TWIST FORMULA OF EPSILON FACTORS OF CONSTRUCTIBLE ÉTALE SHEAVES

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Abstract. This is a talk on joint work with Naoya Umezaki and Yigeng Zhao, viewed as an application of the theory of characteristic cycles.

Contents

Date: June 21, 2019.

0. Notation

We fix some notation for this talk.

- Let k be a finite field of characteristic $p > 0$.
- Let $\ell \neq p$ be a prime.
- Let X be a separated smooth (connected) scheme purely of dimension $d \geqslant 0$ over k.
- Let $K(X, \overline{\mathbb{Q}}_\ell)$ be the Grothendieck group of the triangulated category $D_c^b(X, \overline{\mathbb{Q}}_\ell)$, i.e.,
$$K(X, \overline{\mathbb{Q}}_\ell) = \frac{\begin{array}{c}\text{Free abelian group}\\ \text{generated by objects of } D_c^b(X, \overline{\mathbb{Q}}_\ell)\end{array}}{\left\langle [\mathcal{G}] = [\mathcal{F}] + [\mathcal{H}] \middle| \begin{array}{c}\exists \text{ distinguished triangle}\\ \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to \mathcal{F}[1]\end{array} \right\rangle}$$
Similarly, we can define $K(X, \overline{\mathbb{F}}_\ell)$ and $K(X, \overline{\mathbb{Z}}_\ell)$. We have a surjective homomorphism
$$d_X : K(X, \overline{\mathbb{Q}}_\ell) \xleftarrow[\simeq]{\text{canonical}} K(X, \overline{\mathbb{Z}}_\ell) \xrightarrow{\text{reduction}} K(X, \overline{\mathbb{F}}_\ell),$$
where the first map is an isomorphism and the second map is a surjective map. We call d_X the decomposition homomorphism. For a proof, see a paper by Weizhe Zheng 2015.
- By T. Saito, we have homomorphisms (characteristic class)
$$\begin{tikzcd} K(X, \overline{\mathbb{Q}}_\ell) \arrow[r, two heads, "d_X"] \arrow[rr, bend left, "\mathrm{CC}"] \arrow[dr, "\mathrm{cc}_X"'] & K(X, \overline{\mathbb{F}}_\ell) \arrow[r, "\mathrm{CC}"] \arrow[d, "\mathrm{cc}_X"] & Z_d(T^*X) \arrow[dl, "0_X^!"] \\ & \mathrm{CH}_0(X) & \end{tikzcd}$$
where for $\mathcal{F} \in K(X, \overline{\mathbb{F}}_\ell)$, $\mathrm{cc}_X\mathcal{F} = 0_X^!(\mathrm{CC}\mathcal{F})$ and $X \xrightarrow{0_X} T^*X$ is the zero-section of the cotangent bundle.
- For any $\mathcal{F} \in K(X, \overline{\mathbb{Q}}_\ell)$ or $K(X, \overline{\mathbb{F}}_\ell)$, the global epsilon factor of $\mathcal{F}$ is defined to be
$$\varepsilon(X, \mathcal{F}) = \det(-\mathrm{Frob}_k; R\Gamma_c(X_{\bar{k}}, \mathcal{F}))^{-1}$$
where Frob_k is the geometric Frobenius, i.e., the inverse of the Frobenius substitution $x \mapsto x^{\# k}$ of $\bar{k}$. By linear extension, this defines a multiplicative map
$$\varepsilon(X, -) : K(X, \overline{\mathbb{Q}}_\ell) \to \overline{\mathbb{Q}}_\ell^\times \quad \text{or} \quad \varepsilon(X, -) : K(X, \overline{\mathbb{F}}_\ell) \to \overline{\mathbb{F}}_\ell^\times.$$

1. Kato-T.Saito's Conjecture

1.1. Now we assume that X is projective and smooth over the finite field k. By Kato-S. Saito's unramified class field theory, there is a so-called reciprocity map
$$\begin{aligned} \mathrm{CH}_0(X) &\xrightarrow{\rho_X} \pi_1^{\mathrm{ab}}(X). \\ [s] &\mapsto [\mathrm{Frob}_s] \end{aligned}$$

The following global twist formula for $\varepsilon(X,-)$ is a modification of a conjecture of Kato and T. Saito in 2008.

Theorem 1.2 (Umezaki-Y-Zhao, 2017)**.** *Let $\Lambda = \mathbb{F}_\ell$ or $\Lambda = \overline{\mathbb{Q}}_\ell$. For any $\mathcal{F} \in D_c^b(X,\Lambda)$ and any smooth sheaf $\mathcal{G}$ on X, we have an equality*

$$\varepsilon(X, \mathcal{F}\otimes\mathcal{G}) = \varepsilon(X,\mathcal{F})^{\mathrm{rank}\mathcal{G}} \cdot \det\mathcal{G}(\rho_X(-cc_X\mathcal{F}))$$

where $\mathrm{CH}_0(X) \xrightarrow{\rho_X} \pi_1^{\mathrm{ab}}(X) \xrightarrow{\det\mathcal{G}} \Lambda^*$.

Remark 1.3. *In Kato-T.Saito's Annals paper 2008, for a smooth sheaf $\mathcal{F}$ on a smooth scheme U over a perfect field, they defined the Swan class*

$$\mathrm{Sw}^{ks}(\mathcal{F}) \in \mathrm{CH}_0(\overline{U}\backslash U)\otimes\mathbb{Q} = \varprojlim_{U\subseteq Y \text{ compactification}} \mathrm{CH}_0(Y\backslash U)\otimes Q$$

by using alteration and logarithmic blow-up. In their paper, the global twist formula was written in terms of Sw^{ks}.

Conjecture 1.4 (T. Saito, 2016)**.** *Let k be any perfect field. Let $j: U \to X$ be an open dense subscheme of a smooth scheme X over k. For any smooth sheaf $\mathcal{F}$ on U, we have*

$$\mathrm{Sw}^{ks}(\mathcal{F}) = \mathrm{Sw}^{\mathrm{cc}}(\mathcal{F}) \quad \text{in} \quad \mathrm{CH}_0(X\backslash U).$$

where $\mathrm{Sw}^{\mathrm{cc}}(\mathcal{F}) := 0^!_{X\backslash U}(\mathrm{rank}F\cdot CC(j_!\Lambda) - CC(j_!\mathcal{F}))$ *and* $X\backslash U \xrightarrow{0_{X\backslash U}} T^*X\times_X(X\backslash U)$ *is the zero section.*

If X is projective smooth, then both $\mathrm{Sw}^{ks}(\mathcal{F})$ and $\mathrm{Sw}^{\mathrm{cc}}(\mathcal{F})$ satisfy the higher Grothendieck-Ogg-Shafarevich formula

$$\chi_c(U_{\bar{k}},\mathcal{F}) = \mathrm{rank}\mathcal{F}\cdot\chi_c(U_{\bar{k}},\Lambda) - \deg\mathrm{Sw}^\bullet(\mathcal{F}),$$

which is due to T. Saito for $\mathrm{Sw}^{\mathrm{cc}}$ and due to Kato-T. Saito for Sw^{ks}.

Theorem 1.5 (Weak form of Conjecture 1.4 for surface)**.** *Assuming X is a projective smooth surface over a finite field k, and U an open dense subscheme of X. Then for any smooth sheaf $\mathcal{F}$ on U, we have*

$$\mathrm{Sw}^{ks}(\mathcal{F}) = \mathrm{Sw}^{\mathrm{cc}}(\mathcal{F}) \quad \text{in} \quad \mathrm{CH}_0(X).$$

The proof of Theorem 1.5 is based on Brauer induction and a theorem of T. Saito and Yatagawa (same wild ramification $\Longrightarrow$ same CC).

1.6. **Previous results of twist formulas for epsilon factors.**

(1) (Three kinds of) Local twist formulas, due to Deligne and Henniart in 1981. Theorem 1.2 can be viewed as a globalization of one of their formulas (unramified).
(2) In 1984, S. Saito proved an explicit formula for $\varepsilon(X,\mathcal{F})$ if $\mathcal{F}$ is a smooth sheaf on the projective smooth scheme X over a finite field. Our proof of Theorem 1.2 is based on his method.
(3) In 1993, T. Saito proved an explicit formula for $\varepsilon(X,\mathcal{F})$ if $\mathcal{F}$ is a sheaf tamely ramified along a SNC divisor.
(4) In 2009, I. Vidal (She) proved Theorem 1.2 under the assumption that

  - $\dim X = 2$ and $\mathrm{rank}\mathcal{F} = 1$ (generic rank), smooth on an open dense subscheme with boundary a SNC divisor.
  - Some technical assumptions on the ramification of $\mathcal{F}$, e.g., cleaness. (Swan class can be canonically lifted to the cotangent bundle)

(5) In 2016, Tomoyuki Abe and Deepam Patel proved a K-spectrum version of the twist formula (they called localization formula) for de Rham epsilon factors based on micro-local description of singular supports for $\mathcal{D}_X$-modules:
$$K_S(X, \mathcal{D}_X) \to K_S(T^*X), \quad S \subseteq T^*X,$$
where $K_S(X, \mathcal{D}_X)$ is the K-spectrum of coherent $\mathcal{D}_X$-modules $\mathcal{E}$ with $SS(\mathcal{E}) \subseteq S$, $K_S(T^*X)$ is the K-spectrum of coherent $\mathcal{O}_{T^*X}$-modules with support contained in S.

2. Application to Characteristic Class

2.1. In this section, let $\Lambda = \mathbb{F}_\ell$. Recall that X is a smooth and connected scheme of dimension d over a perfect field k. We first recall the definition of total characteristic class:

$$\begin{tikzcd}
K(X,\Lambda) \arrow[r, "\overline{\mathrm{CC}}"] \arrow[dr, "cc_{X,\bullet}"'] & \mathrm{CH}_d(\mathbb{P}(T^*X \oplus \mathbb{A}^1_X)) \arrow[d, "\simeq"] & \\
& \mathrm{CH}_\bullet(X) \arrow[r, equal] & \bigoplus_{i=0}^{d} \mathrm{CH}_i(X)
\end{tikzcd}$$

where $\overline{\mathrm{CC}\mathcal{F}} = \mathbb{P}\left(\overline{\mathrm{CC}\mathcal{F} \oplus \mathbb{A}^1_X}\right)$. We have

- $cc_{X,0}\mathcal{F} = cc_X\mathcal{F} \in \mathrm{CH}_0(X)$ is the characteristic class.
- $cc_{X,d}\mathcal{F} = (-1)^d \cdot \mathrm{rank}\mathcal{F} \cdot [X] \in \mathrm{CH}_d(X)$.
- $cc_{X,d-1}\mathcal{F} \in \mathrm{CH}_{d-1}(X)$ is the Artin divisor class of $\mathcal{F}$.

2.2. If $k = \mathbb{C}$, a theorem of V. Ginsburg implies that the following diagram is commutative for any projective morphism $f : X \to Y$ between smooth schemes over $\mathbb{C}$:

$$\begin{tikzcd}
K(X,\Lambda) \arrow[r, "cc_{X,\bullet}"] \arrow[d, "f_*"'] \arrow[dr, phantom, "(\blacksquare)"] & \mathrm{CH}_\bullet(X) \arrow[d, "f_*"] \\
K(Y,\Lambda) \arrow[r, "cc_{Y,\bullet}"] & \mathrm{CH}_\bullet(Y).
\end{tikzcd}$$

But in the case where $\mathrm{char}(k) > 0$, the diagram ($\blacksquare$) is not commutative by a philosophy of Grothendieck, except for the degree zero part. We have the following counter example.

Example 2.3. *Let $X = \mathbb{P}^d$ and let $X \xrightarrow{F} X$ be the Frobenius map, which is radical and surjective. We have*

- $F_*(\Lambda) \simeq \Lambda$.
- $cc_{\mathbb{P}^d,\bullet}(\Lambda) = (-1)^d \cdot c(\Omega^1_{\mathbb{P}^d}) = \left((-1)^i \binom{d+1}{i+1}\right)_{\deg i}$.

- *The map* $\mathrm{CH}_\bullet(\mathbb{P}^d) = \bigoplus_{i=0}^{i=d} \mathbb{Z} \xrightarrow{F_*} \mathrm{CH}_\bullet(\mathbb{P}^d)$ *is multiplication by* p^i *on the degree* i*-part.*

Corollary 2.4. *For any projective morphism* $f : X \to Y$ *between smooth projective schemes over a finite field* k*, the degree 0-part of* (■) *is commutative, i.e., for any* $\mathcal{F} \in D^b_c(X, \Lambda)$*, we have an equality*

$$f_* cc_X \mathcal{F} = cc_Y Rf_* \mathcal{F} \quad \text{in} \quad \mathrm{CH}_0(Y).$$

Proof. By using the fact that the decomposition homomorphism $d_X : K(X, \overline{\mathbb{Q}}_\ell) \to K(X, \overline{\mathbb{F}}_\ell)$ is surjective, we may assume that $\Lambda = \overline{\mathbb{Q}}_\ell$. Let χ be a continuous character $\pi_1(Y)^{\mathrm{ab}} \to \overline{\mathbb{Q}}_\ell^\times$. By Functoriality of class field theory, we have a commutative diagram

$$\begin{tikzcd} CH_0(X) \arrow[r] \arrow[d, "f_*"'] & \pi_1^{\mathrm{ab}}(X) \arrow[d, "f_*"] \arrow[dr, "f^*\chi"] & \\ CH_0(Y) \arrow[r] & \pi_1^{\mathrm{ab}}(Y) \arrow[r, "\chi"] & \overline{\mathbb{Q}}_\ell^\times \end{tikzcd} \tag{2.4.1}$$

Now we have

$$\begin{aligned} \chi(-cc_Y Rf_* \mathcal{F}) &\overset{\text{Thm.1.2}}{=} \frac{\varepsilon(Y, Rf_*\mathcal{F} \otimes \chi)}{\varepsilon(Y, Rf_*\mathcal{F})} \\ &\overset{\text{Proj.Formula}}{=} \frac{\varepsilon(X, \mathcal{F} \otimes f^*\chi)}{\varepsilon(X, \mathcal{F})} \\ &\overset{\text{Thm.1.2}}{=} (f^*\chi)(-cc_X\mathcal{F}) \\ &\overset{(2.4.1)}{=} \chi(-f_* cc_X \mathcal{F}). \end{aligned} \tag{2.4.2}$$

Since the equality $\chi(-cc_Y Rf_*\mathcal{F}) = \chi(-f_*(cc_X\mathcal{F}))$ holds for all characters of $\pi_1^{\mathrm{ab}}(Y)$, by the injectivity of the reciprocity map $CH_0(Y) \to \pi_1^{\mathrm{ab}}(Y)$, we have $f_*(cc_X\mathcal{F}) = cc_Y Rf_*\mathcal{F}$. This finishes the proof in this case.

□

2.5. Even though (■) is not commutative in general, we could still expect it commutes under a transversal condition:

- Fix an integer $r \geqslant 0$ and a smooth connected scheme S of dimension r over a perfect field k.
- Let $X \xrightarrow{f} S$ be a smooth morphism purely of relative dimension n.
- Let $D^b_c(X/S, \Lambda) \subseteq D^b_c(X, \Lambda)$ be the thick sub-triangulated category consisting of objects $\mathcal{F}$ such that $X \xrightarrow{f} S$ is $SS(\mathcal{F})$-transversal. Let $K(X/S, \Lambda)$ be the Grothendieck group of $D^b_c(X/S, \Lambda)$.
- By T. Saito, for any proper morphism $h : X \to Y$ between S-smooth schemes:

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

there is a well-defined map

$$K(X/S,\Lambda) \xrightarrow{h_*} K(Y/S,\Lambda).$$

Question 2.6. *We expect that the following diagram is commutative:*

$$\begin{tikzcd} K(X/S,\Lambda) \arrow[r, "\mathrm{cc}_{X,r}"] \arrow[d, "h_*"'] \arrow[dr, phantom, "(\star\star)"] & \mathrm{CH}_r(X) \arrow[d, "h_*"] \\ K(Y/S,\Lambda) \arrow[r, "cc_{Y,r}"] & \mathrm{CH}_r(Y). \end{tikzcd}$$

Some evidences for the above question:

(1) We can prove a cohomological version of $(\star\star)$ following SGA5, Abbes-Saito 2007, or Martin Olsson for a motivic version 2016 (they are working over a field $Y = \mathrm{Spec}k$).
(2) We can prove a relative version of the global twist formula 1.2.

3. Cohomological Characteristic Class

3.1. Let S be a smooth connected scheme of dimension r over a perfect field k and $\Lambda = \mathbb{F}_\ell$. Let $X \xrightarrow{f} S$ be a smooth morphism purely of relative dimension n. Let $\mathcal{K}_{X/S} = Rf^!\Lambda$. We can construct a map, which is compatible with proper push-forward:

$$K(X/S,\Lambda) \xrightarrow{C_{X/S}} H^0(X,\mathcal{K}_{X/S}) = H^{2n}(X,\Lambda(n)),$$

which is called the (relative) cohomological characteristic class. Following a conjecture of T. Saito in 2016, we expect the following diagram commutes

$$\begin{tikzcd} K(X/S,\Lambda) \arrow[d, "(-1)^r cc_{X,r}"'] \arrow[drr, "C_{X/S}"] & & \\ \mathrm{CH}_r(X) = \mathrm{CH}^n(X) \arrow[rr, "\text{cycle class map}"'] & & H^{2n}(X,\Lambda(n)). \end{tikzcd}$$

The construction of $C_{X/S}$ is based on the method of [SGA5, Expose III, Illusie]. See also Abbes-T.Saito 2003, or Olsson 2016 for a motivic version.

Lemma 3.2. *For any $\mathcal{F},\mathcal{G} \in D_c^b(X/S,\Lambda)$, we have an isomorphism*

$$\mathcal{F} \boxtimes_S^L D_{X/S}(\mathcal{G}) \xrightarrow{\simeq} R\mathcal{H}om(pr_2^*\mathcal{G}, Rpr_1^!\mathcal{F})$$

where $D_{X/S}(\mathcal{G}) = R\mathcal{H}om(\mathcal{G},\mathcal{K}_{X/S})$ and

$$\begin{tikzcd} & X \times_S X \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & X \arrow[d] & \mathcal{G} \arrow[l, squiggly] \\ \mathcal{F} \arrow[r, squiggly] & X \arrow[r] & S. & \end{tikzcd}$$

3.3. Now we define $K(X/S,\Lambda) \xrightarrow{C_{X/S}} H^0(X,\mathcal{K}_{X/S})$. Consider the following commutative diagram

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\blacksquare"] & X \arrow[d, "\delta"] \\ X \arrow[r, "\delta"] & X \times_S X. \end{tikzcd}$$

For $\mathcal{F} \in D_c^b(X/S,\Lambda)$, we have

$$\begin{aligned} R\mathcal{H}om(\mathcal{F},\mathcal{F}) &\simeq R\delta^! R\mathcal{H}om(pr_2^*, Rpr_1^! \mathcal{F}) \simeq R\delta^!(\mathcal{F} \boxtimes_S^L D_{X/S}(\mathcal{F})) \\ &\xrightarrow{1\to\delta_*\delta^*} R\delta^!\delta_*\delta^*(\mathcal{F} \boxtimes_S^L D_{X/S}(\mathcal{F})) = R\delta^!\delta_*(\mathcal{F} \otimes^L D_{X/S}(\mathcal{F})) \\ &\xrightarrow{\text{evaluation}} R\delta^!\delta_*\mathcal{K}_{X/S} \xrightarrow[\text{base.change}]{\simeq} \mathcal{K}_{X/S}. \end{aligned}$$

Then we get a map $Hom(\mathcal{F},\mathcal{F}) \xrightarrow{\mathrm{Tr}} H^0(X,\mathcal{K}_{X/S})$. The cohomological characteristic class of $\mathcal{F}$ (relative to $X \to S$) is defined to be

$$C_{X/S}(\mathcal{F}) = \mathrm{Tr}(\mathrm{id}_{\mathcal{F}}).$$

Following SGA5, we have the following formal property:

Proposition 3.4. *For any proper morphism $h : X \to Y$ between smooth schemes over S*

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

we have a commutative diagram

$$\begin{tikzcd} K(X/S,\Lambda) \arrow[r, "C_{X/S}"] \arrow[d, "h_*"'] & H^0(X,\mathcal{K}_{X/S}) \arrow[d, "h_*"] \\ K(Y/S,\Lambda) \arrow[r, "C_{Y/S}"] & H^0(Y,\mathcal{K}_{Y/S}). \end{tikzcd}$$

Remark 3.5. *In the definition of $K(X/S,\Lambda)$ and $C_{X/S}$, we may replace the condition "$SS\mathcal{F}$-transversal" by a weaker condition: "$\mathcal{F}$-transversal". Let $W \xrightarrow{h} X \xrightarrow{f} Y$ be separated k-morphisms of finite type and $\mathcal{F} \in D_c^b(X,\Lambda)$. We say that h is $\mathcal{F}$-transversal if the following canonical morphism is an isomorphism*

$$Rh^!\Lambda \otimes h^*\mathcal{F} \to Rh^!\mathcal{F}.$$

We say that f is $\mathcal{F}$-transversal if the graph $\Gamma_f : X \to X \times_k Y$ is $\mathcal{F} \boxtimes^L \mathcal{G}$-transversal for any $\mathcal{G} \in D_c^b(Y,\Lambda)$.

T. Saito shows that SS-transversal $\implies$ $\mathcal{F}$-transversal.

Remark 3.6. *Recently, using categorical trace, Y.Liu, W. Zheng and Q. Lu can even define the map*

$$C_{X/S} : K(X/S, \Lambda) \to H^0(X, \mathcal{K}_{X/S})$$

without the assumption that $X \to S$ is smooth. (S is a regular J-2 scheme)

4. Relative Twist Formula

4.1. Let k be a finite field, S be a smooth connected scheme of dimension r over k. Let $X \to S$ be a smooth projective morphism of relative dimension $n \geqslant 0$. By a result of T. Saito, there exists a unique way to attach a pairing

$$\mathrm{CH}^n(X) \times \pi_1^{\mathrm{ab}}(S) \to \pi_1^{\mathrm{ab}}(X) \tag{4.1.1}$$

satisfying the following conditions:

(1) When $S = \mathrm{Spec} k$ is a point, for a closed point $x \in |X|$, the pair with the class $[x]$ is the map
$$\mathrm{Gal}(\overline{k}/k) \xrightarrow{\mathrm{transfer}_{k(x)/k}} \mathrm{Gal}(\overline{k(x)}/k(x)) \xrightarrow{i_{x*}} \pi_1^{\mathrm{ab}}(X)$$
where $i : x \to X$ is the inclusion and $\mathrm{transfer}_{k(x)/k}$ is the Galois transfer.

(2) For any point $s \in S$, the following diagram commutes
$$\begin{tikzcd} \mathrm{CH}^n(X) \arrow[d] & \times & \pi_1^{\mathrm{ab}}(S) \arrow[r] & \pi_1^{\mathrm{ab}}(X) \\ \mathrm{CH}^n(X_s) & \times & \pi_1^{\mathrm{ab}}(s) \arrow[r] \arrow[u] & \pi_1^{\mathrm{ab}}(X_s). \arrow[u] \end{tikzcd}$$

Corollary 4.2 (Relative Twist Formula)**.** *Let $\mathcal{F} \in D_c^b(X, \Lambda)$ and let $\mathcal{G}$ be a smooth sheaf of Λ-modules on X. Assume that f is properly $SS(\mathcal{F})$-transversal ("properly" means that for any $s \in |S|$, the fiber $SS\mathcal{F} \times_S s$ is of dimension* $\dim X_s$*). Then there is an isomorphism*

$$\det Rf_*(\mathcal{F} \otimes^L \mathcal{G}) \cong (\det Rf_*\mathcal{F})^{\otimes \mathrm{rank}\mathcal{G}} \otimes^L \det \mathcal{G}((-1)^r \cdot cc_{X,r}(\mathcal{F})) \quad \text{in } K(S, \Lambda), \tag{4.2.1}$$

where $\det \mathcal{G}((-1)^r \cdot cc_{X,r}(\mathcal{F}))$ *is a smooth sheaf of rank 1 on S, defined via the pairing* (4.1.1)

$$\pi_1^{\mathrm{ab}}(S) \xrightarrow{(-1)^r cc_{X,r}\mathcal{F}} \pi_1^{\mathrm{ab}}(X) \xrightarrow{\det \mathcal{G}} \Lambda^\times.$$

Proof. Using Chebotarev density theorem and by pull-back of CC by properly transversal morphism, one reduce to $S = \mathrm{Spec} k$. Then one apply Theorem 1.2. □

5. Sketch: Proof of Kato-T.Saito's Conjecture (following S.Saito)

5.1. Using blow-up and T.Saito-Yatagawa's theorem on the existence of good Lefschetz pencils relative to $SS\mathcal{F}$ (may assume $\mathcal{F} \in D_c^b(X, \overline{\mathbb{Z}}_\ell)$), we can reduce the question to the following case:

(1) There is a fibration $f : X \to C$, where C is a smooth projective curve.
(2) $X \to C$ is $SS\mathcal{F}$-transversal outside a finite set of closed points $\{x_v\}_{v \in \Sigma}$, $\Sigma \subseteq |C|$. Each fiber X_v contains at most one isolated characteristic point x_v.
(3) For $v \in \Sigma$, x_v is k-rational.

(4) Let $\omega \neq 0$ be a meromorphic 1-form on C such that ω has neither poles or zeros at Σ.

(5) For $v \in |C|$ with $\mathrm{ord}_v(\omega) \neq 0$, X_v is smooth and $X_v \xrightarrow{i_v} X$ is properly $SS\mathcal{F}$-transversal.

Then we have an induction formula for $cc_X\mathcal{F}$:

$$cc_X\mathcal{F} = -\sum_{v\in\Sigma} \mathrm{dimtot} R\Phi_{\bar{x}_v}(\mathcal{F}, f) \cdot [x_v] - \sum_{v\in|C\setminus\Sigma|, \mathrm{ord}_v(\omega)\neq 0} \mathrm{ord}_v(\omega) \cdot cc_{X_v}(\mathcal{F}|_{X_v}).$$

5.2. We prove Theorem 1.2 by induction on $\dim X$ and by using Laumon's product formula:

$$\frac{\varepsilon(X, \mathcal{F}\otimes\mathcal{G})}{\varepsilon(X,\mathcal{F})^{\mathrm{rank}\mathcal{G}}} = \frac{\varepsilon(C, Rf_*(\mathcal{F}\otimes\mathcal{G}))}{\varepsilon(C, Rf_*\mathcal{F})^{\mathrm{rank}\mathcal{G}}} = \prod_{v\in|C|} \frac{\varepsilon_v(\omega, Rf_*(\mathcal{F}\otimes\mathcal{G}))}{\varepsilon_v(\omega, Rf_*\mathcal{F})^{\mathrm{rank}\mathcal{G}}},$$

where $\varepsilon_v(\omega, -)$ is the local epsilon factor at v.

Then by induction step and local twist formula for local epsilon factors, we have[1]

$$\frac{\varepsilon_v(\omega, Rf_*(\mathcal{F}\otimes\mathcal{G}))}{\varepsilon_v(\omega, Rf_*\mathcal{F})^{\mathrm{rank}\mathcal{G}}} = \begin{cases} \det\mathcal{G}(\rho_X(\mathrm{ord}_v(\omega)\cdot cc(\mathcal{F}|_{X_v}))), & v \notin |\Sigma| \qquad \text{induction step,} \\ \det\mathcal{G}(\rho_X(\mathrm{dimtot} R\Phi_{u_v}(\mathcal{F}, f)\cdot[u_v])), & v \in |\Sigma| \qquad \text{dist tri of R}\Phi. \end{cases}$$

Therefore we get

$$\begin{aligned} \frac{\varepsilon(X, \mathcal{F}\otimes\mathcal{G})}{\varepsilon(X,\mathcal{F})^{\mathrm{rank}\mathcal{G}}} &= \det\mathcal{G}\left(\rho_X\left(\sum_{v\in|Y|} \mathrm{ord}_v(\omega)\cdot cc(\mathcal{F}|_{X_v}) + \sum_{v\in\Sigma} \mathrm{dimtot} R\Phi_{u_v}(\mathcal{F}, f)\cdot[u_v]\right)\right) \\ &= \det\mathcal{G}(\rho_X(-cc_X\mathcal{F})). \end{aligned}$$

[1] Note that, for $v \in \Sigma$, we have a distinguished triangle $R\Gamma(X_{\bar{v}}, \mathcal{F}) \to R\Gamma(X_{\bar{\eta}_v}, \mathcal{F}) \to R\Phi_{\bar{x}_v}(\mathcal{F}, f) \to \cdot$