Introduction to the Gan-Gross-Prasad and Ichino-Ikeda conjectures III

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- Today, I am going to explain a bit of the proof of the global Gan-Gross-Prasad and Ichino-Ikeda conjectures for unitary groups.
- Recall the setting : K/F quad ext of $\#$ fields, V a n-diml Herm space over K, $V' = V \oplus^{\perp} K.e$, $H = U(V) \hookrightarrow G = U(V) \times U(V')$. For $\pi = \pi_n \boxtimes \pi_{n+1} \hookrightarrow \mathcal{A}_{cusp}([G])$ we define the GGP period

$$\mathcal{P}_H : \phi \in \pi \mapsto \int_{[H]} \phi(h) dh.$$

- Set $L(s, \pi_K) = L(s, \pi_{n,K} \times \pi_{n+1,K})$ where

$$\pi_K = \pi_{n,K} \boxtimes \pi_{n+1,K} \hookrightarrow \mathcal{A}([\mathrm{GL}_{n,K} \times \mathrm{GL}_{n+1,K}])$$

is the base-change of π.
- Recall that we need to vary the gps : for V_0 another Herm space of dim n, $H^{V_0} = U(V_0) \hookrightarrow G^{V_0} = U(V_0) \times U(V_0')$ with $V_0' = V_0 \oplus^{\perp} K.e$.

- Assume that π_K is generic.

Conjecture (Gan-Gross-Prasad)

The following are equivalent :

1. $L(\frac{1}{2}, \pi_K) \neq 0$ *;*
2. *There exist a n-diml Herm space* V_0 *and* $\sigma \hookrightarrow \mathcal{A}_{cusp}(G^{V_0})$ *in the same L-packet as* π *(i.e.* $\sigma_K = \pi_K$*) such that* $\mathcal{P}_{H^{V_0}} |_{\sigma} \not\equiv 0$.

Moreover the pair (V_0, σ) *if it exists is unique.*

Conjecture (Ichino-Ikeda, N.Harris)

For every $\phi = \otimes'_v \phi_v \in \pi = \bigotimes'_v \pi_v$ *we have*

$$|\mathcal{P}_H(\phi)|^2 = \frac{\Delta}{|S_\pi|} \frac{L(\frac{1}{2}, \pi_K)}{L(1, \pi, \mathrm{Ad})} \prod_v \mathcal{P}_{H_v}(\phi_v, \phi_v)$$

where $\Delta = \prod_{k=1}^{n+1} L(k, \eta_{K/F}^k)$, S_π *is "the centralizer of the Langlands parameter of* π*" and* $\mathcal{P}_{H_v}(\phi_v, \phi_v)$ *are 'local (normalized) periods'.*

- Recall also that the local periods are related to 'unnormalized' one (def via integration of matrix coeff) by

$$\mathcal{P}_{H_v}(\phi_v,\phi_v) := \Delta_v^{-1}\frac{L(\frac{1}{2},\pi_{v,K})^{-1}}{L(1,\pi_v,\mathrm{Ad})^{-1}}\mathcal{P}^{\natural}_{H_v}(\phi_v,\phi_v)$$

so that the I-I identity informally reads

$$|\mathcal{P}_H(\phi)|^2 = |S_\pi|^{-1}\prod_v{}' \mathcal{P}^{\natural}_{H_v}(\phi_v,\phi_v).$$

Theorem (W.Zhang, H.Xue, B.-P.)

Both the GGP and the I-I conjectures hold provided there exists a place v st $\pi_{K,v}$ is supercuspidal.

- Remark : $\pi_{K,v}$ supercuspidal $\Rightarrow \pi_K$ is cuspidal $\Rightarrow |S_\pi| = 4$.

Jacquet-Rallis's approach through Relative Trace Formulas

- Recall that the central value $L(\frac{1}{2},\pi_K)$ appears in the factorization of another period : the Rankin-Selberg period $\mathcal{P}_{\mathrm{GL}_{n,K}}$ on $\pi_K = \pi_{n,K} \boxtimes \pi_{n+1,K}$ satisfies (JPSS)

$$\mathcal{P}_{GL_{n,K}}(\phi) = L(\frac{1}{2},\pi_K)\prod_v \mathcal{P}_{GL_{n,K_v}}(\phi_v), \quad \phi = \otimes'_v \phi_v \in \pi_K$$

  for some local (Rankin-Selberg) periods $\mathcal{P}_{GL_{n,K_v}}$.
- Could we relate the GGP period $\mathcal{P}_H\mid_\pi$ to $\mathcal{P}_{GL_{n,K}}\mid_{\pi_K}$? Jacquet and Rallis have proposed a relation between those through a comparison of *Relative Trace Formulas* (RTF).
- RTF are analytic tools invented by Jacquet that relate automorphic periods to more geometric distributions known as *relative orbital integrals*. They are roughly associated to triples $H_1 \hookrightarrow G \hookleftarrow H_2$ with G reductive and can be informally thought as distributions on the double cosets $H_1\backslash G/H_2$.

Jacquet-Rallis's RTF for unitary groups

- RTF for $H = U(V)\backslash G = U(V) \times U(V')/H$: for *nice* test function $f \in C_c^\infty(G(\mathbb{A}))$ we have
$$\sum_{\phi \in \mathcal{A}_{cusp}([G])} \mathcal{P}_H(R(f)\phi)\overline{\mathcal{P}_H(\phi)} = \sum_{\delta \in H(F)\backslash G_{rs}(F)/H(F)} O(\delta, f)$$
where the left sum is over an ONB of $\mathcal{A}_{cusp}([G])$ for the Petersson inner product, $G_{rs}(F)$ is the subset of (relatively) *regular semisimple* elts i.e. those whose stabilizer for the action $(h_1, h_2).\delta = h_1 \delta h_2^{-1}$ is trivial and with (Zariski) closed orbit, and
$$O(\delta, f) := \int_{H(\mathbb{A}) \times H(\mathbb{A})} f(h_1 \delta h_2) dh_1 dh_2$$
are the *relative orbital integrals*.
- The test function should be 'nice' so that it doesn't see noncuspidal autom forms (spectral condition) and singular orbits (geometric condition). This trace formula is called 'simple' both because it is simple to state and simple to prove. Writing a complete RTF in this case without this simplifying assumption is much harder and is currently being considered by Chaudouard and Zydor.

Jacquet-Rallis's RTF for general linear groups

- Set
$$H_1 := \mathrm{GL}_{n,K} \hookrightarrow G' = \mathrm{GL}_{n,K} \times \mathrm{GL}_{n+1,K} \hookleftarrow H_2 = \mathrm{GL}_{n,F} \times \mathrm{GL}_{n+1,F}$$
and $\eta := (\eta_{K/F} \circ \det)^{n+1} \otimes (\eta_{K/F} \circ \det)^n : [H_2] \to \{\pm 1\}$.
- RTF for $H_1 \backslash G' / H_2$: for *nice* test function $f' \in C_c^\infty(G'(\mathbb{A}))$ we have
$$\sum_{\phi \in \mathcal{A}_{cusp}([G'])} \mathcal{P}_{H_1}(R(f')\phi)\overline{\mathcal{P}_{H_2,\eta}(\phi)} = \sum_{\gamma \in H_1(F)\backslash G'_{rs}(F)/H_2(F)} O(\gamma, f')$$
where $\mathcal{P}_{H_1}$ is the Rankin-Selberg period, $\mathcal{P}_{H_2,\eta}$ is the Flicker-Rallis period (integral over $[H_2]$ twisted by η) and this time
$$O(\gamma, f') = \int_{H_1(\mathbb{A}) \times H_2(\mathbb{A})} f'(h_1 \gamma h_2)\eta(h_2) dh_1 dh_2.$$
- As we saw, the period $\mathcal{P}_{H_1}$ detects non-vanishing of Rankin-Selberg L-fns at the central point. On the other hand, non-vanishing of $\mathcal{P}_{H_2,\eta}$ detects the image of base-change i.e. repns of the form π_K (Flicker-Rallis).

Matching

- Recall the two Jacquet-Rallis's RTFs

$$\sum_{\phi \in \mathcal{A}_{cusp}([G])} \mathcal{P}_H(R(f)\phi)\overline{\mathcal{P}_H(\phi)} = \sum_{\delta \in H(F)\backslash G_{rs}(F)/H(F)} O(\delta, f),$$

$$\sum_{\phi \in \mathcal{A}_{cusp}([G'])} \mathcal{P}_{H_1}(R(f')\phi)\overline{\mathcal{P}_{H_2,\eta}(\phi)} = \sum_{\gamma \in H_1(F)\backslash G'_{rs}(F)/H_2(F)} O(\gamma, f').$$

- We compare the two through their geometric sides.
- Correspondence of orbits : $\bigsqcup_{V_0} H^{V_0}(k)\backslash G^{V_0}_{rs}(k)/H^{V_0}(k) \simeq H_1(k)\backslash G'_{rs}(k)/H_2(k)$, $k = F$ or F_v, where V_0 runs over isom classes of n-diml Herm spaces.
- Orbital integrals are local : if $f = \prod_v f_v$, $f' = \prod_v f'_v$ then $O(\delta, f) = \prod_v O(\delta, f_v)$, $O(\gamma, f') = \prod_v O(\gamma, f'_v)$.
- Local matching : $f_v = (f_v^{V_0}) \in \bigoplus_{V_0} C_c^\infty(G^{V_0}(F_v))$ *match* $f'_v \in C_c^\infty(G'_v)$ ($f_v \leftrightarrow f'_v$) if

$$O(\delta, f_v^{V_0}) = \Omega_v(\gamma) O(\gamma, f'_v), \text{ for } \delta \in G^{V_0}_{rs}(F_v) \leftrightarrow \gamma \in G'_{rs}(F_v)$$

  where $(\Omega_v)_v$ are *transfer factors* st $\prod_v \Omega_v(\gamma) = 1$ for $\gamma \in G'_{rs}(F)$.
- To make the comparison effective, we need two local results :
  - Existence of local transfer : for every f_v there exists f'_v st $f_v \leftrightarrow f'_v$ and conversely (W.Zhang p-adic case, H.Xue Archimedean case).
  - Fundamental lemma : $\mathbf{1}_{G(\mathcal{O}_v)} \leftrightarrow \mathbf{1}_{G'(\mathcal{O}_v)}$ for a.a. v (Z.Yun, J. Gordon, B.-P.).

Comparison

- Start with $f = (f^{V_0})_{V_0} = \prod_v f_v \in \bigoplus_{V_0} C_c^\infty(G^{V_0}(\mathbb{A}))$ and $f' = \prod_v f'_v \in C_c^\infty(G'(\mathbb{A}))$ st $f_v \leftrightarrow f'_v$ for every v. We can equal the two RTFs (where we vary the Herm space of unitary gps) to get

$$\sum_{V_0} \sum_{\phi \in \mathcal{A}_{cusp}([G^{V_0}])} \mathcal{P}_{H^{V_0}}(R(f^{V_0})\phi)\overline{\mathcal{P}_{H^{V_0}}(\phi)} = \sum_{\phi \in \mathcal{A}_{cusp}([G'])} \mathcal{P}_{H_1}(R(f')\phi)\overline{\mathcal{P}_{H_2,\eta}(\phi)}.$$

- or regrouping together cusp forms in the same autom repn

$$\sum_{V_0} \sum_{\pi \hookrightarrow \mathcal{A}_{cusp}([G^{V_0}])} J_\pi(f^{V_0}) = \sum_{\Pi \hookrightarrow \mathcal{A}_{cusp}([G'])} I_\Pi(f')$$

where

$$J_\pi(f^{V_0}) = \sum_{\phi \in \pi} \mathcal{P}_{H^{V_0}}(R(f^{V_0})\phi)\overline{\mathcal{P}_{H^{V_0}}(\phi)} \text{ and } I_\Pi(f') = \sum_{\phi \in \Pi} \mathcal{P}_{H_1}(R(f')\phi)\overline{\mathcal{P}_{H_2,\eta}(\phi)}$$

are (global) *relative characters*.
- Separating spectral contributions we get, for a fixed $\pi \hookrightarrow \mathcal{A}_{cusp}([G])$,

$$\sum_{V_0} \sum_{\substack{\sigma \hookrightarrow \mathcal{A}_{cusp}([G^{V_0}]) \\ \sigma_K = \pi_K}} J_\sigma(f^{V_0}) = I_{\pi_K}(f').$$

Application to the GGP conjecture

$$\sum_{V_0} \sum_{\substack{\sigma \hookrightarrow \mathcal{A}_{cusp}([G^{V_0}]) \\ \sigma_K = \pi_K}} J_\sigma(f^{V_0}) = I_{\pi_K}(f')$$

for a fixed $\pi \hookrightarrow \mathcal{A}_{cusp}([G])$, where $f = (f^{V_0})_{V_0} = \prod_v f_v \in \bigoplus_{V_0} C_c^\infty(G^{V_0}(\mathbb{A}))$ and $f' = \prod_v f'_v \in C_c^\infty(G'(\mathbb{A}))$ are st $f_v \leftrightarrow f'_v$.

- Note that the left sum runs over *the automorphic L-packet of* π.
- The F-R period $\mathcal{P}_{H_2,\eta}$ is nonzero on π_K (Flicker-Rallis) and therefore I_{π_K} is nonzero iff the R-S period $\mathcal{P}_{H_1}$ is nonzero on π_K i.e. iff $L(\frac{1}{2}, \pi_K) \neq 0$ (JPSS).
- By linear independence of relative characters, the left hand side is nonzero (as a distribution) iff there exists $\sigma \hookrightarrow \mathcal{A}_{cusp}([G^{V_0}])$ in the same L-packet as π st $J_\sigma \not\equiv 0$ i.e. $\mathcal{P}_{H^{V_0}} |_\sigma \not\equiv 0$.
- From this we deduce the GGP conjecture for π (without the unicity)...except that our test fns should be nice (to use simple versions of RTFs) $\Rightarrow$ we need to impose conditions on π (e.g. $\pi_{K,v}$ supercuspidal) to be able to detect it.

Application to the I-I conjecture

$$\sum_{V_0} \sum_{\substack{\sigma \hookrightarrow \mathcal{A}_{cusp}([G^{V_0}]) \\ \sigma_K = \pi_K}} J_\sigma(f^{V_0}) = I_{\pi_K}(f')$$

- Local GGP conj $\Rightarrow \exists$ at most one $\sigma \hookrightarrow \mathcal{A}_{cusp}([G^{V_0}])$, $\sigma_K = \pi_K$ st $\mathrm{Hom}_{H^{V_0}(\mathbb{A})}(\sigma, \mathbb{C}) \neq 0$ which is a nec. condition to have $\mathcal{P}_{H^{V_0}} |_\sigma \not\equiv 0$ hence $J_\sigma \not\equiv 0$. Thus the LHS contains at most one nonzero term. Wma it is $J_\pi(f)$ to get

$$J_\pi(f) = I_{\pi_K}(f').$$

- By works of Jacquet-Piatetski-Shapiro-Shalika and Flicker-Rallis, we know explicit factorizations of $\mathcal{P}_{H_1}$ and $\mathcal{P}_{H_2,\eta}$ hence of $I_{\pi_K}(f')$. It takes the form

$$I_{\pi_K}(f') = \frac{\Delta}{4} \frac{L(\frac{1}{2}, \pi_K)}{L(1, \pi, \mathrm{Ad})} \prod_v I_{\pi_{K,v}}(f'_v) = \frac{1}{4} \prod_v{}' I^\natural_{\pi_{K,v}}(f'_v)$$

  where $I_{\pi_{K,v}}$ (resp. $I^\natural_{\pi_{K,v}}$) are normalized (resp. unnormalized) local relative chars.
- On the other hand, the Ichino-Ikeda conj can be restated as a factorization

$$J_\pi(f) = \frac{\Delta}{4} \frac{L(\frac{1}{2}, \pi_K)}{L(1, \pi, \mathrm{Ad})} \prod_v J_{\pi_v}(f_v) = \frac{1}{4} \prod_v{}' J^\natural_{\pi_v}(f_v)$$

  where the unnormalized local relative characters are defined by $J^\natural_{\pi_v}(f_v) = \sum_{\phi_v \in \pi_v} \mathcal{P}^\natural_{H_v}(\pi_v(f_v)\phi_v, \phi_v)$.

- To obtain the I-I conj for π we therefore need a comparison of local relative characters (spectral transfer) : $J_{\pi_v}(f_v) = \kappa_v I_{\pi_{K,v}}(f'_v)$ whenever $f_v \leftrightarrow f'_v$ where $(\kappa_v)_v$ is a collection of local signs st $\prod_v \kappa_v = 1$ (Zhang for π_v supercuspidal, B.-P.).

To summarize, we can prove both the GGP and I-I conjectures for unitary gps under a mild local assumption by a comparison of (simple) RTFs provided we have four local ingredients :

- The existence of local transfer (W. Zhang, H.Xue) ;
- The Jacquet-Rallis's fundamental lemma (Z. Yun, J. Gordon, B.-P.) ;
- The local GGP conjecture (B.-P., Gan-Ichino, H.He) ;
- Comparison of local relative characters (W. Zhang, B.-P.).

On the proof of the local transfer (W. Zhang, H.Xue)

- We adapt notation to local case : K/F local quad ext, V n-diml Herm sp$/K$, $V' = V \oplus^{\perp} K.e$, $H = U(V) \hookrightarrow G = U(V) \times U(V')$, $H_1 = \mathrm{GL}_{n,K} \hookrightarrow G' = \mathrm{GL}_{n,K} \times \mathrm{GL}_{n+1,K} \hookleftarrow H_2 = \mathrm{GL}_{n,F} \times \mathrm{GL}_{n+1,F}$, $\eta : H_2(F) \to \{\pm 1\}$.
- Correspondence of orbits : $\bigsqcup_{V_0} H^{V_0}(F) \backslash G^{V_0}_{rs}(F) / H^{V_0}(F) \simeq H_1(F) \backslash G'_{rs}(F) / H_2(F)$.
- Smooth matching : $\bigoplus_{V_0} C_c^\infty(G^{V_0}(F)) \ni f \leftrightarrow f' \in C_c^\infty(G'(F))$ if

$$O(\delta, f) = \Omega(\gamma) O(\gamma, f') \text{ for } \delta \leftrightarrow \gamma.$$

- First step is linearization : $H \backslash G \simeq U(V') \rightsquigarrow H \backslash G_{rs}/H \simeq U(V')_{rs}/U(V) - conj + \mathfrak{u}(V') \simeq U(V')$ birationnally and $U(V)$-equiv.
- Also, $H_1 \backslash G' \simeq \mathrm{GL}_{n+1,K}$ & $\mathrm{GL}_{n+1,K} / \mathrm{GL}_{n+1,F} \simeq S_{n+1} := \{s \in \mathrm{GL}_{n+1,K} \mid ss^c = 1\}$ $\rightsquigarrow H_1 \backslash G'_{rs}/H_2 \simeq S_{n+1,rs}/\mathrm{GL}_{n,F} - conj +$ $\mathfrak{s}_{n+1} := \{X \in \mathfrak{gl}_{n+1,K} \mid X + X^c = 0\} \simeq S_{n+1}$ birationally and $\mathrm{GL}_{n,F}$-equiv.
- This allows to reduce to a problem on the Lie algebras : there is a correspondence

$$\bigsqcup_{V_0} \mathfrak{u}(V'_0)_{rs}/U(V_0) - conj \simeq \mathfrak{s}_{n+1,rs}/\mathrm{GL}_{n,F} - conj$$

and we say $\bigoplus_{V_0} C_c^\infty(\mathfrak{u}(V'_0)) \ni \varphi \leftrightarrow \varphi' \in C_c^\infty(\mathfrak{s}_{n+1})$ if

$$O(X, \varphi) = \omega(Y) O(Y, \varphi') \text{ for } X \leftrightarrow Y.$$

Reduction to a transfer near 0

- Orbital integrals $X \in \mathfrak{u}(V')_{rs} \mapsto O(X, \varphi)$ or $Y \in \mathfrak{s}_{n+1,rs} \mapsto O(Y, \varphi')$ are smooth (i.e. loc. constant in the *p*-adic case) on the regular semi-simple locus but may have (wild) singularities near singular points $X \in \mathfrak{u}(V')$ (resp. $Y \in \mathfrak{s}_{n+1}$).
- This simple remark implies that we can transfer fns $\varphi \in C_c^\infty(\mathfrak{u}(V'))$ (resp. $\varphi' \in C_c^\infty(\mathfrak{s}_{n+1})$) as soon as $\mathrm{Supp}(\varphi) \subset \mathfrak{u}(V')_{rs}$ (resp. $\mathrm{Supp}(\varphi') \subset \mathfrak{s}_{n+1,rs}$).
- More generally, we can 'localize' near $X \in \mathfrak{u}(V')$ (resp. $Y \in \mathfrak{s}_{n+1}$) and show that the singularities at this point are 'the same' as orbital integrals for another linear group action...which is usually 'smaller' and of the same shape except when X (resp. Y) belongs to the *nilpotent cone* i.e. its orbit closure meets the 'center'.
- Thus, using induction, we can transfer any test fns $\varphi \in C_c^\infty(\mathfrak{u}(V') \setminus \mathcal{N})$ or $\varphi' \in C_c^\infty(\mathfrak{s}_{n+1} \setminus \mathcal{N}')$ where $\mathcal{N}$, $\mathcal{N}'$ stand for the nilpotent cones.
- To get rid of the nilpotent cones, we need a way to produce new 'transferable' test fns out of old ones. The idea (which goes back to Waldspurger) is to use the Fourier Transform !

Transfer and Fourier Transforms

- We have a $U(V)$-invt decomposition $\mathfrak{u}(V') = \mathfrak{u}(V) \oplus V \oplus F$ and we can consider (partial) Fourier Transforms w.r.t. each of this three subspaces : $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$.
- Similarly $\mathfrak{s}_{n+1} = \mathfrak{s}_n \oplus (F^n \oplus (F^n)^*) \oplus F$ giving rise to three FT $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$.

Theorem (W.Zhang *p*-adic case, H.Xue Arch case)

There exist (explicit) constants $(c_i)_{i=1}^3$ such that if $C_c^\infty(\mathfrak{u}(V')) \ni \varphi \leftrightarrow \varphi' \in C_c^\infty(\mathfrak{s}_{n+1})$ then

$$\mathcal{F}_i\varphi \leftrightarrow c_i\mathcal{F}_i\varphi'$$

for each $i = 1, 2, 3$.

- Together with an uncertainty principle (Aizenbud) this finishes the proof of the transfer.
- Unlike Waldspurger's theorem that endoscopic transfer commutes with FT (assuming the Fundamental Lemma), the proof of the above theorem is purely local and doesn't assume the Jacquet-Rallis's fundamental lemma.

Outstanding questions ; orthogonal groups

- The original GGP conjecture was formulated for SO gps : $V \subset V' = V \oplus^{\perp} F.e$ quadratic spaces over F : # field, $H = SO(V) \hookrightarrow G = SO(V) \times SO(V')$. Then for $\pi = \pi_1 \boxtimes \pi_2 \hookrightarrow \mathcal{A}_{cusp}([G])$ there is a GGP period

$$\mathcal{P}_H : \phi \in \pi \mapsto \int_{[H]} \phi(h) dh$$

  and the (global) GGP conjecture relates the nonvanishing of $\mathcal{P}_H \mid_\pi$ to the non-vanishing of $L(\frac{1}{2}, \pi_1 \times \pi_2)$. The (original) I-I conj on the other hand gives a precise formula linking the two invariants.
- This case of the conjecture is still wide-open except in small rank : $\dim(V) = 2$ (Waldspurger's formula), $\dim(V) = 3$ (Ichino's triple product formula) and $\dim(V) = 4$ (Gan-Ichino for endoscopic L-packets and some stable ones).
- An approach through comparison of RTF is still missing : Assume that $\dim(V) = 2n$, then there are Langlands functorial lifts (Arthur), $\pi_i \mapsto \pi_i^{\mathrm{GL}} \subset \mathcal{A}_{cusp}([\mathrm{GL}_{2n}])$ $i = 1, 2$, there is a Rankin-Selberg period on $\mathrm{GL}_{2n} \times \mathrm{GL}_{2n}$ giving $L(\frac{1}{2}, \pi_1^{\mathrm{GL}} \times \pi_2^{\mathrm{GL}}) = L(\frac{1}{2}, \pi_1 \times \pi_2)$ but the analogs of the FR periods are more complicated : the one detecting lifts from $SO(2n)$, called Bump-Ginzburg period, is hard to analyze geometrically.
- The local conjecture, for its part, was solved (before unitary gps) by Waldspurger and Mœglin-Waldspurger in the p-adic case.

Outstanding questions ; other groups

- Other conjectures for unitary groups : replace $G=U(n)\times U(n+1)$ by $G=U(n)\times U(m)$, $n\leqslant m$, and $H=U(n)$ by $H=U(n)\ltimes N$ where N is a unipotent subgroup and we twist everything by a 'small' autom repn ξ of H (either a character or a Weil repn).
- Global GGP conj : relates nonvanishing of $\mathcal{P}_{H,\xi}\mid_{\pi}$ $(\pi=\pi_n\boxtimes\pi_m\hookrightarrow\mathcal{A}_{cusp}([G]))$ to the nonvanishing of $L(\frac{1}{2},\pi_{n,K}\times\pi_{m,K})$. There are also extensions of the I-I conjecture to these situations (Y. Liu, H.Xue).
- In those cases here is a RTF approach (Y. Liu) and this was successfully used by H.Xue in the $U(n)\times U(n)$ case (with the same local restriction as in the $U(n)\times U(n+1)$ case).
- When $n\leqslant m-2$ the proof of Zhang seems hard to adapt due to a lack of suitable linearization.
- The local conjecture is also mostly known (B.-P. when $m\not\equiv n[2]$, Gan-Ichino when $m\equiv n[2]$).
- There are also conjectures for symplectic/metaplectic groups whose local component was solved by H. Atobe (p-adic case) and global component is still wide-open.
- Finally, in all these cases the implication $\mathcal{P}_{H,\xi}\mid_{\pi_1\boxtimes\pi_2}\neq 0\Rightarrow L(\frac{1}{2},\pi_1\times\pi_2)\neq 0$ is proved (by other methods) by Ginzburg-Jiang-Rallis/ Jiang-L.Zhang.

Outstanding questions ; arithmetic version

- There is also an arithmetic version of the conjecture generalizing the Gross-Zagier formula on the height pairing of Heegner points.
- More precisely when $\pi \hookrightarrow \mathcal{A}_{cusp}([G])$ is st $\varepsilon(\pi_K) = -1$ then the unique repn σ of the L-packet of π supporting a $H(\mathbb{A})$-invt linear form (which exists by the local conj) is living on a product of local gps $\prod_v G^{V_{0,v}}$ associated to an incoherent family of local Herm spaces $(V_{0,v})_v$ (i.e. not coming from a global Herm space).
- In this situation (and under additional assumption) you can define another "period" $\mathcal{P}_H^{arith}$ on π coming from the height pairing with a special cycle on a Shimura variety $Sh(G)$.
- The arithmetic conjecture says that the non-vanishing of $\mathcal{P}_H^{arith}$ is related to the non-vanishing of $L'(\frac{1}{2}, \pi_K)$. This has been refined to a conjectural exact identity by W. Zhang.
- W. Zhang has also proposed a RTF approach for these conjectures, one crucial ingredient being the Arithmetic Fundamental Lemma...which has been recently proved by W. Zhang over $\mathbb{Q}_p$ when $p > n$.