Recent developments in p-adic Cohomology

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Fix prime number p.

§1 Intro. to p-adic Hodge theory

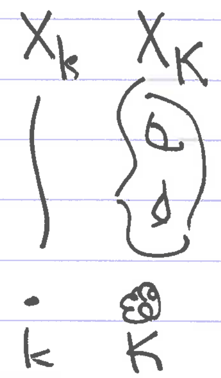
K = cdv field of char 0, with perfect residue field k of char p

eg) a finite ext^n of $\mathbb{Q}_p$

$\mathcal{O}_K \subset K$ its ring of integers

X proper smooth scheme over $\mathcal{O}_K$

$X =$



$\mathrm{spec}\,\mathcal{O}_K =$

special fibre X_k (also $X_{\bar{k}}$) $\overset{?}{\longleftrightarrow}$ generic fibre X_K (also $X_{\bar{K}}$)

How are the two related? eg) cohomologically

Base change in étale cohom $\Rightarrow$

$$H^n_{et}(X_{\bar{k}}, \mathbb{Z}_\ell) \cong H^n_{et}(X_{\bar{K}}, \mathbb{Z}_\ell)$$

for any prime $\ell \neq p$. False if $\ell = p$!

compatible with G_K-actions

$\rightarrow G_K$ action is unramified

Grothendieck (70s): Should replace $H^n_{et}(X_{\bar{k}}, \mathbb{Z}_p)$ by crystalline cohom and try to relate

$$H^n_{crys}(X_k/W) \longleftrightarrow H^n_{et}(X_{\bar{K}}, \mathbb{Z}_p)$$

where $W = W(k)$

Rmk: LHS related to differential forms / de Rham cohom.
RHS is algebraic form of Betti cohom.
So Grothendieck was seeking analogue of Hodge isom.

$$H^n_{dR}(M/\mathbb{C}) \cong H^n_{Betti}(M, \mathbb{C})$$

for M smooth proj $\mathbb{C}$-manifold

Precise answer proposed by Fontaine ('80s):

Crystalline conjecture (Thm of Fontaine, Messing, Bloch, Kato, Faltings, Tsuji, Niziol): There is a natural isom.

$$H^n_{crys}(X_k/W) \otimes_{W(} B_{crys} \cong H^n_{et}(X_{\bar{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{crys}$$

(compatible with G_K-action, filtrations, and Frob. operators)

where

B_{crys} = Fontaine's crystalline period ring
(big $\mathbb{Q}_p$-alg cf) rewrite Hodge isom as

$$H^n_{dR}(M/\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong H^n_{Betti}(M, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$$

and note that $\mathbb{C}$ is big $\mathbb{Q}$-alg.)

Corollary: Standard properties of B_{crys} imply that

$$\left(H^n_{et}(X_{\bar{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{crys}\right)^{G_K} \cong H^n_{crys}(X_k)[\tfrac{1}{p}]$$

ie) $H^n_{et}(X_{\bar{K}}, \mathbb{Q}_p)$ and its G_K-action determine $H^n_{crys}(X_k)[\frac{1}{p}]$!

<u>Problem</u>: Want similar results without inverting p
eg) on torsion and lattices

This is integral p-adic Hodge thy. Goal of course is to survey recent developments (which also simplify §1)

- Bhatt-M-Scholze 2016 + '18
  Bhatt - Scholze 2019
  see also talks of Le Bras and Tsuji

§2 Breuil-Kisin cohomology

$X/\mathcal{O}_K$ as in §1. <u>Qu</u>: how to relate

$$H^n_{et} := H^n_{et}(X_{\bar K}, \mathbb{Z}_p) \qquad H^n_{cris} := H^n_{cris}(X_k/W) \qquad H^n_{dR} := H^n_{dR}(X/\mathcal{O}_K)$$

- H^n_{et}: f.g. $\mathbb{Z}_p$-mod (G_K-action)
- H^n_{cris}: f.g. W-mod (φ-action)
- H^n_{dR}: · hypercohom of the deRham complex $\mathcal{O}_X \xrightarrow{d} \Omega^1_{X/\mathcal{O}_K} \xrightarrow{d} \cdots$
  · f.g. $\mathcal{O}_K$-mod (with filtration)

We use Kisin's ring $W[[u]]$ and maps

$$\begin{tikzcd} & W(C^\flat) & \\ W & W[[u]] \arrow[l, "u \mapsto 0"'] \arrow[u, "u \mapsto [\pi^\flat]"] \arrow[r, "u \mapsto \pi"', "\ker = \langle E(u)\rangle"] & \mathcal{O}_K \end{tikzcd}$$

π, a chosen uniformiser of K

Let $E(u) \in W[u]$ be min. poly.

Side note: here $C := \hat{\bar K}$ is completed alg. closure of K
eg) $K = \mathbb{Q}_p \Rightarrow C = \mathbb{C}_p$
and $C^\flat$ is its <u>tilt</u>, an alg. closed field of char p :

$$C^\flat := \{(x_0, x_1, \dots) : x_i \in C,\ x_{i+1}^p = x_i \ \forall i\}$$

$$(x_0, x_1, \dots)(y_0, y_1, \dots) := (x_0 y_0, x_1 y_1, \dots)$$

$$(x_0, x_1, \dots) + (y_0, y_1, \dots) := \left(\lim_{i\to\infty} (x_i + y_i)^{p^i},\ \lim_{i\to\infty} (x_i + y_i)^{p^{i-1}}, \dots \right)$$

Favourite elements of $C^\flat$:

$$\pi^\flat := (\pi, \pi^{1/p}, \pi^{1/p^2}, \dots)$$

$$\varepsilon := (1, \zeta_p, \zeta_{p^2}, \dots)$$

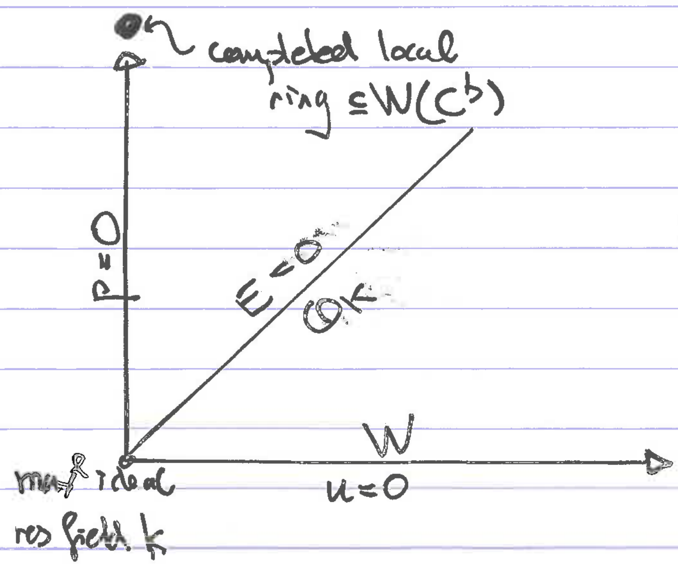
Hopeful 1st answer to question: Maybe there is a f.g. $W[[u]]$-module M and isoms

$$M \otimes_{W[[u]]} W(C^\flat) \cong H^n_{\text{ét}} \otimes_{\mathbb{Z}_p} W(C^\flat)$$

Note: to understand H^n_{et}, we lose no info by $-\otimes W(C^\flat)$, esp. once we introduce a Frob.

$$M/uM \cong H^n_{\text{cryp}} \qquad M/E(u)M \cong H^n_{dR}$$

Picture of spec $W[[u]]$:



Looking for M living on the whole spec which specializes to

- étale cohom at •
- de Rham cohom along $E=0$
- cryp. cohom along $u=0$

This is too naive: it is not necessarily true that

$$H^n_{crys}/p \cong H^n_{dR}/\pi$$

But this IS true in derived sense:

$$R\Gamma_{crys}(X_k/W)/p \simeq R\Gamma_{dR}(X/\mathcal{O}_K)/\pi$$

since both sides are $\simeq R\Gamma_{dR}(X_k/k)$.

This derived/non-derived issue is only obstruction:

Thm 1 (BMS'18) For any proper smooth $X/\mathcal{O}_K$, there exists a natural complex of $W[[u]]$-mods $R\Gamma'_{W[[u]]}(X)$ and equivalences

$$R\Gamma'_{W[[u]]}(X) \otimes_{W[[u]]} W(C^\flat) \simeq R\Gamma_{et}(X_{\bar{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(C^\flat)$$

$$R\Gamma'_{W[[u]]}(X)/u \simeq R\Gamma_{crys}(X_k/W) \qquad R\Gamma'_{W[[u]]}(X)/E \simeq R\Gamma_{dR}(X/\mathcal{O}_K)$$

More explicitly, putting $M^n := H^n(R\Gamma'_{W[[u]]}(X))$, this is a f.g. $W[[u]]$-module, and

$$M^n \otimes_{W[[u]]} W(C^\flat) \cong H^n_{et} \otimes_{\mathbb{Z}_p} W(C^\flat)$$

and there are short exact seqs

$$0 \to M^n/uM^n \to H^n_{crys} \to M^{n+1}[u] \to 0$$

$$0 \to M^n/E\cdot M^n \to H^n_{dR} \to M^{n+1}[E] \to 0$$

and the final two terms are killed by power of p.

<u>Corollary</u>: For each n, H^n_{crys} has "more" torsion than H^n_{et}. More precisely,

$$\text{length}_W\, H^n_{crys}/p^r \geq \text{length}_{\mathbb{Z}_p} H^n_{et}/p^r \quad (\forall r \geq 1)$$

Eg) If torsion in H^n_{crys} is $\cong W/p \oplus W/p$, then torsion in H^n_{et} is $\cong$

$$\mathbb{Z}_p/p \oplus \mathbb{Z}_p/p \quad \text{or} \quad \mathbb{Z}_p/p^2 \quad \text{or} \quad \mathbb{Z}/p \quad \text{or} \quad 0.$$

<u>Rmk</u>: Can also use Corollary to get integral results in $\mathbb{C}$ Hodge theory: given a projective $\mathbb{Z}$ scheme $\mathcal{M}$ which is generically smooth and has good reduction at p, it follows that $H^n_{dR}(\mathcal{M}/\mathbb{Z})$ has more p-primary torsion than $H^n_{Betti}(\mathcal{M}(\mathbb{C}), \mathbb{Z})$

<u>Proof of Corol</u>:
Commutative algebra $\Rightarrow$ for any f.g. $W[[u]]$-module M killed by a power of p, we have

$$\text{length}_W\, M/uM \geq \text{length}_{W(C^b)}\, M \otimes_{W[[u]]} W(C^b)$$

$\square$

In fact, it turns out that $R\Gamma'_{W[[u]]}(X)$ is not the most fundamental invariant: remarkably, it comes from a complex over $W[[u^p]]$. Better, we use the Frobenius

$\varphi \curvearrowright W[[u]]$
- Witt vector Frob. on W
- $u \longmapsto u^p$

Thm 1 ctd.: There is a complex of $W[\![u]\!]$-modules $R\Gamma_{W[\![u]\!]}(X)$ such that

$$R\Gamma_{W[\![u]\!]}(X) \underset{W[\![u]\!],\varphi}{\otimes} W[\![u]\!] \overset{W[\![u]\!]\text{-linear}}{\simeq} R\Gamma'_{W[\![u]\!]}(X)$$

There is moreover an equivalance

$$\text{(BK)} \qquad R\Gamma_{W[\![u]\!]}(X) \underset{W[\![u]\!],\varphi}{\otimes} W[\![u]\!]\left[\tfrac{1}{E}\right] \underset{W[\![u]\!][\frac{1}{E}]\text{-linear}}{\simeq} R\Gamma_{W[\![u]\!]}(X)\left[\tfrac{1}{E}\right]$$

Rmks (1) The distinction between $R\Gamma_{W[\![u]\!]}(X)$ and $R\Gamma'_{W[\![u]\!]}(X)$ has important consequences. For example, suppose that $K = \mathbb{Q}_p(p^{1/p})$, so we may take $\pi = p^{1/p}$. Then the composition

$$W[\![u]\!] \xrightarrow{\varphi} W[\![u]\!] \xrightarrow{u \mapsto \pi} \mathcal{O}_K$$

has image $\mathbb{Z}_p$. It follows that

$$R\Gamma_{dR}(X/\mathcal{O}_K) = R\Gamma_{W[\![u]\!]}(X) \underset{W[\![u]\!],\varphi}{\otimes} W[\![u]\!] \underset{W[\![u]\!]}{\otimes} \mathcal{O}_K$$

$$= \left(\begin{array}{c}\text{a complex}\\ \text{of } \mathbb{Z}_p\text{-mods}\end{array}\right) \underset{\mathbb{Z}_p}{\otimes} \mathcal{O}_K,$$

whence torsion factors in $H^n_{dR}(X/\mathcal{O}_K)$ look like $\mathcal{O}_K/p^m$ for various $m \in \mathbb{N}$, not like $\mathcal{O}_K/p^{1/p}$.

(2) A Breuil-Kisin module (Kisin 2000s) M is a f.g. $W[\![u]\!]$-module equipped with an isom of $W[\![u]\!][\frac{1}{E}]$-modules

$$\varphi_M : M \underset{W[\![u]\!],\varphi}{\otimes} W[\![u]\!]\left[\tfrac{1}{E}\right] \simeq M\left[\tfrac{1}{E}\right]$$

(Rmk: this actually forces $M[\frac{1}{p}]$ to be finite free over $W[[u]][\frac{1}{p}]$)
By taking cohom in the equivalence (BK), we see that each

$$H^n_{W[[u]]}(X) := H^n\left(R\Gamma_{W[[u]]}(X)\right)$$

is a Breuil–Kisin module.

(3) Thm 1 + commutative algebra can be used to reprove the crystalline comparison theorem. Key point is that if M is any B-K module, then there are isoms

$$\left(M \underset{W[[u]]}{\otimes} W(C^\flat)\right)^{\varphi=1} \underset{\mathbb{Z}_p}{\otimes} B_{crys} \cong M \underset{W[[u]]}{\otimes} B_{crys} \cong M/uM \underset{W}{\otimes} B_{crys}$$

(4) History:

- BMS 2016 proved Thm 1 after the "only ramified" base change $W[[u]] \longrightarrow A_{inf}$
  - this is enough for previous cor'l and Rmk(3) above, but not for Rmk(1).
- BMS 2018 proved Thm 1 using topological cyclic homology.
- Bhatt–Scholze 2019 reproved Thm 1 via prismatic cohomology.

The goal for the remaining lecture is to elucidate the various structures in Thm 1. In particular we will see that the equiv (BK) results from a finer result

$$R\Gamma_{W[[u]]}(X) \underset{W[[u]],\varphi}{\otimes} W[[u]] \simeq L\eta_E R\Gamma_{W[[u]]}(X)$$

for affine X (from now on we focus on affines – then glue to get results for general X.)

§3 Décalage functor $L\eta$

<u>Origin</u>: Work of Mazur and Berthelot–Ogus on relating Hodge and Newton polygons (1970s)

<u>Def^n</u>: Let A be a ring, $f \in A$ a non-zero divisor. Given a cochain complex of f-torsion-free A-modules supp. in degrees ≥ 0

$$\mathcal{C} = [\, C^0 \to C^1 \to C^2 \to \cdots \,]$$

we define a subcomplex

$$\eta_f \mathcal{C} \subseteq \mathcal{C} \qquad (\eta_f \mathcal{C})^n := \{ x \in C^n : x \in f^n C^n \text{ and } d(x) \in f^{n+1} C^{n+1} \}$$

<u>Key properties</u> (1) $\eta_f \mathcal{C} \hookrightarrow \mathcal{C}$ is an isomorphism after inverting f.

(2) Bockstein construction: there is a quasi-isom of complexes of A/f-modules

$$(\eta_f \mathcal{C})/f \xrightarrow{\sim} [\, H^0(\mathcal{C}/f) \xrightarrow{\mathrm{Bock}_f} H^1(\mathcal{C}/f) \xrightarrow{\mathrm{Bock}_f} H^2(\mathcal{C}/f) \xrightarrow{\mathrm{Bock}_f} \cdots \,]$$

$(\eta_f \mathcal{C})^n \ni x \longmapsto$ the class in $H^n(\mathcal{C}/f)$ given by the cocycle $x/f^n \in \mathcal{C}^n/f$

where Bock_f denotes the connecting homomorphisms associated to the short exact seq. of complexes

$$0 \to \mathcal{C}/f \xrightarrow{\times f} \mathcal{C}/f^2 \to \mathcal{C}/f \to 0$$

(3) If $C' \to C$ is a quasi-isom, then so is $\eta_f C \to \eta_f C'$.
(Either check directly, or use (1) and (2) since it is enough to check after inverting f and mod f.)

(4) For general $C \in D(A)$ (satisfying following conditions: $H^n(C) = 0$ for $n < 0$ and $H^0(C)$ is f-torsion-free), we define

$$\mathbb{L}\eta_f C \in D(A)$$

to be

$$\mathbb{L}\eta_f C := \eta_f C',$$

where $C' \xrightarrow{\sim} C$ is a resol. of C by a complex C' which satisfies the conditions in the defn of η_f — (this exists by ~~the~~ hypotheses on C and $\mathbb{L}\eta_f$ is well-defined by using (3)).

Main example (crystalline cohom.)
R a smooth k-algebra $\rightsquigarrow$ complex of W-modules

$$R\Gamma_{crys}(R/W)$$

so may apply $\mathbb{L}\eta_p$.

Claim (Berthelot-Ogus) The absolute Frobenius induces a quasi-isom

$$\varphi : R\Gamma_{crys}(R/W) \xrightarrow{\sim} \mathbb{L}\eta_p R\Gamma_{crys}(R/W)$$

Rmk: To make this W-linear should replace lhs by its $- \otimes_{W,\varphi} W$. If we then invert p we get F-isocrystal property

$$R\Gamma_{crys}(R/W) \otimes_{W,\varphi} W[\tfrac{1}{p}] \xrightarrow{\cong} R\Gamma_{crys}(R/W)[\tfrac{1}{p}]$$

cf) the desired Breuil-Kisin property of $R\Gamma_{W[[u]]}$.

Proof of claim: Maybe after localising we may suppose that R lifts to a p-adic formally smooth W-algebra $\tilde{R}$, and that the abs. Frob $\varphi \circlearrowright R$ lifts to an endomorphism $\tilde{\varphi} \circlearrowright \tilde{R}$.
Note then that

$$R\Gamma_{crys}(R/k) = p\text{-adic completion of } \Omega^{\bullet}_{\tilde{R}/W}.$$

Given $f, g \in \tilde{R}$, we have

$$\begin{aligned} \tilde{\varphi}(f\,dg) &\equiv f^p\, dg^p \mod p \\ &= f^p\, p\, g^{p-1}\, dg \\ &\equiv 0 \mod p \end{aligned}$$

ie) $\tilde{\varphi}(\Omega^1_{\tilde{R}/W}) \subseteq p\Omega^1_{\tilde{R}/W}$. More generally, $\tilde{\varphi}(\Omega^n_{\tilde{R}/W}) \subseteq p^n \Omega^n_{\tilde{R}/W}$.

$\Rightarrow \tilde{\varphi}$ induces

$$\tilde{\varphi} : \Omega^{\bullet}_{\tilde{R}/W} \longrightarrow \eta_p \Omega^{\bullet}_{\tilde{R}/W} \quad (*)$$

(which is the desired map in the claim)
When we go mod p this becomes

$$\begin{array}{ccc} \Omega^{\bullet}_{\tilde{R}/W}/p & \longrightarrow & (\eta_p \Omega^{\bullet}_{\tilde{R}/W})/p \\ \| & & \text{is Bock. construction} \\ \Omega^{\bullet}_{R/k} & \xrightarrow{\cong} & [H^0_{dR}(R/k) \to H^1_{dR}(R/k) \to \dots] \end{array}$$

isom. by Cartier isoms

$$\Omega^n_{R/k} \cong H^n_{dR}(R/k)$$

(these are even R-linear isoms if we replace rhs by $H^n(\varphi_* \Omega^{\bullet}_{R/k})$)

So $(*)$ is a quasi-isom mod p, hence a quasi-isom after p-adic completion. $\square$

The key points used to prove the claim were:

- $R\Gamma_{crys}(R|W)/p \simeq \Omega^{\bullet}_{R|k}$ ie) crys cohom deforms de Rham cohom
- Cartier isoms describing cohom groups in terms of differential forms
$$H^n(\varphi_* \Omega^{\bullet}_{R|k}) \cong \Omega^n_{R|k}$$

Prismatic cohomology provides a mixed char version of crystalline cohom with similar properties

ie) instead of deforming de Rham cohom in p-adic directions, will deform in mixed char.

§4 Quick intro to prismatic cohom

Def[n] A (p-torsion free) prism is the data of:

- a ring A with no p-torsion (+ completeness condition)
- $\varphi_A : A \to A$ a ring endo. lifting the abs Frobenius
- a non-zero divisor $d \in A$ s.t
$$p \in (d, \varphi_A(d))$$

Rmk: In fact, Bhatt-Scholze don't pick d but work with the ideal it generates, and even only ask that the ideal be locally generated by a nzd.

To eliminate the p-torsion-free hypothesis, B-S use the δ-ring structure
$$\delta : A \to A, \; f \mapsto \frac{-f^p + \varphi_A(f)}{p}$$

But the above approximate defn of a prism is enough to state what we need.

Examples (i) $W \circlearrowleft$ usual Frob, $d = p$.
(ii) $W[\![u]\!] \circlearrowleft \varphi$ from §2, $d = E(u)$.
(iii) $A_{inf} \circlearrowleft$ usual Frob, $d = \xi$.

Thm 2 (Bhatt-Scholze 2019) A, φ_A, d a prism, and R a (p-adic formally) smooth A/dA-algebra. Then there is a prismatic site

$$(R/A)_{\Delta}$$

equipped with a sheaf of A-algs $\mathcal{O}_{\Delta}$ such that the resulting prismatic cohomology

$$R\Gamma_{\Delta}(R/A) := R\Gamma((R/A)_{\Delta}, \mathcal{O}_{\Delta})$$

has the following properties (among others):

(i) $$H^n(R\Gamma_{\Delta}(R/A)/d) \cong \Omega^n_{R/A} \quad \forall n \geq 0$$
(also, it turns out that $R\Gamma_{\Delta}(R/A)/d$ has the structure of a complex of R-mods, and these isoms are R-linear.)

(ii) $$R\Gamma_{\Delta}(R/A) \underset{A, \varphi_A}{\otimes} A \simeq L\eta_d R\Gamma_{\Delta}(R/A)$$

(iii) A relation to the $\mathbb{Z}_p$ étale cohom of $R[\frac{1}{p}]$.

Rmk: In the case of the prism $W, d = p$, so that R is a smooth k-algebra, we have

$$R\Gamma_{\Delta}(R/W) = \varphi^* R\Gamma_{crys}(R/W)$$

(so we saw properties (i) & (ii) in §3, and property (iii) is empty)

Consequence (of (i)+(ii)):

$$R\Gamma_{\Delta}(R/A) \underset{A, \varphi_A}{\otimes} A/dA \simeq \Omega^{\bullet}_{R/A/dA}$$

re) We get a deformation of de Rham cohom, but only after base change along φ.

To prove this from (i) & (ii), use the Bockstein construction (and identify the differentials via an explicit calculation)

Finally, the desired cohom theory $R\Gamma_{W[[u]]}(X)$ in Thm 1 is given by

$R\Gamma_{W[[u]]}(X) :=$ hypercohomology (re) gluing) of

$$X \supseteq \operatorname{Spec} R \longmapsto R\Gamma_{\Delta}(\hat{R}/W[[u]])$$