ANALYTIC CONTINUATION OF OVERCONVERGENT HILBERT MODULAR FORMS (PRELIMINARY VERSION)

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In these notes, we explain some recent progress on analytic continuation of overconvergent p-adic Hilbert modular forms and applications. We will begin with the classical case of elliptic modular forms to explain the basic ideas and hint at what new ideas are needed in the general case. We then move on to the case of Hilbert modular forms where the prime p is unramified in the relevant totally real field.

1. The classical case

1.1. In [BT99], Buzzard and Taylor proved the modularity of a certain kind of a Galois representation ρ by showing first that ρ arises from an overconvergent modular form f, and then proving that f is indeed a classical modular form. The proofs of classiality in this work and the subsequent generalization by Buzzard [Buz03] were through analytic continuation of f from its original domain of definition (which is an admissible open region in the rigid analytic modular curve) to the entire modular curve. This implies classicality since by the rigid analytic GAGA, any global analytic section of a line bundle over the analytification of a projective variety is, indeed, algebraic.

Earlier, in [Col96], Coleman had proved a criterion for classicality of p-adic overconvergent modular forms in terms of slope, i.e., the p-adic valuation of the eigenvalue of the U_p Hecke operator.

Theorem 1.1.1. (Coleman) *Any overconvergent modular form f of weight k and slope less than $k-1$ is classical.*

Coleman's proof involved calculations with the cohomology of modular curves. We could, however, ask whether this result could be proven by invoking the above principle of analytic continuation. In other words, given the slope condition, could we analytically continue f from its domain of definition to the entire modular curve? In [Kas06], we showed that this is possible and involves the construction of a series whose convergence is guaranteed by the given slope condition. In this section, we will explain the proof by dissecting the method to see what is essential for the application of the method in more general cases. In doing so, we will introduce some ideas of Pilloni which allows for a less explicit and, hence, more general approach.

1.2. The proof of Coleman's theorem via analytic continuation [Kas06]**.** In this section only, we let Y denote the completed modular curve of level $\Gamma_1(N)\cap\Gamma_0(p)$ defined over $\mathbb{Q}_p$, where $N\geq 4$ is an integer prime to p. Its noncuspidal locus classifies the data $(\underline{E},H)$ over $\mathbb{Q}_p$-schemes, where $\underline{E}$ is an elliptic curve with $\Gamma_1(N)$-level structure, and H a finite flat subgroup scheme of E of order p. Let ω be the usual sheaf on Y whose sections are invariant differentials on the universal family of (generalized) elliptic curves on Y. Modular forms of level $\Gamma_1(N)\cap\Gamma_0(p)$, and weight $k\in\mathbb{Z}$ are elements of $H^0(Y,\omega^k)$. We let $Y^{\rm an}$ denote the p-adic rigid analytification of Y, and continue to denote the analytification of ω by ω.

Let $Y^{{\rm an},0}$ denote the modular curve whose noncuspidal locus classifies all $(\underline{E},H,D)$ such that $(\underline{E},H)\neq(\underline{E},D)$ and both are classified by $Y^{\rm an}$. There are two morphisms $\pi_1,\pi_2: Y^{{\rm an},0}\to Y^{\rm an}$ sending $(\underline{E},H,D)$ to $(\underline{E},H)$ and $(\underline{E}/D,\bar{H})$, respectively, where $\bar{H}$ denotes the image of H in $\underline{E}/D$.

To define regions inside $Y^{\rm an}$, we need to recall the notion of degree of a finite flat group scheme over a finite extension of $\mathbb{Q}_p$ and some of its properties.

The degree of a finite flat group scheme. We define the notion of degree and record some properties that we will use later. This useful notion was defined by Illusie and others, and has been more recently studied by Fargues in [Far10].

Definition 1.2.1. Let $\mathcal{O}_K$ be the ring of integers in a finite extension K of $\mathbb{Q}_p$. Let Λ be a finite torsion $\mathcal{O}_K$-module. Choose an isomorphism $\Lambda\cong\oplus_{i=1}^d\mathcal{O}_K/(a_i)$ with $a_i\in\mathcal{O}_K$. We define the degree of Λ to be $\deg(\Lambda)=\sum_{i=1}^d\nu_p(a_i)$. This definition can be shown to be independent of the choice of the above isomorphism.

If G is finite flat group scheme over $\mathcal{O}_K$, we define $\deg(G)=\deg(\omega_G)$, where ω_G is the $\mathcal{O}_K$-module of global invariant differentials on G. It can be shown [Far10, §3] that $\deg(G)$ equals the p-adic valuation of a generator δ_G of $Fitt_0(\omega_G)$, the zeroth fitting ideal of ω_G.

We record some lemmas which we will use later.

Lemma 1.2.2. [Far10, Lemme 4] *Assume that* $0\to G'\to G\to G''\to 0$ *is an exact sequence of finite flat group schemes over* $\mathcal{O}_K$*. We have* $\deg(G)=\deg(G')+\deg(G'')$.

Lemma 1.2.3. [Far10] *Let* $\lambda: A\to B$ *be an isogeny of* p*-power degree between abelian schemes over* $S=\mathrm{Spec}(\mathcal{O}_K)$*. Let* G *be the kernel of* λ*. Let* $\omega_{A/S}$ *and* $\omega_{B/S}$ *denote the conormal sheaves of* A *and* B*, respectively. Then*

$$\deg(G)=\nu_p(\det(\lambda^*:\omega_{B/S}\to\omega_{A/S})).$$

In particular, if A *is an abelian scheme over* $\mathrm{Spec}(\mathcal{O}_K)$ *of dimension* g*, then* $\deg(A[p^n])=ng$.

Proposition 1.2.4. [Far10, Corrolaire 3] *Let* G *and* G' *be two finite flat group schemes over* $S=\mathrm{Spec}(\mathcal{O}_K)$*, and* $\lambda: G\to G'$ *a morphism of group schemes which is generically an isomorphism. Then,* $\deg(G)\leq\deg(G')$ *and the equality happens if and only if* λ *is an isomorphism.*

Proposition 1.2.5. [Pil11, Lemme 2.3.4] *If* G *is a truncated Barsotti-Tate group of level* 1 *defined over a finite extension of* $\mathbb{Q}_p$*, then* $\deg(G)$ *is an integer.*

The degree function can be used to parameterize points on the modular curve, and to cut out rigid analytic subdomains on it.

Definition 1.2.6. If $Q=(\underline{E},H)$ is point on $Y^{\rm an}$, we define $\deg(Q)=\deg(H)$, if Q has good reduction. Otherwise, we define $\deg(Q)=0$ or 1, depending on whether Q has étale or multiplicative reduction. If I is a subinterval of $[0,1]$, we define $Y^{\rm an}I$ to be the admissible open subdomain of $Y^{\rm an}$ consisiting of points Q such that $\deg(Q)\in I$. If a,b are rational numbers, then

$Y^{\mathrm{an}}[a,b]$ is quasi-compact. It is easy to see that the locus of supersingular location is exactly $Y^{\mathrm{an}}(0,1)$. The ordinary locus has two connected components, the multiplicative locus $Y^{\mathrm{an}}[1,1]$, and the étale locus, $Y^{\mathrm{an}}[0,0]$. An overconvergent modular form of weight $k \in \mathbb{Z}$ is a section of ω^k on $Y^{\mathrm{an}}[1-\epsilon,1]$ for some $\epsilon > 0$.

Remark 1.2.7. There is a simple relationship between the degree function and the function v' defined by Buzzard in [Buz03, §4]. We have $v'(\underline{E},H) = 1-\deg(\underline{E},H)$.

Given the above lemma, we can now rephrase the classical theory of canonical subgroups (due to Katz and Lubin) in terms of degrees, as follows:

Proposition 1.2.8. *(Lubin-Katz) Let* $Q = (\underline{E},H) \in Y^{\mathrm{an}}$. *Define* $\mathrm{Sib}(Q) = \{Q' = (\underline{E},H') \in Y^{\mathrm{an}} : Q' \neq Q\}$.

- *If* $\deg(Q) > 1/(p+1)$, *then for any* $Q' \in \mathrm{Sib}(Q)$, *we have* $\deg(Q') = (1-\deg(Q))/p < 1/(p+1)$.
- *If* $\deg(Q) = 1/(p+1)$, *then for any* $Q' \in \mathrm{Sib}(Q)$, *we have* $\deg(Q') = 1/(p+1)$.
- *If* $\deg(Q) < 1/(p+1)$, *then there is a unique* $(E,H') = Q' \in \mathrm{Sib}(Q)$, *such that* $\deg(Q') > 1/(p+1)$*;* H' *is called the (first) canonical subgroup of* $\underline{E}$*, it varies analytically with respect to* Q*, and we have* $\deg(Q') = 1-p\deg(Q)$. *For all other* $Q'' \in \mathrm{Sib}(Q)$, *we have* $\deg(Q'') = \deg(Q) < 1/(p+1)$.

We make a definition:

Definition 1.2.9. If $\deg(\underline{E},H) < \frac{1}{p^{m-1}(p+1)}$, then, for any $1 \leq n \leq m$, we can define a cyclic subgroup C_n of $E[p^n]$ of order p^n, called the n-th canonical subgroup of E, inductively as follows. By Proposition 1.2.8, E has a first canonical subgroup C_1, and $\deg(\underline{E}/C_1,\bar{H}) = 1-\deg(\underline{E},C_1) = p\deg(\underline{E},H) < \frac{1}{p^{m-2}(p+1)}$. Hence, by induction, we can construct C'_n, the n-th canonical subgroup of E/C_1, for all $1 \leq n \leq m-1$. For $2 \leq n \leq m$, we define $C_n = \mathrm{pr}^{-1}(C'_{n-1})$, where $\mathrm{pr} : E \to E/C_1$ is the projection.

The first step of the analytic continuation– the first take. This step is due to Buzzard [Buz03]. Using an iteration of the U_p operator, Buzzard extends f from its initial domain of definition to progressively larger domains, eventually extending f to $Y^{\mathrm{an}}(0,1]$.

Proposition 1.2.10. *(Buzzard) Let* f *be an overconvergent modular form* f *satisfying* $U_p(f) = a_p f$ *with* $a_p \neq 0$. *Then* f *extends analytically to* $Y^{\mathrm{an}}(0,1]$.

We first recall the definition of the U_p operator. Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be admissible opens of Y^{an} such that $\pi_1^{-1}(\mathcal{V}) \subset \pi_2^{-1}(\mathcal{W})$ inside $Y^{\mathrm{an},0}$. We define an operator

$$U_p = U_{\mathcal{W}}^{\mathcal{V}} : \omega^k(\mathcal{W}) \to \omega^k(\mathcal{V}),$$

via the formula

$$U_p(f) = \frac{1}{p}\pi_{1,*}(res(\mathrm{pr}^*\pi_2^*(f))), \tag{1.2.1}$$

where res is restriction from $\pi_2^{-1}(\mathcal{W})$ to $\pi_1^{-1}(\mathcal{V})$, $\pi_{1,*}$ is the trace map associated with the finite flat map π_1, and $\mathrm{pr}^* : \pi_2^*\omega^k \to \pi_1^*\omega^k$ is a morphism of sheaves on Y^{an}, which at $(\underline{E},H,D)$ is induced by $\mathrm{pr}^* : \Omega_{A/D} \to \Omega_A$ coming from the natural projection $\mathrm{pr} : A \to A/D$.

One can also define a set-theoretic U_p correspondence as the map which sends a subset $S \subset Y^{\mathrm{an}}$ to another subset $U_p(S) = \pi_2(\pi_1^{-1}(S))$. The condition $\pi_1^{-1}(\mathcal{V}) \subset \pi_2^{-1}(\mathcal{W})$ is equivalent to $U_p(\mathcal{V}) \subset \mathcal{W}$.

The principle underlying Buzzard's method is the following. Let $\mathcal{W}$ be an admissible open such that $U_p(\mathcal{W}) \subseteq \mathcal{W}$. Suppose f is defined over $\mathcal{W}$ and $U_p(f) = a_p f$ with $a_p \neq 0$. Suppose further that $\mathcal{V} \supset \mathcal{W}$ is an admissible open subset of Y^{an} such that $U_p(\mathcal{V}) \subseteq \mathcal{W}$. Then, f extends from $\mathcal{W}$ to $\mathcal{V}$, and the extended section (which we continue to denote by f) satisfies the same functional equation $U_p(f) = a_p f$. The reason for this is simple: the extension of f shall be taken to be $\frac{1}{a_p}U_p(f)$ and can be checked to satisfy all the desired properties. Therefore, a strategy for extending f to an admissible open $\mathcal{U} \supseteq \mathcal{W}$ is to prove that successive application of U_p sends $\mathcal{U}$ into $\mathcal{W}$. For details of this construction in general, see [Kas09, §3.1].

Proof of Proposition 1.2.10. We invoke the above principle and draw all the shrinkage under U_p we need from the degree calculations in Proposition 1.2.8.

If α is such that $1/(p+1) < 1-\alpha < 1$, then Proposition 1.2.8 shows that

$$U_p(Y^{\mathrm{an}}[1-\alpha, 1]) \subset Y^{\mathrm{an}}[1-\frac{\alpha}{p}, 1].$$

This implies that for M large enough, U_p^M sends $Y^{\mathrm{an}}(1-\alpha, 1]$ inside a domain of definition of f, and, hence, f extends to $Y^{\mathrm{an}}(1-\alpha, 1]$. Repeating this argument for all such α, we can extend f to a section (denoted f again) on $Y^{\mathrm{an}}(\frac{1}{p+1}, 1]$ satisfying $U_p(f) = a_p f$. By Proposition 1.2.8

$$U_p(Y^{\mathrm{an}}[\frac{1}{p+1}, 1]) \subset Y^{\mathrm{an}}(\frac{1}{p+1}, 1].$$

Hence, we deduce that f extends further to $Y^{\mathrm{an}}[\frac{1}{p+1}, 1]$, satisfying still $U_p(f) = a_p f$. Finally, for any $0 < \beta < 1/(p+1)$, Proposition 1.2.8 implies that

$$U_p(Y^{\mathrm{an}}[\beta, 1]) \subset Y^{\mathrm{an}}[p\beta, 1]).$$

Applying this successively, we deduce that a high enough power of U_p will send $Y^{\mathrm{an}}[\beta, 1]$ inside $Y^{\mathrm{an}}[\frac{1}{p+1}, 1]$, and, hence, f can be extended to $Y^{\mathrm{an}}[\beta, 1]$. Applying this to all $\beta > 0$, we get the desired result.

The first step of the analytic continuation– the 2nd take. Proposition 1.2.8 allows a precise calculation of the U_p correspondence in terms of the degree parametrization on the modular curve. This calculation was used in the above proof. In more general situations such calculations could prove difficult to carry out. In this section we explain, à la Pilloni, how Buzzard's proof given above does not really need the full force of the degree calculations under the U_p correspondence.

Looking at the above proof, we can readily see that the correspondence U_p increases degree in the cases considered. This is in fact a general principle.

Proposition 1.2.11. *Let $Q \in Y^{\mathrm{an}}$, and $Q' \in U_p(\{Q\})$. Then* $\deg(Q') \geq \deg(Q)$.

Proof. This is an immediate consequence of Proposition 1.2.4. The morphism $H \to \bar{H}$ induced by $A \to A/D$ is generically an isomorphism as $H \cap D = \{0\}$ generically.

□

Looking more closely at the proof in the last section, we have in fact shown that U_p increases the degree *strictly* on the non-ordinary locus of Y^{an}. This is possible to prove in light of explicit calculations afforded by Proposition 1.2.8. In fact, this is exactly what makes the proof work: by iterating U_p enough times, any point in the non-ordinary locus will eventually land close enough to $Y^{\mathrm{an}}[1,1]$ where f will be defined. The following approach, due to Pilloni, achieves the same without appealing to Proposition 1.2.8. This approach is useful in cases where an analogue of Proposition 1.2.8 is not readily available.

Proposition 1.2.12. *Let $Q = (\underline{E}, H) \in Y^{\mathrm{an}}$ defined over $\mathcal{O}_K$. If there is $Q' \in U_p(\{Q\})$ such that $\deg(Q) = \deg(Q')$, then H is a truncated Barsotti-Tate group of level 1. In particular, $\deg(Q)$ is an integer (by Proposition 1.2.5).*

Proof. Let $Q' = (\underline{E}/D, \bar{H})$, and assume w.l.o.g that Q' is also defined over $\mathcal{O}_K$. Since $H \to \bar{H}$ is generically an isomorphism, we have $\deg(Q') = \deg(\bar{H}) \geq \deg(H) = \deg(Q)$. Since the equality happens, by Proposition 1.2.4, we know that $H \to \bar{H}$ must be an isomorphism over $\mathcal{O}_K$. This implies that $E[p] \cong H \times D$, and hence both H and D are truncated Barsotti-Tate groups of level 1. □

Corollary 1.2.13. *In the situation of Proposition 1.2.12, we have $\deg(Q) \in \{0,1\}$, and hence Q belongs to the ordinary locus. In other words, over the non-ordinary locus of Y^{an}, U_p increases degrees strictly.*

Proof. By Proposition 1.2.12, we must have $\deg(Q) \in \mathbb{Z}$. Since $\deg(Q) \in [0,1]$, the claim follows. □

This gives another proof of the fact that U_p increases degree strictly over the non-ordinary locus of Y^{an}. We can now present a second proof of Proposition 1.2.10, due to Pilloni.

Second proof of Proposition 1.2.10. Assume f is defined on $Y^{\mathrm{an}}[1-\epsilon, 1]$, for some rational $\epsilon > 0$. It is enough to show that for any rational $\alpha \in (0, 1-\epsilon)$, there is $r \in \mathbb{N}$ such that $U_p^r(Y^{\mathrm{an}}[\alpha, 1]) \subset Y^{\mathrm{an}}[1-\epsilon, 1]$. This follows immediately if we show that there is a positive t such that U_p increases degree by t over the entire $Y^{\mathrm{an}}[\alpha, 1-\epsilon]$.

Let $\mathrm{pr} : \mathcal{A} \to \mathcal{A}/\mathcal{H}$ be the universal isogeny over Y^{an}. Let $\omega_{\mathcal{A}}, \omega_{\mathcal{A}/\mathcal{H}}$ denote, respectively, the determinants of the conormal sheaves of $\mathcal{A}, \mathcal{A}/\mathcal{H}$ over Y^{an}. Set $\mathcal{L} = \omega_{\mathcal{A}/\mathcal{H}}^{-1} \otimes \omega_{\mathcal{A}}$, which is an invertible sheaf on Y^{an}. The morphism $\mathrm{pr}^* : \omega_{\mathcal{A}/\mathcal{H}} \to \omega_{\mathcal{A}}$ defines a section δ of $\mathcal{L}$ on Y^{an}. By Lemma 1.2.3, for any $Q \in Y^{\mathrm{an}}$, we have $\deg(Q) = \nu_p(\delta(Q))$.

Consider now the section $\delta^0 = \pi_1^*\delta \otimes (\pi_2^*\delta)^{-1} \in H^0(Y^{\mathrm{an},0}, \pi_1^*\mathcal{L}^{-1} \otimes \pi_2^*\mathcal{L})$. By Corollary 1.2.13, we have $\nu_p(\delta^0) > 0$ over the entire non-ordinary locus. For any rational number $\alpha \in (0, 1-\epsilon)$, $Y^{\mathrm{an}}[\alpha, 1-\epsilon]$ is a quasi-compact rigid analytic domain of Y^{an}, and, hence, $\pi_1^{-1}(Y^{\mathrm{an}}[\alpha, 1-\epsilon])$ is a quasi-compact rigid analytic domain in $Y^{\mathrm{an},0}$. Therefore, by the maximum modulus principle, $\nu_p(\delta^0)$ attains a a minimum t over it. This minimum t must be positive as $Y^{\mathrm{an}}[\alpha, 1-\epsilon]$ lies entirely inside the non-ordinary locus.

The second step of the analytic continuation. So far, we have seen that as long as $\nu(a_p)$ is finite, we can extend f to $Y^{\mathrm{an}}(0,1]$. We now assume that $\nu(a_p) < k-1$, and prove the classicality of f. What is left to show is that under this assumption f can be further extended from $Y^{\mathrm{an}}(0,1]$ to $Y^{\mathrm{an}} = Y^{\mathrm{an}}[0,1]$. The missing locus is $Y^{\mathrm{an}}[0,0]$, i.e., the ordinary étale locus. We will do this by constructing a section on F on $Y^{\mathrm{an}}[0,0]$ and showing that it glues to f on $Y^{\mathrm{an}}(0,1]$ producing a global section.

To motivate the construction of the extension of f to Y^{an}, we assume for now that f is classical of slope less than $k-1$. Assume $(\underline{E},H)$ is in $Y^{\mathrm{an}}[0,0]$. Since $U_p f = a_p f$, we can write

$$f(\underline{E},H) = \frac{1}{pa_p} \sum_{H\cap D_1=0} \mathrm{pr}^* f(\underline{E}/D_1,\bar{H}), \tag{1.2.2}$$

where the sum is over the cyclic subgroups H_1 of rank p which intersect H trivially, and $\bar{H}$ denotes the image of H in $\underline{E}/D_1$. Since H is not canonical, all but one of points appearing on the right hand side of the above formula belong to $Y^{\mathrm{an}}[1,1]$ (by Proposition 1.2.8). The exceptional term corresponds to $D_1 = C_1$, the first canonical subgroup of E. Applying the above formula to $(\underline{E}/C_1,\bar{H})$ we get

$$f(\underline{E}/C_1,\bar{H}) = \frac{1}{pa_p} \sum_{H\cap D_2=0, C_1\subset D_2} \mathrm{pr}^* f(\underline{E}/D_2,\bar{H}), \tag{1.2.3}$$

where the sum is over the cyclic subgroups D_2 of rank p^2 which contain C_1 and intersect H trivially. We, hence, find

$$f(\underline{E},H) = \frac{1}{pa_p} \sum_{H\cap D_1=0, D_1\neq C_1} \mathrm{pr}^* f(\underline{E}/D_1,\bar{H}) + (\frac{1}{pa_p})^2 \sum_{H\cap D_2=0, C_1\subset D_2} \mathrm{pr}^* f(\underline{E}/D_2,\bar{H}).$$

Similarly, we find that the only point appearing in this expression that doesn't belong to $Y^{\mathrm{an}}[1,1]$ is $(\underline{E}/C_2,\bar{H})$. We will repeat this process with $f(\underline{E}/C_2,\bar{H})$, and keep going in the same way. At the n-th step, we separate the term corresponding to the quotient of E by C_n (the n-th canonical subgroup of E) from the rest of the terms, and rewrite the term via the functional equation $U_p f = a_p f$ as above. The result is the following.

Proposition 1.2.14. *Let f be a classical modular form of level $\Gamma_0(N)\cap\Gamma_1(p)$, weight k and slope less than $k-1$. We have*

$$f(\underline{E},H) = \sum_{n=1}^{\infty} (\frac{1}{pa_p})^n \big(\sum_{D_n} \mathrm{pr}^* f(\underline{E}/D_n,\bar{H}) \big), \tag{1.2.4}$$

where D_n runs through all the cyclic subgroups of E of rank p^n which contain C_{n-1}, are different from C_n, and intersect H trivialy. All the points appearing in the above series belong to $Y^{\mathrm{an}}[1,1]$.

Proof. The only thing left to show is that the series converges. By Lemma 1.2.2, $\deg(C_n) = n$, and, hence, Lemma 1.2.3 implies that $\mathrm{pr}^*(\eta)$ is divisible by p^{nk} for any section η of ω^k. Hence the "error term" $(\frac{1}{pa_p})^n \mathrm{pr}^* f(\underline{E}/C_n,\bar{H})$ is divisible by $(\frac{1}{pa_p})^n p^{nk} = (p^{k-1}/a_p)^n$ which tends to zero as n goes to infinity by the assumption on a_p. $\square$

Before we proceed, we would like to make a definition to formalize the above "error term" as a special term among the terms appearing in the definition of the U_p operator.

Definition 1.2.15. For any interval $I \subset [0, 1/(p+1))$, and any $r > 0$, we define I^r to be the interval defined by multiplying all the elements in I by r. Using Proposition 1.2.8, we define

$$U^{sp} : Y^{\mathrm{an}} I^{\frac{1}{p}} \to Y^{\mathrm{an}} I,$$

via $U^{sp}(\underline{E}, H) = (\underline{E}/C_1, \bar{H})$, where C_1 is the canonical subgroup of $\underline{E}$. It induces a morphism

$$U^{sp} : \omega^k(Y^{\mathrm{an}} I) \to \omega^k(Y^{\mathrm{an}} I^{\frac{1}{p}})$$

defined as $U^{sp} f(\underline{E}, H) = \frac{1}{p}\, \mathrm{pr}^* f(\underline{E}/C_1, \bar{H})$. It follows that for any $n \in \mathbb{N}$, the map

$$(U^{sp})^n : \omega^k(Y^{\mathrm{an}} I) \to \omega^k(Y^{\mathrm{an}} I^{\frac{1}{p^n}})$$

is given by $(U^{sp})^n f(\underline{E}, H) = (\frac{1}{p})^n \mathrm{pr}^* f(\underline{E}/C_n, \bar{H})$, where C_n is the n-th canonical subgroup of $\underline{E}$.

Assume $\mathcal{W}, \mathcal{V}$ are admissible opens of Y^{an} satisfying $\pi_2^{-1}(\mathcal{W}) \subset \pi_1^{-1}(\mathcal{V})$ so that we have a U_p operator $U_{\mathcal{W}}^{\mathcal{V}} : \omega^k(\mathcal{W}) \to \omega^k(\mathcal{V})$. Whenever there is a decomposition $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2$, we get a decomposition of the $U_{\mathcal{W}}^{\mathcal{V}}$ operator to a the sum of two operators $U_{\mathcal{W}_1}^{\mathcal{V}} : \omega^k(\mathcal{W}_1) \to \omega^k(\mathcal{V})$ and $U_{\mathcal{W}_2}^{\mathcal{V}} : \omega^k(\mathcal{W}_1) \to \omega^k(\mathcal{V})$. In particular, by virtue of Proposition 1.2.8, we have

$$\pi_1^{-1}(Y^{\mathrm{an}}[0,0]) = \pi_2^{-1}(Y^{\mathrm{an}}[1,1]) \cup \pi_2^{-1}(Y^{\mathrm{an}}[0,0]), \tag{1.2.5}$$

and, correspondingly, the operator $U_p : \omega^k(Y^{\mathrm{an}}[1,1]) \cup \omega^k(Y^{\mathrm{an}}[0,0]) \to \omega^k(Y^{\mathrm{an}}[0,0])$ decomposes as a sum of two operators denoted as follows:

$$U_p = U_p^{nsp} + U_p^{sp}.$$

Unravelling the above construction shows that $U_p^{nsp} : \omega^k(Y^{\mathrm{an}}[1,1]) \to \omega^k(Y^{\mathrm{an}}[0,0])$ is given by

$$U_p^{nsp} f(\underline{E}, H) = \frac{1}{p} \sum_{D \neq C_1, H} \mathrm{pr}^* f(\underline{E}/D, \bar{H}), \tag{1.2.6}$$

and $U_p^{nsp} : \omega^k(Y^{\mathrm{an}}[0,0]) \to \omega^k(Y^{\mathrm{an}}[0,0])$ is the map defined in Definition 1.2.15 for $I = [0,0]$. Using this notation, the discussion above can be summarized as follows: if f is classical, of weight k and of slope less than $k-1$, then

$$f_{|Y^{\mathrm{an}}[0,0]} = \sum_{n=1}^{\infty} (\frac{1}{a_p})^n (U_p^{sp})^{n-1} U_p^{nsp}(f_{|Y^{\mathrm{an}}[1,1]}).$$

Proof of Theorem 1.1.1. Define F on $Y^{\mathrm{an}}[0,0]$ exactly as above.

$$F = \sum_{n=1}^{\infty} (\frac{1}{a_p})^n (U_p^{sp})^{n-1} U_p^{nsp}(f_{|Y^{\mathrm{an}}[1,1]}). \tag{1.2.7}$$

The convergence of the series under the slope assumption follows from the same argument as in Proposition 1.2.14. We want to show that F on $Y^{\mathrm{an}}[0,0]$ can be glued to f on $Y^{\mathrm{an}}(0,1]$. The problem is that the series defining F can not be extended outside $Y^{\mathrm{an}}[0,0]$, as its definition depends on the existence of all C_n's which requires the ordinarity of E. However, as we shall see, the partial sums in the series overconverge outside $Y^{\rm an}[0,0]$ (albeit to extents that vanish in the limit), and it gives us enough information to prove the gluing.

In fact, for any $0 \leq t < 1/(p+1)$, Proposition 1.2.8 gives a decomposition similar to 1.2.5:

$$\pi_1^{-1}(Y^{\rm an}[0,\frac{t}{p}]) = \pi_2^{-1}(Y^{\rm an}[1-\frac{t}{p},1]) \cup \pi_2^{-1}(Y^{\rm an}[0,t]). \tag{1.2.8}$$

Therefore, the above decomposition of U_p extends from $Y^{\rm an}[0,0]$ to $Y^{\rm an}[0,\frac{t}{p}]$, and we can write $U_p = U_p^{sp} + U_p^{nsp}$, where

$$U_p^{nsp} : \omega^k(Y^{\rm an}[1-\frac{t}{p},1]) \to \omega^k(Y^{\rm an}[0,\frac{t}{p}]),$$

is given by the same formulae as 1.2.6, and

$$U_p^{sp} : \omega^k(Y^{\rm an}[0,t]) \to \omega^k(Y^{\rm an}[0,\frac{t}{p}])$$

is as in Definition 1.2.15. Let us fix a rational number $0 < t_0 < 1/(p+1)$, and define

$$\mathcal{V} := Y^{\rm an}(0,t_0]$$

Also, for any $m \geq 0$, define

$$\mathcal{S}_m^\dagger := Y^{\rm an}[0,\frac{t_0}{p^m}].$$

The above decomposition of U_p allows us to define a section of ω^k on $\mathcal{S}_m^\dagger$, for $m \geq 1$, as follows

$$F_m = \sum_{n=1}^{m} (\frac{1}{a_p})^n (U_p^{sp})^{n-1} U_p^{nsp}(f_{|_{Y^{\rm an}[1-\frac{t_0}{p},1]}}).$$

In other words, F_m is the m-th partial sum of F which overconverges from $Y^{\rm an}[0,0]$ to $\mathcal{S}_m^\dagger$. We want to show that these partial series become very close to f outside $Y^{\rm an}[0,0]$. An easy calculation shows us that for $m \geq 1$, we have the following equality on $\mathcal{V} \cap \mathcal{S}_m^\dagger = Y^{\rm an}(0,\frac{t_0}{p^m}]$:

$$f = F_m + (\frac{1}{a_p})^m (U_p^{sp})^m f_{|\mathcal{V}}, \tag{1.2.9}$$

where U_p^{sp} is as in Definition 1.2.15 for $I = (0,t_0]$. We, therefore, need to estimate $(U_p^{sp})^m f_{|\mathcal{V}}$.

Lemma 1.2.16. *The following estimates hold:*

- *The collection of sections $\{f_{|\mathcal{V}}, F_m : m \in \mathbb{N}\}$ is uniformly bounded.*
- *$|f - F_m|$ on $\mathcal{S}_m^\dagger \cap \mathcal{V} = Y^{\rm an}(0,\frac{t_0}{p^m}]$ tends to zero as m goes to infinity.*
- *$|F_{m+1} - F_m|$ on $\mathcal{S}_{m+1}^\dagger = Y^{\rm an}[0,\frac{t_0}{p^{m+1}}]$ tends to zero as m goes to infinity.*

Proof. Let $\mathcal{Z} \subset Y^{\rm an}[0,\frac{1}{p(p+1)})$, and let $h \in \omega^k(\mathcal{Z})$. For any $Q = (\underline{E},H) \in (U_p^{sp})^{-1}(\mathcal{Z})$, we denote the first canonical subgroup of E by C_1. Let $d := \inf\{\deg(C_1) : Q \in (U_p^{sp})^{-1}(\mathcal{Z})\}$. We can write

$$\begin{aligned} |U_p^{sp}h(Q)| &= |\frac{1}{p}\ \mathrm{pr}^* h(U_p^{sp}(Q))| \\ &= p^{-k(\deg(C_1))}|\frac{1}{p}\ h(U_p^{sp}(Q))| \\ &\leq p^{1-kd}|h|_{\mathcal{Z}}, \end{aligned}$$

where, in the second equality, we have used Lemma 1.2.3. This implies that

$$|U_p^{sp}h|_{(U_p^{sp})^{-1}(\mathcal{Z})} \leq p^{1-kd}|h|_{\mathcal{Z}}. \tag{1.2.10}$$

We first show that $|f|_{\mathcal{V}}$ is bounded. For $m \geq 0$, let $\mathcal{Z}_m := \mathcal{S}_m^\dagger - \mathcal{S}_{m+1}^\dagger = Y^{\rm an}(\frac{t_0}{p^{m+1}}, \frac{t_0}{p^m}]$. Proposition 1.2.8 implies that $U_p(\mathcal{Z}_0)$ lies inside (the quasi-compact) $Y^{\rm an}[\frac{1}{p+1}, 1]$, over which f is bounded. This, in turn, implies that $f = \frac{1}{a_p}U_pf$ is bounded on $\mathcal{Z}_0$. On the other hand, we have seen that $f - \frac{1}{a_p}U_p^{sp}f = F_1$ extends to the quasi-compact $\mathcal{S}_1^\dagger$, and, hence, it has a bounded norm. Let M_1 denote a common bound for f on $\mathcal{Z}_0$ and $f - \frac{1}{a_p}U_p^{sp}f$ on $\mathcal{S}_1^\dagger$. We prove, by induction, that f is bounded by $M_1p^{\frac{kt_0}{p}+\cdots+\frac{kt_0}{p^m}}$ on $\mathcal{Z}_m$. Assume this is true for $m-1$, with $m \geq 1$. We have $(U_p^{sp})^{-1}(\mathcal{Z}_{m-1}) = \mathcal{Z}_m$, and the infimum d introduced above equals $1 - \frac{t_0}{p^m}$ on $\mathcal{Z}_m$, by Proposition 1.2.8. Hence, inequality 1.2.10 gives us

$$|\frac{1}{a_p}\ U_p^{sp}f|_{\mathcal{Z}_m} \leq p^{\nu_p(a_p)+1-k(1-\frac{t_0}{p^m})}|f|_{\mathcal{Z}_{m-1}} \leq p^{\frac{kt_0}{p^m}}|f|_{\mathcal{Z}_{m-1}} \leq M_1p^{\frac{kt_0}{p}+\cdots+\frac{kt_0}{p^m}}.$$

Therefore, $|f|_{\mathcal{Z}_m} \leq \max\{|f - \frac{1}{a_p}U_p^{sp}f|_{\mathcal{Z}_m}, |\frac{1}{a_p}U_p^{sp}f|_{\mathcal{Z}_m}\} \leq M_1p^{\frac{kt_0}{p}+\cdots+\frac{kt_0}{p^m}}$, as claimed. Now, since $\mathcal{V} = \bigcup_{m\geq 0}\mathcal{Z}_m$, it follows that

$$|f|_{\mathcal{V}} \leq M := M_1p^{\frac{kt_0}{p-1}}.$$

For the second part of the lemma, we apply Equation 1.2.9 along with inequality 1.2.10 with $\mathcal{Z} = \mathcal{S}_{m-1}^\dagger \cap \mathcal{V}$ to deduce that

$$|(U_p^{sp})^m f_{|\mathcal{V}}|_{\mathcal{S}_m^\dagger\cap\mathcal{V}} \leq p^{1-k(1-\frac{t_0}{p^{m-1}})}|(U_p^{sp})^{m-1}f_{|\mathcal{V}}|_{\mathcal{S}_{m-1}^\dagger\cap\mathcal{V}}.$$

Induction on m gives us

$$|(\frac{1}{a_p})^m(U_p^{sp})^m f_{|\mathcal{V}}|_{\mathcal{S}_m} \leq |\frac{1}{a_p}|^m\ p^{m-k(m-\frac{pt_0}{p-1})} = p^{m(\nu_p(a_p)-(k-1))}p^{\frac{kpt_0}{p-1}} \to 0 \quad \text{as} \quad m \to \infty$$

since $\nu_p(a_p) < k-1$. The third statement of the Lemma can be proven in exactly the same way, as $F_{m+1} - F_m = (\frac{1}{a_p})^{m+1}(U_p^{sp})^m(F_1)$ from the definition. Finally, the uniform boundedness of the collection $\{F_m\}$ follows immediately from the above results, along with the fact that the sequence F_m is convergent on $Y^{\rm an}[0,0]$.

□

Finally, we can use the gluing lemma in [Kas06] to show that f and F glue together to produce a section of ω^k over $Y^{\rm an}[0,t_0]$. While the domains of definitions of F and f do not overlap, we can use the above overconvergence results to prove the gluing.

It is enough to show that f extends from $\mathcal{V} = Y^{\rm an}(0,t_0]$ to $\mathcal{S}_0^\dagger = Y^{\rm an}[0,t_0]$. Let $\check{\mathcal{O}}$ be the sheaf of rigid analytic functions on $Y^{\rm an}$ with norm at most 1. By Lemma 1.2.16, we can rescale (and restrict to a trivializing open cover for ω) to assume that all the F_m's and $f_{|\mathcal{V}}$ are sections of $\check{\mathcal{O}}$. Furthermore, modulo choosing a subsequence, we can assume that

$$|F_m - f|_{\mathcal{S}_m^\dagger\cap\mathcal{V}} \leq (\frac{1}{p})^m.$$

This implies that F_m and $f_{|\mathcal{V}}$ glue mod p^m to give a section h_m of $\check{\mathcal{O}}/p^m\check{\mathcal{O}}$ over $\mathcal{S}_m^\dagger \cup \mathcal{V} = \mathcal{S}_0^\dagger$. A theorem of Bartenwerfer [Bar70] states that for a smooth quasi-compact rigid analytic variety $\mathcal{Z}$, we have $cH^1(\mathcal{Z}, \check{\mathcal{O}}_{\mathcal{Z}}) = 0$ for some scalar c with $|c| \leq 1$. A standard argument shows then that

$$ch_m \in \check{\mathcal{O}}(\mathcal{S}_0^\dagger)/p^m\check{\mathcal{O}}(\mathcal{S}_0^\dagger) \subset (\check{\mathcal{O}}/p^m\check{\mathcal{O}})(\mathcal{S}_0^\dagger).$$

The compatibility of the ch_m's implies that their inverse limit provides an element h of

$$\varprojlim_m ch_m \in \varprojlim_m \check{\mathcal{O}}(\mathcal{S}_0^\dagger)/p^m\check{\mathcal{O}}(\mathcal{S}_0^\dagger) = \check{\mathcal{O}}(\mathcal{S}_0^\dagger).$$

We define h to be $1/c$ times this section. It is immediate from Lemma 1.2.16 that $h_{|\mathcal{V}} = f_{|\mathcal{V}}$ and $h_{|Y^{\mathrm{an}}[0,0]} = F$. This ends the proof of Theorem 1.1.1.

1.3. Discussion: the essential ingredients in the second step of analytic continuation. Since we are interested in applying the above method in more general situations, we would like to discuss some of the ingredients that made the above proof work, in a less case-specific fashion. We first focus on the construction of the series on $Y^{\mathrm{an}}[0,0]$.

As we have seen earlier, the idea of the first step of the analytic continuation does not work on $Y^{\mathrm{an}}[0,0]$ as U_p does not increase degrees strictly on this domain. In fact, another way to characterize $Y^{\mathrm{an}}[0,0]$ is the following:

$$Y^{\mathrm{an}}[0,0] = \{(\underline{E}, H) \in Y^{\mathrm{an}}[0,1) : \exists!(\underline{E}/G_1, \bar{H}) \in U_p\{(\underline{E}, H)\} \text{ s.t. } \deg(\underline{E}/G_1, \bar{H}) = \deg(\underline{E}, H)\}.$$

Again, having Proposition 1.2.8 at our disposal, this is immediate: the unique subgroup of $E[p]$ distinguished above is the first canonical subgroup of E. But more is needed to make possible the writing of the series: it is crucial that for any $(\underline{E}, H)$ in $Y^{\mathrm{an}}[0,0]$ all terms of $U_p(\underline{E}, H)$ apart from $(\underline{E}/G_1, \bar{H})$ lie in a region which is admissibly disjoint from $Y^{\mathrm{an}}[0,0]$, which, in this case, is $Y^{\mathrm{an}}[1,1]$. This is the content of Equation 1.2.5 and is exactly what allows the decomposition of the U_p correspondence as

$$U_p = U_p^{nsp} + U_p^{sp}$$

on $Y^{\mathrm{an}}[0,0]$. This already gives us the first partial sum of the series, i.e.,

$$F_1 = (\frac{1}{a_p})U^{nsp}(f_{|Y^{\mathrm{an}}[1,1]}),$$

defined over $\mathcal{S}_1 := Y^{\mathrm{an}}[0,0]$. To write the second partial sum,

$$F_2 = (\frac{1}{a_p})U_p^{nsp}(f_{|Y^{\mathrm{an}}[1,1]}) + (\frac{1}{a_p})^2 U_p^{sp}U_p^{nsp}(f_{|Y^{\mathrm{an}}[1,1]}),$$

we need to make sense of $U_p^{sp}U_p^{nsp}(f)$ which is formally defined on $\mathcal{S}_2 := (U_p^{sp})^{-1}(\mathcal{S}_1)$ (which, again, happens to be $Y^{\mathrm{an}}[0,0]$ in this case). Similarly, if we define, $\mathcal{S}_m := (U_p^{sp})^{-m}(\mathcal{S}_1)$, we can make sense of

$$F_m = \sum_{n=1}^{m} (\frac{1}{a_p})^n (U_p^{sp})^{n-1} U_p^{nsp}(f_{|Y^{\mathrm{an}}[1,1]})$$

as a section of ω^k on $\mathcal{S}_m$. In the case at hand, $\mathcal{S}_m$ happens to be $Y^{\mathrm{an}}[0,0]$ for all m. But let us forget that knowledge and see what we can deduce about the $\mathcal{S}_m$'s, simply from their definition.

In fact, using Proposition 1.2.11, we can formally see that

$$\mathcal{S}_m = \{(\underline{E}, H) \in Y^{\mathrm{an}}[0,1) : \exists!(\underline{E}/G_m, \bar{H}) \in U_p^m\{(\underline{E}, H)\} \text{ s.t. } \deg(\underline{E}/G_m, \bar{H}) = \deg(\underline{E}, H)\}.$$

Let us call $\mathcal{S}_m$ the *special locus of order m*. Using Proposition 1.2.11, we formally deduce that

$$\mathcal{S}_1 \supseteq \mathcal{S}_2 \supseteq \cdots \supseteq \mathcal{S}_m \supseteq \cdots$$

Therefore, the series given by the partial sums F_m can at least be written down on

$$\mathcal{S}_\infty := \cap_{m\in\mathbb{N}}\mathcal{S}_m,$$

provided it has a rigid analytic structure. The next step would be to show that the series converges on $\mathcal{S}_\infty$ given the slope condition. This boils down to estimating $(\frac{1}{a_p})^m(U_p^{sp})^m(f_{|_{Y^{\mathrm{an}}[1,1]}})$ as in Lemma 1.2.16. This expression involves m iterations of U_p^{sp}, which, in turn, entails m applications of the pullback of differential forms under the map $\mathrm{pr} : \underline{E} \to \underline{E}/G_1$ for various points $(\underline{E}, H) \in \mathcal{S}_\infty$. By Lemma 1.2.3, an estimate can be obtained in terms of the degree of the various distinguished subgroups, i.e., the G_1's that appear in the iterations. In the case at hand, all the G_1's will be canonical subgroups of ordinary elliptic curves, and, hence, will be of degree 1, determining the slope condition $\nu_p(a_p) < k-1$ for the convergence of the series. In general, one expects these degrees to be large enough integers providing estimates which translate into relevant slope conditions.

Some issues remain to be handled. Firstly, if, unlike in the case at hand, $\mathcal{S}_\infty \neq \mathcal{S}_1$, we would still need to analytically continue f to $\mathcal{S}_1$. Secondly, we need to glue the section obtained via the above series to the section defined outside the special locus. Both of these require analytic continuation of F_m outside $\mathcal{S}_m$. To do so, one needs to work in a strict neighborhood of the bad locus $\mathcal{S}_1$, say $\mathcal{S}_0^\dagger$, and construct the *special locus of order m inside* $\mathcal{S}_0^\dagger$, called $\mathcal{S}_m^\dagger$, which will certainly contain $\mathcal{S}_m$. In fact, it would be enough to construct $\mathcal{S}_1^\dagger$, a neighborhood of $\mathcal{S}_1$ inside $\mathcal{S}_0^\dagger$, to have the following properties:

- the decomposition $U_p = U_p^{nsp} + U_p^{sp}$ overconverges from $\mathcal{S}_1$ to $\mathcal{S}_1^\dagger$,
- U_p takes $\mathcal{S}_0^\dagger - \mathcal{S}_1^\dagger$ to a region on which f is already defined.

Having $\mathcal{S}_1^\dagger$ at hand, one can define $\mathcal{S}_m^\dagger := (U_p^{sp})^{-m+1}(\mathcal{S}_1^\dagger)$, and construct F_m, the partial series of order m, on $\mathcal{S}_m^\dagger$, as explained in the previous section. This shall explain the notation used in §1.2. In fact, in that proof one has $\mathcal{S}_m = Y^{\mathrm{an}}[0,0]$, $\mathcal{S}_m^\dagger = Y^{\mathrm{an}}[0, \frac{t_0}{p^m}]$.

In the Hilbert case, explained in §4, the above construction involves one extra step. Since at the m-th step of the argument, we glue F_m and f mod p^m, we need to arrange for the domains of definition of F_m and f to form an admissible covering of $\mathcal{S}_0^\dagger$. The non-explicit nature of the argument does not allow us to rule out, for instance, the possibility that $\mathcal{S}_1^\dagger$ equals $\mathcal{S}_1$. This would certainly cause trouble in the gluing procedure. To remedy this, one needs to enlarge $\mathcal{S}_m^\dagger$ (which already contains the "bad-behaviour" locus for U_p^m) to a strict neighborhood $\mathcal{S}_m^{\dagger\dagger}$ of $\mathcal{S}_m^\dagger$, in a way that the partial series F_m extends from $\mathcal{S}_m^\dagger$ to $\mathcal{S}_m^{\dagger\dagger}$. To arrange this, we essentially need to make sure that the above decomposition of U_p overconverges further yet, from $\mathcal{S}_1^\dagger$ to $\mathcal{S}_1^{\dagger\dagger}$. This step can be done using a general rigid analytic result on overconvergence of sections to finite étale maps between rigid analytic varieties ([Ber96, 1.3.5]).

Our hope is that the rather vague discussion in this section would serve as a "psychological" preparation for the upcoming classicality arguments in the Hilbert case.

2. Hilbert Modular Varieties

In the upcoming sections, we intend to present two types of analytic continuation results for overconvergent Hilbert modular forms. The first will be results on "domains of automatic analytic continuation" for overconvergent Hilbert modular forms as in [Kas13, KST12], where no slope conditions are given (apart from the finiteness of slope). These results have been used in proving cases of the strong Artin conjecture in [Kas13, KST12]. The second type will be classicality results in the presence of slope conditions as in [PS11], where the method presented in §1 is used. In preparation for the above, we will discuss the geometry of Hilbert modular varieties in this section, where the results are mostly from [GK12].

2.1. Notation. Let p be a prime number, $L/\mathbb{Q}$ a totally real field of degree g in which p is unramified, $\mathcal{O}_L$ its ring of integers, $\mathfrak{d}_L$ the different ideal, and N an integer prime to p. Let L^+ denote the elements of L that are positive under every embedding $L \hookrightarrow \mathbb{R}$. For a prime ideal $\mathfrak{p}$ of $\mathcal{O}_L$ dividing p, let $\kappa_\mathfrak{p} = \mathcal{O}_L/\mathfrak{p}$, $f_\mathfrak{p} = \deg(\kappa_\mathfrak{p}/\mathbb{F}_p)$, $f = \mathrm{lcm}\{f_\mathfrak{p} : \mathfrak{p}|p\}$, and κ a finite field with p^f elements. We identify $\kappa_\mathfrak{p}$ with a subfield of κ once and for all. Let $\mathbb{Q}_\kappa$ be the fraction field of $W(\kappa)$. We fix embeddings $\mathbb{Q}_\kappa \subset \mathbb{Q}_p^{\mathrm{ur}} \subset \overline{\mathbb{Q}}_p$.

Let $[\mathrm{Cl}^+(L)]$ be a complete set of representatives for the strict (narrow) class group $\mathrm{Cl}^+(L)$ of L, chosen so that its elements are ideals $\mathfrak{a} \lhd \mathcal{O}_L$, equipped with their natural positive cone $\mathfrak{a}^+ = \mathfrak{a} \cap L^+$. Let

$$\mathbb{B} = \mathrm{Emb}(L, \mathbb{Q}_\kappa) = \coprod_\mathfrak{p} \mathbb{B}_\mathfrak{p},$$

where $\mathfrak{p}$ runs over prime ideals of $\mathcal{O}_L$ dividing p, and $\mathbb{B}_\mathfrak{p} = \{\beta \in \mathbb{B} \colon \beta^{-1}(pW(\kappa)) = \mathfrak{p}\}$. Let σ denote the Frobenius automorphism of $\mathbb{Q}_\kappa$, lifting $x \mapsto x^p$ modulo p. It acts on $\mathbb{B}$ via $\beta \mapsto \sigma \circ \beta$, and transitively on each $\mathbb{B}_\mathfrak{p}$. For $S \subseteq \mathbb{B}$ we let

$$\ell(S) = \{\sigma^{-1} \circ \beta \colon \beta \in S\}, \qquad r(S) = \{\sigma \circ \beta \colon \beta \in S\},$$

and

$$S^c = \mathbb{B} - S.$$

The decomposition

$$\mathcal{O}_L \otimes_\mathbb{Z} W(\kappa) = \bigoplus_{\beta \in \mathbb{B}} W(\kappa)_\beta,$$

where $W(\kappa)_\beta$ is $W(\kappa)$ with the $\mathcal{O}_L$-action given by β, induces a decomposition,

$$M = \bigoplus_{\beta \in \mathbb{B}} M_\beta,$$

on any $\mathcal{O}_L \otimes_\mathbb{Z} W(\kappa)$-module M.

Let A be an abelian scheme over a scheme S, equipped with real multiplication $\iota \colon \mathcal{O}_L \to \mathrm{End}_S(A)$. Then the dual abelian scheme $A^\vee$ has a canonical real multiplication, and we let $\mathcal{P}_A = \mathrm{Hom}_{\mathcal{O}_L}(A, A^\vee)^{\mathrm{sym}}$. It is a projective $\mathcal{O}_L$-module of rank 1 with a notion of positivity; the positive elements correspond to $\mathcal{O}_L$-equivariant polarizations.

For a $W(\kappa)$-scheme S we shall denote by $\underline{A}/S$, or simply $\underline{A}$ if the context is clear, a quadruple:

$$\underline{A}/S = (A/S, \iota, \lambda, \alpha),$$

comprising the following data: A is an abelian scheme of relative dimension g over a $W(\kappa)$-scheme S, $\iota\colon \mathcal{O}_L \hookrightarrow \mathrm{End}_S(A)$ is a ring homomorphism. The map λ is a polarization as in [DP94], namely, an isomorphism $\lambda\colon (\mathcal{P}_A, \mathcal{P}_A^+) \to (\mathfrak{a}, \mathfrak{a}^+)$ for a representative $(\mathfrak{a}, \mathfrak{a}^+) \in [\mathrm{Cl}^+(L)]$ such that $A \otimes_{\mathcal{O}_L} \mathfrak{a} \cong A^\vee$. The existence of λ is equivalent, since p is unramified, to $\mathrm{Lie}(A)$ being a locally free $\mathcal{O}_L \otimes \mathcal{O}_S$-module. Finally, α is a rigid $\Gamma_{00}(N)$-level structure, that is, $\alpha\colon \mu_N \otimes_{\mathbb{Z}} \mathfrak{d}_L^{-1} \to A$ is an $\mathcal{O}_L$-equivariant closed immersion of group schemes.

Let $X/W(\kappa)$ be the Hilbert modular scheme classifying such data $\underline{A}/S = (A/S, \iota, \lambda, \alpha)$. Let $Y/W(\kappa)$ be the Hilbert modular scheme classifying $(\underline{A}/S, H)$, where $\underline{A}$ is as above and H is a finite flat isotropic $\mathcal{O}_L$-subgroup scheme of $A[p]$ of rank p^g, where isotropic means relative to the μ-Weil pairing for some $\mu \in \mathcal{P}_A^+$ of degree prime to p. Let

$$\pi\colon Y \to X$$

be the natural morphism, whose effect on points is $(\underline{A}, H) \mapsto \underline{A}$.

Let $\overline{X}, \mathfrak{X}, \mathfrak{X}_{\mathrm{rig}}$ be, respectively, the special fibre of X, the completion of X along $\overline{X}$, and the rigid analytic space associated to $\mathfrak{X}$ in the sense of Raynaud. We use similar notation $\overline{Y}, \mathfrak{Y}, \mathfrak{Y}_{\mathrm{rig}}$ for Y and let π denote any of the induced morphisms. These spaces have models over $\mathbb{Z}_p$ or $\mathbb{Q}_p$, denoted $X_{\mathbb{Z}_p}, \mathfrak{X}_{\mathrm{rig},\mathbb{Q}_p}$, etc. For a point $P \in \mathfrak{X}_{\mathrm{rig}}$ we denote by $\overline{P} = \mathrm{sp}(P)$ its specialization in $\overline{X}$, and similarly for Y. We denote the ordinary locus in $\overline{X}$ (respectively, $\overline{Y}$) by $\overline{X}^{\mathrm{ord}}$ (respectively, $\overline{Y}^{\mathrm{ord}}$). Let $\mathfrak{Y}^0_{\mathrm{rig}}$ be the rigid analytic variety over $W(\kappa)$ which classifies all $(\underline{A}, H, D)$ such that $\underline{A} \in \mathfrak{X}_{\mathrm{rig}}$, H, D are two subgroups of $\underline{A}$ of the type classified by $\mathfrak{Y}_{\mathrm{rig}}$, and $H \cap D = 0$. There are two morphisms $\pi_1, \pi_2 : \mathfrak{Y}^0_{\mathrm{rig}} \to \mathfrak{Y}_{\mathrm{rig}}$, where π_1 forgets D, and π_2 quotients out by D.

2.2. The (φ, η)-invariant on $\overline{Y}$. Let $\overline{Q} \in \overline{Y}$ correspond to $(\underline{A}, H)$ defined over a field $k \supseteq \kappa$. Let $f\colon A \to A/H$ be the natural projection and $f^t : A/H \to A$ be the map induced by multiplication by p. We have $f^t \circ f = [p]_A$ and $f \circ f^t = [p]_{A/H}$. The natural maps induced by f, f^t between the Lie algebras decompose as

$$\begin{aligned} \bigoplus_{\beta\in\mathbb{B}} \mathrm{Lie}(f)_\beta &\colon \bigoplus_{\beta\in\mathbb{B}} \mathrm{Lie}(\underline{A})_\beta \longrightarrow \bigoplus_{\beta\in\mathbb{B}} \mathrm{Lie}(\underline{A}/H)_\beta, \\ \bigoplus_{\beta\in\mathbb{B}} \mathrm{Lie}(f^t)_\beta &\colon \bigoplus_{\beta\in\mathbb{B}} \mathrm{Lie}(\underline{A}/H)_\beta \longrightarrow \bigoplus_{\beta\in\mathbb{B}} \mathrm{Lie}(\underline{A})_\beta. \end{aligned} \tag{2.2.1}$$

We define the following invariants of $\overline{Q}$ using these maps:

$$\begin{aligned} \varphi(\overline{Q}) &= \varphi(\underline{A}, H) = \{\beta \in \mathbb{B}\colon \mathrm{Lie}(f)_{\sigma^{-1}\circ\beta} = 0\}, \\ \eta(\overline{Q}) &= \eta(\underline{A}, H) = \{\beta \in \mathbb{B}\colon \mathrm{Lie}(f^t)_\beta = 0\}, \\ I(\overline{Q}) &= I(\underline{A}, H) = \ell(\varphi(\overline{Q})) \cap \eta(\overline{Q}) = \{\beta \in \mathbb{B}\colon \mathrm{Lie}(f)_\beta = \mathrm{Lie}(f^t)_\beta = 0\}. \end{aligned} \tag{2.2.2}$$

The elements of $I(\overline{Q})$ are the *critical indices* of [Sta97]. By assumption $\underline{A}$ satisfies the Rapoport condition, and, hence, for any $\beta \in \mathbb{B}$, both $\mathrm{Lie}(\underline{A})_\beta$ and $\mathrm{Lie}(\underline{A}/H)_\beta$ are one-dimensional. Since

$f \circ f^t$ is multiplication by $p = 0$ on the Lie algebras, it follows that always at least one of the maps $\mathrm{Lie}(f)_\beta$ and $\mathrm{Lie}(f^t)_\beta$ is zero for any $\beta \in \mathbb{B}$. This leads to the following definition.

Definition 2.2.1. A pair (φ, η) of subsets of $\mathbb{B}$ is called *admissible* if $\ell(\varphi^c) \subseteq \eta$. Given another admissible pair (φ', η') we say that

$$(\varphi', \eta') \geq (\varphi, \eta),$$

if both inclusions $\varphi' \supseteq \varphi, \eta' \supseteq \eta$ hold.

In the above definition, it is clear that if (φ, η) is admissible, then so is (φ', η'), and that the admissibility of (φ, η) is equivalent to $r(\eta^c) \subseteq \varphi$. It is also easy to see that there are 3^g admissible pairs.

Remark 2.2.2. If $H = \mathrm{Ker}(\mathrm{Fr}_{\underline{A}})$, then $\varphi(\underline{A}, H) = \mathbb{B}$. Similarly, If $H = \mathrm{Ker}(\mathrm{Ver}_{\underline{A}})$, then $\eta(\underline{A}, H) = \mathbb{B}$. The invariant (φ, η) can be thought of as telling us for every direction $\beta \in \mathbb{B}$ whether H is $\mathrm{Ker}(\mathrm{Fr}_{\underline{A}})$, $\mathrm{Ker}(\mathrm{Ver}_{\underline{A}})$, or neither, even though the subgroup H does not necessarily decompose.

2.3. The type invariant on $\overline{X}$. Let k be a perfect field of positive characteristic p. Let $\mathbb{D}$ denote the contravariant Dieudonné functor, $G \mapsto \mathbb{D}(G)$, from finite commutative p-primary group schemes G over k, to finite length $W(k)$-modules M equipped with two maps $\mathrm{Fr}\colon M \to M$, and $\mathrm{Ver}\colon M \to M$, such that $\mathrm{Fr}(\alpha m) = \sigma(\alpha)\mathrm{Fr}(m), \mathrm{Ver}(\sigma(\alpha)m) = \alpha \mathrm{Ver}(m)$ for $\alpha \in W(k), m \in M$ and $\mathrm{Fr} \circ \mathrm{Ver} = \mathrm{Ver} \circ \mathrm{Fr} = [p]$. This functor is an anti-equivalence of categories and commutes with base change. It follows that if G has rank p^ℓ the length of $\mathbb{D}(G)$ is ℓ. Applying the Dieudonné functor to the Frobenius morphism $\mathrm{Fr}_G : G \to G^{(p)}$ gives $\mathbb{D}(\mathrm{Fr}_G) : \mathbb{D}(G^{(p)}) \to \mathbb{D}(G)$. This map, in view of $\mathbb{D}(G^{(p)}) = \mathbb{D}(G) \otimes_{W(k),\sigma} W(k)$, results in a σ-linear map $\mathbb{D}(G) \to \mathbb{D}(G)$, which is nothing but the Frobenius morphism Fr of the Dieudonné module $\mathbb{D}(G)$. A similar statement is true for Ver. Let $g_i : G \to H_i$ be morphisms, for $i = 1, 2$. By considering $g_1 \times g_2 : G \to H_1 \times H_2$, and applying the exactness of $\mathbb{D}$, it follows that

$$\mathbb{D}(\mathrm{Ker}(g_1) \cap \mathrm{Ker}(g_2)) = \mathbb{D}(G)/(\mathrm{Im}(\mathbb{D}(g_1) + \mathrm{Im}(\mathbb{D}(g_2))). \tag{2.3.1}$$

Definition 2.3.1. Let k be a perfect field of characteristic p. For an abelian scheme $\underline{A}/k$ classified by $\overline{X}$, the *type* of $\underline{A}$ is a subset of $\mathbb{B}$ defined by

$$\tau(\underline{A}) = \{\beta \in \mathbb{B} : \mathbb{D}\left(\mathrm{Ker}(\mathrm{Fr}_A) \cap \mathrm{Ker}(\mathrm{Ver}_A)\right)_\beta \neq 0\}. \tag{2.3.2}$$

If $\overline{P}$ is a point on $\overline{X}$ corresponding to $\underline{A}$, we define $\tau(\overline{P}) = \tau(\underline{A})$.

2.4. The relationship between the type and the (φ, η) invariants.

Lemma 2.4.1. *Let $\overline{Q} = (\underline{A}, H)$ be a k-point of $\overline{Y}$.*

1. *$\beta \in \tau(\underline{A})$ if and only if one of the following equivalent statements hold:*
   (a) $\mathrm{Im}(\mathbb{D}(\mathrm{Fr}_A))_\beta = \mathrm{Im}(\mathbb{D}(\mathrm{Ver}_A))_\beta$.
   (b) $\mathrm{Im}(\mathrm{Fr})_\beta = \mathrm{Im}(\mathrm{Ver})_\beta$.
   (c) $\mathrm{Ker}(\mathrm{Fr})_\beta = \mathrm{Ker}(\mathrm{Ver})_\beta$.
2. $\beta \in \varphi(\underline{A}, H) \iff \mathrm{Im}(\mathbb{D}(\mathrm{Fr}_A))_\beta = \mathrm{Im}(\mathbb{D}(f))_\beta$.
3. $\beta \in \eta(\underline{A}, H) \iff \mathrm{Im}(\mathbb{D}(\mathrm{Ver}_A))_\beta = \mathrm{Im}(\mathbb{D}(f))_\beta$.

Proof. Basic properties of Dieudonné modules recalled in §2.3 imply that

$$\mathbb{D}(\mathrm{Ker}(\mathrm{Fr}_A)\cap\mathrm{Ker}(\mathrm{Ver}_A)) = \mathbb{D}(A[p])/(\mathrm{Im}\,\mathbb{D}(\mathrm{Fr}_A)+\mathrm{Im}\,\mathbb{D}(\mathrm{Ver}_A)) = \mathbb{D}(A[p])/(\mathrm{Im}\,\mathrm{Fr}+\mathrm{Im}\,\mathrm{Ver}).$$

The modules $\mathbb{D}(A[p])$, $\mathrm{Im}(\mathbb{D}(\mathrm{Fr}_A))$, and $\mathrm{Im}(\mathbb{D}(\mathrm{Ver}_A))$ all have actions of $\mathcal{O}_L\otimes_{\mathbb{Z}}W(\kappa)$, and, hence, decompose as a direct sum of their β-components in the usual way. Each $\mathbb{D}(A[p])_\beta$ is a two-dimensional k-vector space, and both $\mathrm{Im}(\mathbb{D}(\mathrm{Fr}_A))_\beta$ and $\mathrm{Im}(\mathbb{D}(\mathrm{Ver}_A))_\beta$ are one dimensional. This proves (1).

To prove (2) we recall the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & H^0(A,\Omega^1_{A/k}) \arrow[r] \arrow[d, equal] & H^1_{\mathrm{dR}}(A/k) \arrow[r] \arrow[d, equal] & H^1(A,\mathcal{O}_A) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{D}(\mathrm{Ker}(\mathrm{Fr}_A))\otimes_k k \arrow[r] \arrow[d, equal] & \mathbb{D}(A[p]) \arrow[r] & \mathbb{D}(\mathrm{Ker}(\mathrm{Ver}_A)) \arrow[r] & 0, \\
& \mathbb{D}(\mathrm{Ker}(\mathrm{Fr}_{A^{(1/p)}})) & & &
\end{tikzcd}$$

which is functorial in A. By duality, the map $\mathrm{Lie}(f)\colon \mathrm{Lie}(A)\to\mathrm{Lie}(B)$ induces the map

$$f^*\colon \mathrm{Lie}(B)^* = H^0(B,\Omega^1_{B/k}) \to \mathrm{Lie}(A)^* = H^0(A,\Omega^1_{A/k}),$$

which is precisely the pull-back map f^* on differentials. The map f^* has isotypic decomposition relative to the $\mathcal{O}_L\otimes k$-module structure.

Now, $\beta\in\varphi(f) \iff \mathrm{Lie}(f)_{\sigma^{-1}\circ\beta}=0 \iff f^*_{\sigma^{-1}\circ\beta}=0$. Via the identifications in the above diagram, the map f^* can also be viewed as a map

$$f^*\colon \mathbb{D}(\mathrm{Ker}(\mathrm{Fr}_{B^{(1/p)}}))\to\mathbb{D}(\mathrm{Ker}(\mathrm{Fr}_{A^{(1/p)}})),$$

which is equal to the linear map $\mathbb{D}(f^{(1/p)}|_{\mathrm{Ker}(\mathrm{Fr}_{A^{(1/p)}})})$. So,

$$\begin{aligned} f^*_{\sigma^{-1}\circ\beta}=0 &\iff \mathbb{D}(f^{(1/p)}|_{\mathrm{Ker}(\mathrm{Fr}_{A^{(1/p)}})})_{\sigma^{-1}\circ\beta}=0 \\ &\iff \mathbb{D}(f|_{\mathrm{Ker}(\mathrm{Fr}_A)})_\beta=0. \end{aligned}$$

We therefore have,

$$\beta\in\varphi(f)\iff\mathbb{D}(f|_{\mathrm{Ker}(\mathrm{Fr}_A)})_\beta=0.$$

Now, $\mathbb{D}(f|_{\mathrm{Ker}(\mathrm{Fr}_A)})_\beta=0$ if and only if $[\mathbb{D}(\mathrm{Ker}(\mathrm{Fr}_A))/\mathbb{D}(f)(\mathbb{D}(\mathrm{Ker}\,\mathrm{Fr}_B))]_\beta\neq 0$ and that is equivalent to $\mathbb{D}(A[p])_\beta/\left[\mathbb{D}(f)(\mathbb{D}(B[p]))+\mathbb{D}(\mathrm{Fr}_A)(\mathbb{D}(A^{(p)}[p]))\right]_\beta\neq 0$. By considering dimensions over k we see that this happens if and only if $\mathrm{Im}(\mathbb{D}(f))_\beta=\mathrm{Im}(\mathbb{D}(\mathrm{Fr}_A))_\beta$, as the lemma states.

We first show that

$$\mathrm{Lie}(f^t)_\beta=0\iff H^1(f)_\beta=0.$$

Let $\gamma \in \mathcal{P}_A$ be an isogeny of degree prime to p. In [GK12, Proof of Lemma 2.1.2], it is shown that there is an $i\gamma \in \mathcal{P}_B$ such that the following diagram is commutative:

$$\begin{tikzcd} A \arrow[r, "\gamma"] & A^\vee \\ B \arrow[u, "f^t"] \arrow[r, "i\gamma"'] & B^\vee \arrow[u, "f^\vee"'] \end{tikzcd} \tag{2.4.1}$$

Applying $\mathrm{Lie}(\cdot)_\beta$ to the diagram, we obtain

$$\begin{tikzcd} \mathrm{Lie}(A)_\beta \arrow[r, "\cong"] & \mathrm{Lie}(A^\vee)_\beta \\ \mathrm{Lie}(B)_\beta \arrow[u, "\mathrm{Lie}(f^t)_\beta"] \arrow[r, "\cong"] & \mathrm{Lie}(B^\vee)_\beta, \arrow[u, "\mathrm{Lie}(f^\vee)_\beta"'] \end{tikzcd}$$

and, hence,

$$\mathrm{Lie}(f^t)_\beta = 0 \Longleftrightarrow \mathrm{Lie}(f^\vee)_\beta = 0.$$

Since we have a commutative diagram:

$$\begin{tikzcd} \mathrm{Lie}(A^\vee) \arrow[r, "\cong"] & H^1(A, \mathcal{O}_A) \\ \mathrm{Lie}(B^\vee) \arrow[u, "\mathrm{Lie}(f^\vee)"] \arrow[r, "\cong"] & H^1(B, \mathcal{O}_B), \arrow[u, "H^1(f)"'] \end{tikzcd}$$

where we can pass to β-components, we conclude that $\mathrm{Lie}(f^t)_\beta = 0 \Longleftrightarrow H^1(f)_\beta = 0$.

The map $H^1(f)$ can be viewed as $\mathbb{D}(f|_{\mathrm{Ker}(\mathrm{Ver}_A)}) : \mathbb{D}(\mathrm{Ker}(\mathrm{Ver}_B)) \to \mathbb{D}(\mathrm{Ker}(\mathrm{Ver}_A))$, and hence, $H^1(f)_\beta = 0$ if and only if $\mathbb{D}(f|_{\mathrm{Ker}(\mathrm{Ver}_A)})_\beta = 0$. This is equivalent to

$$\mathbb{D}(A[p])_\beta / \Big[\mathbb{D}(f)(\mathbb{D}(B[p])) + \mathbb{D}(\mathrm{Ver}_A)(\mathbb{D}(A^{(p)}[p]))\Big]_\beta \neq 0.$$

Dimension considerations show that this happens if and only if $\mathrm{Im}(\mathbb{D}(f))_\beta = \mathrm{Im}(\mathbb{D}(\mathrm{Ver}_A))_\beta$. $\square$

We can now write down the relationship between the (φ, η) and τ.

Corollary 2.4.2. *Let $\overline{Q} = (\underline{A}, H)$ be a point of $\overline{Y}$, and $\overline{P} = \pi(\overline{Q}) = \underline{A}$ a point of $\overline{X}$. The following inclusions hold.*

$$\varphi(\overline{Q}) \cap \eta(\overline{Q}) \subseteq \tau(\overline{P}) \subseteq (\varphi(\overline{Q}) \cap \eta(\overline{Q})) \cup (\varphi(\overline{Q})^c \cap \eta(\overline{Q})^c)$$

2.5. Definition of the strata. We define the stratum $W_{\varphi,\eta}$ on $\overline{Y}$. We will show later that $\{W_{\varphi,\eta}\}$ is, indeed, a stratification of $\overline{Y}$. First, we need a lemma.

Lemma 2.5.1. *Given $\varphi \subseteq \mathbb{B}$ (respectively, $\eta \subseteq \mathbb{B}$), there is a locally closed subset U_φ, and a closed subset U_φ^+ (respectively, V_η and V_η^+) of $\overline{Y}$, such that U_φ (respectively, V_η) consists of the closed points $\overline{Q}$ of $\overline{Y}$ with $\varphi(\overline{Q}) = \varphi$ (respectively, $\eta(\overline{Q}) = \eta$), and U_φ^+ (respectively, V_η^+) consists of the closed point $\overline{Q}$ with $\varphi(\overline{Q}) \supseteq \varphi$ (respectively, $\eta(\overline{Q}) \supseteq \eta$).*

Proof. We will show the existence of U_φ and U_φ^+, as the rest can be done similarly. It suffices to prove that U_φ^+ is closed, because

$$U_\varphi = U_\varphi^+ - \bigcup_{\varphi' \supsetneq \varphi} U_{\varphi'}^+.$$

Furthermore, since $U_\varphi^+ = \bigcap_{\beta\in\varphi} U_{\{\beta\}}^+$, we reduce to the case where $\varphi = \{\beta\}$ is a singleton. Let $\overline{Q} = (\underline{A}, H)$, and consider the natural map $f\colon \underline{A} \to \underline{A}/H$. By definition, $\varphi(\overline{Q}) \supseteq \{\beta\}$, if and only if $\mathrm{Lie}(f)_{\sigma^{-1}\circ\beta} = 0$. Let $(\underline{A}^{\mathrm{univ}}, H^{\mathrm{univ}})$ be the universal object over $\overline{Y}$. Then, $\mathrm{Lie}(\underline{A}^{\mathrm{univ}})_{\sigma^{-1}\circ\beta}$ and $\mathrm{Lie}(\underline{A}^{\mathrm{univ}}/H^{\mathrm{univ}})_{\sigma^{-1}\circ\beta}$ are line bundles over $\overline{Y}$, and

$$\mathrm{Lie}(f)_{\sigma^{-1}\circ\beta}\colon \mathrm{Lie}(\underline{A}^{\mathrm{univ}})_{\sigma^{-1}\circ\beta} \longrightarrow \mathrm{Lie}(\underline{B}^{\mathrm{univ}})_{\sigma^{-1}\circ\beta}$$

is a morphism of line bundles and consequently its degeneracy locus $U_{\{\beta\}}^+ := \{\mathrm{Lie}(f)_{\sigma^{-1}\circ\beta} = 0\}$ is closed.

□

Definition 2.5.2. For an admissible pair (φ, η), we define

$$W_{\varphi,\eta} = U_\varphi \cap V_\eta.$$

By Lemma 2.5.1, $W_{\varphi,\eta}$ is a locally closed subset of $\overline{Y}$ with the property that a closed point $\overline{Q}$ of $\overline{Y}$ has invariants (φ, η) if and only if $\overline{Q} \in W_{\varphi,\eta}$. Similarly, we define

$$Z_{\varphi,\eta} = U_\varphi^+ \cap V_\eta^+.$$

It is a closed subset of $\overline{Y}$, and we have $Z_{\varphi,\eta} = \bigcup_{(\varphi',\eta')\geq(\varphi,\eta)} W_{\varphi',\eta'}$.

2.6. The infinitesimal nature of $\overline{Y}$. In this section, we will discuss the infinitesimal nature of $\overline{X}$, $\overline{Y}$. First we define the partial Hasse invariants on $\overline{X}$.

Definition 2.6.1. Let $\mathrm{Ver} = \mathrm{Ver}_{\underline{A}^{\mathrm{univ},(p)}} : \underline{A}^{\mathrm{univ},(p)} \to \underline{A}^{\mathrm{univ}}$ be the Verschiebung morphism. Pulling back by Ver, induces a morphism of sheaves $\mathrm{Ver}^* : \omega \to \omega^{(p)}$, which takes ω_β into $\omega^{(p)}_{\sigma^{-1}\circ\beta}$. This gives

$$h_\beta \in \mathrm{Hom}_{\overline{X}}(\omega_\beta, \omega^{(p)}_{\sigma^{-1}\circ\beta}) = H^0(\overline{X}, \omega_\beta^{-1} \otimes \omega^p_{\sigma^{-1}\circ\beta})$$

called the β-th partial Hasse invariant.

We will use the following result proven in [GO00].

Theorem 2.6.2. *(Goren-Oort) Let $\overline{P}$ be a closed k-rational point of $\overline{X}$. There is a choice of isomorphism*

$$\widehat{\mathcal{O}}_{X,\overline{P}} \cong W(k)[\![t_\beta : \beta \in \mathbb{B}]\!], \tag{2.6.1}$$

inducing an isomorphism

$$\widehat{\mathcal{O}}_{\overline{X},\overline{P}} \cong k[\![t_\beta : \beta \in \mathbb{B}]\!], \tag{2.6.2}$$

such that for all $\beta \in \tau(\overline{P})$, t_β is the image of h_β, the β-th partial Hasse invariant, in $\widehat{\mathcal{O}}_{\overline{X},\overline{P}}$.

In [Sta97] the infinitesimal nature of $\overline{Y}$ is studied. We recall a more specific version of Stamm's result here, and sketch a proof.

Theorem 2.6.3. *(Stamm) Let $\overline{Q} = (\underline{A}, H)$ be a point of $\overline{Y}$, defined over a field $k \supseteq \kappa$. Let $\varphi = \varphi(\overline{Q}), \eta = \eta(\overline{Q})$ and $I = I(\overline{Q}) = \ell(\varphi) \cap \eta$. Then, there is an isomorphism*

$$\widehat{\mathcal{O}}_{Y,\overline{Q}} \cong W(k)[\![\{x_\beta, y_\beta : \beta \in I\}, \{z_\gamma : \gamma \in I^c\}]\!]/(\{x_\beta y_\beta - p : \beta \in I\}). \tag{2.6.3}$$

inducing an isomorphism

$$\widehat{\mathcal{O}}_{\overline{Y},\overline{Q}} \cong k[\![\{x_\beta, y_\beta : \beta \in I\}, \{z_\gamma : \gamma \in I^c\}]\!]/(\{x_\beta y_\beta : \beta \in I\}). \tag{2.6.4}$$

such that, the following holds: if $\overline{Q} \in U^+_{\{\beta\}}$, then $U^+_{\{\beta\}} \cap \mathrm{Spf}(\widehat{\mathcal{O}}_{\overline{Y},\overline{Q}})$ is equal to $\mathrm{Spf}(\widehat{\mathcal{O}}_{\overline{Y},\overline{Q}})$ if $\beta \notin r(I)$, and is otherwise given by the vanishing of $y_{\sigma^{-1}\circ\beta}$. Similarly, if $\overline{Q} \in V^+_{\{\beta\}}$, then $V^+_{\{\beta\}} \cap \mathrm{Spf}(\widehat{\mathcal{O}}_{\overline{Y},\overline{Q}})$ is equal to $\mathrm{Spf}(\widehat{\mathcal{O}}_{\overline{Y},\overline{Q}})$ if $\beta \notin I$, and is otherwise given by the vanishing of x_β.

Proof. We sketch a proof. As in [DP94], one constructs a morphism from a Zariski-open neighborhood $T \subset \overline{Y}$ of $\overline{Q}$ to the Grassmann variety $\mathbf{G}$ associated to the data: $H = (\mathcal{O}_L \otimes k)^2$, two free $\mathcal{O}_L \otimes k$-sub-modules of H, say W_1, W_2, such that under the $\mathcal{O}_L \otimes k$ map $h : H \to H$ given by $(x, y) \mapsto (y, 0)$, we have $h(W_1) \subseteq W_2, h(W_2) \subseteq W_1$. Notice that we can perform the usual decomposition according to $\mathcal{O}_L$-eigenspaces to get

$$h = \oplus_\beta h_\beta : \oplus_\beta k^2_\beta \to \ \oplus_\beta \, k^2_\beta,$$

such that each h_β is the linear transformation corresponding to two-by-two matrix $M = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Furthermore, $W_i = \oplus_\beta (W_i)_\beta$, and $(W_i)_\beta$ is a one-dimensional k-vector space contained in k^2. We have $MW_1 \subseteq W_2, MW_2 \subseteq W_1$.

The basis for this construction is Grothendieck's crystalline theory. Let $f \colon \underline{A} \to \underline{B} = \underline{A}/H$ correspond to $\overline{Q}$. The $\mathcal{O}_L \otimes k$-module H is isomorphic to $H^1_{\mathrm{dR}}(\underline{A}/k)$. By the elementary divisors theorem, we can then identify $H^1_{\mathrm{dR}}(\underline{B}/k)$ with H, and possibly adjust the identification of $H^1_{\mathrm{dR}}(\underline{A}/k)$ with H, such that the induced maps f^* and $(f^t)^*$ are both the map h defined above. Let $W_A = H^0(A, \Omega^1_{A/k}) = \mathrm{Lie}(A)^* \subset H$ be the Hodge flitration, and similarly for W_B. Then we have $h(W_A) \subseteq W_B, h(W_B) \subseteq W_A$, and so we get a point $\mathbf{Q}$ of the Grassmann variety $\mathbf{G}$ described above. Let $\mathcal{O} = \widehat{\mathcal{O}}_{\overline{Y},\overline{Q}}$ and $\mathfrak{m}$ be the maximal ideal. By Grothendieck's theory, the deformations of $(f \colon \underline{A} \to \underline{B})$ over $R := \mathcal{O}/\mathfrak{m}^p$ (which carries a canonical divided power structure) are given by deformation of the Hodge filtration over that quotient ring. Namely, are in bijection with free, direct summands, $\mathcal{O}_L \otimes R$-modules (W^R_A, W^R_B) of rank one of $H \otimes_k \mathcal{O}/\mathfrak{m}^p = (\mathcal{O}_L \otimes R)^2$ such that $h(W^R_A) \subseteq W^R_B, h(W^R_B) \subseteq W^R_A$, and $W^R_A \otimes k = W_A, W^R_B \otimes k = W_B$. This, by the universal property of the Grassmann variety is exactly $\widehat{\mathcal{O}}_{\mathbf{G},\mathbf{Q}}/\mathfrak{m}^p_{\mathbf{G},\mathbf{Q}}$. A boot-strapping argument as in [DP94] furnishes an isomorphism of the completed local rings themselves, even in the arithmetic setting.

Therefore, to study the singularities and uniformization of the completed local rings, we can do so on the above Grassman variety. By considering each $\beta \in \mathbb{B}$ separately, we may reduce to the case of the Grassmann variety parameterizing two one-dimensional subspaces $\Lambda_1 = W_{1,\beta}, \Lambda_2 = W_{2,\beta}$ of k^2 satisfying $M\Lambda_1 \subseteq \Lambda_2, M\Lambda_2 \subseteq \Lambda_1$. Fix such a pair (Λ_1, Λ_2). If $\Lambda_1 \neq \mathrm{Ker}(M)$, then $\Lambda_2 = M\Lambda_1 = \mathrm{Ker}(M)$. It is an easy calculation to show that the same holds true for any deformation of Λ_1, Λ_2 to a local artinian k-algebra. Therefore the deformation is determined by choice of $\Lambda_1 = \mathrm{Span}\{(1,d)\}$, which implies that the local deformation ring is $k[\![z_\beta]\!]$. Here z_β is the parameter whose values for the particular discussed deformation is d . If $\Lambda_1 = \mathrm{Ker}(M)$ and $\Lambda_2 \neq \mathrm{Ker}(M)$ then the situation is similar and we see that the local deformation ring is $k[\![z_\beta]\!]$, where, in this case, the choice of letter z_β determines the deformation of Λ_2. Finally, suppose both $\Lambda_1 = \mathrm{Ker}(M)$ and $\Lambda_2 = \mathrm{Ker}(M)$. The subspace Λ_i is spanned by $(1,0)$ and a deformation of it to a local artinian k-algebra D is uniquely described by a basis vector $(1, d_i)$ where $d_i \in \mathfrak{m}_D$. The condition that the deformations are compatible under f is precisely $d_1 d_2 = 0$, and we see that the local deformation ring is $k[\![x_\beta, y_\beta]\!]/(x_\beta y_\beta)$.

Returning to the situation of abelian varieties, let $\overline{Q} = (\underline{A}, H)$, giving the two usual maps $f : \underline{A} \to \underline{B} = \underline{A}/H$, and $f^t : B \to A$. We have

$$(W_{1,\beta}, W_{2,\beta}) = (H^0(A, \Omega^1_{A/k})_\beta, H^0(B, \Omega^1_{B/k})_\beta),$$

and the condition $W_{1,\beta} = \mathrm{Ker}((f^t)^*)_\beta$ is the the same as $\beta \in \eta(\overline{Q})$, while $W_{2,\beta} = \mathrm{Ker}(f^*)_\beta$ is the condition that $\sigma \circ \beta \in \varphi(\overline{Q})$. The first case considered above, i.e.,

$$W_{1,\beta} \neq \mathrm{Ker}((f^t)^*)_\beta \qquad \text{and} \qquad W_{2,\beta} = \mathrm{Ker}(f^*)_\beta,$$

corresponds to the case $\beta \in \eta^c \cap \ell(\varphi) = \ell(\varphi) - I$. In this case, any deformation of $W_{1,\beta}$ and $W_{2,\beta}$ satisfy the same conditions as above, and, therefore, the condition "$\beta \in \ell(\varphi) - I$" continues to hold for every deformation. In particular, if $\beta \in \ell(\varphi) - I$, then $U^+_{\{\beta\}} \cap \mathrm{Spf}(\widehat{\mathcal{O}}_{\overline{Y},\overline{Q}}) = \mathrm{Spf}(\widehat{\mathcal{O}}_{\overline{Y},\overline{Q}})$. Similarly, studying the second case considered above gives us that if $\beta \in \eta - I$, then, we have $V^+_{\{\beta\}} \cap \mathrm{Spf}(\widehat{\mathcal{O}}_{\overline{Y},\overline{Q}}) = \mathrm{Spf}(\widehat{\mathcal{O}}_{\overline{Y},\overline{Q}})$. Finally, the third case

$$W_{1,\beta} = \mathrm{Ker}((f^t)^*)_\beta \qquad \text{and} \qquad W_{2,\beta} = \mathrm{Ker}(f^*)_\beta,$$

corresponds to $\beta \in \eta \cap \ell(\varphi)$. In this case, β belongs to the η-invariant of the deformation $(\tilde{W}_{1,\beta}, \tilde{W}_{2,\beta})$ if and only if $\tilde{W}_{1,\beta} = \mathrm{Ker}((f^t)^*)_\beta$, which corresponds to $d_1 = 0$. In terms of the parameters, this translates to $x_\beta = 0$. In other words, $V^+_\beta \cap \mathrm{Spf}(\widehat{\mathcal{O}}_{\overline{Y},\overline{Q}})$ is given by the vanishing of x_β. The remaining case can be done similarly. $\square$

2.7. The geometry of $\overline{Y}$. We now study the stratification $\{W_{\varphi,\eta}\}$ on $\overline{Y}$ defined in [GK12] and recall some of its properties. For an admissible pair (φ, η), we have defined the locally closed subset $W_{\varphi,\eta}$, and the closed subset $Z_{\varphi,\eta}$ in Definition 2.5.2.

Theorem 2.7.1. *Let (φ, η) be an admissible pair, $I = \ell(\varphi) \cap \eta$.*

(1) *$W_{\varphi,\eta}$ is nonempty and its Zariski closure equals $Z_{\varphi,\eta}$. The collection $\{W_{\varphi,\eta}\}$ is a stratification of $\overline{Y}$ by 3^g strata.*
(2) *The starata $W_{\varphi,\eta}$ and $Z_{\varphi,\eta}$ are equi-dimensional, and we have*
$$\dim(W_{\varphi,\eta}) = \dim(Z_{\varphi,\eta}) = 2g - (\sharp\, \varphi + \sharp\, \eta).$$
(3) *The irreducible components of $\overline{Y}$ are exactly the irreducible components of the strata $Z_{\varphi,\ell(\varphi^c)}$ for $\varphi \subseteq \mathbb{B}$.*
(4) *Let $\overline{Q}$ be a closed point of $\overline{Y}$ with invariants (φ, η), $I = \ell(\varphi) \cap \eta$. For an admissible pair (φ', η'), we have $\overline{Q} \in Z_{\varphi',\eta'}$ if and only if we have:*
$$\varphi \supseteq \varphi' \supseteq \varphi - r(I), \qquad \eta \supseteq \eta' \supseteq \eta - I.$$

In that case, write $\varphi' = \varphi - J, \eta' = \eta - K$ *(so that* $\ell(J) \subseteq I, K \subseteq I$ *and* $\ell(J) \cap K = \emptyset$*). We have:*

$$\widehat{\mathcal{O}}_{Z_{\varphi',\eta'},\overline{Q}} = \widehat{\mathcal{O}}_{\overline{Y},\overline{Q}}/\mathcal{I},$$

where $\mathcal{I}$ *is the ideal*

$$\mathcal{I} = \langle \{x_\beta : \beta \in I - K\}, \{y_\gamma : \gamma \in I - \ell(J)\} \rangle .$$

This implies that each stratum in the stratification $\{Z_{\varphi,\eta}\}$ *is non-singular.*

Proof. We first prove assertion (4). We keep the notation $\varphi = \varphi(\overline{Q}), \eta = \eta(\overline{Q}), I = I(\overline{Q})$. By definition, $\overline{Q} \in Z_{\varphi',\eta'}$, exactly when (φ', η') is an admissible pair satisfying $(\varphi', \eta') \leq (\varphi, \eta)$. Writing $\varphi' = \varphi - J$, and $\eta' = \eta - K$, the admissibility condition can be easily seen to be equivalent to $\ell(J) \subseteq I, K \subseteq I$ and $\ell(J) \cap K = \emptyset$. This implies that $\varphi \supseteq \varphi' \supseteq \varphi - r(I)$, and $\eta \supseteq \eta' \supseteq \eta - I$.

To write down the ideal of $Z_{\varphi',\eta'}$ at $\overline{Q}$, we use Theorem 2.6.3. To impose the condition that φ' is included in the φ-invariant at $\overline{Q}$, is to demand the vanishing of $y_{\sigma^{-1}\circ\beta}$ for all $\beta \in \varphi' \cap r(I) = r(I) - J$. In other words, we need to impose $y_\beta = 0$ for all $\beta \in I - \ell(J)$. Similarly, to require that η' is included in the η-invariant at $\overline{Q}$, is equivalent to the vanishing of x_β for all $\beta \in \eta' \cap I = I - K$. This proves the assertion (4).

If $\underline{A}$ is superspecial and H is the kernel of Frobenius, then $\overline{Q} = (\underline{A}, H)$ has invariants $(\mathbb{B}, \mathbb{B})$, and belongs to every stratum $Z_{\varphi,\eta}$, and hence each $Z_{\varphi,\eta}$ is non-empty. Assertion (4) also shows that $Z_{\varphi,\eta}$ is pure dimensional and $\dim(Z_{\varphi,\eta}) = 2g - (\sharp\, \varphi + \sharp\, \eta)$.

Since $Z_{\varphi,\eta} - W_{\varphi,\eta} = \bigcup_{(\varphi',\eta') \gneqq (\varphi,\eta)} Z_{(\varphi',\eta')}$ is a union of lower-dimensional strata, it follows that $W_{\varphi,\eta}$ is non-empty for all admissible (φ, η). The computations above show that $W_{\varphi,\eta}$ is pure-dimensional and $\dim(W_{\varphi,\eta}) = 2g - (\sharp\, \varphi + \sharp\, \eta)$ as well.

The stratum $Z_{\varphi,\eta}$ is closed and contains $W_{\varphi,\eta}$, and, hence, it contains $\overline{W}_{\varphi,\eta}$. Dimension considerations imply that $\overline{W}_{\varphi,\eta}$ must be a union of irreducible components of $Z_{\varphi,\eta}$. If $\overline{W}_{\varphi,\eta} \neq Z_{\varphi,\eta}$, then the remaining components of $Z_{\varphi,\eta}$ are contained in $\bigcup_{(\varphi',\eta') \gneqq (\varphi,\eta)} Z_{(\varphi',\eta')}$, which is not possible by dimension considerations.

It remains to prove assertion (3). First note that, by admissibility, $\dim(Z_{\varphi,\eta}) = g$ exactly when $\eta = \ell(\varphi)^c$. Let C be an irreducible component of $\overline{Y}$. Since C is contained in the union of all g-dimensional closed strata, it must be contained in a single one, i.e., $C \subseteq Z_{\varphi,\eta}$ for some (φ, η). Therefore, C must be an irreducible component of $Z_{\varphi,\eta}$ with $\eta = \ell(\varphi)^c$. Conversely, every irreducible component of $Z_{\varphi,\ell(\varphi)^c}$ is g-dimensional, and hence an irreducible component of $\overline{Y}$. In particular, $\overline{Y}$ is of pure dimension g. □

Lemma 2.7.2. *Let* $\overline{P}$ *be a closed point of* $\overline{X}$*. Then,* $\pi^{-1}(\overline{P}) \cap W_{\varphi,\eta}$ *has dimension at most* $g - \sharp\,(\varphi \cup \eta)$*, for any admissible pair* (φ, η)*.*

Proof. Fix a closed point $\overline{P}$ of $\overline{X}$, corresponding to $\underline{A}$ defined over an algebraically closed field k. Let $\mathbb{D} = \mathbb{D}(\underline{A}[p]) = \oplus_{\beta\in\mathbb{B}}\mathbb{D}_\beta$, and recall that each $\mathbb{D}_\beta$ is a 2-dimensional vector space over k on which $\mathcal{O}_L$ acts via β. Recall also that Ker(Ver) and Ker(Fr) in $\mathbb{D}$ decompose as

$$\mathrm{Ker}(\mathrm{Fr}) = \oplus_{\beta\in\mathbb{B}}\mathrm{Ker}(\mathrm{Fr})_\beta,$$

$$\mathrm{Ker}(\mathrm{Ver}) = \oplus_{\beta\in\mathbb{B}}\mathrm{Ker}(\mathrm{Ver})_\beta,$$

where each $\mathrm{Ker}(\mathrm{Fr})_\beta, \mathrm{Ker}(\mathrm{Ver})_\beta$ is a one dimensional subspace of $\mathbb{D}_\beta$. By part (1) of Lemma 2.4.1, we have

$$\beta \in \tau(\underline{A}) \Longleftrightarrow \mathrm{Ker}(\mathrm{Fr})_\beta = \mathrm{Ker}(\mathrm{Ver})_\beta.$$

Consider the variety $\mathscr{G} = \mathscr{G}(\overline{P})$ parameterizing subspaces $\mathbb{H} = \oplus_{\beta\in\mathbb{B}}\mathbb{H}_\beta$ of $\mathbb{D}$ satisfying the conditions:

- $\mathbb{H}_\beta \subset \mathbb{D}_\beta$ is 1-dimensional,
- $\mathrm{Fr}(\mathbb{H}(\beta)) \subseteq \mathbb{H}_{\sigma\circ\beta}$,
- $\mathrm{Ver}(\mathbb{H}(\beta)) \subseteq \mathbb{H}_{\sigma^{-1}\circ\beta}$.

We view $\mathscr{G}$ as a closed reduced subscheme of $(\mathbb{P}^1_k)^g$. Define a morphism

$$h\colon \pi^{-1}(\overline{P})_{\mathrm{red}} \longrightarrow \mathscr{G},$$

as follows. We use the identification $\mathbb{D} = H^1_{\mathrm{dR}}(\underline{A}, \mathcal{O}_{\underline{A}})$. The universal family $(f\colon \underline{A}^{\mathrm{univ}} \to \underline{B}^{\mathrm{univ}})$ over the reduced fibre $\pi^{-1}(\overline{P})_{\mathrm{red}}$ produces a sub-vector bundle of $\mathbb{D}\times\pi^{-1}(\overline{P})_{\mathrm{red}}$ by considering $f^*\mathbb{H}^1_{\mathrm{dR}}(\underline{B}, \mathcal{O}_{\underline{B}})$, which point-wise is $f^*\mathbb{H}^1_{\mathrm{dR}}(\underline{B}_x, \mathcal{O}_{\underline{B}_x}) = \mathbb{D}(f)(\mathbb{D}(\underline{B}_x[p]))$ $(x \in \pi^{-1}(\overline{P})_{\mathrm{red}})$, and so is a subspace of the kind parameterized by $\mathscr{G}$. By the universal property of Grassmann variety $(\mathbb{P}^1_k)^g = \mathrm{Grass}(1,2)^g$, we get a morphism $h\colon\ :\pi^{-1}(\overline{P})_{\mathrm{red}} \to (\mathbb{P}^1_k)^g$ that factors through $\mathscr{G}$, because it does so at every closed point of $\pi^{-1}(\overline{P})_{\mathrm{red}}$. We note that for every x as above $\mathbb{D}/f^*\mathbb{H}^1_{\mathrm{dR}}(\underline{B}_x, \mathcal{O}_{\underline{B}_x}) = \mathbb{D}(\mathrm{Ker}(f_x))$ and so it is clear that h is injective on geometric points and in fact, by the theory of Dieudonné modules, bijective. We have therefore constructed a bijective morphism

$$h\colon \pi^{-1}(\overline{P})_{\mathrm{red}} \longrightarrow \mathscr{G}.$$

Since h is a morphism between projective varieties, it is closed and hence it is a homeomorphism. For $\mathbb{H}\subset\mathbb{D}$ as above, define

$$\varphi(\mathbb{H}) = \{\beta\in\mathbb{B} : \mathbb{H}_\beta = \mathrm{Ker}(\mathrm{Ver})_\beta\},$$

$$\eta(\mathbb{H}) = \{\beta\in\mathbb{B} : \mathbb{H}_\beta = \mathrm{Ker}(\mathrm{Fr})_\beta\}.$$

Now, let $H \subset \underline{A}[p]$ be a subgroup scheme such that $(\underline{A}, H) \in \pi^{-1}(\overline{P})$, and $f : A \to A/H$ be the canonical map. Let $\mathbb{H} = \mathrm{Im}[\mathbb{D}(f)] = \mathrm{Ker}[\mathbb{D}(\underline{A}[p]) \to \mathbb{D}(H)]$. Then, by Lemma 2.4.1, we have

$$\begin{aligned}\varphi(\underline{A}, H) &= \{\beta\in\mathbb{B} : \mathrm{Im}(\mathbb{D}(f))_\beta = \mathrm{Im}(\mathbb{D}(\mathrm{Fr}_A))_\beta\}\\ &= \{\beta\in\mathbb{B} : \mathbb{H}_\beta = (\mathrm{Im}(\mathrm{Fr}))_\beta\}\\ &= \{\beta\in\mathbb{B} : \mathbb{H}_\beta = \mathrm{Ker}(\mathrm{Ver})_\beta\}\\ &= \varphi(\mathbb{H}).\end{aligned}$$

Similarly, we find that $\eta(\underline{A}, H) = \eta(\mathbb{H})$. It now follows that $\pi^{-1}(\overline{P})_{\mathrm{red}} \cap W_{\varphi,\eta}$ is homeomorphic to the locally closed subset of $\mathscr{G}$ parameterizing subspaces $\mathbb{H}$ with $\varphi(\mathbb{H}) = \varphi$ and $\eta(\mathbb{H}) = \eta$. Its dimension is thus at most $g - \sharp\,(\varphi\cup\eta)$. □

Remark 2.7.3. One can show that the equality holds if the fibre is non-empty. See [GK12, Cor. 2.6.7].

In the following we will define the generic locus of a stratum.

Definition 2.7.4. Let (φ,η) be an admissible pair. We define

$$W_{\varphi,\eta}^{gen}=\{(\underline{A},H)\in W_{\varphi,\eta}:\tau(\underline{A})=\varphi\cap\eta\}=\pi^{-1}(W_{\varphi\cap\eta})\cap W_{\varphi,\eta}.$$

Similarly, one can define $Z_{\varphi,\eta}^{gen}$.

Proposition 2.7.5. *$W_{\varphi,\eta}^{gen}$ is a Zariski dense open subset of $W_{\varphi,\eta}$, and, hence, of $Z_{\varphi,\eta}$.*

Proof. Since $W_{\varphi\cap\eta}$ is a Zariski dense open subset of $Z_{\varphi\cap\eta}$, it is enough to prove the following: if C is an irreducible component of $Z_{\varphi,\eta}$, then, $\pi(C)$ is an irreducible component of $Z_{\varphi\cap\eta}$. We argue as follows.

For every point $\overline{P}\in\pi(C)$, $\dim(\pi^{-1}(\overline{P})\cap C)\leq g-\sharp\,(\varphi\cup\eta)$, by Lemma 2.7.2. Therefore,

$$\begin{aligned}\dim(\pi(C))&\geq\dim(C)-(g-\sharp\,(\varphi\cup\eta))\\&=2g-(\sharp\,\varphi+\sharp\,\eta)-(g-\sharp\,(\varphi\cup\eta))\\&=g-\sharp\,(\varphi\cap\eta).\end{aligned}$$

On the other hand, since $\tau(\overline{P})\supseteq\varphi\cap\eta$, we have $\pi(C)\subseteq Z_{\varphi\cap\eta}$. Moreover, $\dim(Z_{\varphi\cap\eta})=g-\sharp\,(\varphi\cap\eta)$. Since π is proper, $\pi(C)$ is closed and irreducible. By comparing the dimensions, we conclude that $\pi(C)$ is an irreducible component of $Z_{\varphi\cap\eta}$. □

Remark 2.7.6. In fact, one can show that

$$\pi(W_{\varphi,\eta})=\bigcup_{\substack{(\varphi\cap\eta)\cup(\varphi^c\cap\eta^c)\supseteq\tau'\\ \tau'\supseteq\varphi\cap\eta}}W_{\tau'},$$

and

$$\pi(Z_{\varphi,\eta})=Z_{\varphi\cap\eta}.$$

See [GK12, Props. 2.6.4, 2.6.16].

In [GK12], the geometry of $\overline{Y}$ is studied in more detail. We would like to recall the following geometric result, which is used in the proof of the upcoming key lemma.

Theorem 2.7.7. *Let C be an irreducible component of $Z_{\varphi,\eta}$. Then*

$$C\cap W_{\mathbb{B},\mathbb{B}}\neq\emptyset.$$

Proof. See [GK12,]. □

2.8. The Key Lemma. In [GK12, Lemma 2.8.1], we prove a result which describes the morphism π infinitesimally. This result is crucial for many of the analytic continuation results in the Hilbert case, by giving the relationship between directional degrees and directional Hodge heights defined ahead. Here, we present the mod-p version of the lemma, and later we give a p-adic reformulation.

Let k be a finite field containing κ, and $\overline{Q}$ a closed point of $\overline{Y}$ with residue field k. Let $\overline{P}=\pi(\overline{Q})$; let $\varphi=\varphi(\overline{Q}),\eta=\eta(\overline{Q}),I=I(\overline{Q})$, and $\tau=\tau(\overline{P})$. We choose isomorphisms

$$\widehat{\mathcal{O}}_{\overline{Y},\overline{Q}}\cong k[\![\{x_\beta,y_\beta:\beta\in I\},\{z_\beta:\beta\in I^c\}]\!]/(\{x_\beta y_\beta:\beta\in I\}), \tag{2.8.1}$$

$$\widehat{\mathcal{O}}_{\overline{X},\overline{P}}\cong k[\![t_\beta:\beta\in\mathbb{B}]\!], \tag{2.8.2}$$

as in Theorems 2.6.2 and 2.6.3.

Lemma 2.8.1. *(The Key Lemma) Let $\beta\in\varphi\cap\eta$, and $\pi^*\colon\widehat{\mathcal{O}}_{\overline{X},\overline{P}}\to\widehat{\mathcal{O}}_{\overline{Y},\overline{Q}}$ the induced ring homomorphism.*

(1) *If $\sigma\circ\beta\in\varphi,\sigma^{-1}\circ\beta\in\eta$, then*
$$\pi^*(t_\beta)=ux_\beta+vy^p_{\sigma^{-1}\circ\beta},$$
for some units $u,v\in\widehat{\mathcal{O}}_{\overline{Y},\overline{Q}}$.

(2) *If $\sigma\circ\beta\in\varphi,\sigma^{-1}\circ\beta\notin\eta$, then*
$$\pi^*(t_\beta)=ux_\beta,$$
for some unit $u\in\widehat{\mathcal{O}}_{\overline{Y},\overline{Q}}$.

(3) *If $\sigma\circ\beta\notin\varphi,\sigma^{-1}\circ\beta\in\eta$, then*
$$\pi^*(t_\beta)=vy^p_{\sigma^{-1}\circ\beta},$$
for some unit $v\in\widehat{\mathcal{O}}_{\overline{Y},\overline{Q}}$.

(4) *If $\sigma\circ\beta\notin\varphi,\sigma^{-1}\circ\beta\notin\eta$, then*
$$\pi^*(t_\beta)=0.$$

Proof. We only give a rough sketch of the idea of the proof given in [GK12]. For another proof see [Tia11, §4]. □

2.9. The p-adic geometry of Y. Let $\mathbb{C}_p$ be the completion of an algebraic closure of $\mathbb{Q}_p$. It has a valuation $\mathrm{val}:\mathbb{C}_p\to\mathbb{Q}\cup\{\infty\}$ normalized so that $\mathrm{val}(p)=1$. Define the truncated p-adic valuation

$$\nu(x)=\min\{\mathrm{val}(x),1\}.$$

Recall the definitions of $\mathfrak{X}_{\mathrm{rig}}$, $\mathfrak{Y}_{\mathrm{rig}}$, $\mathfrak{Y}^0_{\mathrm{rig}}$, etc, given in §2.1.

Directional Hodge heights. Let $P\in\mathfrak{X}_{\mathrm{rig}}$ be a rigid point corresponding to $\underline{A}$ defined over the ring of integers $\mathcal{O}_K$ of a finite extension $K/\mathbb{Q}_\kappa$. Let $\bar{A}$ be the base change of A to $\mathcal{O}_K/p$. For each $\beta\in\mathbb{B}$, let $h_\beta\in\Gamma(\overline{X},\omega^p_{\sigma^{-1}\circ\beta}\otimes\omega_\beta)$ be the β-th partial Hasse invariant given in Definition 2.6.1. Using a basis of $\omega_{\bar{A}/\mathcal{O}_K}$, we may represent $h_\beta(\bar{A})$ as an element of $\mathcal{O}_K/p$. We define the β-th partial Hodge height of A to be

$$w_\beta(A)=\nu(\tilde{h}_\beta(\bar{A}))\in\mathbb{Q}\cap[0,1],$$

where $\tilde{h}_\beta(\bar{A})$ is any lift of $h_\beta(\bar{A})$ to $\mathcal{O}_K$. In our main reference [GK12, 4.2], these are denoted $\nu_\beta(A)$ and are defined using the geometry as follows:

Lemma 2.9.1. *Let $P \in \mathfrak{X}_{\mathrm{rig}}$ correspond to $\underline{A}$ defined over $\mathcal{O}_F$. Let $\{t_\beta\}_{\beta\in\mathbb{B}}$ be a sets of parameters at $\overline{P} \in \overline{X}$ as in (2.6.1). For any $\beta \in \tau(\overline{P})$, we have $w_\beta(A) = \nu(t_\beta(P))$. For any $\beta \notin \tau(\overline{P})$, we have $w_\beta(A) = 0$.*

Proof. This follows immediately from Theorem 2.6.2. □

Directional degrees. Let $Q = (\underline{A}, H)$ be a rigid point of $\mathfrak{Y}_{\mathrm{rig}}$ defned over a finite extension $K/\mathbb{Q}_\kappa$. Then $\underline{A}$ and H can be defined over $\mathcal{O}_K$. Let ω_H be the module of invariant differential 1-forms of H. We have a canonical decomposition

$$\omega_H = \bigoplus_{\beta\in\mathbb{B}} \omega_{H,\beta},$$

where each $\omega_{H,\beta}$ is a finitely generated torsion $\mathcal{O}_F$-module. We define

$$\deg_\beta(Q) = \deg_\beta(H) = \deg(\omega_{H,\beta}),$$

where the function deg is given in Definition 1.2.1.

We explain the relation to Raynaud group schemes. Let K be a finite extension of $\mathbb{Q}_\kappa$, and $Q = (\underline{A}, H)$ a point defined over $\mathcal{O}_K$. Then, for any $\mathfrak{p}|p$, the subgroup scheme $H[\mathfrak{p}]$ is a $\kappa_\mathfrak{p}$-Rayunad group scheme over $\mathcal{O}_K$, i.e., a $\kappa_\mathfrak{p}$-vector space scheme of dimension 1 over $\mathcal{O}_K$. By Raynaud's work, we have an isomorphism of schemes

$$H[\mathfrak{p}] \cong \mathrm{Spec}(\mathcal{O}_K)[T_\beta : \beta \in \mathbb{B}_\mathfrak{p}]/(T^p_{\sigma^{-1}\circ\beta} - a_\beta T_\beta : \beta \in \mathbb{B}_\mathfrak{p}),$$

such that $a_\beta \in \mathcal{O}_K$ and $0 \leq \mathrm{val}(a_\beta) \leq 1$ for all $\beta \in \mathbb{B}_\mathfrak{p}$. It is easy from this explicit description to see that if $\beta \in \mathbb{B}_\mathfrak{p}$, then

$$\deg_\beta(H) = \mathrm{val}(a_\beta) \in \mathbb{Q} \cap [0, 1].$$

We define $\underline{\deg}(H[\mathfrak{p}]) = (\mathrm{val}(a_\beta) : \beta \in \mathbb{B})$.

Remark 2.9.2. Raynuad shows that there is a one-to-one correspondence between the isomorphism classes of $\kappa_\mathfrak{p}$-Raynaud group schemes defined over $\overline{K}$ and $[0,1]^{\mathbb{B}_\mathfrak{p}} \cap \mathbb{Q}^{\mathbb{B}_\mathfrak{p}}$ given by

$$G \mapsto \underline{\deg}(G).$$

Definition 2.9.3. Let $Q = (\underline{A}, H)$ be a point of $\mathfrak{Y}_{\mathrm{rig}}$ defined over $\mathcal{O}_K$. We define

$$\underline{\deg}(Q) := \underline{\deg}(H) := (\underline{\deg}(H[\mathfrak{p}]) : \mathfrak{p}|p) = (\deg_\beta(H) : \beta \in \mathbb{B}).$$

We obtain a parametrization of $\mathfrak{Y}_{\mathrm{rig}}$ by the directional degrees

$$\underline{\deg} : \mathfrak{Y}_{\mathrm{rig}} \to [0,1]^{\mathbb{B}} =: \Theta.$$

In [GK12, 4.2], the above parametrization is given using the vector of valuations $(\nu_\beta(Q) : \beta \in \mathbb{B})$. These valuations are defined using the geometry of $\overline{Y}$, and the relationship between directional degrees and geometry can be summarized as follows:

Lemma 2.9.4. *Let Q be point of $\mathfrak{Y}_{\mathrm{rig}}$. Let $\{x_\beta, y_\beta\}_{\beta\in I(\overline{Q})}$ be parameters as in (2.6.3). We have*

$$\deg_\beta(Q) = \begin{cases} 0 & \beta \notin \ell(\varphi(\overline{Q})), \\ \nu(y_\beta(Q)) & \beta \in I(\overline{Q}), \\ 1 & \beta \notin \eta(\overline{Q}). \end{cases}$$

Proof. This follows easily from the defintions. See [Tia11, Prop 4.8] or [Kas06, Prop 3.1] □

The following lemma is useful in estimating the norm of the $U_{\mathfrak{p}}$ operators. For any $\underline{A} \in \mathfrak{X}_{\mathrm{rig}}$ defined over $\mathcal{O}_K$, a finite extension of $W(\kappa)$. Let $\omega_{\underline{A}} = \oplus_{\beta\in\mathbb{B}}\, \omega_{\underline{A},\beta}$ be the decomposition of the conormal sheaf of $\underline{A}/\mathrm{Spec}(\mathcal{O}_K)$ as an $\mathcal{O}_L \otimes_{\mathbb{Z}} W(\kappa)$-module.

Lemma 2.9.5. *Let $\underline{A}, \underline{B} \in \mathfrak{X}_{\mathrm{rig}}$ be defined over $\mathcal{O}_K$, and $\lambda : A \to B$ be an $\mathcal{O}_L$-morphism with kernel G. Then, $\deg_\beta(G) = \mathrm{val}(\lambda^* : \omega_{\underline{B},\beta} \to \omega_{\underline{A},\beta})$.*

Proof. This follows from the definition of $\deg_\beta$, and that $0 \to \omega_{\underline{B},\beta} \to \omega_{\underline{A},\beta} \to \omega_{G,\beta} \to 0$ is exact. □

We now present a refinement of Fargues's degree-increasing principle (Proposition 1.2.4), in terms of directional degrees.

Proposition 2.9.6. *Let H_1, H_2 be two $\kappa_{\mathfrak{p}}$-Raynaud group schemes over $\mathcal{O}_K$ as above. Then, there is a homomorphism $H_1 \to H_2$ which is generically an isomorphism if and only if*

$$\sum_{j=0}^{f_{\mathfrak{p}}-1} p^{f_{\mathfrak{p}}-1-j} \deg_{\sigma^j\circ\beta}(H_1) \le \sum_{j=0}^{f_{\mathfrak{p}}-1} p^{f_{\mathfrak{p}}-1-j} \deg_{\sigma^j\circ\beta}(H_2)$$

Proof. Let $H_i = \mathrm{Spec}(\mathcal{O}_K)[T_\beta : \beta \in \mathbb{B}_{\mathfrak{p}}]/(T^p_{\sigma^{-1}\circ\beta} - a_{i,\beta}T_\beta : \beta \in \mathbb{B}_{\mathfrak{p}})$ for $i = 1, 2$. By Raynaud's work, giving a homomorphism $H_1 \to H_2$ which is generically an isomorphism is equivalent to finding a collection of elements $u_\beta \in \mathcal{O}_K$ for $\beta \in \mathbb{B}_{\mathfrak{p}}$, such that $a_{1,\beta}u_\beta = a_{2,\beta}u^p_{\sigma^{-1}\circ\beta}$. This implies that $\Pi_{j=0}^{f_{\mathfrak{p}}-1}(a_{1,\sigma^{-j}\circ\beta}/a_{2,\sigma^{-j}\circ\beta})^{p^j} = u_\beta^{p^{f_{\mathfrak{p}}}-1}$, and shows that the existence of the u_β's is equivalent to the given condition. □

For convenience, we make the following definition.

Definition 2.9.7. Let G be a $\kappa_{\mathfrak{p}}$-Raynuad group scheme. For any $\beta \in \mathbb{B}_{\mathfrak{p}}$, we define

$$\tilde{\deg}_\beta(G) := \sum_{j=0}^{f_{\mathfrak{p}}-1} p^{f_{\mathfrak{p}}-1-j} \deg_{\sigma^j\circ\beta}(G).$$

We define $\underline{\tilde{\deg}}(G) = (\tilde{\deg}_\beta(G))_{\beta\in\mathbb{B}_{\mathfrak{p}}}$. Let $Q = (\underline{A}, H)$ be a point of $\mathfrak{Y}_{\mathrm{rig}}$. For any $\beta \in \mathbb{B}_{\mathfrak{p}} \subset \mathbb{B}$, we set $\tilde{\deg}_\beta(Q) := \tilde{\deg}_\beta(H) := \tilde{\deg}_\beta(H[\mathfrak{p}])$. We set $\underline{\tilde{\deg}}(Q) = (\tilde{\deg}_\beta(Q))_{\beta\in\mathbb{B}}$. We also set

$$\tilde{\Theta} := \tilde{\deg}(\mathfrak{Y}_{\mathrm{rig}}),$$

a "skewed hypercube" which can evidently be obtained by applying a linear transformatin to the standard hypercube $\Theta = [0, 1]^{\mathbb{B}}$.

In the above notation, if H_1 and H_2 are two $\kappa_{\mathfrak{p}}$-Raynuad group schemes, a morphism $H_1 \to H_2$ which is generically an isomorphism exists if and only if $\tilde{\deg}_\beta(H_1) \le \tilde{\deg}_\beta(H_2)$, for all $\beta \in \mathbb{B}_{\mathfrak{p}}$.

2.10. The Key Lemma revisited. The Key Lemma can be lifted mod p to give information about the relationship between the directional Hodge heights and the directional degrees.

Lemma 2.10.1. *(The Key Lemma II) Let $Q = (\underline{A}, H)$ be a point of $\mathfrak{Y}_{\rm rig}$, and $P = \underline{A}$ its image under π in $\mathfrak{X}_{\rm rig}$. We have*

$$w_\beta(P) \geq \min\{p \deg_{\sigma^{-1}\circ\beta}(Q), 1 - \deg_\beta(Q)\},$$

and the equality happens if $p \deg_{\sigma^{-1}\circ\beta}(Q) \neq 1 - \deg_\beta(Q)$.

Proof. The lemma follows directly from Lemma 2.8.1. We give some indications. Let $\overline{Q}$, $\overline{P}$ denote, respectively, the specializations of Q and P. Let $\varphi = \varphi(\overline{Q})$, $\eta = \eta(\overline{Q})$, $I = I(\overline{Q})$, and $\tau = \tau(\overline{P})$.

First, if $\beta \notin \varphi \cap \eta$, then the above statement holds trivially: if $\beta \notin \eta$, Lemma 2.9.4 implies that $1 - \deg_\beta(Q) = 0$, and if $\beta \notin \varphi$, then we have $p \deg_{\sigma^{-1}\circ\beta}(Q) = 0$. In both cases the statement is equivalent to the trivial $w_\beta(P) \geq 0$. Besides, in this case, we have $p \deg_{\sigma^{-1}\circ\beta}(Q) \neq 1 - \deg_\beta(Q)$ exactly when $\beta \in \varphi \cap \eta^c$ or $\beta \in \eta \cap \varphi^c$. In both cases, by Corollary 2.4.2, $\beta \notin \tau$. Lemma 2.9.1 implies that $w_\beta(P) = 0$, i.e., the equality holds in these cases.

Assume now that $\beta \in \varphi \cap \eta$, so that the Key Lemma applies. We will only prove the result in the first case, i.e, under the assumption $\sigma \circ \beta \in \varphi, \sigma^{-1} \circ \beta \in \eta$. The other cases follow similarly. In this case, we have $\{\sigma^{-1} \circ \beta, \beta\} \subset I$. Assuming $\overline{Q}$ is defined over k, pick isomorphisms

$$\widehat{\mathcal{O}}_{Y,\overline{Q}} \cong W(k)[\![\{x_\beta, y_\beta : \beta \in I\}, \{z_\beta : \beta \in I^c\}]\!]/(\{x_\beta y_\beta : \beta \in I\}), \tag{2.10.1}$$

$$\widehat{\mathcal{O}}_{X,\overline{P}} \cong W(k)[\![t_\beta : \beta \in \mathbb{B}]\!], \tag{2.10.2}$$

as in Theorems 2.6.2 and 2.6.3. Applying the Key Lemma, and lifting mod p, we find

$$\pi^*(t_\beta) = u x_\beta + v y^p_{\sigma^{-1}\circ\beta} + pG,$$

for some units $u, v \in \widehat{\mathcal{O}}_{Y,\overline{Q}}$, and some $G \in \widehat{\mathcal{O}}_{Y,\overline{Q}}$. Applying Lemmas 2.9.1 and 2.9.4, we find $\nu(x_\beta(Q)) = 1 - \deg_\beta(Q)$, $\nu(y^p_{\sigma^{-1}\circ\beta}) = \min\{1, p \deg_{\sigma^{-1}\circ\beta}(Q)\}$, and $\nu(\pi^*(t_\beta(Q))) = w_\beta(P)$. The statement now follows immediately.

□

3. Domains of automatic analytic continuation

In the following, we will present a result which roughly states that every overconvergent Hilbert modular form of finite slope automatically extends to a "big" region of the Hilbert modular variety. We will hint at what "big" refers to at the end of this section. The results we present here have been used in [Kas13], [KST12] to prove certain cases of the Strong Artin Conjecture over totally real fields. We will perform the analytic continuation in two steps: first we prove automatic analytic continuation to the *canonical locus* as in [GK12], and, then, we show further analytic continuation to a "big" region Σ.

3.1. The preliminaries. Let us officially define overconvergent Hilbert modular forms. Let $\underline{k} = (k_\beta)_{\beta\in\mathbb{B}} \in \mathbb{Z}^{\mathbb{B}}$. Define

$$\omega^{\underline{k}} := \otimes_{\beta\in\mathbb{B}}\, \omega_\beta^{k_\beta}$$

which is line bundle on $\mathfrak{Y}_{\mathrm{rig}}$. For any interval $J \subset [0, g]$, define $\mathfrak{Y}_{\mathrm{rig}}J$ to be the admissible open of $\mathfrak{Y}_{\mathrm{rig}}$ whose points are characterized by $\deg \in J$.

Definition 3.1.1. An overconvergent Hilbert modular forms of level $\Gamma_1(N) \cap \Gamma_0(p)$ and weight $\underline{k}$, is a section of $\omega^{\underline{k}}$ on $\mathfrak{Y}_{\mathrm{rig}}[g-\epsilon, g]$ for some $\epsilon > 0$. The space of such forms is denoted $\mathcal{M}^{\dagger}_{\underline{k}}$.

We define the $U_{\mathfrak{p}}$-operators for $\{\mathfrak{p}|p\}$. Let $U_{\mathfrak{p}} : \mathfrak{Y}_{\mathrm{rig}} \to \mathfrak{Y}_{\mathrm{rig}}$ be the correspondence which sends $(\underline{A}, H)$ to $\{(\underline{A}/D, \bar{H}) : (\underline{A}, D) \in \mathfrak{Y}_{\mathrm{rig}}, D \neq H\}$. If $\mathcal{V}$, $\mathcal{W}$ are opens in $\mathfrak{Y}_{\mathrm{rig}}$ such that $U_{\mathfrak{p}}(\mathcal{W}) \subset \mathcal{V}$, then $U_{\mathfrak{p}} : \omega^{\underline{k}}(\mathcal{V}) \to \omega^{\underline{k}}(\mathcal{W})$ can be defined via

$$U_{\mathfrak{p}}(f)(\underline{A}, H) = \frac{1}{p^{f_{\mathfrak{p}}}} \sum_{(\underline{A}/D, \bar{H}) \in U_{\mathfrak{p}}(\underline{A}, H)} \mathrm{pr}^* f(\underline{A}/D, \bar{H}),$$

where pr^* is induced by pulling back differentials under $\mathrm{pr} : \underline{A} \to \underline{A}/D$. We define $U_p = \circ_{\mathfrak{p}|p} U_{\mathfrak{p}}$.

We record the degree-increasing principle for the U operators, which can be proven exactly as Proposition 1.2.11.

Proposition 3.1.2. *Let $\mathfrak{p}|p$. Let $Q \in \mathfrak{Y}_{\mathrm{rig}}$, and $Q' \in U_{\mathfrak{p}}(\{Q\})$. Then,* $\deg(Q') \geq \deg(Q)$.

3.2. Analytic continuation, the first step. We first show that any overconvergent Hilbert modular form of finite slope extends to the canonical locus defined in [GK12]. We define the following admissible opens in $\mathfrak{Y}_{\mathrm{rig}}$ and $\mathfrak{X}_{\mathrm{rig}}$

$$\mathcal{V}_{can} = \{Q \in \mathfrak{Y}_{\mathrm{rig}} : \deg_{\beta}(Q) + p \deg \sigma^{-1} \circ \beta(Q) > 1, \forall \beta \in \mathbb{B}\},$$

$$\mathcal{U}_{can} = \{P \in \mathfrak{X}_{\mathrm{rig}} : w_{\beta}(P) + p w_{\sigma^{-1} \circ \beta}(P) < p, \forall \beta \in \mathbb{B}\}.$$

The following result was essentially proved in [GK12].

Proposition 3.2.1. *Let $f \in \mathcal{M}^{\dagger}_{\underline{k}}$ be such that $U_{\mathfrak{p}}(f) = a_{\mathfrak{p}} f$ and $a_{\mathfrak{p}} \neq 0$ for all $\mathfrak{p}|p$. Then, f extends analytically to $\mathcal{V}_{can}$.*

Proof. We sketch a proof. For simplicity, we assume p is inert in $\mathcal{O}_L$. The general case can be done by repeating the same argument in "all directions." We have, therefore, $U_p(f) = a_p f$, and $a_p \neq 0$. Define

$$\mathcal{W}_{can} = \{Q \in \mathfrak{Y}_{\mathrm{rig}} : \deg_{\beta}(Q) + p \deg \sigma^{-1} \circ \beta(Q) > 1, \forall \beta \in \mathbb{B}\}.$$

In [GK12, §5], using several applications of the Key Lemma, it is shown that

$$\pi^{-1}(\mathcal{U}_{can}) = \mathcal{V}_{can} \cup \mathcal{W}_{can},$$

and that $\pi : \mathcal{V}_{can} \to \mathcal{U}_{can}$ is an isomorphism. We skip this rather lengthy proof and refer the reader to [GK12, §5]. Let $Q = (\underline{A}, H) \in \mathcal{V}_{can}$, and $P = \underline{A} \in \mathcal{U}_{can}$. By definition of $\mathcal{V}_{can}$, Lemma 2.10.1 implies that

$$w_{\beta}(P) = \min\{p \deg_{\sigma^{-1} \circ \beta}(Q), 1 - \deg_{\beta}(Q)\} = 1 - \deg_{\beta}(Q).$$

Consider now any $Q' = (\underline{A}, D) \neq Q$. It follows from the above facts that $Q' \in \mathcal{W}_{can}$, and applying Lemma 2.10.1 implies that

$$w_{\beta}(P) = \min\{p \deg_{\sigma^{-1} \circ \beta}(Q'), 1 - \deg_{\beta}(Q')\} = p \deg_{\sigma^{-1} \circ \beta}(Q').$$

This shows that if $\deg_\beta(H) = 1-\lambda_\beta$, then $\deg_\beta(A[p]/D) = 1-\deg_\beta(D) = 1-\frac{\lambda_{\sigma\circ\beta}}{p}$, for any $D \neq H$. In particular, there is $M \gg 0$ such that $U_p^M(Q)$ lies in the domain of definition of f. We now proceed as in the proof of Proposition 1.2.10 (and the discussion before it) to extend f to $\mathcal{V}_{can}$.

□

3.3. Analytic continuation, the second step. We will now use Proposition 3.2.1 to prove further automatic analytic continuation to a region of $\mathfrak{Y}_{\rm rig}$ denoted Σ. We will explain the significance of Σ at the end of this section.

We first make some definitions. Let $W_{\varphi,\eta}$ be a stratum of $\overline{Y}$. For $\mathfrak{p}|p$, define $\varphi_\mathfrak{p} = \varphi \cap \mathbb{B}_\mathfrak{p}$, and $\eta_\mathfrak{p} = \eta \cap \mathbb{B}_\mathfrak{p}$. We say that $W_{\varphi,\eta}$ is *not étale at* $\mathfrak{p}$ if $(\varphi_\mathfrak{p}, \eta_\mathfrak{p}) \neq (\emptyset, \mathbb{B}_\mathfrak{p})$. We say that $W_{\varphi,\eta}$ *has no étale part*, if $W_{\varphi,\eta}$ is not étale at any $\mathfrak{p}|p$.

Let $W_{\varphi,\eta}$ be a stratum of codimension 1. By admissibility and part (2) of Theorem 2.7.1, there is a unique $\beta_0 \in \mathbb{B}$ such that $\ell(\varphi) \cap \eta = \{\beta_0\}$. We say that $W_{\varphi,\eta}$ is *bad*, if $\sigma \circ \beta_0 \in \eta$; otherwise, we say that $W_{\varphi,\eta}$ is *good*.

Definition 3.3.1. Let W be a stratum of codimension 0 or 1 with no étale part. We will define an admissible open subset $]W^{\rm gen}['$ of $\mathfrak{Y}_{\rm rig}$ as follows:

Case 1: $\mathrm{codim}(W) = 0$. We put $]W^{\rm gen}['=]W^{\rm gen}[$, where $W^{\rm gen}$ is the generic part of W defined in Definition 2.7.4.

Case 2: $\mathrm{codim}(W) = 1$ *and W is good.* We put $]W^{\rm gen}['=]W^{\rm gen}[$.

Case 3: $\mathrm{codim}(W) = 1$ *and $W = W_{\varphi,\eta}$ is bad.* Let $\ell(\varphi) \cap \eta = \{\beta_0\}$ and $\mathfrak{p}_0|p$ be such that $\beta_0 \in \mathbb{B}_{\mathfrak{p}_0}$. We distinguish two cases:

Case 3a: $\varphi_{\mathfrak{p}_0} = \{\sigma \circ \beta_0\}$ *and* $\eta_{\mathfrak{p}_0} = \mathbb{B}_{\mathfrak{p}_0}$. We set

$$]W^{\rm gen}['= \begin{cases} \{Q \in]W[\,:\, \deg_{\beta_0}(Q) > \frac{1}{p+1}\} & \text{if } f_{\mathfrak{p}_0} = 1; \\ \{Q \in]W^{\rm gen}[\,:\, \deg_{\beta_0}(Q) \in (\sum_{i=1}^{f_{\mathfrak{p}_0}-1} \frac{1}{p^i}, 1)\} & \text{if } f_{\mathfrak{p}_0} > 1. \end{cases}$$

Case 3b: otherwise. We put

$$]W^{\rm gen}['= \{Q \in]W^{\rm gen}[\,:\, \deg_{\beta_0}(Q) \in (0, \frac{1}{p}) \cup (\sum_{i=1}^{f_{\mathfrak{p}_0}-1} \frac{1}{p^i}, 1)\}.$$

We define an admissible open subset of $\mathfrak{Y}_{\rm rig}$:

$$\Sigma = \bigcup_W \,]W^{\rm gen}[',$$

where W runs though all the strata of codimension 0 and 1 with no étale part.

Now we can state the main result.

Theorem 3.3.2. *Let $f \in M_{\underline{k}}^\dagger$ be such that $U_\mathfrak{p}(f) = a_\mathfrak{p} f$, and $a_\mathfrak{p} \neq 0$, for all $\mathfrak{p}|p$. Then, f extends analytically to Σ.*

We refer the reader to [KST12] for a complete proof, however, in the following we give some indications of how the proof proceeds. In view of Proposition 3.2.1, it enough to show that f extends from $\mathcal{V}_{can}$ to Σ. It turns out that we can indeed prove the following.

Proposition 3.3.3. *We have $U_p(\Sigma) \subset \mathcal{V}_{can}$. In other words, if $(\underline{A}, H) \in \Sigma$, then, for any $(\underline{A}, D) \in \mathfrak{Y}_{\rm rig}$ such that $D \neq H$, and for any $\beta \in \mathbb{B}$, we have*

$$p \deg_\beta(D) + \deg_{\sigma\circ\beta}(D) < p.$$

Theorem 3.3.2 follows from this, since, following our usual approach, we can define the extension of f from $\mathcal{V}_{can}$ to Σ to be $\frac{U_p(f)}{a_p}$.

One needs a lemma.

Lemma 3.3.4. *Let $\mathfrak{p}|p$. Let $(\underline{A}, H) \in \mathfrak{Y}_{\rm rig}$, and $(\underline{A}, D) \in \mathfrak{Y}_{\rm rig}$ be such that $D[\mathfrak{p}] \neq H[\mathfrak{p}]$. Then, for any $\beta \in \mathbb{B}_\mathfrak{p}$, we have*

$$\deg_\beta(H) + \deg_\beta(D) \leq 1 + \sum_{j=1}^{f_\mathfrak{p}-1} \frac{1}{p^j}.$$

Proof. Suppose A, H, D are defined over $\mathcal{O}_K$, a finite extension of $W(\kappa)$. Consider the homomorphism $D[\mathfrak{p}] \to A[\mathfrak{p}]/H[\mathfrak{p}]$ of $\kappa_\mathfrak{p}$-Raynaud group schemes over $\mathcal{O}_K$, which is generically an isomorphism. By Proposition 2.9.6, we have

$$\sum_{j=0}^{f_\mathfrak{p}-1} p^{f_\mathfrak{p}-1-j} \deg_{\sigma^j\circ\beta}(D) \leq \sum_{j=0}^{f_\mathfrak{p}-1} p^{f_\mathfrak{p}-1-j} \deg_{\sigma^j\circ\beta}(A/H).$$

The result now follows by a simple estimation, using the fact that $\deg_\beta(A/H) = 1 - \deg_\beta(H)$ for all $\beta \in \mathbb{B}$.

□

Let us assume p is inert in $\mathcal{O}_L$ from this point on for simplicity of presentation. As usual, the arguments easily work in all "directions." Under this assumption, we have one relevant Hecke operator at p called U_p, and we have $f_{p\mathcal{O}_L} = g$.

The proof of Proposition 3.3.3 involves many applications of the Key Lemma as well as information from the mod p geometry of $\overline{Y}$. We will not reproduce the proof here, but to give the reader an idea of the type of arguments involved, we will prove Proposition 3.3.3 in the simpler case when the specialization of $Q = (\underline{A}, H)$ to $\overline{Y}$ belongs to a codimension-0 stratum $W_{(\varphi,\eta)}$ with no étale part. By assumption, we have $\eta = \ell(\varphi^c)$, and $\eta \neq \mathbb{B}$. In this case, Proposition 3.3.3 follows evidently from the following lemma.

Lemma 3.3.5. *Let assumptions be as above, and notation as in Proposition 3.3.3 . For any $\beta \in \ell(\eta) \cap \eta^c$, we have $\deg_\beta(D) \leq \sum_{j=1}^{g-1} \frac{1}{p^j}$, and, for any $\beta \in \ell(\eta)^c \cup \eta$, we have $\deg_\beta(D) = 0$.*

Proof. We first prove a sublemma.

Sublemma. Let $\beta \in \eta^c$. Then $\deg_\beta(D) \leq \sum_{j=1}^{g-1} \frac{1}{p^j}$, and $\deg_{\sigma^{-1}\circ\beta}(D) = 0$.

Proof. Since $\beta \in \eta^c(\overline{Q})$, it follows that $\deg_\beta(Q) = \deg_\beta(H) = 1$ by Lemma 2.9.4. Lemma 3.3.4, then, implies that $\deg_\beta(D) \leq \sum_{j=1}^{g-1} \frac{1}{p^j}$. Since $\overline{Q} \in W_{\varphi,\eta}^{\rm gen}$, we have $\tau(\pi(\overline{Q})) = \varphi \cap \eta$, and hence $\beta \notin \tau(\pi(\overline{Q}))$. This implies that $w_\beta(\underline{A}) = 0$. Applying Lemma 2.10.1, it follows that

$$0 = w_\beta(\underline{A}) \geq \min\{p \deg_{\sigma^{-1}\circ\beta}(D), 1 - \deg_\beta(D) > 0\},$$

and, hence, we must have $\deg_{\sigma^{-1}\circ\beta}(D) = 0$. □

We now return to the proof of Lemma 3.3.5. If $\beta \in \ell(\eta) \cap \eta^c$, then, in particular, $\beta \in \eta^c$, and $\deg_\beta(D) \leq \sum_{j=1}^{g-1} \frac{1}{p^j}$ by the Sublemma. Assume, now, that $\beta \in \ell(\eta)^c \cup \eta$. If $\beta \in \ell(\eta^c)$, then $\sigma \circ \beta \in \eta^c$, and applying the Sublemma to $\sigma \circ \beta$ we obtain $\deg_\beta(D) = 0$. Finally, assume that $\beta \in \eta$. Since $\eta \neq \mathbb{B}$, there is a smallest $m > 0$ such that $\sigma^m \circ \beta \notin \eta$, but $\sigma^j \circ \beta \in \eta$ for all $0 \leq j \leq m-1$. We prove by induction that $\deg_{\sigma^j \circ \beta}(D) = 0$ for all $0 \leq j \leq m-1$. The statement is true for $j = m-1$ by applying the Sublemma to $\sigma^m \circ \beta$. Now assume that $\deg_{\sigma^j \circ \beta}(D) = 0$ for some j satisfying $1 \leq j \leq m-1$. Then, $\sigma^{j-1} \circ \beta \in \eta$, and, hence, $\sigma^j \circ \beta \notin \varphi = r(\eta^c)$. Since $\overline{Q} \in W^{\rm gen}_{\varphi,\eta}$, this implies that $\sigma^j \circ \beta \notin \tau(\pi(\overline{Q}))$, or $w_{\sigma^j \circ \beta}(\underline{A}) = 0$. We apply the Key Lemma II to conclude that

$$0 = w_{\sigma^j \circ \beta}(\underline{A}) \geq \min\{p \deg_{\sigma^{j-1} \circ \beta}(D), 1 - \deg_{\sigma^j \circ \beta}(D) = 1\},$$

whence, $\deg_{\sigma^{j-1} \circ \beta}(D) = 0$. This completes the induction, proving that $\deg_\beta(D) = 0$. □

To finish the proof of Proposition 3.3.3, we need to consider the remaining cases of 2, 3a, 3b in Definition 3.3.1. In these cases, since $\varphi \neq r(\eta)^c$, the above induction gets interrupted, and one needs to consider several possibilities. The Key Lemma continues to be an essential ingredient for the rest of the proof.

We point out that this result extends to overconvergent Hilbert modular forms of level $\Gamma_1(Np)$, where an analogue of Σ is defined, simply, by pulling back Σ under the natural forgetful map to level $\Gamma_1(N) \cap \Gamma_0(p)$.

In the end, we would like to explain the sense in which Σ is "big". Let us assume that p is inert in $\mathcal{O}_L$ for simplicity. Let $w : \mathfrak{Y}_{\rm rig} \to \mathfrak{Y}_{\rm rig}$ be the Atkin-Lehner involution $(\underline{A}, H) \mapsto (\underline{A}/H, \underline{A}[p]/H)$. In [KST12], we prove the following result.

Proposition 3.3.6. *Assume that p is inert in $\mathcal{O}_L$ (for simplicity). Then, $\Sigma \cup w(\Sigma)$ contains a region of the form $]C[\subset \mathfrak{Y}_{\rm rig}$, such that $\operatorname{codim}_{\overline{Y}}(\overline{Y} - C) \geq 2$. In particular, applying the rigid analytic Koecher principle, we have*

$$H^0(\Sigma \cup w(\Sigma), \omega^{\underline{k}}) = H^0(\mathfrak{Y}_{\rm rig}, \omega^{\underline{k}}).$$

In other words, if f, g are overconvergent Hilbert modular forms of finite slope such that $f = w(g)$ on $\Sigma \cap w(\Sigma)$, then both f and g are classical. In [KST12] this result is used (albeit in level $\Gamma_1(Np)$) to prove certain cases of the strong Artin conjecture over totally real fields. The corresponding result in the case of level $\Gamma_1(N) \cap \Gamma_0(p)$ is proven in [Kas13], where analytic continuation to a smaller region $\mathfrak{Y}_{\rm rig}^{|\tau| \leq 1}[\sum_{j=1}^{g-1} \frac{1}{p^j}, g] \subset \Sigma$ suffices for proving the classicality of the forms involved. See also [Pil12].

4. Classicality

4.1. In this section, we present the proof of the following Theorem.

Theorem 4.1.1. *Let f be an overconvergent Hilbert modular form of weight $\underline{k} = (k_\beta)_{\beta \in \mathbb{B}}$. Assume that for all $\mathfrak{p}|p$, $U_\mathfrak{p}(f) = a_\mathfrak{p} f$ with $\operatorname{val}(a_\mathfrak{p}) < \inf\{k_\beta : \beta \in \mathbb{B}_\mathfrak{p}\} - f_\mathfrak{p}$. Then, f is classical.*

In the following, we will present Pilloni-Stroh's proof in [PS11]. The theroem was proved by Sasaki [Sas10] in the case $f_\mathfrak{p} \leq 1$ for all $\mathfrak{p}|p$, and by Tian [Tia11] in the case $f_\mathfrak{p} \leq 2$ for all $\mathfrak{p}|p$. Recently, Tian-Xiao have given a proof providing better bounds using an approach similar to Coleman's original proof.

From this point on, we will always assume that p is inert in $\mathcal{O}_L$. The general case can be done by applying the arguments in "all directions" as in Sasaki's proof. Under our assumption, the notation simplifies as follows. We index the elements of $\mathbb{B}$, by picking an arbitrary β_1, and matching

$$\sigma^{i-1} \circ \beta_1 \leftrightarrow i,$$

for $1 \leq i \leq g-1$. We will write $M_i, x_i, \deg_i$, etc., in place of $M_{\sigma^{i-1}\circ\beta_1}, x_{\sigma^{i-1}\circ\beta_1}, \deg_{\sigma^{i-1}\circ\beta_1}$, etc. We have $\kappa = \kappa_{p\mathcal{O}_L}$, $f_{p\mathcal{O}_L} = g$, and $U_p = U_{p\mathcal{O}_L}$.

The proof follows the method of [Kas06] as presented in §1. We begin by determining the degrees of the various κ-Raynaud subgroups of an abelian scheme $\underline{A}$ which has two κ-Raynaud subgroups whose degree vectors are two opposing vertices of the cube $[0,1]^g$. This can be thought of as a partial analogue of Proposition 1.2.8.

Let G_1 and G_2 be κ-Raynaud group schemes over $\mathcal{O}_K$, a finite extension of $W(\kappa)$. We say $G_1 \leq G_2$, if there is a morphism $G_1 \to G_2$ which is generically an isomorphism. We write $G_1 < G_2$, if $G_1 \leq G_2$ and G_1 is not isomorphic to G_2. By Proposition 2.9.6,

$$G_1 \leq G_2 \Longleftrightarrow \tilde{\deg}_i(G_1) \leq \tilde{\deg}_i(G_2),\ \forall 1 \leq i \leq g. \tag{4.1.1}$$

Lemma 4.1.2. *Let G_1, G_2 be as above. There is a κ-Raynaud group scheme $\inf(G_1, G_2)$ over $\mathcal{O}_K$ with the property that for any κ-Raynaud scheme H, we have*

$$H \leq G_1 \text{ and } H \leq G_2 \Longleftrightarrow H \leq \inf(G_1, G_2).$$

Proof. In view of the equivalence (4.1.1) and Remark 2.9.2, it is enough to take $\inf(G_1, G_2)$ to be such that $\tilde{\deg}_i(\inf(G_1, G_2)) = \inf\{\tilde{\deg}_i(G_1), \tilde{\deg}_i(G_2)\} := d_i$ for all i. It is an elementary argument to see that $(d_i)_i$ belongs to $\tilde{\Theta}$ (Definition 2.9.7), and, hence, $\inf(G_1, G_2)$ exists. □

The following lemma is key in calculating the degrees of the various κ-Raynaud subgroup schemes of an $\underline{A} \in \mathfrak{X}_{\mathrm{rig}}$ in certain cases.

Lemma 4.1.3. *Let $\underline{A} \in \mathfrak{X}_{\mathrm{rig}}$, and $(\underline{A}, H_1), (\underline{A}, H_2) \in \mathfrak{Y}_{\mathrm{rig}}$. Assume $\tilde{\deg}_i(H_1) \neq \tilde{\deg}_i(H_2)$ for all $1 \leq i \leq g$. Then, for any $(\underline{A}, H) \in \mathfrak{Y}_{\mathrm{rig}}$ such that $H \neq H_1, H_2$, we have*

$$H \cong \inf(H_1, H_2).$$

In other words, $\tilde{\deg}_i(H) = \inf\{\tilde{\deg}_i(H_1), \tilde{\deg}_i(H_2)\}$, for all i.

Proof. We first prove a Sublemma.

Sublemma: Let G_1, G_2, G_3 be κ-Raynaud subgroup schemes of $\underline{A}[p]$ defined over $\mathcal{O}_K$. We have $\inf(G_1, G_2) \leq G_3$.

Proof. Consider $\underline{A}[p](\bar{K})$, a κ-vector space of dimension 2. We can arrange that $G_j(\bar{K})$ is generated by $x_j \in \underline{A}[p](\bar{K})$, for $j = 1, 2$, and $G_3(\bar{K})$ is generated by $x_1 + x_2$. The morphism $\inf(G_1, G_2) \to G_j$ can be modified, using the $\mathcal{O}_L$-action, to ensure that a generator of $\inf(G_1, G_2)$ is mapped to x_j, for $j = 1, 2$. Consider the composite morphism $\inf(G_1, G_2) \to G_1 \times G_2 \to A[p]$, where the first morphism is induced by the above morphisms, and the second one is multiplication inside $\underline{A}[p]$. This morphism has image G_3, and is generically an isomorphism. □

We continue with the proof of Lemma 4.1.2. Fix i, and assume that $\tilde{\deg}_i(H_2) < \tilde{\deg}_i(H_1)$. The sublemma implies that $\inf(H_1, H_2) \leq H$, whence $\tilde{\deg}_i(H_2) \leq \tilde{\deg}_i(H)$. Similarly, we have $\inf(H_1, H) \leq H_2$, which implies that $\inf\{\tilde{\deg}_i(H_1), \tilde{\deg}_i(H)\} \leq \tilde{\deg}_i(H_2) < \tilde{\deg}_i(H_1)$. It follows that $\tilde{\deg}_i(H) = \tilde{\deg}_i(H_2) = \inf\{\tilde{\deg}_i(H_1), \tilde{\deg}_i(H_2)\}$. $\square$

Let us record the relevant degree-increasing principle for U_p in this context.

Proposition 4.1.4. *Let $Q \in \mathfrak{Y}_{\rm rig}$, and $Q' \in U_p(\{Q\})$. Then, $\tilde{\deg}_i(Q') \geq \tilde{\deg}_i(Q)$ for $1 \leq i \leq g$. In particular, $\deg(Q') \geq \deg(Q)$.*

Proof. Let $Q = (\underline{A}, H)$ and $Q' = (\underline{A}/D, \bar{H})$. The result follows from Proposition 2.9.6, by considering the natural morphism $H \to \bar{H}$. $\square$

Before proceeding, we would like to give the reader some indication of how the proof proceeds from here. We invite the reader to read §1.3 closely. The notation in the general discussion presented there is chosen to match the following presentation.

The key step in the analytic continuation of an overconvergent f is to understand the behavior of U_p on the *special locus* $\mathcal{S}_1$ where U_p does not strictly increase degrees. In fact, it is crucial to understand this behavior over a strict neighborhood $\mathcal{S}_1^\dagger$ of $\mathcal{S}_1$ for gluing purposes. The method in [Kas06] works well, if we can find $\mathcal{S}_1^\dagger$ over which U_p decomposes as $U_p^{sp} + U_p^{nsp}$, where U_p^{sp} is dividing by a special subgroup, and U_p^{nsp}, the complementary correspondence, takes $\mathcal{S}_1^\dagger$ into a locus where f is already defined. This implies that the only undetermined term in the extension-to-be of f, i.e., "$\frac{U_p(f)}{a_p}$", would be $\frac{U_p^{sp}(f)}{a_p}$. Repeating this process, as explained in §1, we can construct a series which provides the extension of f over a strict neighborhood of the special locus.

This construction will be done around each vertex $\underline{x}$ of $[0,1]^g$ separately, and, in fact, inside a strict neighborhood $\mathcal{S}_0^\dagger(\underline{x})$ of $\underline{\deg}^{-1}(\underline{x})$. In the following, we will first construct $\mathcal{S}_0^\dagger(\underline{x})$, $\mathcal{S}_1^\dagger(\underline{x})$, and, then, define $\{\mathcal{S}_m^\dagger(\underline{x})\}_{m\geq 2}$ formally from this data.

Let us fix, then, the vertex $\underline{x} = (x_i)$, given by $x_i = 1$ if $i \in T \subset \{1, \cdots, g\}$, and $x_i = 0$ otherwise. We assume $r := |T| < g$. Let $\underline{1} = (1, 1, \cdots, 1)$. For any rational $\delta > 0$, we define $W_\delta(\underline{x}) = \{Q \in \mathfrak{Y}_{\rm rig} : |\deg_i(Q) - x_i| \leq \delta\}$, a quasi-compact open subset of $\mathfrak{Y}_{\rm rig}$.

Lemma 4.1.5. *Let notation be as above.*

(1) *Let $\underline{A} \in \mathfrak{X}_{\rm rig}$, and $(\underline{A}, H_1), (\underline{A}, H_2), (\underline{A}, H)$, be three distinct points of $\mathfrak{Y}_{\rm rig}$. If $\underline{deg}(H_1) = \underline{x}$ and $\underline{deg}(H_2) = \underline{1} - \underline{x}$, then $H \cong \inf(H_1, H_2)$. Furthermore, If $r \neq 0$, we have $H < H_1$ and $H < H_2$, and, in particular, $\deg(H) < \deg(H_2) = g - r$.*

(2) *Given $\epsilon > 0$, there exists a positive rational $\delta < \epsilon$, such that for $W := W_\delta(\underline{x})$, we have*
   (a) *If $Q = (\underline{A}, H_1) \in W$ is such that $U_p(\{Q\}) \cap W \neq \emptyset$, then*
$$U_p(\{Q\}) \cap W = \{(\underline{A}/H_2, \bar{H}_1)\},$$
   where $(\underline{A}, H_2) \in \mathfrak{Y}_{\rm rig}$ is called the special subgroup of Q, and for any κ-Raynaud subgroup $H \neq H_1, H_2$ of $\underline{A}$, we have $H \cong \inf(H_1, H_2)$.
   (b) *If $r \neq 0$, and D_1, D_2 are κ-Raynaud group schemes satisfying*
$$|\deg_i(D_1) - x_i| \leq \delta \quad \text{and} \quad |\deg_i(D_2) - (1 - x_i)| \leq \delta, \qquad (\dagger)$$

for all $1 \leq i \leq g$*, then, there is a rational* $\epsilon_0 > 0$ *such that*

$$\deg(\inf(D_1, D_2)) \leq \alpha := (g-r)(1-\delta) - \epsilon_0.$$

(3) *There is a quasi-compact open subset* $W_1 \subset W$ *such that*

$$W_1 = \{Q \in W : U_p(Q) \cap W \neq \emptyset\} = \{Q \in W : |U_p(Q) \cap W| = 1\}.$$

(4) *There is an analytic family of subgroups of* $\underline{A}^{univ}[p]$ *over* W_1 *which at every* $Q \in W_1$ *gives the special subgroup of* Q*.*

Proof. If $r = 0$, all the statements follow from the proof of proposition 3.2.1. In the following, therefore, we assume $r \neq 0$.

We first prove (1). We show that $\tilde{\deg}_i(H_1) \neq \tilde{\deg}_i(H_2)$ for all i. If not, then

$$\sum_{i+j \in T} p^{g-1-j} = \sum_{i+j \notin T} p^{g-1-j}$$

which is impossible as one side is at least p^{g-1}, and the other at most $1 + p + ... + p^{g-2} < p^{g-1}$. The claim now follows from Lemma 4.1.3. Now assume $r \neq 0$. To show the second statement in (1), we must show that $H_1 \not\leq H_2$ and $H_2 \not\leq H_1$. Since $r \neq 0$, there is i such that $x_i = 1$, and arguing as above, we find that $\tilde{\deg}_i(H_1) \geq p^{g-1} > \tilde{\deg}_i(H_2)$. The other direction follows similarly, using $r \neq g$.

For part (2) of the Lemma, one can show, by a simple continuity argument, that $\delta < \epsilon$ can be chosen such that for D_1, D_2, two κ-Raynaud group schemes satisfying (†), the statement (b) holds true and we have $\tilde{\deg}_i(D_1) \neq \tilde{\deg}_i(D_2)$, for all $1 \leq i \leq g$. Now, assume we are in the situation of part (a), and H_2 is a Raynaud subgroup of $\underline{A}[p]$ such that $(\underline{A}/H_2, \bar{H}_1) \in U_p(\{Q\}) \cap W$. It follows, then, from Lemma 4.1.3, that for any Raynaud subgroup $H \neq H_1, H_2$, we have $H \cong \inf(H_1, H_2)$. Furthermore, no such H satisfies $(\underline{A}/H, \bar{H}_1) \in U_p(\{Q\}) \cap W$, since, otherwise, we would have $\deg(\underline{A}/H, \bar{H}_1) \in \deg(W)$, or $g - \deg(H) = g - \deg(\inf(H_1, H_2)) \leq r + (g-r)\delta$, which contradicts part (b).

For part (3), we note that $W_1 = \pi_1(\pi_1^{-1}(W) \cap \pi_2^{-1}(W))$ is a quasi-compact open in $\mathfrak{Y}_{\mathrm{rig}}$, as $\pi_1, \pi_2 : \mathfrak{Y}^0_{\mathrm{rig}} \to \mathfrak{Y}_{\mathrm{rig}}$ are finite-flat maps, and it satisfies the desired property.

For the last statement, note that, by part (2), we have

$$\pi_1^{-1}(W_1) \subset \pi_2^{-1}(W) \bigsqcup \pi_2^{-1}(\mathfrak{Y}_{\mathrm{rig}}[g-\alpha, g]),$$

where the right side is an admissible disjoint union since $\deg(W) \leq r + (g-r)\delta = g - \alpha - \epsilon_0$, i.e., $W \subset \mathfrak{Y}_{\mathrm{rig}}[0, g-\alpha-\epsilon_0]$. This implies that $\pi_1 : \pi_1^{-1}(W) \cap \pi_2^{-1}(W) \to W_1$ is a finite-flat morphism, which, by part 2a), has degree 1. Hence, the map is an isomorphism, and its inverse provides a family of special subgroups on W_1 as desired.

□

Let us keep in mind that we will prove the desired analytic continuation by an induction process going from $\deg = r+1$ to $\deg = r$. In particular, we will apply the above results at a stage of the induction where f has been extended to $\mathfrak{Y}_{\mathrm{rig}}(r, g]$, and we intend to further extend f to a strict neighborhood $\mathcal{S}_0^\dagger(\underline{x})$ of $\underline{\deg}^{-1}(\underline{x})$. Why can't we take $\mathcal{S}_0^\dagger(\underline{x}) = W$, so that the special locus $\mathcal{S}_1^\dagger(\underline{x})$ equals W_1? It appears this should work, as we understand the behavior of U_p on W_1, as follows: for any point $Q \in W_1$, there is a unique point $Q' \in U_p(Q)$ which lies in W_1, and the rest fall in $\mathfrak{Y}_{\rm rig}(r,g]$ where f has been defined. The problem is that we do not yet understand the bahavior of U_p on $W - W_1$. To fix this problem, we will take $\mathcal{S}_0^\dagger(\underline{x})$ to be a subset of W on which we can determine the bahavior of U_p. We will define this below.

Given any $\underline{\tilde{z}} = (\tilde{z}_i) \in \tilde{\Theta}$, define $\mathfrak{Y}_{\rm rig}^{\geq \underline{\tilde{z}}} = \{Q \in \mathfrak{Y}_{\rm rig} : \tilde{\deg}_i(Q) \geq \tilde{z}_i, \forall\, 1 \leq i \leq g\}$. For $t \in [0,g] \cap \mathbb{Q}$, define

$$\mathfrak{Y}_{\rm rig}^{\geq \underline{\tilde{z}}}[0,t] = \{Q \in \mathfrak{Y}_{\rm rig}^{\geq \underline{\tilde{z}}} : \deg(Q) \leq t\}.$$

We have the following elementary result.

Lemma 4.1.6. *The collection of the regions $\{\mathfrak{Y}_{\rm rig}^{\geq \underline{\tilde{z}}}[0,t] : t > r, , \underline{\tilde{z}} \in \tilde{\Theta}\}$ contains a fundamental system of strict neighborhoods of $\underline{\deg}^{-1}(\underline{x})$.*

Definition 4.1.7. Lemma 4.1.6 implies that there is a rational $r < t_{\underline{x}} < r + \epsilon_0$ (where ϵ_0 is as in part (b) of Lemma 4.1.5 if $r \neq 0$, and can be taken 1 if $r = 0$), and $\underline{\tilde{z}} \in \tilde{\Theta}$, such that $\mathfrak{Y}_{\rm rig}^{\geq \underline{\tilde{z}}}[0, t_{\underline{x}}]$ is a strict neighborhood of $\underline{\deg}^{-1}(\underline{x})$ contained in W. We define $\mathcal{S}_0^\dagger(\underline{x})$ to be this strict neighborhood of $\underline{\deg}^{-1}(\underline{x})$.

Lemma 4.1.8. *The region $\mathcal{S}_0^\dagger(\underline{x}) \cup \mathfrak{Y}_{\rm rig}[t_{\underline{x}}, g]$ is U_p-stable.*

Proof. We have $\mathcal{S}_0^\dagger(\underline{x}) \cup \mathfrak{Y}_{\rm rig}[t_{\underline{x}}, g] = \mathfrak{Y}_{\rm rig}^{\geq \underline{\tilde{z}}} \cup \mathfrak{Y}_{\rm rig}[t_{\underline{x}}, g]$, which is stable under U_p by Proposition 4.1.4 □

We can now identify the *spcial locus* of $\mathcal{S}_0^\dagger(\underline{x})$ as $\mathcal{S}_1^\dagger(\underline{x})$, and study its properties.

Proposition 4.1.9. *There is a quasi-compact open subset $\mathcal{S}_1^\dagger(\underline{x}) \subset \mathcal{S}_0^\dagger(\underline{x})$ such that*

$$\begin{aligned} \mathcal{S}_1^\dagger(\underline{x}) &= \{Q \in \mathcal{S}_0^\dagger(\underline{x}) : U_p(\{Q\}) \cap \mathcal{S}_0^\dagger(\underline{x}) \neq \emptyset\} \\ &= \{Q \in \mathcal{S}_0^\dagger(\underline{x}) : U_p(\{Q\}) \cap \mathcal{S}_0^\dagger(\underline{x}) = \{Q'\}\}. \end{aligned}$$

If $Q = (\underline{A}, H)$, $Q' = (\underline{A}/G_1, \bar{H})$, we call G_1 the special subgroup of $(\underline{A}, H)$. The following hold:

1. *There is an analytic family of subgroups of $\underline{A}^{univ}[p]$ over $\mathcal{S}_1^\dagger(\underline{x})$ which at every $Q \in \mathcal{S}_1^\dagger(\underline{x})$ gives the special subgroup of Q.*
2. *Over $\mathcal{S}_1^\dagger(\underline{x})$, we have $U_p = U_p^{sp} + U_p^{nsp}$ such that $U_p^{sp} : \mathcal{S}_1^\dagger(\underline{x}) \to \mathcal{S}_0^\dagger(\underline{x})$ is given by dividing by the special subgroup, and $U_p^{nsp} : \mathcal{S}_1^\dagger(\underline{x}) \to \mathfrak{Y}_{\rm rig}[t_{\underline{x}}, g]$ is the complement of U_p^{sp}.*
3. *$U_p(\mathcal{S}_0^\dagger(\underline{x}) - \mathcal{S}_1^\dagger(\underline{x})) \subset \mathfrak{Y}_{\rm rig}[t_{\underline{x}}, g]$.*

Proof. We simply take $\mathcal{S}_1^\dagger(\underline{x}) = W_1 \cap \mathcal{S}_0^\dagger(\underline{x})$. All statements but the last follow from Lemma 4.1.5. The last statement follows from Lemma 4.1.8 and the characterization of $\mathcal{S}_1^\dagger(\underline{x})$. □

Remark 4.1.10. The special locus at vertex $\underline{x}$ is $\mathcal{S}_1(\underline{x}) = \mathcal{S}_1^\dagger(\underline{x}) \cap \underline{\deg}^{-1}(\underline{x})$.

The following is a classical result on automatic overconvergence of sections. See [Ber96, 1.3.5].

Lemma 4.1.11. *Let $\lambda : \mathcal{Y}^0 \to \mathcal{Y}$ be a finite étale morphism between quasi-compact rigid analytic spaces. Assume that λ admits a section s over $\mathcal{S} \subset \mathcal{Y}$. Then s extends to a section $s^\dagger : \mathcal{S}^\dagger \to \mathcal{Y}^0$ to λ, where $\mathcal{S}^\dagger$ is a strict neighborhood of $\mathcal{S}$ inside $\mathcal{Y}$.*

Corollary 4.1.12. *There is a quasi-compact strict neighborhood $\mathcal{S}_1^{\dagger\dagger}(\underline{x})$ of $\mathcal{S}_1^{\dagger}(\underline{x})$ inside $\mathcal{S}_0^{\dagger}(\underline{x})$, such that the family of special subgroups extends analytically from $\mathcal{S}_1^{\dagger}(\underline{x})$ to $\mathcal{S}_1^{\dagger\dagger}(\underline{x})$. In particular, on $\mathcal{S}_1^{\dagger\dagger}(\underline{x})$, we have $U_p = U_p^{sp} + U_p^{nsp}$, where*

$$U_p^{sp} : \mathcal{S}_1^{\dagger\dagger}(\underline{x}) \to \mathcal{S}_0^{\dagger}(\underline{x}) \cup \mathfrak{Y}_{\mathrm{rig}}[t_{\underline{x}}, g]$$

is dividing by the special subgroup, and

$$U_p^{nsp} : \mathcal{S}_1^{\dagger\dagger}(\underline{x}) \to \mathfrak{Y}_{\mathrm{rig}}[t_{\underline{x}}, g]$$

is the complement. Furthermore, by characterization of $\mathcal{S}_1^{\dagger}(\underline{x})$, we have

$$U_p^{sp}(\mathcal{S}_1^{\dagger\dagger}(\underline{x}) - \mathcal{S}_1^{\dagger}(\underline{x})) \subset \mathfrak{Y}_{\mathrm{rig}}[t_{\underline{x}}, g] - \mathcal{S}_0^{\dagger}(\underline{x}).$$

Shrinking $\mathcal{S}_1^{\dagger\dagger}(\underline{x})$, if necessary, we may assume that $U_p^{sp}(\mathcal{S}_1^{\dagger\dagger}(\underline{x})) \subset W = W_\delta(\underline{x})$ (see Lemma 4.1.5, and Definition 4.1.7).

We conclude that $\mathcal{S}_1^{\dagger}(\underline{x}) = (U_p^{sp})^{-1}(\mathcal{S}_0^{\dagger}(\underline{x}))$. We now define $\mathcal{S}_m^{\dagger}(\underline{x})$ for $m \geq 2$.

Definition 4.1.13. Recall $U_p^{sp} : \mathcal{S}_1^{\dagger\dagger}(\underline{x}) \to \mathcal{S}_0^{\dagger}(\underline{x}) \cup \mathfrak{Y}_{\mathrm{rig}}[t_{\underline{x}}, g]$. For any $m \geq 2$, define

$$\mathcal{S}_m^{\dagger}(\underline{x}) := (U_p^{sp})^{-m}(\mathcal{S}_0^{\dagger}(\underline{x})).$$

It follows, immediately, that for all $m \geq 0$,

$$\mathcal{S}_{m+1}^{\dagger}(\underline{x}) \subset \mathcal{S}_m^{\dagger}(\underline{x}).$$

Successive application of part (2) of Proposition 4.1.9 shows that there is a family of cyclic $\mathcal{O}_L/p^m$-group schemes $\mathcal{G}_m$ on $\mathcal{S}_m^{\dagger}(\underline{x})$, with the property that $\mathcal{G}_1 = \mathcal{G}$, and $\mathcal{G}_m|_{\mathcal{S}_{m+1}^{\dagger}(\underline{x})} = \mathcal{G}_{m+1}[p]$. By Lemma 4.1.11, and arguing as in Corollary 4.1.12, we can find a strict neighborhood $\mathcal{S}_m^{\dagger\dagger}(\underline{x})$ of $\mathcal{S}_m^{\dagger}(\underline{x})$ in $\mathcal{S}_0^{\dagger}$, such that $\mathcal{G}_m$ extends to a family of subgroups to this strict neighborhood. After possibly shrinking these strict neighborhoods, one can arrange to have

- $\mathcal{S}_{m+1}^{\dagger\dagger}(\underline{x}) \subset \mathcal{S}_m^{\dagger\dagger}(\underline{x})$,
- $U_p^{sp}(\mathcal{S}_{m+1}^{\dagger\dagger}(\underline{x})) \subset \mathcal{S}_m^{\dagger\dagger}(\underline{x})$.

We now begin the analytic continuation process following the method presented in §1.

Proof of Theorem 4.1.1: We prove the classicality of f by induction: assume f is defined on $\mathfrak{Y}_{\mathrm{rig}}[r + 1 - \epsilon_1, g]$ for an integer $0 \leq r \leq g - 1$ and some rational $\epsilon_1 > 0$, and show f extends to $\mathfrak{Y}_{\mathrm{rig}}[r - \epsilon_2, g]$ for some positive rational $\epsilon_2 > 0$. As in Proposition 1.2.12, we have the following result.

Proposition 4.1.14. *Let $Q = (\underline{A}, H) \in \mathfrak{Y}_{\mathrm{rig}}$. Assume that there is $Q' \in U_p(\{Q\})$ such that $\deg(Q) = \deg(Q')$. Then, H is a truncated Barsotti-Tate group of level 1, and $\deg_i(Q)$ is an integer for all $1 \leq i \leq g$.*

Pick a rational $\alpha > 0$ such that $r + \alpha < r + 1 - \epsilon_1$. Then, on $\mathfrak{Y}_{\mathrm{rig}}[r + \alpha, r + 1 - \epsilon_1]$ degree is never an integer, and, hence, U_p increases degree strictly, by Propositions 4.1.4 and 4.1.14 . Since $\mathfrak{Y}_{\mathrm{rig}}[r + \alpha, r + 1 - \epsilon_1]$ is quasi-compact, using the Maximum Modulus Principle as in the

Second proof of Proposition 1.2.10 implies that there is a positive lower bound for the increase in degree under U_p over the entire $\mathfrak{Y}_{\rm rig}[r+\alpha, r+1-\epsilon_1]$. In particular, there is $M>0$, such that $U_p^M(\mathfrak{Y}_{\rm rig}[r+\alpha,g]) \subset \mathfrak{Y}_{\rm rig}[r+1-\epsilon_1,g]$. As usual, we will extend f to $\mathfrak{Y}_{\rm rig}[r+\alpha,g]$ via $\frac{U_p^M(f)}{a_p^M}$. Allowing $\alpha>0$ to vary, we get compatible sections which glue together to provide an extension of f to $\mathfrak{Y}_{\rm rig}(r,g]$.

Now, we want to extend f from $\mathfrak{Y}_{\rm rig}(r,g]$ to $\mathfrak{Y}_{\rm rig}[r-\epsilon_2,g]$, for some $\epsilon_2>0$. We will first extend f to strict neighborhoods of the tubes of all vertices of degree r. Fix such a vertex $\underline{x}$. Recall that $t_{\underline{x}}>r$, which implies that f is defined on $\mathfrak{Y}_{\rm rig}[t_{\underline{x}},g]$. We will extend f to $\mathfrak{Y}_{\rm rig}[t_{\underline{x}},g] \cup \mathcal{S}_0^\dagger(\underline{x})$ (which contains $\underline{\deg}^{-1}(\underline{x})$).

By Proposition 4.1.9 and the definition of $\mathcal{S}_m^\dagger(\underline{x})$, for all $m \geq 1$, we have

$$U_p((\mathcal{S}_0^\dagger(\underline{x})-\mathcal{S}_m^\dagger(\underline{x})) \cup \mathfrak{Y}_{\rm rig}[t_{\underline{x}},g]) \subset (\mathcal{S}_0^\dagger(\underline{x})-\mathcal{S}_{m-1}^\dagger(\underline{x})) \cup \mathfrak{Y}_{\rm rig}[t_{\underline{x}},g].$$

Let $f_0 = f|_{\mathfrak{Y}_{\rm rig}[t_{\underline{x}},g]}$. For $m \geq 1$, define f_m, recursively, on $(\mathcal{S}_0^\dagger(\underline{x})-\mathcal{S}_m^\dagger(\underline{x})) \cup \mathfrak{Y}_{\rm rig}[t_{\underline{x}},g]$ via $\frac{U_p(f_{m-1})}{a_p}$. It is easy to see that the f_m's are compatible. Also, define F_m on $\mathcal{S}_m^{\dagger\dagger}(\underline{x})$ via

$$F_m = \sum_{j=0}^{m-1} (\frac{1}{a_p})^{j+1} (U_p^{sp})^j U_p^{nsp}(f_{|_{\mathfrak{Y}_{\rm rig}[t_{\underline{x}},g]}}).$$

Note that $\mathcal{S}_0^\dagger(\underline{x}) = (\mathcal{S}_0^\dagger(\underline{x})-\mathcal{S}_m^\dagger(\underline{x})) \cup \mathcal{S}_m^{\dagger\dagger}(\underline{x})$ is an admissible covering. At step m, we plan to glue f_m on $\mathcal{S}_0^\dagger(\underline{x})-\mathcal{S}_m^\dagger(\underline{x})$ to F_m on $\mathcal{S}_m^{\dagger\dagger}(\underline{x})$ modulo p^m to create a section mod p^m on $\mathcal{S}_0^\dagger(\underline{x})$. The limit of this sequence of sections mod p^m would provide the sought-after analytic continuation of f to $\mathcal{S}_0^\dagger(\underline{x})$. To make this argument work, we will need several norm estimates:

(1) $\{|F_m|_{\mathcal{S}_m^{\dagger\dagger}(\underline{x})}, |f_m|_{\mathcal{S}_0^\dagger(\underline{x})-\mathcal{S}_m^\dagger(\underline{x})}\}_{m\geq 1}$ is bounded: having this, we can simultaneously rescale all the sections involved in the argument to have norm at most 1. Using a trivializing open cover for ω, then, this reduces the problem at hand to one involving sections of $\check{\mathcal{O}} = \{h \in \mathcal{O}_{\mathcal{S}_0^\dagger(\underline{x})} : |h|_{sup} \leq 1\}$.

(2) $|F_m - f_m|_{\mathcal{S}_m^{\dagger\dagger}(\underline{x})-\mathcal{S}_m^\dagger(\underline{x})} \to 0$, as $m \to \infty$: having this, up to choosing a subsequence, we can assume

$$F_m \equiv_{p^m} f_m \quad \text{over } (\mathcal{S}_0^\dagger(\underline{x})-\mathcal{S}_m^\dagger(\underline{x})) \cap \mathcal{S}_m^{\dagger\dagger}(\underline{x}) = \mathcal{S}_m^{\dagger\dagger}(\underline{x})-\mathcal{S}_m^\dagger(\underline{x}),$$

which would imply that F_m and f_m glue mod p^m to give a section h_m of $\check{\mathcal{O}}/p^m\check{\mathcal{O}}$ on $\mathcal{S}_0^\dagger(\underline{x})$. Applying Bartenwerfer's result [Bar70], and arguing just as in the final passage of the proof of Theorem 1.1.1, we find that for some c with $|c| \leq 1$, we have $ch_m \in \check{\mathcal{O}}(\mathcal{S}_0^\dagger(\underline{x}))/p^m\check{\mathcal{O}}(\mathcal{S}_0^\dagger(\underline{x}))$.

(3) $|F_{m+1}-F_m|_{\mathcal{S}_{m+1}^{\dagger\dagger}(\underline{x})} \to 0$, as $m \to \infty$: having this, after possibly choosing a subsequence, we can deduce that the sections ch_m are compatible, and, hence, we can define

$$f = c^{-1} \varprojlim_m ch_m \in \varprojlim_m \check{\mathcal{O}}(\mathcal{S}_0^\dagger(\underline{x}))/p^m\check{\mathcal{O}}(\mathcal{S}_0^\dagger(\underline{x})) = \check{\mathcal{O}}(\mathcal{S}_0^\dagger(\underline{x})),$$

which glues to f on $\mathfrak{Y}_{\mathrm{rig}}(r,g]$, providing an extension of f to $\mathfrak{Y}_{\mathrm{rig}}(r,g] \cup \mathcal{S}_0^\dagger(\underline{x})$.

Let us assume the above norm estimates for the moment, and finish the rest of the proof. Let

$$\mathcal{S}_0^\dagger := \bigcup_{\underline{x} \text{ vertex of deg } r} \mathcal{S}_0^\dagger(\underline{x}).$$

We have shown that f extends to a section on $\mathfrak{Y}_{\mathrm{rig}}(r,g] \cup \mathcal{S}_0^\dagger$. Let $\mathcal{V}$ be a quasi-compact open of $\mathfrak{Y}_{\mathrm{rig}}$ disjoint from $\underline{\deg}^{-1}(\underline{x})$ for all vertices $\underline{x}$ of degree r, such that

$$\mathfrak{Y}_{\mathrm{rig}}[r-1/2,g] = \mathfrak{Y}_{\mathrm{rig}}(r,g] \cup \mathcal{S}_0^\dagger \cup \mathcal{V}.$$

It follows that degree is never an integer on $\mathcal{V}$, and, hence, U_p increases degrees strictly on $\mathcal{V}$. Arguing as usual, we deduce that there is $M>0$ such that $U_p^M(\mathcal{V}) \subset \mathfrak{Y}_{\mathrm{rig}}(r,g] \cup \mathcal{S}_0^\dagger$. Therefore, $\frac{U_p^M(f)}{a_p^M}$ provides the analytic continuation of f from $\mathfrak{Y}_{\mathrm{rig}}(r,g] \cup \mathcal{S}_0^\dagger$ to $\mathfrak{Y}_{\mathrm{rig}}[r-1/2,g]$, completing the induction step.

4.2. The norm estimates. Since ϵ appearing in Lemma 4.1.5 can be taken arbitrarily small, we can and will assume that

$$\mathrm{val}(a_p) < \inf\{k_i\}_i - g - \epsilon \sum_i k_i.$$

All the norm estimates follow essentially from the following Lemma. Let $\underline{x} = (x_i)$ be a vertex of degree r.

Lemma 4.2.1. *Let $\mathcal{Z} \subset \mathcal{S}_1^{\dagger\dagger}(\underline{x})$. Let $h \in \omega^{\underline{k}}(U_p^{sp}(\mathcal{Z}))$. Then,*

$$|U_p^{sp}(h)|_{\mathcal{Z}} \leq p^{g-\sum_{i=1}^g k_i(1-x_i-\epsilon)} |h|_{U_p^{sp}(\mathcal{Z})},$$

where ϵ is as in part (2) of Lemma 4.1.5.

Proof. Let $(\underline{A},H)$ be a point in $\mathcal{Z}$. We write

$$\begin{aligned} |U_p^{sp}(h)(\underline{A},H)| &= |\frac{1}{p^g}\mathrm{pr}^* h(\underline{A}/G_1,\bar{H})| \\ &= p^{g-\sum_{i=0}^g k_i \deg_i(G_1)} |h(\underline{A}/G_1,\bar{H})| \\ &\leq p^{g-\sum_{i=0}^g k_i(1-x_i-\epsilon)} |h|_{U_p^{sp}(\mathcal{Z})}, \end{aligned}$$

where, for the second equality, we have used Lemma 2.9.5, and, for the last inequality, we have used the fact that by choice of ϵ in part (2) of Lemma 4.1.5, and by the last statement in Corollary 4.1.12, we have $|\deg_i \bar{H} - x_i| \leq \epsilon$, implying $\deg_i(G_1) \geq 1 - x_i - \epsilon$, for all i. $\square$

Corollary 4.2.2. *Let $\mathrm{val}(a_p) < \inf\{k_i\}_i - g - \epsilon\sum_i k_i$, where ϵ is as in part (2) of Lemma 4.1.5. Assume that we have a collection $\{\mathcal{Z}_m\}_{m\geq 1}$ of quasi-compact open subsets of $\mathcal{S}_1^{\dagger\dagger}(\underline{x})$, such that $U_p^{sp}(\mathcal{Z}_{m+1}) \subset \mathcal{Z}_m$ for all $m \geq 1$. Assume h is a section of $\omega^{\underline{k}}$ on $\mathcal{Z}_1$. Then*

$$|\frac{1}{a_p^m}(U_p^{sp})^m(h)|_{\mathcal{Z}_{m+1}} \to 0 \quad \text{as} \quad m \to \infty.$$

We now prove the norm estimates (1), (2), (3) presented above.

Lemma 4.2.3. *For $1 \leq j \leq m$, we have $f_m - \frac{1}{a_p^j}(U_p^{sp})^j(f_m) = F_j$ on $\mathcal{S}_j^{\dagger\dagger}(\underline{x}) - \mathcal{S}_m^{\dagger}(\underline{x})$.*

Proof. This is a simple calculation from the definitions. It is worth mentioning that $(U_p^{sp})^j(f_m)$ is defined on $\mathcal{S}_j^{\dagger\dagger}(\underline{x}) - \mathcal{S}_{m+j}^{\dagger}(\underline{x})$, since

$$(U_p^{sp})^j(\mathcal{S}_j^{\dagger\dagger}(\underline{x}) - \mathcal{S}_{m+j}^{\dagger}(\underline{x})) \subset (\mathcal{S}_0^{\dagger}(\underline{x}) - \mathcal{S}_m^{\dagger}(\underline{x})) \cup \mathfrak{Y}_{\mathrm{rig}}[0, t_{\underline{x}}],$$

by definitions. □

We can now prove the estimates. We first show that $\{|f_m|_{\mathcal{S}_0^{\dagger}(\underline{x})-\mathcal{S}_m^{\dagger}(\underline{x})}\}_{m\geq 1}$ is bounded. Since $\mathcal{S}_1^{\dagger}(\underline{x})$ is quasi-compact, $|F_1|_{\mathcal{S}_1^{\dagger}(\underline{x})}$ is bounded. Also, $|f_1|_{\mathcal{S}_0^{\dagger}(\underline{x})-\mathcal{S}_1^{\dagger}(\underline{x})}$ is bounded, as f_1 is obtained by applying $\frac{1}{a_p}U_p$ to $f|_{\mathfrak{Y}_{\mathrm{rig}}[t_{\underline{x}},g]}$ which has finite norm. Let C be a common bound for the above two norms. We claim that $|f_m|_{\mathcal{S}_0^{\dagger}(\underline{x})-\mathcal{S}_m^{\dagger}(\underline{x})} \leq C$, for all $m \geq 1$. By compatibility of the f_m's, it is enough to show that $|f_m|_{\mathcal{S}_{m-1}^{\dagger}(\underline{x})-\mathcal{S}_m^{\dagger}(\underline{x})} \leq C$ for all $m \geq 1$. By Lemma 4.2.3, on $\mathcal{S}_1^{\dagger}(\underline{x}) - \mathcal{S}_m^{\dagger}(\underline{x})$,

$$f_m = F_1 + \frac{1}{a_p}U_p^{sp}(f_m).$$

It is, therefore, enough to show that $|\frac{1}{a_p}U_p^{sp}(f_m)|_{\mathcal{S}_{m-1}^{\dagger}(\underline{x})-\mathcal{S}_m^{\dagger}(\underline{x})} \leq C$. By Lemma 4.2.1, for $m \geq 2$, we have

$$\begin{aligned} |\frac{1}{a_p}U_p^{sp}(f_m)|_{\mathcal{S}_{m-1}^{\dagger}(\underline{x})-\mathcal{S}_m^{\dagger}(\underline{x})} &\leq p^{\mathrm{val}(a_p)+g-\sum_{i=1}^{g} k_i(1-x_i-\epsilon)}|f_m|_{\mathcal{S}_{m-2}^{\dagger}(\underline{x})-\mathcal{S}_{m-1}^{\dagger}(\underline{x})} \\ &\leq |f_{m-1}|_{\mathcal{S}_{m-2}^{\dagger}(\underline{x})-\mathcal{S}_{m-1}^{\dagger}(\underline{x})}, \end{aligned}$$

using the compatibility of the f_m's and the bound on $\mathrm{val}(a_p)$. Therefore, the claim follows by induction. Next, we show $\{|F_m|_{\mathcal{S}_m^{\dagger\dagger}(\underline{x})}\}_{m\geq 1}$ is bounded. We write

$$|F_m|_{\mathcal{S}_m^{\dagger\dagger}(\underline{x})} \leq \sup_{0\leq j\leq m-1} |a_p^{-j-1}(U_p^{sp})^j U_p^{nsp}(f_{|\mathfrak{Y}_{\mathrm{rig}}[t_{\underline{x}},g]})|_{\mathcal{S}_m^{\dagger\dagger}(\underline{x})}$$

Since $F_1 = U_p^{nsp}(f_{|\mathfrak{Y}_{\mathrm{rig}}[t_{\underline{x}},g]})$, applying Lemma 4.2.1, we obtain $|F_m|_{\mathcal{S}_m^{\dagger\dagger}(\underline{x})} \leq |F_1|_{\mathcal{S}_1^{\dagger\dagger}(\underline{x})} < \infty$.

Now, we prove estimates (2), (3). By Lemma 4.2.3, we have $F_m - f_m = \frac{1}{a_p^m}(U_p^{sp})^m(f_m)$ on $\mathcal{S}_m^{\dagger\dagger}(\underline{x}) - \mathcal{S}_m^{\dagger}(\underline{x})$. The claim now follows from Corollary 4.2.2. Similarly, we have $F_{m+1} - F_m = \frac{1}{a_p^{m+1}}(U_p^{sp})^m(F_1)$ on $\mathcal{S}_{m+1}^{\dagger\dagger}(\underline{x})$, which tends to zero by Corollary 4.2.2.

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