1 Singular support

1.1 Closed conical subsets and the transversality

Definition 1.1.1. *Let C be a closed conical subset of the cotangent bundle T^*X and let $h\colon W \to X$ be a morphism of smooth schemes over k.*

We say that h is C-transversal if the intersection of the subsets $h^*C = W \times_X C$ and $\mathrm{Ker}(W \times_X T^*X \to T^*W)$ of $W \times_X T^*X$ is a subset of the 0-section.

The intersection $C \cap T^*_X X$ with the 0-section $X = T^*_X X$ is called the base of C.

If h is smooth, then h is C-transversal for any C.

If C is a subset of the 0-section, any h is C-transversal.

If $C \subset C'$, the C'-transversality implies the C-transversality.

The transversality is an open condition.

Lemma 1.1.2. *Assume that $h\colon W \to X$ is C-transversal. Then, $W \times_X T^*X \to T^*W$ is finite on h^*C.*

Lemma 1.1.3. $\dim h^*C \geqq \dim C + \dim W - \dim X$.

Lemma 1.1.4. *Assume that $h\colon W \to X$ is C-transversal. For a morphism $g\colon V \to W$ of smooth schemes over k, the following conditions are equivalent:*

(1) *g is $h^\circ C$-transversal.*

(2) *$h \circ g$ is C-transversal.*

Definition 1.1.5. *Let C be a closed conical subset of the cotangent bundle T^*X and C' be a closed conical subset of the cotangent bundle T^*Y. Let $h\colon W \to X$ and $f\colon W \to Y$ be morphisms of smooth schemes over k.*

1. *We say that (h, f) is (C, C')-transversal if $(h, f)\colon W \to X \times Y$ is $C \times C'$-transversal.*

2. *If $h = 1_X$ and $C' = T^*Y$, we say that f is C-transversal if $(1_X, f)$ is (C, T^*Y)-transversal.*

Lemma 1.1.6. 1. *The following conditions are equivalent:*

(1) *$h\colon W \to X$ is C-transversal.*

(2) *$(h, 1_W)$ is $(C, T^*_W W)$-transversal.*

1. *The following conditions are equivalent:*

(1) *$f\colon X \to Y$ is C-transversal.*

(2) *The inverse image of C by $X \times_Y T^*Y \to T^*X$ is a subset of the 0-section.*

2. *The following conditions are equivalent:*

(1) *(h, f) is (C, T^*Y)-transversal.*

(2) *$h\colon W \to X$ is C-transversal and $f\colon W \to X$ is $h^\circ C$-transversal.*

$f\colon X \to Y$ is $T^*_X X$-transversal if and only if f is smooth.

If $f\colon X \to Y$ is C-transversal, then f is smooth on a neighborhood of the base of C.

Definition 1.1.7. *Let $C \subset T^*X$ be a closed conical subset and $f\colon X \to Y$ be a morphism of smooth schemes over k. Assume that f is proper on the base of C. Then, we define a closed conical subset $f_\circ C \subset T^*Y$ by the algebraic correspondence $T^*X \leftarrow X \times_Y T^*Y \to T^*Y$.*

Proposition 1.1.8. *Let $g\colon X' \to X$ be a morphism of smooth schemes over k and let $C \subset T^*X'$ be a closed conical subset. Assume that g is proper on the basis B' of C' and define $C = g_\circ C' \subset T^*X$.*

1. *Let $h\colon W \to X$ be a morphism of smooth schemes over k and*

$$\begin{tikzcd} X' \arrow[d, "g"'] & W' \arrow[l, "h'"'] \arrow[d, "g'"] \\ X & W \arrow[l, "h"'] \end{tikzcd}$$

be a cartesian diagram. Assume that h is C-transversal. Then, there exists an open neighborhood U' of the inverse image $B'_{W'} = h'^{-1}(B') \subset W'$ smooth over W.

2. *For a morphism $f\colon W \to Y$ of smooth schemes over k, the following conditions are equivalent:*

(1) *(h, f) is C-transversal.*

(2) *$(h'|_{U'}, f \circ g'|_{U'})$ is C'-transversal.*

1.2 Legendre transform

Let $\mathbf{P}$ be a projective space, $\mathbf{P}^\vee$ be the dual projective space and $Q \subset \mathbf{P} \times \mathbf{P}^\vee$ be the universal hyperplane. The kernel $\mathrm{Ker}((T^*\mathbf{P} \times T^*\mathbf{P}^\vee) \times_{\mathbf{P}\times\mathbf{P}^\vee} Q \to T^*Q$ equals the conormal bundle $T^*_Q(\mathbf{P} \times \mathbf{P}^\vee)$.

We identify Q as the projective space bundle $\mathbf{P}(T^*\mathbf{P})$ associated to the vector bundle $T^*\mathbf{P}$. Symmetrically, Q is identified with $\mathbf{P}(T^*\mathbf{P}^\vee)$.

Definition 1.2.1. *Let C be a closed conical subset $C \subset T^*\mathbf{P}$. We consider the projectivization $E = \mathbf{P}(C) \subset \mathbf{P}(T^*\mathbf{P}) = Q$ as a closed subset of Q. Define the Legendre transform $C^\vee = LC$ by $C^\vee = p^\vee_\circ p^\circ C$.*

Lemma 1.2.2. *The intersection of $C \times T^*\mathbf{P}^\vee$ with $\mathrm{Ker}((T^*\mathbf{P} \times T^*\mathbf{P}^\vee) \times_{\mathbf{P}\times\mathbf{P}^\vee} Q \to T^*Q = T^*_Q(\mathbf{P} \times \mathbf{P}^\vee)$ equals the union of $T^*_Q(\mathbf{P} \times \mathbf{P}^\vee) \times_Q E$ with the 0-section on $p^{-1}B$.*

Proof. Since the image of the conormal bundle $T^*_Q(\mathbf{P} \times \mathbf{P}^\vee) \subset (T^*\mathbf{P} \times T^*\mathbf{P}^\vee) \times_{\mathbf{P}\times\mathbf{P}^\vee} Q$ in $T^*\mathbf{P} \times_{\mathbf{P}} Q$ by the first projection is the tautological line bundle, the assertion follows.

Proposition 1.2.3. 1. *The complement $Q - E$ is the largest open subset where $(p, p^\vee)$ is C-transversal.*

2. *C is equal to the image of the intersection of $(C \times T^*\mathbf{P}^\vee) \cap T^*_Q(\mathbf{P} \times \mathbf{P}^\vee)$ by the composition $(T^*\mathbf{P} \times T^*\mathbf{P}^\vee) \times_{\mathbf{P}\times\mathbf{P}^\vee} Q \to T^*\mathbf{P} \times_{\mathbf{P}} Q \to T^*\mathbf{P}$.*

Proof. 1. Clear from Lemma.

2.

Corollary 1.2.4. $\mathbf{P}(C) = \mathbf{P}(C^\vee)$.

Proof. Since $C^\vee$ is equal to the image of the intersection of $(C \times T^*\mathbf{P}^\vee) \cap T^*_Q(\mathbf{P} \times \mathbf{P}^\vee)$ by the composition $(T^*\mathbf{P} \times T^*\mathbf{P}^\vee) \times_{\mathbf{P}\times\mathbf{P}^\vee} Q \to T^*\mathbf{P}^\vee \times_{\mathbf{P}^\vee} Q \to T^*\mathbf{P}^\vee$, it follows from Lemma and Proposition.

Proposition 1.2.5. *Let $C^+ = C \subset T^*_{\mathbf{P}}\mathbf{P}$ be the union with the 0-section. Then, we have*

$$C^+ = p_\circ(p^{\vee\circ}T^*\mathbf{P}^\vee \times_Q E) \cup T^*_{\mathbf{P}}\mathbf{P}.$$

Proof. By Lemma and Proposition, we have $C \subset p_\circ(p^{\vee\circ}T^*\mathbf{P}^\vee \times_Q E) \cup T^*_{\mathbf{P}}\mathbf{P} \subset C^+$.

Corollary 1.2.6. *We consider a cartesian diagram*

$$\begin{tikzcd}
\mathbf{P}^\vee & Q \arrow[l, "p^\vee"'] \arrow[d, "p"'] \arrow[dr, phantom, "\square"] & Q_W \arrow[l, "h_Q"'] \arrow[d, "p_W"] & \\
& \mathbf{P} & W \arrow[l, "h"'] \arrow[r, "f"] & Y
\end{tikzcd}$$

of smooth schemes over k. For a closed conical subset $C \subset T^*\mathbf{P}$ and its Legendre transform $C^\vee \subset T^*\mathbf{P}^\vee$ and the union $C^+ = C \cup T^*_{\mathbf{P}}\mathbf{P}$ with the 0-section, the following conditions are equivalent:

(1) *(h, f) is C^+-transversal.*

(2) *$f : W \to Y$ is smooth and $Q_W \to \mathbf{P}^\vee \times Y$ is smooth of the inverse image $E_W = E \times_Q Q_W$.*

Proof. Since $C^+ = p_\circ(p^{\vee\circ}T^*\mathbf{P}^\vee \times_Q E) \cup T^*_{\mathbf{P}}\mathbf{P}$ by Lemma, the condition (1) is equivalent to the combination of the following conditions.

($1'$) (h, f) is $T^*_{\mathbf{P}}\mathbf{P}$-transversal.

($1''$) (h, f) is $p_\circ(p^{\vee\circ}T^*\mathbf{P}^\vee \times_Q E)$-transversal.

The condition ($1'$) is equivalent to that $f : W \to Y$ is smooth. Since p is proper and smooth, by Lemma, the condition ($1''$) is equivalent to $(h_Q, f \circ p_W)$ is $p^{\vee\circ}T^*\mathbf{P}^\vee \times_Q E$-transversal. Since the transversality is an open condition, this is equivalent to that $(h_Q, f \circ p_W)$ is $p^{\vee\circ}T^*\mathbf{P}^\vee$-transversal on a neighborhood of E_W. By Lemma, this is further equivalent to that $(p \vee \circ h_Q, f \circ p_W)$ is $T^*\mathbf{P}^\vee$-transversal on a neighborhood of E_W. This means that $Q_W \to \mathbf{P}^\vee \times Y$ is smooth of the inverse image $E_W = E \times_Q Q_W$.

Let $h : W \to \mathbf{P}$ be an immersion and $f : W \to Y$ be a smooth morphism. Define sub vector bundles $C_W \subset C_f \subset T^*\mathbf{P} \times_{\mathbf{P}} W$ by $C_W = T^*_W\mathbf{P}$ and C_f as the inverse image of $W \times_Y T^*Y \subset T^*W$ by the surjection $T^*\mathbf{P} \times_{\mathbf{P}} W \to T^*W$.

Lemma 1.2.7. *Let $C^\vee \subset T^*\mathbf{P}^\vee$ be a closed conical subset and let $C = L^\vee C^\vee \subset T^*\mathbf{P}$ be the inverse Legendre transform.*

1. *The following conditions are equivalent:*

(1) *h is C-transversal.*

(2) *The intersection of $\mathbf{P}(C) \subset \mathbf{P}(T^*\mathbf{P}) = Q$ and $\mathbf{P}(C_W) \subset \mathbf{P}(T^*\mathbf{P} \times_{\mathbf{P}} W) = Q \times_{\mathbf{P}} W \subset Q$ is empty.*

2. *Assume that $h : W \to \mathbf{P}$ is C-transversal. Then $Q \times_{\mathbf{P}} W \to \mathbf{P}^\vee$ is $C^\vee$-transversal. The complement $Q \times_{\mathbf{P}} W - \mathbf{P}(C \cap C_f)$ equals the largest open subset $U \subset Q \times_{\mathbf{P}} W$ where $(p^\vee : Q \times_{\mathbf{P}} W \to \mathbf{P}^\vee, fp : Q \times_{\mathbf{P}} W \to W \to Y)$ is $C^\vee$-transversal. Further $\mathbf{P}(C \cap C_f)$ is a subset of the inverse image of the complement of the largest open subset where f is $h^\circ C$-transversal.*

3. *Further if $\dim Y = 1$, the closed subset $\mathbf{P}(C \cap C_f) \subset Q \times_{\mathbf{P}} W$ is finite over W.*

Proof. 1. (1) means $C \cap C_W$ is a closed subset of the zero-section and is equivalent to (2).

2. By Proposition 1.1.8, the C-transversality of $h : W \to \mathbf{P}$ implies the $C^\vee$-transversality of $Q \times_{\mathbf{P}} W \to Q$. Since $p^\vee : Q \to \mathbf{P}^\vee$ is smooth, the first assertion follows.

The largest open subset $U \subset Q \times_{\mathbf{P}} W$ is the same as that where $(p^\vee, p)$ is $C^\vee \times C_f$-transversal. Hence, it equals the complement of $\mathbf{P}(C^\vee) \cap \mathbf{P}(C_f) = \mathbf{P}(C) \cap \mathbf{P}(C_f) = \mathbf{P}(C \cap C_f)$.

If f is $h^\circ C$-transversal, then $(p^\vee, fp)$ is $C^\vee$-transversal and the last assertion follows.

3. Since $\dim Y = 1$, the subvector bundle $C_W \subset C_f$ is of codimension 1 and the complement $\mathbf{P}(C_f) - \mathbf{P}(C_W)$ is a vector bundle over W. Since $\mathbf{P}(C \cap C_W)$ is empty by 1, the intersection $\mathbf{P}(C \cap C_f)$ is a closed subset of $\mathbf{P}(C_f - C_W)$. Hence its closed subset $\mathbf{P}(C \cap C_f)$ proper over W is finite over W.

1.3 Local acyclicity

Let $f\colon X \to S$ be a morphism of schemes. Let $x \to X$ and $t \to S$ be geometric points and let $S_{(s)}$ be the strict localization at the image $s = f(x) \to S$ of x. Then a specialization $x \leftarrow t$ is a lifting of $t \to S$ to $t \to S_{(s)}$.

Definition 1.3.1. *Let $f\colon X \to S$ be a morphism of schemes and $\mathcal{F}$ be a complex of torsion sheaves on X. We say that f is locally acyclic relatively to $\mathcal{F}$ if for each geometric points $x \to X$ and $t \to S$ and each specialization $x \leftarrow t$, the canonical morphism $\mathcal{F}_x \to R(X_{(x)} \times_{S_{(s)}} t, \mathcal{F})$ is an isomorphism.*

We say that f is universally locally acyclic relatively to $\mathcal{F}$, if for every morphism $S' \to S$, the base change of f is locally acyclic relatively to the pull-back of $\mathcal{F}$.

For geometric points s, t of S and a specialization $t \to S_{(s)}$, let $i\colon X_s \to X \times_S S_{(s)}$ and $j\colon X_t \to X \times_S S_{(s)}$ denote the canonical morphisms. Then, the local acyclity is equivalent to that the canonical morphism $i^*\mathcal{F} \to i^*Rj_*\mathcal{F}$ is an isomorphism for each s, t and $s \leftarrow t$.

If $\mathcal{F}$ is a constructible sheaf on X, $\mathcal{F}$ is locally constant if and only if 1_X is locally acyclic relatively to $\mathcal{F}$.

The local acyclicity is preserved by quasi-finite base change $S' \to S$. Hence for constructible $\mathcal{F}$, the universal local acyclicity is reduced to smooth base change.

Theorem 1.3.2. 1. (local acyclicity of smooth morphism) *Assume that $f\colon X \to S$ is smooth and that $\mathcal{F}$ is locally constant killed by an integer invertible on S. Then f is ula relatively to $\mathcal{F}$.*

2. (generic local acyclicity) *Assume that $f\colon X \to S$ is of finite type and that $\mathcal{F}$ is constructible. Then, there exists a dense open subscheme $U \subset S$ such that the base change of f to U is ula relatively to the restriction of $\mathcal{F}$.*

Corollary 1.3.3. *Assume that $g\colon Y \to S$ is smooth, that $f\colon X \to Y$ is la relatively to $\mathcal{F}$ and $\mathcal{F}$ is killed by an integer invertible on S. Then, gf is locally acyclic relatively to $\mathcal{F}$.*

Lemma 1.3.4. *Let $f\colon X \to Y$ be a proper morphism of schemes over S and assume that $X \to S$ is locally acyclic relatively to $\mathcal{F}$. Then $Y \to S$ is locally acyclic relatively to $Rf_*\mathcal{F}$.*

Proof. Proper base change theorem.

1.4 Micro support

Definition 1.4.1. *Let $\mathcal{F}$ be a constructible complex on X and $C \subset T^*X$ be a closed conical subset. We say that $\mathcal{F}$ is micro supported on C, if for every C-transversal pair (h, f) of $h\colon W \to X$ and $f\colon W \to Y$, f is (universally) locally acyclic relatively to $h^*\mathcal{F}$.*

If $\mathcal{F}$ is micro supported on $C \subset C'$, then $\mathcal{F}$ is micro supported on C'.

Lemma 1.4.2. *If $\mathcal{F}$ is micro supported on C, then the support of $\mathcal{F}$ is a subset of the base B of C.*

Proof. Let $U = X - B$. It suffices to show that $\mathcal{F}|_U = 0$. The pair $U \to X, U \to 0 \subset \mathbf{A}^1$ is C-transversal. Hence $U \to \mathbf{A}^1$ is locally acyclic relatively to $\mathcal{F}|_U$ and $\mathcal{F}|_U = 0$.

Lemma 1.4.3. *Let $U \subset X$ be an open subscheme and A be the complement. Assume that $\mathcal{F}$ is micro supported on C and assume that $\mathcal{F}|_U$ is micro supported on C'_U. Then $\mathcal{F}$ is micro supported on the union of $C|_A$ and the closure C' of C'_U.*

Lemma 1.4.4. *Let $\to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to$ be a distinguished triangle and suppose that $\mathcal{F}'$ and $\mathcal{F}''$ are micro supported on C' and on C'' respectively. Then $\mathcal{F}$ is micro supported on $C = C' \cup C''$.*

Lemma 1.4.5. *The following conditions are equivalent:*

(1) *$\mathcal{F}$ is locally constant.*

(2) *$\mathcal{F}$ is micro supported on the 0-section $T^*_X X$.*

Proof. (h, f) is $T^*_X X$-transversal if and only if f is smooth.

(1)⇒(2): f is universally locally acyclic relatively to locally constant $h^*\mathcal{F}$.

(2)⇒(1): $(1_X, 1_X)$ is $T^*_X X$-transversal. Hence, 1_X is locally acyclic relatively to $\mathcal{F}$ and $\mathcal{F}$ is locally constant.

Lemma 1.4.6. *Any constructible $\mathcal{F}$ is micro supported on T^*X.*

Proof. Suppose (h, f) is T^*X-transversal. Then $W \to X \times Y$ is smooth. Locally, $W \to Y$ is the composition of an étale morphism $W \to X \times \mathbf{A}^n \times Y$ with the projection $X \times \mathbf{A}^n \times Y \to Y$. Hence the local acyclicity follows from the generic local acyclicity and Corollary 1.3.3.

Lemma 1.4.7. *Assume that $\mathcal{F}$ is micro supported on C.*

1. *If $h \colon W \to X$ is C-transversal, then $h^*\mathcal{F}$ is micro supported on $h^\circ C$.*
2. *If $f \colon X \to Y$ is proper on the base of C, then $Rf_*\mathcal{F}$ is micro supported on $f_\circ C$.*

Proof. 1. Suppose $g \colon V \to W, f \colon V \to Y$ is $h^\circ C$-transversal. Then, (hg, f) is C-transversal and f is locally acyclic relatively to $(hg)^*\mathcal{F}$.

2. Suppose $h \colon W \to Y, g \colon W \to Z$ is $f_\circ C$-transversal. Then, $h_X \colon W \times_Y X \to X, g \circ f_W \colon W \times_Y X \to W \to Z$ is C-transversal and $h_X^*\mathcal{F}$ is locally acyclic relatively to $g \circ f_W$. Hence $h^* Rf_*\mathcal{F} = Rf_{W*} h_X^* \mathcal{F}$ is locally acyclic relatively to g.

1.5 Singular support

Definition 1.5.1. *We say that $C \subset T^*X$ is the singular support of $\mathcal{F}$ if for $C' \subset T^*X$, the inclusion $C \subset C'$ is equivalent to the condition that $\mathcal{F}$ is micro supported on C.*

Lemma 1.5.2. *Let $\mathcal{F}$ be a constructible sheaf on X.*

1. *Let $U \subset X$ be an open subscheme. Assume that $C \subset T^*X$ is the singular support of $\mathcal{F}$. Then, $C|_U$ is the singular support of $\mathcal{F}|_U$.*
2. *Let (U_i) be an open covering of X and C_i be the singular support of $\mathcal{F}|_{U_i}$. Then, $C = \bigcup_i C_i$ is the singular support of $\mathcal{F}$.*

Lemma 1.5.3. *Let $i \colon X \to P$ be a closed immersion. Assume that $C_P \subset T^*P$ is the singular support of $i_*\mathcal{F}$.*

1. *C_P is a subset of $T^*P|_X$ and its image $C \subset T^*X$ is the singular support of $\mathcal{F}$.*
2. *We have $C_P = i_\circ C$.*

Proof. 1. By Lemma 1.4.3, C_P is a subset of $T^*P|_X$.

To show $C = SS\mathcal{F}$, it suffices to show the following:

(1) If $\mathcal{F}$ is micro supported on C', we have $C \subset C'$.

(2) C is closed and $\mathcal{F}$ is micro supported on C.

We show (1). Suppose $\mathcal{F}$ is micro supported on C'. Then by Lemma ??, $i_*\mathcal{F}$ is micro supported on $i_\circ C'$. Since C_P is the smallest, we have $C_P \subset i_\circ C'$ and hence $C \subset C'$.

We show (2). Since the assertion is local, we may assume that there exists a cartesian diagram

$$\begin{tikzcd} P & X \arrow[l, "i"'] \arrow[d] \\ \mathbf{A}_k^n \arrow[u, leftarrow] & \mathbf{A}_k^m \arrow[l] \end{tikzcd}$$

such that the vertical arrows are isomorphism. Then, by choosing a projection $\mathbf{A}_k^n \to \mathbf{A}_k^m$ inducing the identity on $\mathbf{A}_k^m$, we obtain a cartesian diagram

$$\begin{tikzcd} P \arrow[d] & Q \arrow[l] \arrow[d, "r"] \\ \mathbf{A}_k^m & X \arrow[l] \end{tikzcd}$$

where the horizontal arrows are étale. The immersion $X \to P$ induces a section $i' \colon X \to Q$. Since $h \colon Q \to P$ is étale, $i'_*\mathcal{F}$ is micro supported on $h^\circ C_P$. By Lemma ??, $\mathcal{F} = r_* j_* \mathcal{F}$ is micro supported on $C_r = r_\circ h^\circ C_P$. Hence by (1), we have $C \subset C_r$. Since $C_r \subset C$, we have $C_r = C$ and C is closed and $\mathcal{F}$ is micro supported on $C = C_r$.

2. By the proof of (2), we have $C = C_{r'}$ for any projection r'. If k is infinite, this implies $C_P = i_\circ C$.

Theorem 1.5.4. (Beilinson) *$SS\mathcal{F}$ exists.*

Proof will be given at the end of next section.

Theorem 1.5.5. (Beilinson) 1. $\dim E \leqq \dim \mathbf{P} - 1$.

2. *Every irreducible component of E has* $\dim \mathbf{P} - 1$.

1.6 Radon transform

We define the naive Radon transform $R\mathcal{F}$ to be $Rp^\vee_* p^* \mathcal{F}$ and the naive inverse Radon transform $R^\vee \mathcal{G}$ to be $Rp_* p^{\vee *} \mathcal{G}$.

Proposition 1.6.1. *There exists a distinguished triangle*

$$\to \bigoplus_{q=0}^{n-2} R\Gamma(\mathbf{P}_{\bar{k}}, \mathcal{F})(q)[2q] \to R^\vee R\mathcal{F} \to \mathcal{F}(n-1)[2(n-1)] \to .$$

Proof. By the cartesian diagram

$$\begin{tikzcd}
\mathbf{P} & Q \arrow[l, "p"'] \arrow[d, "p^\vee"'] & Q \times_{\mathbf{P}^\vee} Q \arrow[l, "pr_1"'] \arrow[d, "pr_2"] \\
& \mathbf{P}^\vee & Q \arrow[l, "p^\vee"'] \arrow[d, "p"] \\
& & \mathbf{P}
\end{tikzcd}$$

and the proper base change theorem, we have a canonical isomorphism

$$R^\vee R\mathcal{F} \to Rpr_{2*}\big(pr_1^*\mathcal{F} \otimes R(p\times p)_*\Lambda_{Q\times_{\mathbf{P}^\vee}Q}\big)$$

for $p \times p \colon Q \times_{\mathbf{P}^\vee} Q \to \mathbf{P} \times \mathbf{P}$.

We compute $R(p\times p)_*\Lambda_{Q\times_{\mathbf{P}^\vee}Q}$. The closed scheme $Q \times_{\mathbf{P}^\vee} Q \subset \mathbf{P} \times \mathbf{P} \times \mathbf{P}^\vee$ is the $\mathbf{P}^{n-1}$-bundle Q on the diagonal $\mathbf{P} \subset \mathbf{P} \times \mathbf{P}$. On the complement $\mathbf{P} \times \mathbf{P} - \mathbf{P}$, it is a sub $\mathbf{P}^{n-2}$-bundle. Hence, we have a distinguished triangle

$$\to \tau_{\leqq 2(n-2)} R\Gamma(\mathbf{P}^\vee_{\bar k}, \Lambda) \otimes \Lambda_{\mathbf{P}\times\mathbf{P}} \to R(p\times p)_*\Lambda_{Q\times_{\mathbf{P}^\vee}Q} \to \Lambda_{\mathbf{P}}(n-1)[2(n-1)] \to .$$

Proposition 1.6.2. *For $\mathcal{G}$ on $\mathbf{P}^\vee$ and $C^\vee \subset T^*\mathbf{P}^\vee$, we have implications* (1)⇒(2)⇒(3)*.*

(1) *$\mathcal{G}$ is micro supported on $C^\vee$.*

(2) *p is universally locally acyclic relatively to $p^{\vee *}\mathcal{G}$ outside $E = \mathbf{P}(C^\vee)$.*

(3) *$R^\vee\mathcal{G}$ is micro supported on C^+.*

Proof. (1)⇒(2): Since $p^\vee \colon Q \to \mathbf{P}^\vee, p \colon Q \to \mathbf{P}$ is $C^\vee$-transversal outside $E = \mathbf{P}(C^\vee)$, p is universally locally acyclic relatively to $p^{\vee *}\mathcal{G}$ outside E.

(2)⇒(3): Assume $h \colon W \to \mathbf{P}, f \colon W \to Y$ is C^+-transversal. We consider the cartesian diagram

$$\begin{tikzcd}
\mathbf{P}^\vee & Q \arrow[l, "p^\vee"'] \arrow[d, "p"'] \arrow[dr, phantom, "\square"] & Q_W \arrow[l, "h'"'] \arrow[d, "p'"] \\
& \mathbf{P} & W \arrow[l, "h"'] \arrow[d, "f"] \\
& & Y.
\end{tikzcd}$$

We first show that $fp' \colon Q_W \to Y$ is locally acyclic relatively to $\mathcal{G}_{Q_W} = h'^* p^{\vee *}\mathcal{G}$. By (2), $p' \colon Q_W \to W$ is locally acyclic relatively to $\mathcal{G}_{Q_W}$ outside the inverse image $E_W \subset Q_W$ of E. By Corollary 1.2.6, $f \colon W \to Y$ is smooth and $Q_W \to \mathbf{P}^\vee \times Y$ is smooth on the inverse image E_W.

Hence by Corollary 1.3.3, $fp' \colon Q_W \to Y$ is locally acyclic relatively to $\mathcal{G}_{Q_W}$ outside E_W. Further by the generic local acyclicity and Corollary 1.3.3, $fp' \colon Q_W \to Y$ is locally acyclic relatively to $\mathcal{G}_{Q_W}$ on a neighborhood of E_W. Thus, $fp' \colon Q_W \to Y$ is locally acyclic relatively to $\mathcal{G}_{Q_W}$. Hence by Lemma, $f \colon W \to Y$ is locally acyclic relatively to $Rp'_*\mathcal{G}_{Q_W} = h^* R^\vee \mathcal{G}$.

Proof of Theorem 1.5.4. It is reduced to the case X is affine, an affine space and then a projective space.

Let $E \subset Q$ be the smallest closed subset such that $p\colon Q \to \mathbf{P}$ is universally locally acyclic relatively to $p^{\vee *}R\mathcal{F}$ on the complement $Q - E$. Let $C \subset T^*\mathbf{P}$ be the closed conical subset defined by E. Then, by ??, $R^\vee R\mathcal{F}$ is micro supported on C^+. Hence by ??, $\mathcal{F}$ is also micro supported on C^+.

Let $U = \mathbf{P} - B$ be the complement of the base of C. Then, since $C^+ \cap T^*U = T^*_U U$, the restriction $\mathcal{F}|_U$ is locally constant. If $\mathcal{F}|_U = 0$, $\mathcal{F}$ is micro supported on C. We show that C is the singular support of $\mathcal{F}$ if $\mathcal{F}|_U = 0$ and that C^+ is the singular support of $\mathcal{F}$ if otherwise.

Suppose $\mathcal{F}$ is micro supported on C'. Then by (1)⇒(3), $\mathcal{G} = R\mathcal{F}$ is micro supported on $C'^{\vee +}$. Hence by (1)⇒(2), $p\colon Q \to \mathbf{P}$ is universally locally acyclic relatively to $p^{\vee *}\mathcal{G}$ outside $E' = \mathbf{P}(C'^\vee) = \mathbf{P}(C')$. Since E is the smallest, we have $E \subset E'$ and hence $C \subset C'$. If $\mathcal{F}|_U \neq 0$, we have $\operatorname{supp}\mathcal{F} = \mathbf{P}$ and hence $T^*_{\mathbf{P}}\mathbf{P} \subset C'$ and $C^+ \subset C'$.

2 Characteristic cycle

2.1 Characteristic cycles

Theorem 2.1.1. *There exists a unique way to attach a* $\mathbf{Z}$*-linear combination* $CC\mathcal{F} = \sum_a m_a C_a$ *of irreducible components* $SS\mathcal{F} = \bigcup_a C_a$ *for each constructible complex* $\mathcal{F}$ *of* Λ*-modules on a smooth scheme* X *over* k*, satisfying the following axioms:*

(1) *(normalization) For* $X = \operatorname{Spec} k$ *and* $\mathcal{F} = \Lambda$*, we have*

$$CC\Lambda = T^*_X X. \tag{2.1}$$

(2) *(additivity) For distringuished triangle* $\to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to$*, we have*

$$CC\mathcal{F} = CC\mathcal{F}' + CC\mathcal{F}''. \tag{2.2}$$

(3) *(pull-back) For* $SS\mathcal{F}$*-transversal morphism* $h\colon W \to X$ *of smooth schemes over* k*, we have*

$$CCh^*\mathcal{F} = h^!CC\mathcal{F}. \tag{2.3}$$

(4) *(closed immersion) For closed immersion* $i\colon X \to P$ *of smooth schemes over* k*, we have*

$$CCi_*\mathcal{F} = i_!CC\mathcal{F}. \tag{2.4}$$

(5) *(Radon transform) For* $X = \mathbf{P}^n$ *and for the Radon transform, we have*

$$CCR\mathcal{F} = LCC\mathcal{F}. \tag{2.5}$$

Corollary 2.1.2. *(index formula) Assume that* X *is projective and smooth. Then, we have*

$$\chi(X_{\bar{k}}, \mathcal{F}) = (CC\mathcal{F}, T^*_X X). \tag{2.6}$$

Proof. By (1), (2) and (3), if $\mathcal{F}$ is locally constant, we have

$$CC\mathcal{F} = (-1)^n \mathrm{rank}\, \mathcal{F} \cdot T^*_X X. \tag{2.7}$$

By (4), we may assume that $X = \mathbf{P}^n$ and $n \geqq 2$. Then, we have

$$CCR^\vee R\mathcal{F} = CC\mathcal{F} + (n-1) \cdot \chi(\mathbf{P}^n_{\bar{k}}, \mathcal{F})[T^*_{\mathbf{P}^n}\mathbf{P}^n]. \tag{2.8}$$

By (5) and (2), we have $CC(R^\vee R\mathcal{F}) - CC\mathcal{F} = L^\vee LCC\mathcal{F} - CC\mathcal{F}$. Hence, we have $(n-1)\chi(\mathbf{P}^n_{\bar{k}}, \mathcal{F}) = (n-1)(CC\mathcal{F}, T^*_{\mathbf{P}^n}\mathbf{P}^n)$ and (2.6).

We will deduce Theorem 2.1.1 from the following variant.

Theorem 2.1.3. *There exists a unique way to attach a* $\mathbf{Q}$*-linear combination* $CC\mathcal{F} = \sum_a m_a C_a$ *of irreducible components* $SS\mathcal{F} = \bigcup_a C_a$ *for each constructible complex* $\mathcal{F}$ *of* Λ*-modules on smooth smooth scheme* X *over* k*, satisfying the following axioms:*

(1) *(Milnor formula) Let* $f\colon X \to Y$ *be a proper morphism over* k *to a smooth curve* Y *over* k *and* $x \in X$ *be a closed point such that* f *is* $SS\mathcal{F}$*-transversal on the complement of* x*. Then, the coefficient of the fiber* $T^*_y Y$ *at* $y = f(x)$ *in* $f_\circ CC\mathcal{F}$ *is minus the Artin conductor* $-a_x Rf_*\mathcal{F}$*.*

(3) *For étale morphism* $h\colon W \to X$ *of smooth schemes over* k*, we have* (2.3)*.*

(4) *For closed immersion* $i\colon X \to P$ *of smooth schemes over* k*, we have* (2.4)*.*

Outline and key points of proof of theorems.

Proof of Theorem 2.1.3. We show the uniqueness. By (3), we may assume X is affine. By (4), we may assume $X = \mathbf{A}^n$. By (3), we may assume X is projective. We may take a Lefschetz pencil. Since it suffices to determine the coefficient m_a for each C_a, we may assume that $f\colon W \to L$ is C_b-transversal for $C_b \neq C_a$ and C_a-transversal except at x and is not C_a-transversal at x. Then, by (1), we have

$$m_a(C_a, df)_x = -a_x \tag{2.9}$$

and the uniqueness follows.

To show the existence, first we show that the coefficient m_a determined by (2.9) is well-defined. This follows from the (semi-)continuity of Swan conductor and the formalism of vanishing cycles over general base. Then $CC\mathcal{F}$ characterized by (2.9) satisfies the conditions (3) and (4) by standard properties of usual vanishing cycles.

Proof of the uniqueness in Theorem 2.1.1. By Corollary 2.1.2, we have the index formula (2.6) for projective and smooth X. By comparing the index formula (2.6) for proper smooth curve X and the Grothendieck-Ogg-Shafarevich formula and using (3) for étale morphism of smooth curves and (2.7), we obtain (1) in Theorem 2.1.3 for $f = 1_X\colon X \to X$.

Similarly as in the proof of Theorem 2.1.3, it is reduced to the case where X is projective and smooth. Then by taking a Lefschetz pencil, it follows from (5), (3) and (1) in Theorem 2.1.3.

Proof of the existence in Theorem 2.1.1. We deduce the existence from Theorem 2.1.3. We show that $CC\mathcal{F}$ satisfying the conditions in Theorem 2.1.3 also satisfies those in Theorem 2.1.1. The conditions (1) and (2) in Theorem 2.1.1 follow from (1) in Theorem 2.1.3. The condition (4) in Theorem 2.1.1 is the same as (4) in Theorem 2.1.3. Hence it remains to show the conditions (3), (5) and the integrality.

The condition (3) for smooth morphism is a consequence of the Thom-Sebastiani formula. The integrality in the case $p \neq 2$ or non-exceptional case in $p = 2$ follows from (1) in Theorem 2.1.3. In the exceptional case, it is reduced to the non-exceptional case using the condition (3) for $X \times \mathbf{A}^1 \to X$.

To show (3) in the case where h is an immersion, we first consider the case where X is an projective space $\mathbf{P}^n$.

Lemma 2.1.4. *Let $h\colon W \to P = \mathbf{P}^n$ be an immersion and*

$$\begin{tikzcd} W \arrow[d, "h"'] & W \times_P Q \arrow[l, "p_W"'] \arrow[r, "p_W^\vee"] \arrow[d] & P^\vee \\ P & Q \arrow[l, "p"'] & \end{tikzcd}$$

be the cartesian diagram. Let $\mathcal{G}$ be a constructible complex on $P^\vee$ micro supported on $C^\vee$ and assume that h is properly C-transversal for $C = L^\vee C^\vee$. Then, we have

$$\mathbf{P}(CCRp_{W*}p_W^{\vee *}\mathcal{G}) = \mathbf{P}(p_{W!}p_W^{\vee !}CC\mathcal{G}).$$

Proof. Since the characteristic cycle is characterized by the Milnor formula, it suffices to show that $p_{W!}p_W^{\vee !}CC\mathcal{G}$ satisfies the Milnor formula for $Rp_{W*}p_W^{\vee *}\mathcal{G}$ and for *smooth* morphisms $f\colon W \to Y$ to a curve defined locally on W. Since h is C-transversal, $p_W^\vee\colon Q \times_{\mathbf{P}} W \to \mathbf{P}^\vee$ is $C^\vee$-transversal by Lemma 1.2.7.2 and $p_W^*\mathcal{G}$ is micro supported on $p_W^\vee C^\vee$. Since $p_W^\vee\colon Q \times_{\mathbf{P}} W \to \mathbf{P}^\vee$ is smooth outside $\mathbf{P}(C_W)$, we have $CCp_W^{\vee *}\mathcal{G} = p_W^{\vee\circ}CC\mathcal{G}$ outside $\mathbf{P}(C_W)$ as (3) is already proved for smooth morphisms.

Assume that f is smooth and has only isolated characteristic point. Then, by Lemma 1.2.7.2, the composition fp_W is $p^\vee C$-transversal outside the inverse images of the characteristic points. Further it is $p^\vee C$-transversal outside of finitely many closed points in the inverse images by Lemma 1.2.7.3 and these points are not contained in $\mathbf{P}(C_W)$ by Lemma 1.2.7.1. Hence the assertion follows.

Lemma 2.1.4 implies also $\mathbf{P}(CCh^*\mathcal{F}) = \mathbf{P}(h^!CC\mathcal{F})$. Since the coefficient of the 0-section is determined by the generic rank as in (2.7), we deduce (3) in the case $X = \mathbf{P}$. In the general case, since the assertion is local, we may assume that there exists an open subscheme $U \subset \mathbf{P}$ and a cartesian diagram

$$\begin{tikzcd} W \arrow[r, "h"] \arrow[d, "j"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "i"] \\ V \arrow[r, "g"] & U \subset P \end{tikzcd}$$

where $i\colon X \to U$ and $g\colon V \to U$ are closed immersions of smooth subschemes meeting transversely. Then, since h is properly C-transversal, g is properly $i_\circ C$-transversal. Hence the case where $X = \mathbf{P}$ implies $CCg^*i_*\mathcal{F} = g^!CCi_*\mathcal{F} = g^!i_!CC\mathcal{F}$. This implies $j_!CCh^*\mathcal{F} = CCj_*h^*\mathcal{F} = j_!h^!CC\mathcal{F}$ and (2.3).

We show (5). The case $W = \mathbf{P}$ in Lemma 2.1.4 means the projectivization

$$\mathbf{P}(CCR\mathcal{F}) = \mathbf{P}(LCC\mathcal{F}) \tag{2.10}$$

of (5). Hence it remains to show that the coefficients of the 0-section in $CCR\mathcal{F} = LCC\mathcal{F}$ are the same. Similarly as in the proof of Corollary 2.1.2, this is equivalent to the index formula (2.6) for $X = \mathbf{P}^n$. To prove this, we introduce the characteristic class.

2.2 Characteristic class

We identify the Chow group of the projective completion $\mathbf{P}(T^*X\oplus\mathbf{A}^1_X)$ by the canonical isomorphism

$$\mathrm{CH}_\bullet(X)=\bigoplus_{i=0}^{n}\mathrm{CH}_i(X)\to\mathrm{CH}_n(\mathbf{P}(T^*X\oplus\mathbf{A}^1_X)). \tag{2.11}$$

For a constructible complex $\mathcal{F}$ on X with the characteristic cycle $CC\mathcal{F}=\sum_a m_aC_a$, we define the characteristic class

$$cc_X(\mathcal{F})\in\mathrm{CH}_\bullet(X) \tag{2.12}$$

to be the class of $\sum_a m_a\bar{C}_a\in\mathrm{CH}_n(\mathbf{P}(T^*X\oplus\mathbf{A}^1_X))$.

Let $K(X,\Lambda)$ denote the Grothendieck group of the category of constructible complexes of Λ-modules on X. By the additivity, we have a morphism

$$cc_X\colon K(X,\Lambda)\to\mathrm{CH}_\bullet(X) \tag{2.13}$$

sending the class $\mathcal{F}$ to $cc_X\mathcal{F}$. In characteristic 0, we recover the MacPherson Chern class.

The pull-back by the immersion $\mathbf{P}(T^*X)\to\mathbf{P}(T^*X\oplus\mathbf{A}^1_X)$ and the push-forward by $\mathbf{P}(T^*X\oplus\mathbf{A}^1_X)\to X$ induce an isomorphism

$$\mathrm{CH}_n(\mathbf{P}(T^*X\oplus\mathbf{A}^1_X))\to\mathrm{CH}_{n-1}(\mathbf{P}(T^*X))\oplus\mathrm{CH}_n(X).$$

For $A=\sum_a m_aC_a$, the images of $\bar{A}=\sum_a m_a\bar{C}_a$ is the pair of $\mathbf{P}(A)=\sum_a m_a\mathbf{P}(C_a)$ and the coefficient of the 0-section.

End of Proof of Theorem 2.1.3. Under (2.10), the equality (2.5) is equivalent to the condition that the diagram

$$\begin{array}{ccc} K(\mathbf{P}^n,\Lambda) & \xrightarrow{cc_{\mathbf{P}^n}} & \mathrm{CH}_\bullet(\mathbf{P}^n)\\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle L}\\ K(\mathbf{P}^{n\vee},\Lambda) & \xrightarrow{cc_{\mathbf{P}^{n\vee}}} & \mathrm{CH}_\bullet(\mathbf{P}^{n\vee}) \end{array} \tag{2.14}$$

gets commutative after composed with the projection $\mathrm{CH}_\bullet(\mathbf{P}^{n\vee})\to\mathrm{CH}_n(\mathbf{P}^{n\vee})$ and also to the commutativity of the diagram (2.14) itself.

We prove the commutativity of (2.14) (CD n) and the index formula (2.6) for $\mathbf{P}^n$ (IF n) by a simultaneous induction on n along the diagram; (IF $n-1$) $\Rightarrow$ (CD n) $\Rightarrow$ (IF n). For $n\leqq 1$, the commutativity of (2.14) is obvious. For $n=0$, the index formula follows from (2.7). For $n=1$, this is nothing but the Grothendieck-Ogg-Shafarevich formula.

We prove (IF $n-1$) $\Rightarrow$ (CD n). Let $i\colon H\to\mathbf{P}^n$ be the immersion of a hyperplane. Then, the right square in

$$\begin{array}{ccccc} K(\mathbf{P}^n,\Lambda) & \xrightarrow{cc_{\mathbf{P}^n}} & \mathrm{CH}_\bullet(\mathbf{P}^n) & \xrightarrow{i^!} & \mathrm{CH}_{n-1}(H)\\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle L} & & \downarrow{\scriptstyle \deg}\\ K(\mathbf{P}^{n\vee},\Lambda) & \xrightarrow{cc_{\mathbf{P}^{n\vee}}} & \mathrm{CH}_\bullet(\mathbf{P}^{n\vee}) & \longrightarrow & \mathrm{CH}_n(\mathbf{P}^{n\vee})=\mathbf{Z} \end{array} \tag{2.15}$$

is commutative. Hence it suffices to show that the long rectangle is commutative. For $\mathcal{F}$ on $\mathbf{P}^n$, the generic rank of $R\mathcal{F}$ equals the Euler number $\chi(H_{\bar{k}}, \mathcal{F})$ for a generic H. Hence the composition via lower left sends the class of $\mathcal{F}$ to $\chi(H_{\bar{k}}, \mathcal{F})$. By (3) for the immersion $i\colon H \to \mathbf{P}^n$ and (IF $n-1$), we have $\chi(H_{\bar{k}}, \mathcal{F}) = (CCi^*\mathcal{F}, T^*_H H) = \deg i^! cc_{\mathbf{P}^n}\mathcal{F}$ and the long rectangle is commutative.

We prove (CD n) $\Rightarrow$ (IF n). Let $\chi\colon K(\mathbf{P}^n, \Lambda) \to \mathbf{Z}$ be the morphism sending the class of $\mathcal{F}$ to the Euler number $\chi(\mathbf{P}^n_{\bar{k}}, \mathcal{F})$. We show that there is a commutative diagram

$$\begin{tikzcd} K(\mathbf{P}^n, \Lambda) \arrow[r, "cc_{\mathbf{P}^n}"] \arrow[dr, "\chi"'] & \mathrm{CH}_\bullet(\mathbf{P}^n) \arrow[d] \\ & \mathbf{Z}. \end{tikzcd} \tag{2.16}$$

Since $cc_{\mathbf{P}^n}$ is a surjection, it suffices to show that $cc_{\mathbf{P}^n}\mathcal{F} = 0$ implies $\chi(\mathbf{P}^n_{\bar{k}}, \mathcal{F}) = 0$. By (2.8), (CD n) and the assumption $cc_{\mathbf{P}^n}\mathcal{F} = 0$ imply $\chi(\mathbf{P}^n_{\bar{k}}, \mathcal{F}) = 0$ for $n-1 \neq 0$. Thus, there exists a unique morphism $\mathrm{CH}_\bullet(\mathbf{P}^n) \to \mathbf{Z}$ making the diagram (2.16) commutative. We show that the morphism $\mathrm{CH}_\bullet(\mathbf{P}^n) \to \mathbf{Z}$ equals the degree mapping. This is reduced to the case where $\mathcal{F} = \Lambda_{\mathbf{P}^i}, i = 0, \ldots, n$ generating $\mathrm{CH}_\bullet(\mathbf{P}^n) = \mathbf{Z}^{n+1}$.