mapsto (p\gamma, \gamma^{-1}u),$$

and $P(x)$ is the quotient space. This is a right *torsor* for U with respect to the action

$$[(p, u)]u' = [(p, uu')],$$

[4]We remind the reader of the elementary fact that in the compact case, $\mathcal{X}(\mathbb{Z}) = X(\mathbb{Q})$, so that the discussion subsumes the consideration of rational points.

sometimes called the *push-out* of the right $\pi_1^{et}(\bar{X}, b)$-torsor $\pi_1^{et}(\bar{X}; b, x)$ via the homomorphism

$$\pi_1^{et}(\bar{X}, b) \longrightarrow U.$$

Both U and $P(x)$ admit compatible actions of G that factor through $G_T = \mathrm{Gal}(\mathbb{Q}_T/\mathbb{Q})$. The induced action of the local Galois group $G_p = \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ is also *crystalline*. (For all these facts, we refer to [11].)

We get thereby a *global* $\mathbb{Q}_p$*-pro-unipotent Albanese map* (op. cit.):

$$\mathcal{X}(\mathbb{Z}) \xrightarrow{j} H^1_f(G_T, U);$$

$$x \mapsto [P(x)];$$

to the points of a $\mathbb{Q}_p$-moduli scheme

$$H^1_f(G_T, U)$$

of torsors for U that are unramified outside T and crystalline at p. Similarly, for each prime v, we get local Albanese maps [16]

$$\mathcal{X}(\mathbb{Z}_v) \xrightarrow{j_v} H^1(G_v, U).$$

such that the component at p factors

$$\mathcal{X}(\mathbb{Z}_p) \xrightarrow{j_p} H^1_f(G_p, U) \subset H^1(G_p, U)$$

through the subscheme consisting of crystalline classes. Here and in the following, we will allow ourselves some sloppiness in notationally confusing the $\mathbb{Q}_p$ points of the moduli schemes and the schemes themselves.

It is best to view these maps as towers, which in the global case looks like

$$\begin{tikzcd}
 & \vdots \arrow[d, phantom] \\
\vdots & H^1_f(G_T, U_3) \arrow[d] \\
 & H^1_f(G_T, U_2) \arrow[d] \\
\mathcal{X}(\mathbb{Z}) \arrow[r, "j"] \arrow[ur, "j"] \arrow[uur, "j"] & H^1_f(G_T, U_1)
\end{tikzcd}$$

(We will denote the maps at various levels with the same j or j_v.) Each of the schemes $H^1_f(G_T, U_n)$ is affine and of finite type.

Meanwhile, the Albanese maps are compatible with localization

$$\begin{tikzcd}
\mathcal{X}(\mathbb{Z}) \arrow[r, hook] \arrow[d, "j"'] & \mathcal{X}(\mathbb{Z}_v) \arrow[d, "j_v"'] \\
H^1_f(G_T, U_n) \arrow[r, "\mathrm{loc}_v"] & H^1(G_v, U_n)
\end{tikzcd}$$

and each loc_v is an algebraic map.

Now, we come to the important definition of the *Selmer scheme*:

$$H^1_{\mathbb{Z}}(G, U_n) := \cap_{v \neq p} \mathrm{loc}_v^{-1}[Im(j_v)] \subset H^1_f(G_T, U_n)$$

These are the torsors that are *locally geometric*, that is, come from points for each place $v \neq p$. Of course we are still imposing the condition that they be crystalline at p. The reader should be warned that the terminology Selmer scheme (or variety) was used in previous papers with somewhat looser defining conditions. However, the current usage seems to be the most consistent with the situation for elliptic curves. According to [16], each image

$$j_v(\mathcal{X}(\mathbb{Z}_v)) \subset H^1(G_v, U_n)$$

is finite for $v \neq p$, so that $H^1_{\mathbb{Z}}(G, U_n)$ is a closed subscheme of $H^1_f(G_T, U_n)$. For $n = 1$,

$$Im(j_v) \subset H^1(G_v, U_1)$$

is actually zero for $v \neq p$. We also have

$$Im(j_v) \subset H^1(G_v, U_n) = 0$$

even for higher n when $v \notin T$.

To formulate the conjecture, we need to focus on the localization diagram at p:

$$\begin{tikzcd}
\mathcal{X}(\mathbb{Z}) \arrow[r, hook] \arrow[d, "j"] & \mathcal{X}(\mathbb{Z}_p) \arrow[d, "j_p"] \\
H^1_{\mathbb{Z}}(G, U_n) \arrow[r, "\mathrm{loc}_p"] & H^1_f(G_p, U_n)
\end{tikzcd}$$

The main problem we wish to address is that of understanding the upper horizontal arrow, that is, recognizing in $\mathcal{X}(\mathbb{Z}_p)$ the global points $\mathcal{X}(\mathbb{Z})$. The strategy to be employed is to focus on the lower horizontal arrow instead, which is an algebraic map of affine schemes.

Define

$$\mathcal{X}(\mathbb{Z}_p)_n := j_p^{-1}(\mathrm{loc}_p[H^1_{\mathbb{Z}}(G, U_n)]),$$

the p-adic points that are *cohomologically global of level* n. For brevity, we will refer also to points in $\mathcal{X}(\mathbb{Z}_p)_n$ as *weakly global* of level n.

From the commutative diagrams

$$\begin{tikzcd}
 & H^1_f(G_p, U_{n+1}) \arrow[d] & H^1_{\mathbb{Z}}(G, U_{n+1}) \arrow[l, "\mathrm{loc}_p"'] \arrow[d] \\
\mathcal{X}(\mathbb{Z}_p) \arrow[ur, "j_p"] \arrow[r, "j_p"] & H^1_f(G_p, U_n) & H^1_{\mathbb{Z}}(G, U_n) \arrow[l, "\mathrm{loc}_p"']
\end{tikzcd}$$

we see that there is a non-increasing sequence of refinements

$$\mathcal{X}(\mathbb{Z}_p) \supset \mathcal{X}(\mathbb{Z}_p)_1 \supset \mathcal{X}(\mathbb{Z}_p)_2 \supset \cdots \supset \mathcal{X}(\mathbb{Z}_p)_n \supset \mathcal{X}(\mathbb{Z}_p)_{n+1} \supset \cdots \supset \mathcal{X}(\mathbb{Z}),$$

each level of which contains the set $\mathcal{X}(\mathbb{Z})$ of global points. Finally, we arrive at the

Conjecture 0.1.

$$\mathcal{X}(\mathbb{Z}_p)_n = \mathcal{X}(\mathbb{Z})$$

for $n >> 0$.

The point of this conjecture is that $\mathcal{X}(\mathbb{Z}_p)_n$ should be computable in principle, suggesting a method of computing $\mathcal{X}(\mathbb{Z})$[5]. Note that an appropriate generalization of this conjecture to number fields would open up also the possibility of providing a priori the initial requirement of a base-point. Such a generalization will be discussed in a subsequent paper.

The idea behind computability has an easy local portion coming from p-adic Hodge theory and the theory of the De Rham/crystalline fundamental group ([11], section 2), whereby we get a description of

$$\mathcal{X}(\mathbb{Z}_p) \xrightarrow{j_p} H^1_f(G_p, U_n) \simeq U_n^{DR}/F^0 \simeq \mathbb{A}^{r_n}$$

as

$$j_p(z) = (j_p^w(z))$$

where each coordinate $j_p^w(z)$ is a p-adic analytic function defined by iterated Coleman integrals

$$\int_b^z \alpha_1\alpha_2\cdots\alpha_n$$

of differential forms on $\mathcal{X}$. That is to say, the triangle on the right of the diagram

$$\begin{tikzcd}
\mathcal{X}(\mathbb{Z}) \arrow[r, hook] \arrow[d, "j"'] & \mathcal{X}(\mathbb{Z}_p) \arrow[d, "j_p"'] \arrow[dr, "(j_p^w)"] & \\
H^1_{\mathbb{Z}}(G, U_n) \arrow[r, "\mathrm{loc}_p"] & H^1_f(G_p, U_n) \arrow[r, "\simeq"] & \mathbb{A}^{r_n}
\end{tikzcd}$$

is essentially completely understood.

The difficult part is to compute

$$H^1_{\mathbb{Z}}(G, U_n) \longrightarrow H^1_f(G_p, U_n),$$

which should be accessible in principle, being an algebraic map between affine $\mathbb{Q}_p$-schemes. More precisely, we would like to compute the defining ideal

$$\mathcal{L}(n)$$

for the image of the global Selmer scheme at level n:

$$\mathrm{loc}_p[H^1_{\mathbb{Z}}(G, U_n)] \subset H^1_f(G_p, U_n) = \mathbb{A}^{r_n},$$

from which we wish to get the set

$$\mathcal{X}(\mathbb{Z}_p)_n$$

as zeros of $f \circ j_p$, as f runs over the generators of $\mathcal{L}(n)$. Now, as soon as $\mathcal{L}(n) \neq 0$, $\mathcal{X}(\mathbb{Z}_p)_n$ must be finite ([11], section 3), so we are in any case investigating a finite discrepancy

$$\mathcal{X}(\mathbb{Z}) \subset \mathcal{X}(\mathbb{Z}_p)_n.$$

As discussed elsewhere (loc. cit.), standard motivic conjectures (Fontaine-Mazur-Jannsen, Bloch-Kato) imply that for n large,

$$j^*\mathcal{L}(n+1) \supsetneq j^*\mathcal{L}(n),$$

and, in fact, the larger ideal contains elements that are algebraically independent of the elements in $j^*\mathcal{L}(n)$. So something in the common zero set for all n should be there for a good reason, hopefully, that of being a global point. Somewhat unfortunately, this vague expectation is all that can be offered at present as theoretical basis for the conjecture. Numerical evidence for the conjecture will be presented in a forthcoming paper with J. Balakrishnan, I. Dan-Cohen, and S. Wewers.

Incidentally, we remark that the said implication of motivic conjectures, in particular, $\mathcal{L}(n) \neq 0$ for $n >> 0$, has been proved in a range of cases, including $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, CM elliptic curves minus their origins, and curves of higher genus whose Jacobians have CM [4, 10, 13] .

[5] Cf. [12], as well as the usual method of descent on elliptic curves.

References

[1] Abrashkin, V. A. Good reduction of two-dimensional Abelian varieties. Izv. Akad. Nauk SSSR Ser. Mat. 40 (1976), no. 2, 262–272, 460.

[2] Baker, Matthew; Ribet, Kenneth A. Galois Theory and Torsion Points on Curves. Journal de Théorie des Nombres de Bordeaux 15 (2003), 11–32.

[3] Balakrishnan, Jennifer; Kedlaya, Kiran; Kim, Minhyong Appendix and erratum to "Massey products for elliptic curves of rank 1". J. Amer. Math. Soc. 24 (2011), no. 1, 281–291.

[4] Coates, John; Kim, Minhyong Selmer varieties for curves with CM Jacobians. Kyoto J. Math. 50 (2010), no. 4, 827–852.

[5] Deligne, Pierre Le groupe fondamental de la droite projective moins trois points. Galois groups over $\mathbb{Q}$ (Berkeley, CA, 1987), 79–297, Math. Sci. Res. Inst. Publ., 16, Springer, New York, 1989.

[6] Faltings, G. Endlichkeitssätze fÃijr abelsche Varietäten über Zahlkörpern. Invent. Math. 73 (1983), no. 3, 349–366.

[7] Fontaine, Jean-Marc Il n'y a pas de variété abélienne sur Z. Invent. Math. 81 (1985), no. 3, 515–538.

[8] Invent. Math. 155 (2004), no. 2, 253–286. Grothendieck, Alexander Brief an G. Faltings. London Math. Soc. Lecture Note Ser., 242, *Geometric Galois actions*, 1, 49âĂŞ58, Cambridge Univ. Press, Cambridge, 1997.

[9] Kato, Kazuya Lectures on the approach to Iwasawa theory for Hasse-Weil L-functions via B_{dR}. I. Arithmetic algebraic geometry (Trento, 1991), 50–163, Lecture Notes in Math., 1553, Springer, Berlin, 1993.

[10] Kim, Minhyong The motivic fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and the theorem of Siegel. Invent. Math. 161 (2005), no. 3, 629–656.

[11] Kim, Minhyong The unipotent Albanese map and Selmer varieties for curves. The unipotent Albanese map and Selmer varieties for curves. Publ. Res. Inst. Math. Sci. 45 (2009), no. 1, pp. 89–133. (Proceedings of special semester on arithmetic geometry, Fall, 2006.)

[12] Kim, Minhyong Remark on fundamental groups and effective Diophantine methods for hyperbolic curves. Number Theory, Analysis, and Geometry, in memory of Serge Lang. D. Goldfeld et. al. (ed.), Springer-verlag (2012).

[13] Kim, Minhyong p-adic L-functions and Selmer varieties associated to elliptic curves with complex multiplication. Annals of Math. 172 (2010), no. 1, 751–759.

[14] Kim, Minhyong Galois theory and Diophantine geometry. Non-abelian fundamental groups and Iwasawa theory, 162–187, London Math. Soc. Lecture Note Ser., 393, Cambridge Univ. Press, Cambridge, 2012.

[15] Kim, M.; Sujatha, R.; Lafforgue, L.; Genestier, A.; Ngô, B. C. Autour des motifs—École d'été Franco-Asiatique de Géométrie Algébrique et de Théorie des Nombres/Asian-French Summer School on Algebraic Geometry and Number Theory. Volume I. Lecture notes of the school held in Bures-sur-Yvette and at the Université Paris-Sud XI, Orsay, July 17–29, 2006. Edited by J.-B. Bost and J.-M. Fontaine. Panoramas et Synthèses. 29.

[16] Kim, Minhyong, and Tamagawa, Akio The l-component of the unipotent Albanese map. Math. Ann. 340 (2008), no. 1, 223–235.

[17] Kolyvagin, Victor A. On the Mordell-Weil group and the Shafarevich-Tate group of modular elliptic curves. Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), 429–436, Math. Soc. Japan, Tokyo, 1991.

[18] Langlands, R. P. L-functions and automorphic representations. Proceedings of the International Congress of Mathematicians (Helsinki, 1978), pp. 165–175, Acad. Sci. Fennica, Helsinki, 1980.

[19] Narasimhan, M. S.; Seshadri, C. S. Stable and unitary vector bundles on a compact Riemann surface. Ann. of Math. (2) 82 1965 540–567.

[20] Simpson, Carlos T. Higgs bundles and local systems. Inst. Hautes Études Sci. Publ. Math. No. 75 (1992), 5–95.

[21] Skorobogatov, Alexei Torsors and rational points. Cambridge Tracts in Mathematics, 144. Cambridge University Press, Cambridge, 2001. viii+187 pp.

[22] Toën, Bertrand Champs affines. Selecta Math. (N.S.) 12 (2006), no. 1, 39–135.

[23] Weil, André L'arithmétique sur les courbes algeébriques. Acta Math. 52 (1929), no. 1, 281–315.

[24] Weil, André Généralisation des fonctions abéliennes. J. Math Pur. Appl. 17 (1938), no. 9, 47âĂŞ87.