Carthage

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Companion forms for p-adic automorphic forms

This mini course aims to illustrate how p-adic Hodge theory influences p-adic variation of automorphic forms.

1. Automorphic forms on (definite) unitary groups

$E/\mathbb{Q}$ quadratic imaginary, $q : E^m \to \mathbb{Q}$ >0 hermitian form

$G = U(q) = \{ g \in GL_{m,E} \mid q \circ g = q \}$ unitary group.

$G \times E \simeq GL_{m,E}$, $G(\mathbb{R}) \simeq U_m(\mathbb{R})$ is compact, p split in E, $G(\mathbb{Q}_p) \simeq GL_m(\mathbb{Q}_p)$.

Weight: V irreducible (algebraic) representation of G.

$\underline{k} = (k_1 \geq k_2 \geq \cdots \geq k_m) \in \mathbb{Z}^m_+ \longleftrightarrow$ Weights

$\underline{k} \longmapsto \mathrm{Res}_{E/\mathbb{Q}}\left(\begin{array}{c}\text{highest weight rep of}\\ \text{wt } \underline{k} \text{ of } GL_{m,E}\end{array}\right) =: V_{\underline{k}}$.

Level: $K \subset G(\mathbb{A}_f)$ compact open subgroup.

The space of automorphic forms of weight V and level K is

$$S_V(K) = \{ f : G(\mathbb{A}_f)/K \to V \mid \forall \gamma \in G(\mathbb{Q}),\ f(\gamma \cdot) = \gamma \cdot f(-) \}.$$

$$\dim_{\mathbb{Q}} S_V(K) < \infty \quad \Big(\Leftarrow S_V(K) = \bigoplus_{g \in G(\mathbb{Q}) \backslash G(\mathbb{A}_f)/K} V^{G(\mathbb{Q}) \cap g K g^{-1}} \Big).$$

finite set

Assume that $K = \prod_\ell K_\ell$, $K_\ell \subset G(\mathbb{Q}_\ell)$ compact ouvert.

Fix p unramified split in E.

$\mathcal{S} = \{ v \mid \ell,\ \ell \neq p \text{ split unramified and } K_\ell \text{ maximal} \}$.

$v \in \mathcal{S} \rightsquigarrow G(\mathbb{Q}_\ell) \xrightarrow{\ \iota_v\ } GL_m(E_v) \simeq GL_m(\mathbb{Q}_\ell)$

$\mathcal{H}^{sph} = \mathbb{Z}_p[T_{i,v} \mid v \in \mathcal{S},\ 1 \leq i \leq m]$ free commutative $\mathbb{Z}_p$-algebra.

$\mathcal{H}^{sph}$ acts on $S_V(K)$ $\quad T_{i,v} \longleftrightarrow K_\ell \, \iota_v^{-1}\Big(\underbrace{\ell \cdots \ell}_{i}\ 1 \cdots 1\Big) K_\ell$

$$\mathbb{T}_V(K) := \mathrm{Im}(\mathcal{H}^{sph} \to \mathrm{End}(S_V(K)))$$

It is a reduced finite $\mathbb{Z}_p$-algebra.

(Hecke) eigensystem of weight V, level K is a character
$\lambda : \mathbb{T}_V(K) \longrightarrow \overline{\mathbb{Q}_p}$.

The following theorem is one of the achievments of a huge collective work.

<u>Theorem</u> $\lambda : \mathbb{T}_V(K) \longrightarrow \overline{\mathbb{Q}_p}$ eigensystem.

$\exists !\ \rho_\lambda : \mathrm{Gal}(\overline{E}/E) \longrightarrow GL_n(\overline{\mathbb{Q}_p})$ cts semisimple such that

- $v \in \mathcal{S} \Rightarrow \rho_{\lambda, v} := \rho_\lambda|_{G_{E_v}}$ unramified and

$$\det(X - \rho_\lambda(\mathrm{Frob}_v)) = \underbrace{P_{\lambda, v}(X) := X^n - \lambda(T_{1,v})X^{n-1} + \cdots + (-1)^i \ell_v^{\frac{i(i-1)}{2}} \lambda(T_{i,v}) X^{n-i} + \cdots}_{\text{Hecke polynomial}}$$

Moreover, we have

- $\rho_\lambda^\vee \simeq \rho_\lambda^c \otimes \chi_{cycl}^{n-1}$ where $\mathrm{Gal}(E/\mathbb{Q}) = \{1, c\}$.
- K_p maximal, $v | p \Rightarrow \rho_\lambda|_{G_{E_v}}$ crystalline of HT weights $(n-1-k_n > n-2-k_{n-1} > \cdots > -k_1)$ where $V = V_{\underline{k}}$ and $\left(\varphi \in \mathrm{End}(D_{crys}(\rho_\lambda|_{G_{E_v}}))\right)$, $\det(X - \varphi) = P_{\lambda, v}(X)$.

2. The p-adic eigenvariety

Assume K_p is maximal. Fix $v | p$. $K^p = \prod_{\ell \neq p} K_\ell$ tame level.

To interpolate eigensystems, we have to take care of possible congruences between themselves.

$\mathcal{V}$ finite set of weights. $\mathbb{T}_{\mathcal{V}}(K) = \mathrm{Im}\left(\mathcal{H}^{sph} \longrightarrow \prod_{V \in \mathcal{V}} \mathrm{End}(S_V(K))\right)$

$\mathbb{T}(K^p) = \varprojlim_{\mathcal{V}} \mathbb{T}_{\mathcal{V}}(K)$. It is the "big" Hecke algebra of tame level K^p. $\forall V$, $\mathbb{T}(K^p) \twoheadrightarrow \mathbb{T}_V(K)$.

For the projective limit topology, this is a reduced complete noetherian semilocal $\mathbb{Z}_p$-algebra.

We can consider $\mathcal{X} = \mathcal{X}_{Hecke} = (\mathrm{Spf}\, \mathbb{T}(K^p))^{rig}$ its generic fiber. It is a reduced rigid analytic space over $\mathbb{Q}_p$.

$\mathcal{X}(\overline{\mathbb{Q}_p}) \simeq \mathrm{Hom}_{cts}(\mathbb{T}(K^p), \overline{\mathbb{Q}_p}) \supset \mathrm{Hom}(\mathbb{T}_V(K^pK_p), \overline{\mathbb{Q}_p}) \ni \lambda$.

λ is called a <u>classical</u> point.

λ classical. A <u>refinement</u> of λ is an ordering $R = (\varphi_1, -, \varphi_m)$ on roots of $P_{\lambda,v}(X)$.

(λ, R) refined eigensystem of weight $V_{\underline{k}}'$

$$\delta_R : (\mathbb{Q}_p^\times)^m \longrightarrow \overline{\mathbb{Q}_p}^\times$$
$$= ur(\varphi_1) \otimes ur(p\varphi_2) \otimes \cdots \otimes ur(p^{m-1}\varphi_m)$$

smooth (unramified) character

$$\delta_{\underline{k}} : (\mathbb{Q}^\times)^m \longrightarrow \overline{\mathbb{Q}_p}^\times \qquad \delta_{\underline{k},R} = \delta_{\underline{k}} \cdot \delta_R.$$
$$= x^{k_1} \otimes \cdots \otimes x^{k_m}.$$

Let $\mathcal{T} = \widehat{\mathbb{Q}_p^\times}$ the rigid analytic variety of characters of $\mathbb{Q}_p^\times$.

$$\mathbb{Q}_p^\times \simeq \mathbb{Z}_p^\times \times p^{\mathbb{Z}} \Rightarrow \mathcal{T} \simeq \underbrace{\widehat{\mathbb{Z}_p^\times}}_{\mathcal{W}} \times \mathbb{G}_m^{rig}$$

$\mathcal{W}$ the "weight space".

$$\mathcal{W} \simeq \coprod_{p-1} \{|z|<1\}.$$

We <u>define</u> the eigenvariety of (G, K^p) as

$$\mathcal{E} = \overline{\left\{ (\lambda, \underbrace{\delta_{\underline{k}} \cdot \delta_R}_{\delta_{\underline{k},R}}) \in \mathcal{X}_{Heck} \times \mathcal{T}^m \;\middle|\; (\lambda, R) \text{ refined eigensystem of weight } V_{\underline{k}}' \text{ (with } k_i > k_{i+1}) \right\}}$$

+ techn condition.

Zariski closure in $\mathcal{X}_{Hecke} \times \mathcal{T}^m$.

<u>Remark</u>: There is an analogy with the case of modular forms.

$f \in S_k(SL_2(\mathbb{Z}))$ eigenform $X^2 - a_p X + p^{k-1} = (X-\alpha)(X-\beta)$.

Coleman (Hida): the pairs (f, α) can be p-adically interpolated

$\alpha \leftrightarrow$ refinement $\qquad \beta p^{-(k-2)} = p\alpha^{-1}$.

<u>Theorem</u> (Chenevier) $\mathcal{E}$ is a rigid analytic space of (equidim) dimension n. The map $\kappa : \mathcal{E} \longrightarrow \mathcal{T}^n \longrightarrow \mathcal{W}^n$ is quasi-finite.

Moreover if $(\lambda, \delta_{\underline{k}} \delta_R) \in \mathcal{E}$ and $k_i - k_{i+1} > v_p\left(\delta\begin{pmatrix} p^{n-1} & & \\ & \ddots & \\ & & 1 \end{pmatrix}\right)$, then (λ, R) is a refined classical eigensystem.

We can consider $\mathcal{E} \xrightarrow{\alpha} \mathcal{X}_{Hecke}$. The points in a single fiber are called <u>companion points</u>.

(λ, R) refined eigensystem with λ φ-regular ($\in \mathcal{E}$).

$$P_{\lambda,v}(X) = \prod (X - \varphi_i) \qquad \varphi_i \varphi_j^{-1} \notin \{1, p^{\pm 1}\} \text{ for } i \neq j.$$

$\rightarrow$ $\mathcal{F}_R = (F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \dots \subsetneq F_n = F)$ complete φ-stable flag of $D_{\mathrm{cris}}(\rho_{\lambda,v})$ s.t. $\varphi|_{F_i/F_{i-1}} = \varphi_i \, \mathrm{Id}$.

$\rightarrow$ $\mathcal{F}_{dR}$ Hodge filtration on $D_{dR} = D_{\mathrm{cris}}$, complete flag.

$\exists!$ $w_R \in \mathfrak{S}_n$ such that $(\mathcal{F}_R, \mathcal{F}_{dR}) \in GL_n \cdot (1, w_R) \subset GL_n/B \times GL_n/B$.

<u>Conjecture</u> (Breuil) λ φ-regular of weight $V_{\underline{k}}'$

Then $\alpha^{-1}(\lambda) = \left\{ (\lambda, \delta_R \, \delta_{w \cdot \underline{k}}) \;\middle|\; \begin{array}{l} R \text{ refinement of } \lambda \\ w \leq w_R w_0 \end{array} \right\}$.

$\leq$: Bruhat order; w_0: longest element

$$w \cdot \underline{k} = w(\underline{k} + \rho) - \rho \qquad \rho = (n-1, n-2, \dots, 0).$$

<u>Remark</u>: • generic case: $w_R = w_0$ $\forall R$, companion points are classical refined eigensystems.

• points of non dominant weights can appear when $\mathcal{F}_{dR}$ is not in generic position.

$\sim$ Coleman criterion: $f \in S_k$, $v_p(\alpha) = 0$ $\quad v_p(\beta) = k-1$

$\rho_{f,p}$ reducible/split $\iff f_\beta = \theta^{k-1}(g)$, g weight $2-k$.

Everything can be generalized to the case where E CM, $[E:F] = 2$, F tot real, G/F.

<u>Theorem</u> (Breuil-Hellmann-S.) Assume E/F unramified and G quasisplit at finite places, K_v hyperspecial if v inert in E, $p \geq 2(n+1)$ and $\bar{\rho}_\lambda : \mathrm{Gal}(\bar{E}/E(\zeta_p)) \rightarrow GL_n(\mathbb{F}_p)$ irreducible.

Then the conjecture is true for λ.

Moreover $(\lambda, \delta_{\underline{k},R}) \in \mathcal{E}$, $\underline{k} \in \mathbb{Z}_+^n$ and $\rho_{\lambda,v}$ cryst

$\Rightarrow \lambda$ classical

3 Trianguline representations

I need to introduce (φ, Γ)-modules over the Robba ring.

$r < 1$, $\mathcal{R}^{]r,1[} = \left\{ \sum a_n z^n \;\middle|\; a_n \in \mathbb{Q}_p,\ \text{CV for } r < |z| < 1 \right\}$.

$\mathcal{R} = \varinjlim_{r<1} \mathcal{R}^{]r,1[}$. $\quad \varphi : \mathcal{R} \hookrightarrow \mathcal{R}$, $z \longmapsto (1+z)^p - 1$ $\qquad \Gamma = \mathbb{Z}_p^\times \to \mathrm{Aut}(\mathcal{R})$, $a \longmapsto [a](z) = (1+z)^a - 1 = \sum_{n \geq 1} \binom{a}{n} z^n$

A typical element of $\mathcal{R}$ is $t = \log(1+z)$.

We have $\varphi(t) = pt$ $\quad [a](t) = at$. "$2\pi i$ of Fontaine".

$\underline{\text{Def}}$ A (φ, Γ)-module (over $\mathcal{R}_L$) is a finite free $\mathcal{R}_L$-module D

$L/\mathbb{Q}_p$ finite, $\mathcal{R}_L = \mathcal{R} \otimes_{\mathbb{Q}_p} L$

$+ \left.\begin{array}{l} \varphi \in \mathrm{End}(D) \\ \Gamma \to \mathrm{Aut}(D) \end{array}\right\}$ semilinear, commuting, continuous and such that $D = \mathcal{R}_L \varphi(D)$.

$\rightsquigarrow$ category $\varphi\Gamma_{\mathcal{R}_L}$ it is additive L-linear but not abelian ($\sim$ category of vector bundles).

Fontaine, Cherbonnier-Colmez, Berger:

$D_{\mathrm{rig}} : \mathrm{Rep}_L\, G_{\mathbb{Q}_p} \hookrightarrow \varphi\Gamma_{\mathcal{R}_L}$ fully faithful L linear functor.

$\underline{\text{Example}}$. $\delta : \mathbb{Q}_p^\times (= W_{\mathbb{Q}_p}^{ab}) \longrightarrow L^\times$ continuous character

$\mathcal{R}(\delta) := \mathcal{R}_L \cdot e$ with $\varphi(e) = \delta(p)\, e$ $\quad [a](e) = \delta(a)\, e$.

Colmez: $\mathrm{Hom}_{ct}(\mathbb{Q}_p^\times, L^\times) \simeq \left\{ \begin{array}{c} \text{rk 1 objects} \\ \text{in } \varphi\Gamma_{\mathcal{R}_L} \end{array} \right\} / \simeq$

compatible with local class field theory.

$\underline{\text{Definition}}$ (Colmez) $(\rho, V) \in \mathrm{Rep}_L\, G_{\mathbb{Q}_p}$ is $\underline{\text{trianguline}}$

if $D_{\mathrm{rig}}(\rho, V) \simeq \begin{pmatrix} \mathcal{R}(\delta_1) & & * \\ & \ddots & \\ 0 & & \mathcal{R}(\delta_n) \end{pmatrix}$ $\quad (\delta_1, \ldots, \delta_n) \in \mathcal{T}^n$ is a $\underline{\text{parameter}}$ of (ρ, V).

We say that $(\delta_1, \ldots, \delta_n) \in \mathcal{T}^n$ is $\underline{\text{regular}}$ if

$\forall i \neq j$, $\quad \delta_i \delta_j^{-1} \notin \{ x^n, x^{n+1}|x| \mid n \in \mathbb{N} \}$.

<u>Example</u> A crystalline representation is (φ,Γ) trianguline

$$D_{rig}(V)[\tfrac{1}{t}] \simeq D_{cris}(V) \otimes_L \mathcal{R}_L[\tfrac{1}{t}] \quad \varphi, \Gamma\text{-compatible}$$

(Berger) ↖ φ triangular for L big enough.

$\mathcal{X}_{Gal}(A) = \mathrm{Hom}_{cont}(G_{\mathbb{Q}_p}, GL_n(A))$ A $\mathbb{Q}_p$-aff

↳ rigid analytic space $\mathcal{X}_{Gal}(\overline{\mathbb{Q}}_p) = \mathrm{Hom}(G_{\mathbb{Q}_p}, GL_n(\overline{\mathbb{Q}}_p))$.

$U_{sat} = \{ (\rho, \underline{\delta}) \in \mathcal{X}_{Gal} \times \mathcal{T}^n \mid \rho$ trianguline of <u>regular</u> parameter $\underline{\delta}$ $\}$.

X_{tri} = Zariski closure of U_{sat} in $\mathcal{X}_{Gal} \times \mathcal{T}^n$.

<u>Theorem</u> $U_{sat} \subset X_{tri}$ Zariski open and dense.

Moreover $U_{sat} \xrightarrow{\kappa} \mathcal{T}^n$ smooth of rel dimension $n^2 + \frac{n(n-1)}{2}$.

OTA
$$\begin{tikzcd} U_{sat} \arrow[r] \arrow[dr, "\kappa"'] & \mathcal{T}^n \arrow[d] \\ & \mathcal{W}^n \end{tikzcd}$$

$\Rightarrow X_{tri}$ equidimensional of dim $n^2 + \frac{n(n+1)}{2}$.

We can consider $\alpha : X_{tri} \to \mathcal{X}_{Gal}$ and study <u>local companion points</u>.

The local analogue is a theorem. | ρ cryst, $\alpha^{-1}(\rho) \ni (\rho, \delta)$ $\Rightarrow \delta$ loc alg unramified

<u>Theorem</u> Let $\rho \in \mathcal{X}_{Gal}$ crystalline regular (ie φ-regular and HT weights pairwise $\neq 0$).

Then $\alpha^{-1}(\rho) = \{ (\rho, \delta_{w(\underline{k})}\delta_R) \mid R$ refinement of ρ, $w \leq w_R w_0 \}$.

where $HT(\rho) = \{ k_1 > k_2 > \dots > k_n \}$.

4. The local structure of X_{tri} at regular crystalline points

ρ cryst, δ parameter s.t. $x := (\rho, \delta) \in U_{sat} \Rightarrow \kappa$ smooth at x

In general that's not the case and fibers have interesting properties.

<u>Example</u> $n=2$. $HT(\rho) = (k_1 > k_2)$

$\varphi \sim \begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix}$ $\quad D_{cris}(\rho) = Le_1 \oplus Le_2$.

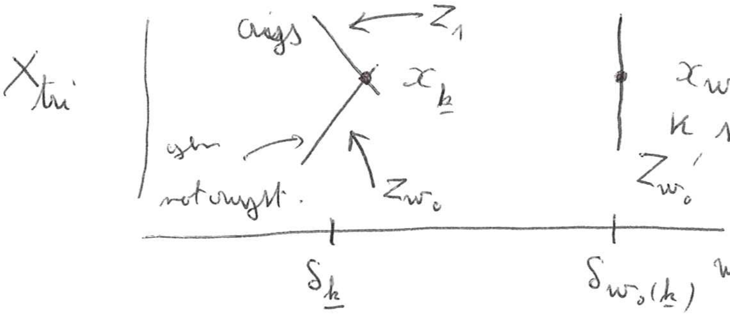
$R = (\varphi_1, \varphi_2)$ $\quad w_R = \begin{cases} 1 & \text{if } e_1 \notin \mathrm{Fil}^{-k_2} \\ w_0 & \text{if } e_1 \in \mathrm{Fil}^{-k_2}. \end{cases}$

$w_R = 1$ $\quad \underline{\delta = \delta_{\underline{k}}\delta_\varphi} = x^{k_1} \mathrm{nr}(\varphi_1) \otimes x^{k_2} \mathrm{nr}(\varphi_2)$ is a parameter of ρ.

$w_R = w_0$ $\quad \underline{\delta}$ is not a parameter.

Actually $D_{rig}(\rho) = \mathcal{R}_L(x^{k_1}\mathrm{nr}(\varphi_2)) \oplus \mathcal{R}_L(x^{k_2}\mathrm{nr}(\varphi_1))$.

$\Longrightarrow$ $\delta_{w_0(k)}\delta_R = x^{k_2}\mathrm{nr}(\varphi_1) \otimes x^{k_1}\mathrm{nr}(\varphi_2)$ is a parameter $\in U_{sat}$.



$x_{w_0} = \delta_{w_0(k)}\delta_R$ $\quad \downarrow \kappa$.

κ smooth.

$\alpha(Z_{w_0}) = \alpha(Z'_{w_0})$

w^2 $\cancel{\mathcal{C}}^2$ $\quad Z_{w_0} \cap U_{sat} = \emptyset$.

There is a non saturated filtration: $\mathcal{R}_L(x^{k_1}\mathrm{nr}(\varphi_1)) \hookrightarrow \mathcal{R}(x^{k_2}\mathrm{nr}(\varphi_1))$.

The general result is largely inspired from the Breuil-Mézard conjectures.

<u>Notation</u> X rigid space, $x \in X(\overline{\mathbb{Q}}_p)$, $\widehat{X}_x := \mathrm{Spec}\,\widehat{\mathcal{O}}_{X,x}$.

$\mathfrak{X}_\rho = \widehat{\mathfrak{X}_{Gal,\rho}}$ it is the deformation space of ρ.

<u>Theorem</u> $\rho \in \mathfrak{X}_{Gal}$ crystalline regular. R refinement of ρ.

$\forall w \leq w_R w_0$, there exists $Z_w \subset \mathfrak{X}_\rho = \widehat{\mathfrak{X}_{Gal,\rho}}$

some $d = n^2 + \frac{n(n-1)}{2}$ - dim cycle such that

$\forall w' \leq w_R w_0$, we have

- α is a closed immersion at $x = (\rho, \delta_{w'(\underline{k})}\delta_R)$

and $\alpha\left(\widehat{\kappa^{-1}(\delta_{w'(k)})_x}\right) = \sum_{w \in \mathfrak{S}_n} [M(w') : L(w)]\, Z_w$ in $CH_d(\mathfrak{X}_\rho)$,

where $M(w') = U(\mathfrak{gl}_n) \otimes_{U(\mathfrak{b})} w' \cdot 0 \twoheadrightarrow L(w')$.

<u>Remark</u> cycles Z_w can be reducible if $n \geq 8$.

$\boxed{\bullet\ X_{tri} \text{ is irreducible at } x}$

Strategy of the proof:

- construct a scheme $X = \cup X_w$

$$\begin{tikzcd} X = \cup X_w \arrow[d, "\kappa"] \\ \mathbb{A}^n \end{tikzcd} \qquad \kappa^{-1}(0)^{red} = \bigcup_{w'} \widetilde{Z}_{w'}$$

and

$$\begin{tikzcd} \widehat{X}_{x_R} \arrow[r, hook] & \mathfrak{X}_e \\ \widehat{X}_{w,x_R} \arrow[u, phantom, "\cup"] \arrow[d] & \widehat{X}_{tri, x_{w,R}} \arrow[l, "\sim"'] \arrow[u] \arrow[dl, "\kappa - S_{w(k)}"] \\ \mathbb{A}^n & \end{tikzcd}$$

Have to understand $\kappa^{-1}(0) \cap X_w$.

- $Z_{w'}$ will be the characteristic support of an holonomic D-module associated to $L_{w'}$

and $\kappa^{-1}(0) \cap X_w \xrightarrow[\text{(formally)}]{} M_w$.

We will work with a smooth modification of $\mathfrak{X}_e$.

5. The local model (*).

A B_{dR}-representation is a finite-dim B_{dR} vector space W with a cts semilinear action of $G_{\mathbb{Q}_p}$.

A B_{dR}^+ ——— finite free B_{dR}^+-module W^+.

W is de Rham iff it is trivial $\Leftrightarrow \dim_{\mathbb{Q}_p} W^{G_{\mathbb{Q}_p}} = \dim_{B_{dR}} W$.

W is almost de Rham iff it is a finite extension of de Rham representation.

Theorem (Fontaine). $\exists$ exact equivalence

$$\{\text{almost de Rham}\} \xrightarrow[D_{pdR}]{\sim} \operatorname{Rep} \mathbb{G}_a \xrightarrow{\sim} \{(D, A) \mid A \in \operatorname{End}_{\mathbb{Q}_p} D\}.$$

- $\mathbb{G}_a$ W almost de Rham.

$$\left\{ \begin{array}{c} G_{\mathbb{Q}_p}\text{-stable } B_{dR}^+\text{-lattices} \\ \text{in } W \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{separated + exhaustive} \\ \text{decreasing filtrations} \\ \text{on } D_{pdR}(W) \\ \text{stable under } A. \end{array} \right\}$$

Corollary. Let W^+ be a B_{dR}^+-rep of $\overset{adB}{\text{HT}}$ weights pairwise $\neq 0$.
$\underline{e}$ basis of $D_{pdR}(W^+[\tfrac{1}{t}])$.
Then the deformation functor of $(W^+, \underline{e})$ is pro-representable

9

by $\widehat{\tilde{g}_x}$ where $\tilde{g} = \{ (\sigma_\bullet, A) \in GL_n/B \times gl_n \mid A\sigma_\bullet \subset \sigma_\bullet \}$ is Grothendieck simultaneous resolution and

$x = (\underbrace{\sigma_{dR}, A}_{\text{in } e}) \in \tilde{g}$.

The starting point is the following result.

(*) **Theorem** (Kedlaya - Pottharst - Xiao, Liu, (Kisin)).
$x = (\rho, \delta) \in X_{tri}$ (*). Then ρ_A is trianguline and $D_{rig}(\rho_A)[\frac{1}{t}]$ has a filtration by finite free $R_A[\frac{1}{t}]$
(*) for any $Sp\, A \hookrightarrow X_{tri}$ inf neighborhood of x)
submodules (φ, Γ)-stable $D_{i,A} \subset D_{i+1,A} \subset \dots$ st.
$D_{i,A}/D_{i-1,A} \simeq R(\delta_{i,A})[\frac{1}{t}]$. ($R \otimes_{\mathbb{Q}_p} A$ - (φ,Γ)-module).

R refinement σ_R ...
We define $\mathfrak{X}_{\rho,R}(A) = \{ \rho_A \in \mathfrak{X}_\rho(A) + \sigma_A$ triangulation of $D_{rig}(\rho_A)[\frac{1}{t}]$ st $\sigma_A \otimes_A L = \sigma_R \}$.

ρ crystalline regular $\Rightarrow$ $\mathfrak{X}_{\rho,R} \subset \mathfrak{X}_\rho$ closed subscheme.

$$\begin{tikzcd} \forall\, w \leq w_R w_0, \quad \widehat{X}_{tri, x_{w,R}} \arrow[r] \arrow[dr] & \mathfrak{X}_\rho \\ & \mathfrak{X}_{\rho,R} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

Berger:
$$\begin{tikzcd} \{(\varphi,\Gamma)\text{-modules}/R\} \arrow[r, "W_{dR}^+"] \arrow[d, "{[\frac{1}{t}]}"] & \{B_{dR}^+\text{-rep}\} \arrow[d, "{[\frac{1}{t}]}"] \\ \{(\varphi,\Gamma)\text{-modules}/R[\frac{1}{t}]\} \arrow[r, "W_{dR}"] & \{B_{dR}\text{-rep}\} \end{tikzcd}$$

$$\begin{tikzcd} \mathfrak{X}_{\rho,R} \arrow[r] & \mathfrak{X}_{W_{dR}^+(\rho),R} \\ \widetilde{\mathfrak{X}}_{\rho,R} \arrow[u, "\text{formally smooth}"] \arrow[r] \arrow[ur, phantom, "\square"] & \mathfrak{X}_{W_{dR}^+(\rho),R,e} \arrow[u] \arrow[dl] \\ & \end{tikzcd}$$

$\Rightarrow$ $\mathfrak{X}_{W_{dR}^+(\rho),R}(A) = \{ W_A^+$ deformation of $W_{dR}^+(\rho)$ + σ_A filtration of $W_{dR}(\rho)$ deforming $W_{dR}(\sigma_R) \}$.

$\Rightarrow$ pro-represented by the completion of $\tilde{g} \times_g \tilde{g}$

at the point $(\sigma_R, 0, \sigma_{dR})$.

5. Application to the global situation

$G = U(q)$ unitary group (definite) $K^p \subset G(\mathbb{A}_f^p)$ tame level.

$\mathbb{T}(K^p)$ $\quad \mathcal{E} \subset \mathcal{X}_{Hecke} \times \mathcal{T}^n \quad \mathcal{T} = \widehat{\mathbb{Q}_p^\times}$.

Conjecture $(\lambda, \underline{k})$ ~~refined~~ classical eigensystem of wt $V_{\underline{k}}$'

$$\alpha^{-1}(\lambda) = \left\{ (\lambda, \delta_{w\cdot \underline{k}}\, \delta_R) \;\middle|\; \begin{array}{l} R \text{ refinement} \\ w \leq w_R w_0 \end{array} \right\}.$$

Similar conjecture if we replace $\mathbb{Q}$ by F tot real and E by CM ext of F.

Theorem (BHS) Assume E/F unramified, G q split at finite places

K_v hyperspecial if v inert in E and $p \geq 2(n+1)$.

If λ is such that $\rho_\lambda|_{Gal(\bar{E}/E(\zeta_p))}$ is irreducible, the conj is true for λ.

I need to say a bit more about $\mathcal{E}$. $(F = \mathbb{Q})$.

$$\mathbb{T}(K^p) \curvearrowright S^{an}(K^p) = \Big\{ f : \underbrace{G(\mathbb{Q}) \backslash G(\mathbb{A}_f) / K^p}_{\simeq GL_n(\mathbb{Z}_p)^{\oplus s}} \longrightarrow \mathbb{Q}_p \text{ locally analytic} \Big\}.$$

for K^p small enough.

$\circlearrowleft$ $GL_n(\mathbb{Q}_p)$

locally analytic representation of $GL_n(\mathbb{Q}_p)$.

$\mathcal{A} = S^{an}(K^p)^{N_0}$ with $N_0 = \begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$.

$\circlearrowleft$ Hecke action

$T_+ = \{ t \in T(\mathbb{Q}_p) \mid t N_0 t^{-1} \subset N_0 \}$

$\mathrm{Hom}_{K_p}(V, S^{an}(K^p)) \simeq S_{V^\vee}(K_p K^p)$.

$x = (\lambda, \delta) \in \mathcal{E} \iff \mathcal{A}[\lambda, \delta] \neq 0$.

Better : $\exists\, \mathcal{M}$ coherent sheaf on $\mathcal{E}$ (maximal CM)

such that $\mathcal{M} \otimes k(x) \simeq \mathcal{A}[\lambda, \delta]'$.

Special case $\quad x = (\lambda, \delta_{\underline{k}}\, \delta_R) \quad \underline{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$.

$\quad$ ↳ unr.

$$\mathcal{A}[\lambda,\delta] \simeq \mathrm{Hom}_{U(\mathfrak{g})}\left(M(\underline{k}), \cancel{\ } S^{an}\right)^{N_0}[\lambda, \delta_P].$$

the functor $M \longmapsto \underbrace{\mathrm{Hom}(M, S^{an})^{N_0}[\delta_P]'}_{\text{finite type on } \mathbb{T}(K^p)}$ is exact.

$\mathcal{M}(M) =$

$$M(\underline{k}) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \delta_{\underline{k}}.$$

$[s = w_0]$.

$\underline{n=2}$ $\quad 0 \to M(s\cdot\underline{k}) \to M(\underline{k}) \to V_{\underline{k}} \to 0$

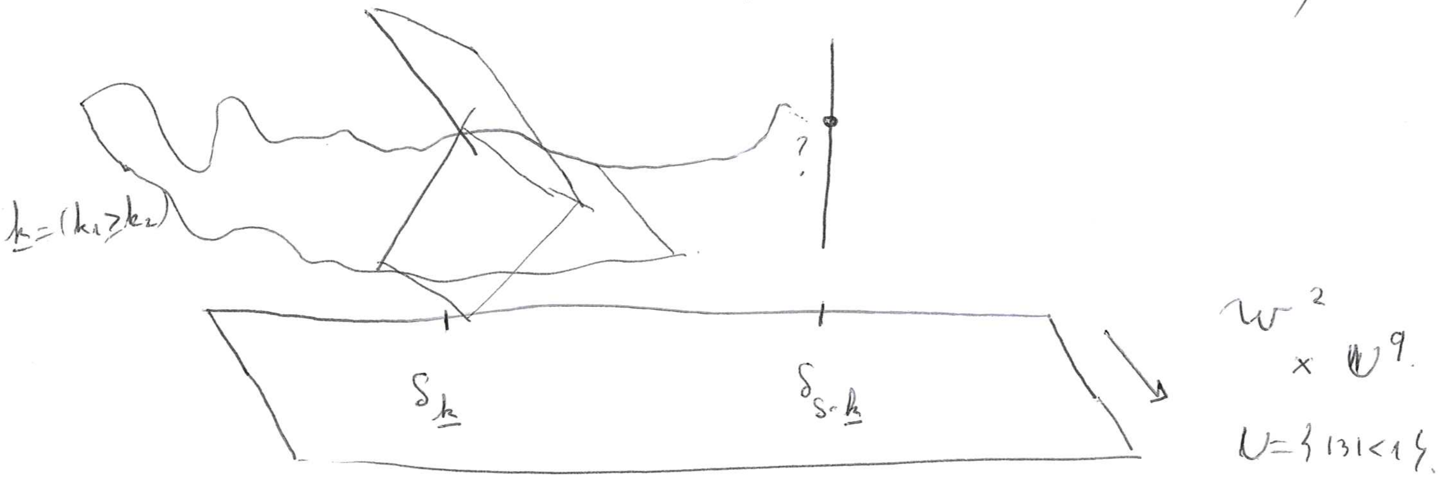
$$0 \to \mathcal{M}(M(s\cdot\underline{k})) \to \mathcal{M}(M(\underline{k})) \to \mathcal{M}(V_{\underline{k}}) \to 0$$

$\overset{?}{\underset{0}{\shortparallel}}$ $\qquad$ module of classical forms.

if $w_R = s$ (depends on $\rho_{\lambda,v}$).

$$\mathcal{E}(K^p) \longrightarrow X_{tri}$$

$$(\rho, \delta) \longmapsto \left(\rho|_{G_{\mathbb{Q}_p}}, \delta\cdot(1, (x|x|)^{-1}, \dots, (x|x|)^{1-n})\right).$$



We can do it and obtain some patched eigenvariety.

$$S^{an}_{\mathfrak{m}} \hookrightarrow \underset{\substack{\circlearrowleft \\ GL_n(\mathbb{Q}_p)}}{\Pi_\infty} \circlearrowleft R_\infty \twoheadrightarrow \mathbb{T}(K^p)_{\mathfrak{m}} \qquad (*).$$

\begin{tikzcd}
\mathcal{E}_\infty \arrow[d, "\kappa_\infty"] \arrow[dr, phantom, "\square"] & \mathcal{E} \arrow[l, hook'] \arrow[d, "\kappa"] \\
\mathcal{W}_\infty = \mathcal{W}^2 \times \mathbb{U}^q & \mathcal{W}^2 \arrow[l, hook']
\end{tikzcd}

$\mathcal{M}_\infty$ coherent sheaf on $\mathcal{E}_\infty$ such that

$$\mathcal{M}_\infty \otimes k(x) \simeq \left(\Pi_\infty^{N_0}[\lambda, \delta]\right)'.$$

Moreover $\mathcal{E}_\infty \hookrightarrow X_{tri} \times \mathbb{U}^g$.

$\downarrow$ $\mathcal{W}^2$ $\qquad$ union of irreducible components.

I forgot to precise: $x \in X_{tri}$ crystalline regular, X_{tri} is irreducible at x.

$\mathcal{M}_\infty|_{\kappa^{-1}(S_{\underline{k},R})}$ is the pullback along α of.

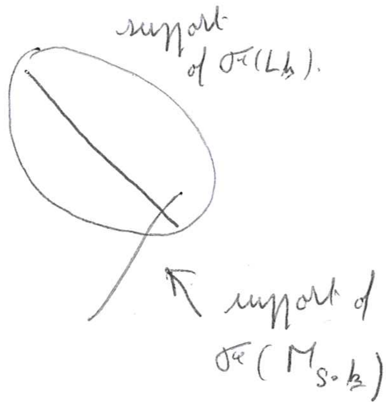
$$\mathrm{Hom}_{U(\mathfrak{g})}(M(\underline{k}), \Pi_\infty)^{N_0}[S_R]'$$

$$= \sigma_\infty(M_{\underline{k}}).$$

$\circlearrowleft$ R_∞-module of finite type.

$$0 \to \sigma(M_{s\cdot k}) \to \sigma(M_{\underline{k}}) \to \sigma(L_{\underline{k}}) \to 0$$

$$\sigma(L_{\underline{k}}) \cong \underbrace{S_{V_{\underline{k}}'}(K^p I)[S_R]'}_{\text{supported on crystalline points.}}$$



$\Longrightarrow$ is in the support of $\mathcal{E}_\infty$.

$\Longrightarrow x_{w_0 ; R} \in \mathcal{E}$. $\square$

$(*)$ The map $R_\infty \twoheadrightarrow \mathbb{T}(K^p)$

$\mathbb{T}(K^p)$ semilocal choose $\mathfrak{m}$ max ideal st $\mathbb{T}(K^p) \xrightarrow{\lambda} \overline{\mathbb{Q}}_p$, $\mathbb{T}(K^p) \to \mathbb{T}(K^p)_{\mathfrak{m}} \to \overline{\mathbb{Q}}_p$

Theory of pseudo-representations:

$$\Longrightarrow \mathrm{Gal}(\overline{E}/E) \xrightarrow{e^{\mathrm{un}}} GL_n(\mathbb{T}(K^p)_{\mathfrak{m}})$$

st $e^{\mathrm{un}} \otimes k(\lambda) \simeq e_\lambda$.

$$G_{\mathbb{Q}_p} \to GL_n(\mathbb{T}(K^p)_{\mathfrak{m}}) \rightsquigarrow R_{\overline{\rho}_{\tilde{v}}} \to \mathbb{T}(K^p)_{\mathfrak{m}}.$$

$$R_{\overline{\rho}_{\tilde{v}}} \hookrightarrow R_\infty \to \mathbb{T}(K^p)_{\mathfrak{m}}$$

6. Some word on the local model

$$X_{tri} = \{(e,\delta) \mid e \text{ trianguline of regular parameter } \delta\} \subset \mathfrak{X}_{\bar\rho} \times \mathcal{T}^n.$$

Ψ

$x = (e,\delta)$ $\quad$ e crystalline regular.

$\delta = \delta_{W(\underline{k})}\delta_R$ $\quad$ R refinement $\quad$ $\underline{k} = HT(e) = (k_1 > \dots > k_n)$

$\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}} = \bigcup_{w \in \mathfrak{S}_n} X_w$ $\quad$ X_{ww_0} is a local model for X_{tri} at x.

$$\begin{tikzcd} & y \arrow[dl, "\text{form smooth}"'] \arrow[dr] & \\ \widehat{X}_{tri,x} \arrow[dr, "\kappa"'] & & \widehat{X}_{ww_0,y} \arrow[dl] \\ & \widehat{\mathcal{W}}^n_{\kappa(x)} \simeq \widehat{\mathbb{A}}^n_0 & \end{tikzcd}$$

Theorem (Kedlaya - Pottharst - Xiao) $\quad$ $\operatorname{Sp} A \hookrightarrow X_{tri}$ some infinitesimal neighborhood of x. Then

e_A deformation of e.

$D_{rig}(e_A)$ ~~$[\frac{1}{t}]$~~ has a filtration by sub $\mathcal{R}_A$-modules stable under φ, Γ and $\exists$ maps $F_{i,A}/F_{i-1,A} \longrightarrow \mathcal{R}(\delta_{i,A})$ which are iso after inverting t. $\rightsquigarrow$ triangulation σ_R/refinement of e.

$\Rightarrow$ we define $\mathfrak{X}_{e,R} \subset \mathfrak{X}_e$ as being

$$\mathfrak{X}_{e,R}(A) = \left\{ e_A \in \mathfrak{X}_e(A) + \sigma_A \text{ triangulation of } D_{rig}(e_A)[\tfrac{1}{t}] \text{ such that } \sigma_A \bmod m_A = \sigma_R \right\}$$

B_{dR}-representation : W B_{dR}-v.s + semilinear action of $G_{\mathbb{Q}_p}$.

B_{dR}^+ - ——— : W B_{dR}^+-lattice ———

W B_{dR}-rep. $\quad$ W de Rham $\Longleftrightarrow$ $W \simeq B_{dR}^{\dim W}$ $\Longleftrightarrow$ $\dim_{\mathbb{Q}_p} W^{G_{\mathbb{Q}_p}} = \dim_{B_{dR}} W$.

<u>Fontaine</u>: W almost de Rham iff W is an extension of de Rham representations

Theorem (Fontaine)

$$\mathrm{Rep}^{\mathrm{adR}}_{B_{dR}} G_{\mathbb{Q}_p} \xrightarrow{\sim} \mathrm{Rep}\, \mathbb{G}_a \xrightarrow{\sim} \left\{ (D,N) \;\middle|\; \begin{array}{l} D \text{ fin dim } \mathbb{Q}_p\text{-vs} \\ N \in \mathrm{End}_{\mathbb{Q}_p}(V) \\ \text{nilpotent} \end{array} \right\}$$

$$W \longmapsto D_{pdR}(W)$$

W almost de Rham.

$$\left\{ \begin{array}{l} G_{\mathbb{Q}_p}\text{-stable } B_{dR}^+\text{-lattice} \\ \text{in } W \end{array} \right\} \simeq \left\{ \begin{array}{l} \text{rep + exhaustive} \\ \text{filtrations on} \\ D_{pdR}(W) \text{ stable by } N \end{array} \right\}.$$

Corollary W^+ B_{dR}^+-rep st $W^+[\frac{1}{t}]$ almost de Rham of HT weights pairwise $\neq 0$. $\underline{e}$ basis of $D_{pdR}(W)$. Deformation functor of $(W^+, \underline{e})$ is pro-represented by the completion of $\tilde{\mathfrak{g}}$ at (N, σ^e_{dR}) using $D_{pdR}(W^+[\frac{1}{t}]) \underset{\underline{e}}{\simeq} \mathbb{Q}_p^m$.

Use

$$\begin{tikzcd}
V \arrow[r, mapsto] & V \otimes B_{dR}^+ \\
\mathrm{Rep}_{\mathbb{Q}_p} G_{\mathbb{Q}_p} \arrow[r] \arrow[d, hook] & \mathrm{Rep}_{B_{dR}^+} G_{\mathbb{Q}_p} \arrow[d] \\
\varphi\Gamma_{\mathcal{R}_{\mathbb{Q}_p}} \arrow[ur, dashed, "W_{dR}^+", "\text{Berger}"'] \arrow[d] & \mathrm{Rep}_{B_{dR}} G_{\mathbb{Q}_p} \\
\varphi\Gamma_{\mathcal{R}[\frac{1}{t}]} \arrow[ur, dashed, "W_{dR}"] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{X}_{e,R} \arrow[r] & \mathrm{Def}\left(W_{dR}^+(e), W_{dR}(\sigma^e_{dR}) \right). \\
\mathcal{X}_{e,R,\underline{e}} \arrow[r, "\text{formally smooth}"'] \arrow[u] \arrow[u, bend left=60, "\text{formally smooth}"] & \mathrm{Def}\left(W_{dR}^+(e), W_{dR}(\sigma^e_{dR}), \underline{e} \right) \simeq \widehat{\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}}_{(W_{dR}(\sigma^e), W_{dR}(\sigma^e_{dR}))} \arrow[u]
\end{tikzcd}$$