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References

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- I. Vidal. Critères valuatifs (d'après O. Gabber). Appendice à "Courbes nodales et ramification sauvage virtuelle", *Manuscripta Math.*, 2005.

Plan of the talk

1 Serre's conjectures on ℓ -independence

2 Compatible systems along the boundary

3 Relation with wild ramification

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References

- J.-P. Serre, J. Tate. Good reduction of abelian varieties. *Ann. Math.* (1968).
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Serre proposed conjectures C1–C8 related to the definition of the Hasse-Weil zeta functions of projective smooth varieties over global fields.

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Arithmetic zeta function

• Riemann zeta function:

$$\zeta(s) = \sum_{n \ge 1} \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}}.$$

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Arithmetic zeta function

• Riemann zeta function:

$$\zeta(s) = \sum_{n \ge 1} \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}}.$$

• Let \mathcal{X} be a scheme of finite type over Spec(**Z**). Arithmetic zeta function:

$$\zeta_{\mathcal{X}}(s) = \sum_{C \in Z_0^{\text{eff}}(\mathcal{X})} \frac{1}{(NC)^s} = \prod_{x \in |\mathcal{X}|} \frac{1}{1 - (Nx)^{-s}}$$
$$= \prod_{v \in |V|} Z_{\mathcal{X}_v}((Nv)^{-s})$$

for \mathcal{X} over V of finite type over Spec(**Z**).

Cohomological interpretation

Let X be a variety (= scheme separated of finite type) over a field k. For each $\ell \neq \operatorname{char}(k)$, Grothendieck defined a finite-dimensional \mathbf{Q}_{ℓ} -vector space $H^i_{\ell,c} = H^i_c(X_{\bar{k}}, \mathbf{Q}_{\ell})$, equipped with a continuous action of $\operatorname{Gal}(\bar{k}/k)$.

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Theorem (Grothendieck)

Let X be a variety over $k = \mathbf{F}_q$. For each $\ell \nmid q$,

$$Z_X(t) = \prod_i P_{i,\ell}(t)^{(-1)^{i+1}},$$

where

$$P_{i,\ell}(t) = \det(1 - \operatorname{Fr} t, H^i_{\ell,c}).$$

Weil conjectures (continued)

Let X be a proper smooth variety over $k = \mathbf{F}_q$.

Theorem (Deligne, C2)

The reciprocal roots of $P_{i,\ell}$ are of weight *i* (algebraic numbers with all complex conjugates of absolute value $q^{i/2}$).

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Corollary (C1) $P_{i,\ell} \in \mathbf{Z}[t]$ and is independent of ℓ .

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Corollary (C1) $P_{i,\ell} \in \mathbf{Z}[t]$ and is independent of ℓ .

Corollary

Let X be a proper smooth variety over an arbitrary field k. Then the Betti number dim $H^i(X_{\bar{k}}, \mathbf{Q}_{\ell})$ is independent of $\ell \neq \operatorname{char}(k)$.

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Hasse-Weil zeta function

Let X be a proper smooth variety over a global field F.

$$\zeta_X(s) = \prod_i L_i(s)^{(-1)^{i+1}},$$

 $L_i(s) = \prod_v \det(1 - \operatorname{Fr} q_v^{-s}, (H_\ell^i)^{I_v}),$

where v runs over finite places of F, and I_v denotes the inertia group at v.

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ℓ -independence

Let K be a local field: a complete discrete valuation field of finite residue field \mathbf{F}_{q} . Let X be a proper smooth variety over K.

Conjecture

• (Serre, C5) det $(1 - \operatorname{Fr} t, (H_{\ell}^i)^{I_{\kappa}}) \in \mathbf{Z}[t]$ and is independent of $\ell \nmid q$.

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Conjecture

- (Serre, C5) det $(1 \operatorname{Fr} t, (H^i_{\ell})^{I_{K}}) \in \mathbf{Z}[t]$ and is independent of $\ell \nmid q$.
- (Serre-Tate, C8) For each lifting $F \in \text{Gal}(\overline{K}/K)$ of Fr, det $(1 - Ft, H_{\ell}^i) \in \mathbf{Z}[t]$ and is independent of $\ell \nmid q$.

Monodromy Weight Conjecture

Let M denote the monodromy filtration.

Conjecture

Eigenvalues of F lifting Fr on $gr_n^M H_{\ell}^i$ are of weight i + n.

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C8 + Monodromy Weight Conjecture $\Rightarrow \det(1 - Ft, \operatorname{gr}_n^M H_{\ell}^i) \in \mathbb{Z}[t] \text{ and is independent of } \ell$ $\Rightarrow C5$

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 $\begin{array}{l} \mathsf{C8} + \mathsf{Monodromy} \; \mathsf{Weight} \; \mathsf{Conjecture} \\ \Rightarrow \; \mathsf{det}(1 - \mathsf{Ft}, \mathsf{gr}_n^M \mathsf{H}_\ell^i) \in \mathbf{Z}[t] \; \mathsf{and} \; \mathsf{is} \; \mathsf{independent} \; \mathsf{of} \; \ell \\ \Rightarrow \; \mathsf{C5} \end{array}$

(Monodromy Weight Conjecture \Rightarrow C6 + C7)

General residue field

Let K be a complete discrete valuation field of residue field k. Let X be a proper smooth variety over K.

Conjecture (Serre-Tate, C4)

For each $F \in I_K$, $det(1 - Ft, H_{\ell}^i) \in \mathbf{Z}[t]$ and is independent of $\ell \neq char(k)$.

Local monodromy theorem

Let X be a variety over K.

Theorem

• (Grothendieck) An open subgroup of I_K acts on $H^i_{\ell,c}$ unipotently.

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Local monodromy theorem

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Theorem

- (Grothendieck) An open subgroup of I_K acts on $H_{\ell,c}^i$ unipotently.
- (Deligne, Gabber, Illusie) There exists an open subgroup I' of I_K , independent of ℓ , such that for every $g \in I'$, $(g 1)^{i+1}$ acts by 0 on H^i_{ℓ} and $H^i_{\ell,c}$.

Serre's conjectures on *l*-independence

Equal characteristic case

Theorem (Deligne, Terasoma, Ito) Monodromy Weight Conjecture holds in equal characteristic.

Equal characteristic case (continued)

Let K be a complete discrete valuation field of residue field k, both of characteristic p > 0. Let X be a proper smooth variety over K.

Theorem

- (Lu-Z., C4) For each $F \in I_K$, det $(1 Ft, H_{\ell}^i) \in \mathbf{Z}[t]$ and is independent of $\ell \neq p$.
- (Deligne, Terasoma, Lu-Z., C8) Assume $k = \mathbf{F}_q$. For each lifting $F \in \text{Gal}(\bar{K}/K)$ of Fr, $\det(1 Ft, H^i_\ell) \in \mathbf{Z}[t]$ and is independent of $\ell \neq p$.

Equal characteristic case (continued)

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Corollary (C5)

Assume $k = \mathbf{F}_q$. For each lifting $F \in \text{Gal}(\overline{K}/K)$ of Fr, det $(1 - Ft, (H_{\ell}^i)^{I_K}) \in \mathbf{Z}[t]$ and is independent of $\ell \nmid q$.

General characteristic: alternating sums

Let X be a variety over a field K.

Theorem

• (Gabber, C1') Assume $K = \mathbf{F}_q$. For each $F \in W(\bar{K}/K)$, $\sum_i (-1)^i \operatorname{tr}(F, H^i_\ell) \in \mathbf{Q}$ and is independent of $\ell \nmid q$.

General characteristic: alternating sums

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- (Vidal, C4') Assume K is a complete discrete valuation field of residue characteristic p > 0. For each $F \in I_K$, $\sum_i (-1)^i \operatorname{tr}(F, H^i_\ell) \in \mathbf{Z}$ and is independent of $\ell \neq p$.
- (Ochiai, Z., C8') Assume K is a local field of residue field \mathbf{F}_q . For each $F \in W(\bar{K}/K)$, $\sum_i (-1)^i \operatorname{tr}(F, H^i_\ell) \in \mathbf{Q}$ and is independent of $\ell \nmid q$.

Plan of the talk



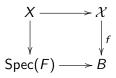
2 Compatible systems along the boundary

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Spreading out

Let X be a proper smooth variety over a field F of characteristic p > 0. There exists a scheme B of finite type over \mathbf{F}_p and a Cartesian square



with f proper smooth. We have

$$H^i(X_{\overline{F}}, \mathbf{Q}_\ell) \simeq (R^i f_* \mathbf{Q}_\ell)_{\overline{F}}$$

This leads us to study the system $(R^i f_* \mathbf{Q}_{\ell})_{\ell}$ of (lisse) \mathbf{Q}_{ℓ} -sheaves on B.

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Compatible systems

Let $\mathcal{O}_{\mathcal{K}}$ be an excellent Henselian discrete valuation ring of residue field $k = \mathbf{F}_q$ (no restriction on the characteristic of the fraction field \mathcal{K}). Let X be a scheme of finite type over $S = \text{Spec}(\mathcal{O}_{\mathcal{K}})$. Let $\mathcal{K}(X, \overline{\mathbf{Q}_{\ell}})$ be the Grothendieck group of $\overline{\mathbf{Q}_{\ell}}$ -sheaves on X. Fix ℓ_i , $i \in I$.

Definition

 $(L_i) \in \prod_i K(X, \overline{\mathbf{Q}_{\ell_i}})$ is compatible if for every $x \in |X|$, and every $F \in W(\bar{x}/x)$, tr $(F, (L_i)_{\bar{x}}) \in \mathbf{Q}$ and is independent of *i*. Here $|X| := |X_K| \cup |X_k|$ denotes the set of locally closed points of *X*.

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More general notion with fixed embeddings $Q \hookrightarrow \overline{\mathbf{Q}_{\ell_i}}$.

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Gabber's theorem

Theorem (Gabber, Z.)

Over S, compatible systems are preserved by duality and Grothendieck's six operations:

 $f^*, f_*, f_!, f^!, \otimes, R\mathcal{H}om.$

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Local fundamental groups

Let \overline{C} be a smooth curve over \mathbf{F}_q and let $C \subseteq \overline{C}$ be a Zariski dense open. For $x \in \overline{C} \setminus C$, we have $\text{Spec}(K_x) = \overline{C}_{(x)} \times_{\overline{C}} C \to C$, where $\overline{C}_{(x)}$ denotes the Henselization of \overline{C} at x. Short exact sequence:

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$$1 \rightarrow I_x \rightarrow \mathsf{Gal}(\overline{K_x}/K_x) \rightarrow \mathsf{Gal}(\bar{x}/x) \rightarrow 1.$$

More generally, let \bar{X} be a normal scheme of finite type over S and let $X \subseteq \bar{X}$ be a Zariski dense open. For $x \in \bar{X}$, the open immersion $\bar{X}_{(x)} \times_{\bar{X}} X \subseteq \bar{X}_{(x)}$ induces a surjection

$$\pi_1(\bar{X}_{(x)} imes_{\bar{X}} X) o \pi_1(\bar{X}_{(x)}) \simeq \operatorname{Gal}(\bar{x}/x).$$

Definition

 $(L_i) \in \prod_i \mathcal{K}_{\text{lisse}}(X, \overline{\mathbf{Q}}_{\ell_i})$ is compatible on \bar{X} if for every $x \in |\bar{X}|$, for every $F \in W(\bar{X}_{(x)} \times_{\bar{X}} X, \bar{a})$ (where \bar{a} is a geometric point), $\text{tr}(F, (L_i)_{\bar{a}}) \in \mathbf{Q}$ and is independent of i.

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Question

Assume $(L_i) \in \prod_i K_{\text{lisse}}(X, \overline{\mathbf{Q}_{\ell_i}})$ compatible on X. Is (L_i) compatible on \overline{X} ?

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Assume $(L_i) \in \prod_i K_{\text{lisse}}(X, \overline{\mathbf{Q}_{\ell_i}})$ compatible on X. Is (L_i) compatible on \overline{X} ?

Yes up to stratification or modification.

Compatible \Rightarrow Compatible along the boundary up to ...

Theorem (Lu-Z.)

Let X be a scheme of finite type over S and let $(L_i) \in \prod_{i \in I} K_{\text{lisse}}(X, \overline{\mathbf{Q}_{\ell_i}})$ compatible with I finite. There exists a partition $X = \bigcup_{\alpha} X_{\alpha}$ into locally closed subschemes such that each X_{α} admits a normal compactification $X_{\alpha} \subseteq \overline{X}_{\alpha}$ over S with $(L_i|_{X_{\alpha}})$ compatible on \overline{X}_{α} .

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Theorem (Lu-Z.)

Let \overline{X} be a reduced scheme separated of finite type over S and let $X \subseteq \overline{X}$ be a Zariski dense open. Let $(L_i) \in \prod_{i \in I} K_{\text{lisse}}(X, \overline{\mathbf{Q}_{\ell_i}})$ compatible with I finite. There exists a proper birational transformation $f : \overline{Y} \to \overline{X}$ such that $(L_i|_{f^{-1}(X)})$ is compatible on \overline{Y} .

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Theorem (Lu-Z.)

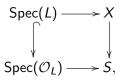
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Due to Deligne in the case where X is a curve over \mathbf{F}_q .

Valuative criterion

Corollary

Let X be a scheme of finite type over S and let $(L_i) \in \prod_{i \in I} K(X, \overline{\mathbf{Q}_{\ell_i}})$. Consider commutative squares



where \mathcal{O}_L is a Henselian valuation ring and $L = \operatorname{Frac}(\mathcal{O}_L)$.

• $(L_i)_{i \in I}$ compatible \Leftrightarrow for every square with closed point of Spec (\mathcal{O}_L) quasi-finite over S, tr $(F, (L_i)_{\overline{L}}) \in \mathbf{Q}$ and is independent of ℓ for all $F \in W(\overline{L}/L)$.

② $(L_i)_{i \in I}$ compatible ⇒ for every square with \mathcal{O}_L strictly Henselian, tr $(F, (L_i)_{\bar{L}}) \in \mathbf{Q}$ and is independent of ℓ for all $F \in \text{Gal}(\bar{L}/L)$.

Serre's conjectures in equal characteristic

Let \mathcal{O}_L be a Henselian (not necessarily discrete) valuation field ring of residue field k and characteristic p > 0. Let $L = \operatorname{Frac}(\mathcal{O}_L)$. Let X be a proper smooth variety over L.

Corollary

- (C4) For each $F \in I_L$, det $(1 Ft, H_{\ell}^i) \in \mathbb{Z}[t]$ and is independent of $\ell \neq p$.
- (C8) Assume $k = \mathbf{F}_q$. For each lifting $F \in \text{Gal}(\overline{L}/L)$ of Fr, $\det(1 Ft, H^i_{\ell}) \in \mathbf{Z}[t]$ and is independent of $\ell \neq p$.

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The valuative criterion was inspired by Gabber's valuative criterion for the ramified part of π_1 .

Plan of the talk



2 Compatible systems along the boundary



Relation with wild ramification

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Ramified part of π_1

Let $\mathcal{O}_{\mathcal{K}}$ be an encellent Henselian discrete valuation ring of residue characteristic p > 0.

Definition (Vidal)

Let X be a integral normal scheme separated of finite type over

- $S = \text{Spec}(\mathcal{O}_{\mathcal{K}})$. Closed subsets $\pi_1^{\text{wr}}(X) \subseteq \pi_1^{\text{r}}(X) \subseteq \pi_1(X)$:

 - (ramified part) $\pi_1^r(X) = \bigcap_{\bar{X}} \pi_1^r(X)_{\bar{X}}$.

Ramified part of π_1

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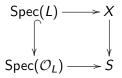
- (ramified part) $\pi_1^r(X) = \bigcap_{\bar{X}} \pi_1^r(X)_{\bar{X}}$.
- (wildly ramified part) $\pi_1^{wr}(X) = \pi_1^r(X) \cap \bigcup_H H$, where H runs through pro-p-Sylows of $\pi_1(X)$.

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Gabber's valuative criterion

Theorem (Gabber)

 $\pi_1^r(X)$ is the closure of the union of the conjugates of $\operatorname{Im}(\operatorname{Gal}(\overline{L}/L) \to \pi_1(X))$, indexed by commutative squares



where \mathcal{O}_L is a strictly Henselian valuation ring.

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Compatible wild ramification

Let X be a scheme of finite type over S.

Definition

 $(L_i) \in \prod_{i \in I} \mathcal{K}(X, \overline{\mathbf{F}}_{\ell_i})$ has compatible wild ramification if for every separated integral normal subscheme Y and every $g \in \pi_1^{\mathrm{wr}}(Y, \bar{a})$ (where \bar{a} is a geometric point), $\mathrm{tr}^{\mathrm{Br}}(g, (L_i)_{\bar{a}}) \in \mathbf{Q}$ and is independent of ℓ (as long as $L_i \in \mathcal{K}_{\mathrm{lisse}}$).

Saito-Yatagawa and Yatagawa studied a weaker condition "same wild ramification".

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Saito-Yatagawa and Yatagawa studied a weaker condition "same wild ramification" .

Theorem (Deligne, Vidal, Saito-Yatagawa, Yatagawa, Guo)

- "Compatible wild ramification" is preserved by $f^*, f_*, f_!, f^!, \otimes, R\mathcal{H}om$.
- "Same wild ramification" is preserved by $f^*, f_*, f_!, f^!$.

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Compatible \Rightarrow Compatible wild ramification

Assume that the residue field of \mathcal{O}_K is finite. The decomposition map d_ℓ is the composition

$$\mathcal{K}(X,\overline{\mathbf{Q}_{\ell}}) \xleftarrow{\sim} \mathcal{K}(X,\overline{\mathbf{Z}_{\ell}}) o \mathcal{K}(X,\overline{\mathbf{F}_{\ell}}),$$

where both arrows are given by extension of scalars. Combining Gabber's valuative criterion with ours, we get:

Corollary

 $(L_i) \in \prod_i K(X, \overline{\mathbf{Q}_{\ell_i}})$ compatible $\Rightarrow (d_{\ell_i}L_i) \in \prod_i K(X, \overline{\mathbf{F}_{\ell_i}})$ has compatible wild ramification.



Thank you!

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Compatible systems along the boundary

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