Coefficients in integral p-adic Hodge theory

(joint work with Matthew Morrow) June 27, 2019

In Hodge theory and p-adic Hodge theory before Bhatt-Morrow-Scholze, we compare cohomology groups by taking scalar extensions to big rings $\mathbb{C}$, B_{crys}, B_{dR}, $\underline{A_{crys}}$

integral p-adic Hodge only for H^i ($i \le p-2$)

In integral p-adic Hodge theory by BMS, they introduced a new cohomology, which gives p-adic cohomology groups by scalar extensions.

K: complete dvf of mixed char. $(0,p)$ with perfect res. field

$C = \hat{\bar{K}} \supset \mathcal{O}_C$, $\mathcal{O}_C^\flat = \varprojlim (\mathcal{O}_C/p \xleftarrow{\phi} \mathcal{O}_C/p \xleftarrow{\phi} \cdots)$ ϕ = abs. Frob.

$\mathcal{O}_C^\flat \ni p^\flat = (p^{\frac{1}{p^n}} \bmod p)_{n\ge 0}$ $C^\flat = \mathcal{O}_C^\flat[\frac{1}{p^\flat}]$

$A_{inf} := W(\mathcal{O}_C^\flat)$ (alg. closed valuation field and $\mathcal{O}_C^\flat$ is the valuation ring)

We have the following homomorphisms

$A_{inf} \longrightarrow W(k) =: W$ k: res. field of C

$A_{inf} \xrightarrow{\theta} \mathcal{O}_C$

$A_{inf} \longrightarrow W(C^\flat)$

(In their first paper) $\mathfrak{X}/\mathcal{O}_C$ proper smooth formal scheme

BMS defined $R\Gamma_{A_{inf}}(\mathfrak{X}) \in D_{perf}(A_{inf})$ s.t. the derived scalar extensions to W, $\mathcal{O}_C$ and $W(C^\flat)$ give, (Frob. inv. gives $R\Gamma_{et}(X, \mathbb{Z}_p)$)

$R\Gamma_{crys}(\mathfrak{X}_0/W)$, $R\Gamma_{dR}(\mathfrak{X}/\mathcal{O}_C)$ and $R\Gamma_{et}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(C^\flat)$

$\mathfrak{X}_0 = \mathfrak{X} \otimes_W k$, X = rigid analytic generic fiber of $\mathfrak{X}$

Remark $- \otimes^L_{A_{inf}} A_{crys}$ gives $R\Gamma_{crys}(\mathfrak{X}/A_{crys})$ ($\rightsquigarrow$ rational p-adic Hodge theory)

Today A theory of coefficients for $R\Gamma_{A_{inf}}$.

We introduce categories

$BKF(\mathfrak{X})$ rigid symmetric monoidal A_{inf}-linear

$BKF(\mathfrak{X}, \varphi)$ ―――――――――――――― $\mathbb{Z}_p$-linear

s.t.

[$\Rightarrow$ $\mathrm{length}_W H^i_{crys}(-)_{tor}/p^n \ge \mathrm{length}_{\mathbb{Z}_p} H^i_{et}(\ )_{tor}/p^n$ etc.]

(The names D_{crys}, D_{dR}, … are temporary)

<u>Theorem 1</u> $\exists$ specialization functors

$$\mathrm{BKF}(\mathfrak{X}) \longrightarrow (\text{loc. free crystals on } \mathfrak{X}_0/W)$$
$$\xrightarrow[D_{dR}]{D_{crys}} \left(\begin{array}{l}\text{vector bundles with}\\ \text{integrable connections on } \mathfrak{X}/\mathcal{O}_C\end{array}\right)$$

$$\mathrm{BKF}(\mathfrak{X}, \varphi) \longrightarrow (p\text{-torsion free lisse } \mathbb{Z}_p\text{-sheaves on } X)$$
$$\xrightarrow[D_{A_{crys}}]{V_{\acute{e}t}} (\text{filtered crystals with Frobenius on } \mathfrak{X}/A_{crys})$$

BKF = Breuil - Kisin - Fargues

<u>Remark</u> (1) We deal only with locally free coefficients.

(2) The properness of $\mathfrak{X}/\mathcal{O}_C$ is not necessary to define $\mathrm{BKF}(\mathfrak{X})$ and $\mathrm{BKF}(\mathfrak{X},\varphi)$.

We have A_{inf}-coh. $R\Gamma_{A_{inf}}(\mathfrak{X}, M)$ for $M \in \mathrm{BKF}(\mathfrak{X}, \varphi)$ and

<u>Theorem 2</u> the derived scalar extensions to W, $\mathcal{O}$, and $W(\mathcal{O}_C^b)$ gives

$R\Gamma_{crys}(\mathfrak{X}_0/W, D_{crys}(M))$, $R\Gamma_{dR}(\mathfrak{X}/\mathcal{O}_C, D_{dR}(M))$ and $R\Gamma_{\acute{e}t}(X, V_{\acute{e}t}(M)) \otimes_{\mathbb{Z}_p} W(C^b)$

<u>Remark</u> $- \overset{L}{\otimes}_{A_{inf}} A_{crys}$ should give $R\Gamma_{crys}(\mathfrak{X}/A_{crys}, D_{A_{crys}}(M))$, but not proven yet.

<u>Examples</u>

1) $p \geq 3$. $\exists$ equiv. of categories

$$\mathrm{BKF}_{[-1,0]}(\mathfrak{X}, \varphi) \xrightarrow{\sim} (p\text{-divisible gps}/\mathfrak{X}) = BT(\mathfrak{X})$$

$\mathfrak{X} = \mathrm{Spf}(\mathcal{O}_C)$ Fargues "Frob. slopes $\in [-1,0]$"

2) Y/W proper smooth

Relative Fontaine - Laffaille theory by Faltings

$$MF^{\nabla}_{[0,p-2],\text{loc. free}}(Y/W) \xrightarrow[\exists \text{ canonical}]{} \mathrm{BKF}(\mathfrak{X}, \varphi)$$
$$\mathfrak{X} = Y \times \mathrm{Spf}(\mathcal{O}_C)$$

3) (in progress) $f: \mathfrak{Y} \to \mathfrak{X}$ proper smooth

Assume $R^q f_* \Omega_{\mathfrak{Y}/\mathfrak{X}}$ are locally free.

$\exists$? relative A_{inf}-coh $\underline{H}^q_{A_{inf}}(\mathfrak{Y}/\mathfrak{X}) \in \mathrm{BKF}(\mathfrak{X}, \varphi)$

4) Tensor products and duals of the above examples.

Definition of BKF($\mathfrak{X}$)

$A_{\inf}$ $\quad S$ flat $\mathcal{O}_K$-alg. $\quad S \subset S[\frac{1}{p}]$ int. closed.

$\phi : S/p \to S/p$ surj. $\quad S^\flat = \varprojlim (S/pS \xleftarrow{\phi} S/pS \xleftarrow{\phi} \cdots)$

$A_{\inf}(S) := W(S^\flat) \circlearrowleft \varphi$

We have a sheaf $A_{\inf}$ on $X_{\text{proét}}$

s.t. $A_{\inf}(U) = A_{\inf}(\mathcal{O}_X^+(U))$ for $\forall\, U \in X_{\text{proét}}$ affinoid perfectoid.
$\circlearrowleft \varphi$

(Some elements of $A_{\inf}$)

$\varepsilon := (1, \zeta_p, \zeta_{p^2}, \cdots) \in \mathcal{O}_C^\flat \subset S^\flat$

$q = [\varepsilon] = (\varepsilon, 0, \cdots) \in A_{\inf} \qquad \varphi(q) = q^p$

$\mu = q - 1 \qquad [p]_q = \dfrac{\varphi(\mu)}{\mu} = 1 + q + \cdots + q^{p-1}$

Definition $\quad$ A relative BKF-module M is a loc. free $A_{\inf}$-module of finite type "trivial mod $\langle\mu\rangle$" ($=$ free mod $\mu/\varphi^{-r}(\mu)$ Zariski locally on $\mathfrak{X}$ for $\forall r > 0$)

A Frob. str. on M is an isom

$$\Phi_M : \varphi^*(M)\left[\tfrac{1}{[p]_q}\right] \xrightarrow{\cong} M\left[\tfrac{1}{[p]_q}\right]$$

Frobenius slopes $\in [a,b] \underset{\text{def}}{\iff} ([p]_q)^b M \subset \Phi_M(\varphi^*(M)) \subset ([p]_q)^a M$

(Remark $\quad$ We have no idea how to generalize the condition "trivial mod $\langle\mu\rangle$" to perfect complexes)

Local description

$\mathfrak{X} = \mathrm{Spf}(R) \xrightarrow[\text{ét}]{\square} \mathrm{Spf}\, \mathcal{O}_C\langle T_1^{\pm 1}, \cdots, T_d^{\pm 1}\rangle$

$R_\infty = R \mathbin{\hat{\otimes}}_{R\langle \underline{T}^{\pm 1}\rangle} \mathcal{O}_C\langle \underline{T}^{\pm 1}\rangle \circlearrowleft \Gamma = \mathbb{Z}_p(1)^d \qquad \wedge :$ p-adic completion

$$\begin{tikzcd}
R & A^{\square}(R) \arrow[l, two heads] \arrow[r, hook, dashed] & A_{\inf}(R_\infty) \\
 & & {[t_i^\flat]} \arrow[u, phantom, sloped, "\in"] \\
\mathcal{O}_C\langle \underline{T}^{\pm 1}\rangle \arrow[uu] & A_{\inf}\langle \underline{U}^{\pm 1}\rangle \arrow[l, two heads] \arrow[uu, "\text{étale lift}"'] \arrow[uur, hook] & \\
 & U_i \arrow[uur, mapsto] &
\end{tikzcd}$$

$t_i^\flat = (t_i^{\frac{1}{p^n}} \bmod p) \in R_\infty^\flat$

$A^{\square}(R) \subset A_{\inf}(R_\infty)$ is stable under Γ-action and φ.

Theorem 3 $\quad BKF(\mathrm{Spf}(R)) \xrightarrow[\text{equiv.}]{\sim} \underbrace{\mathrm{Rep}^{\mu}_{\Gamma}(A_{\inf}(R_\infty))}_{\uparrow} \xleftarrow[\text{equiv.}]{\sim} \mathrm{Rep}^{\mu}_{\Gamma}(A^{\square}(R))$

(finite proj. $A_{\inf}(R_\infty)$-module with semilinear Γ-action trivial mod μ
$(M/\mu)^\Gamma$ is a fin. gen proj. $(A_{\inf}(R_\infty)/\mu)^\Gamma$-mod
$(M/\mu)^\Gamma \otimes_{(A_{\inf}(R_\infty)/\mu)^\Gamma} A_{\inf}(R_\infty)/\mu \xrightarrow{\cong} M/\mu$)

$A_{\inf}$-cohomology

$\nu : X_{\text{proét}} \to \mathfrak{X}_{\text{Zar}}$ $\qquad\qquad$ ← derived p-adic completion

Definition $\quad A\Omega_{\mathfrak{X}}(\mathfrak{M}) = L\eta_\mu \widehat{R\nu_*\mathfrak{M}} \qquad \mathfrak{M} \in BKF(\mathfrak{X})$.

$R\Gamma_{A_{\inf}}(\mathfrak{X}, \mathfrak{M}) = R\Gamma(\mathfrak{X}, A\Omega_{\mathfrak{X}}(\mathfrak{M}))$

($C^\bullet$ $\quad$ μ-tor. free complex
$\eta_\mu C^\bullet = \mu^q C^q \cap d^{-1}(\mu^{q+1} C^{q+1})$
$L\eta_\mu$ derived version)

Proposition $\quad \mathfrak{X} = \mathrm{Spf}(R)$, $\mathfrak{M} \in BKF(\mathfrak{X}) \overset{\text{Thm.3}}{\rightsquigarrow} M^{\square} \in \mathrm{Rep}^{\mu}_{\Gamma}(A^{\square}(R))$.

$$A\Omega_{\mathfrak{X}}(\mathfrak{M}) \cong K\left(\frac{\gamma_1 - 1}{\mu}, \dots, \frac{\gamma_d - 1}{\mu}; M^{\square}\right)$$

($\gamma_i = \varepsilon \cdot \underline{1}_i \in \Gamma = \mathbb{Z}_p(1)^d \quad (i = 1, \dots, d)$
$K(\ ) =$ Koszul complex)

$A^{\square}/\mu \circlearrowleft \Gamma$ trivial $\Rightarrow (\gamma_i - 1)(M^{\square}) \subset \mu M^{\square}$

$\frac{\gamma_i - 1}{\mu}$ gives $\overbrace{\text{connections}}^{\text{integrable}}$ on $M^{\square} \hat{\otimes} \mathcal{O}_C$, which are $M^{\square} \hat{\otimes} W$

local descriptions of Theorems 1 and 2 for crys and dR.

- $V_{\text{ét}}(\mathfrak{M}) \overset{\text{def}}{:=} \mathfrak{M}[\frac{1}{\mu}]^{\varphi = 1}$ on $X_{\text{proét}}$
- $D_{A_{\text{crys}}}(\mathfrak{M})$: $M \in \mathrm{Rep}^{\mu}_{\Gamma}(A_{\inf}(R_\infty))$ is admissible w.r.t. a period ring $\mathcal{O}A_{\text{crys}}(R_\infty)$ with Γ-action ∇, Fil, φ defined by $A^{\square}(R)$ and R_∞

- p-divisible groups

  <u>Thm</u> (Lau, Scholze-Weinstein)

  $$BT(R_\infty) \xrightarrow[\exists\ \text{equiv.}]{\sim} BKF(R_\infty)$$

  We obtain $BT(R) \to \mathrm{Rep}_\Gamma^\mu(A_{\inf}(R_\infty))$ fully faithful.

  ess. surj : $M \rightsquigarrow G_\infty \in BT(R_\infty)$

  $$\begin{array}{c} \updownarrow \text{Thm. 3} \\ M^D \rightsquigarrow \text{descent data on } G_\infty. \end{array}$$

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Notes added after the talk.

- The last two brief explanations on $D_{A_{crys}}(M)$ and p-divisible groups were not mentioned in the talk.
- On the remark after the proposition in p. 4.

  $(\log(1+\mu))^{-1} \log \gamma_i$ gives an integrable connection on $M^D \hat{\otimes}_{A_{\inf}} A_{crys}$, which is a local description of $D_{A_{crys}}$ in Theorem 1.