Compatible systems along the boundary

Weizhe Zheng

Morningside Center of Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences

June 15, 2018

Plan of the talk

1. Serre’s conjectures on \( \ell \)-independence

2. Compatible systems along the boundary

3. Relation with wild ramification
Serre proposed conjectures C1–C8 related to the definition of the Hasse-Weil zeta functions of projective smooth varieties over global fields.

References

Arithmetic zeta function

- Riemann zeta function:

\[ \zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}}. \]
Arithmetic zeta function

- Riemann zeta function:
  \[
  \zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}}.
  \]

- Let \( \mathcal{X} \) be a scheme of finite type over \( \text{Spec}(\mathbb{Z}) \).
  Arithmetic zeta function:
  \[
  \zeta_{\mathcal{X}}(s) = \sum_{C \in Z^0_{\text{eff}}(\mathcal{X})} \frac{1}{(NC)^s} = \prod_{x \in |\mathcal{X}|} \frac{1}{1 - (Nx)^{-s}}
  = \prod_{v \in |V|} Z_{\mathcal{X}_v}((Nv)^{-s})
  \]
  for \( \mathcal{X} \) over \( V \) of finite type over \( \text{Spec}(\mathbb{Z}) \).
Let $X$ be a variety (≡ scheme separated of finite type) over a field $k$. For each $\ell \neq \text{char}(k)$, Grothendieck defined a finite-dimensional $\mathbb{Q}_\ell$-vector space $H^i_{\ell,c} = H^i_c(X_{\bar{k}}, \mathbb{Q}_\ell)$, equipped with a continuous action of $\text{Gal}(\bar{k}/k)$. 

Theorem (Grothendieck) Let $X$ be a variety over $k = \mathbb{F}_q$. For each $\ell \nmid q$, $Z_X(t) = \prod P_{i,\ell}(t)^{(-1)^i+1}$, where $P_{i,\ell}(t) = \det(1 - F_{\ell}t, H^i_{\ell,c})$. 

Weizhe Zheng 
Compatible systems along the boundary June 15, 2018 6 / 30
Let $X$ be a variety (= scheme separated of finite type) over a field $k$. For each $\ell \neq \text{char}(k)$, Grothendieck defined a finite-dimensional $\mathbb{Q}_\ell$-vector space $H^i_{\ell,c} = H^i_c(X_{\bar{k}}, \mathbb{Q}_\ell)$, equipped with a continuous action of $\text{Gal}(\bar{k}/k)$.

**Theorem (Grothendieck)**

Let $X$ be a variety over $k = \mathbb{F}_q$. For each $\ell \nmid q$,

$$Z_X(t) = \prod_i P_{i,\ell}(t)^{(-1)^i+1},$$

where

$$P_{i,\ell}(t) = \det(1 - \text{Fr}_t, H^i_{\ell,c}).$$
Let $X$ be a proper smooth variety over $k = \mathbf{F}_q$.

**Theorem (Deligne, C2)**

*The reciprocal roots of $P_{i,\ell}$ are of weight $i$ (algebraic numbers with all complex conjugates of absolute value $q^{i/2}$).*
Let $X$ be a proper smooth variety over $k = \mathbb{F}_q$.

**Theorem (Deligne, C2)**

The reciprocal roots of $P_{i,\ell}$ are of weight $i$ (algebraic numbers with all complex conjugates of absolute value $q^{i/2}$).

**Corollary (C1)**

$P_{i,\ell} \in \mathbb{Z}[t]$ and is independent of $\ell$. 
Weil conjectures (continued)

Let $X$ be a proper smooth variety over $k = \mathbb{F}_q$.

Theorem (Deligne, C2)

The reciprocal roots of $P_{i,\ell}$ are of weight $i$ (algebraic numbers with all complex conjugates of absolute value $q^{i/2}$).

Corollary (C1)

$P_{i,\ell} \in \mathbb{Z}[t]$ and is independent of $\ell$.

Corollary

Let $X$ be a proper smooth variety over an arbitrary field $k$. Then the Betti number $\dim H^i(X_{\overline{k}}, \mathbb{Q}_\ell)$ is independent of $\ell \neq \text{char}(k)$. 
Let $X$ be a proper smooth variety over a global field $F$.

$$\zeta_X(s) = \prod_i L_i(s)^{(-1)^{i+1}},$$

$$L_i(s) = \prod_v \det(1 - \text{Fr}_q^{-s}, (H^i_{\ell})^{I_v}),$$

where $v$ runs over finite places of $F$, and $I_v$ denotes the inertia group at $v$. 

"Hasse-Weil zeta function"
$\ell$-independence

Let $K$ be a local field: a complete discrete valuation field of finite residue field $F_q$. Let $X$ be a proper smooth variety over $K$.

Conjecture

- $(\text{Serre}, \text{C5}) \det(1 - Frt, (H^i_l)^{lK}) \in \mathbb{Z}[t]$ and is independent of $\ell \nmid q$. 

Weizhe Zheng
Compatible systems along the boundary
June 15, 2018
Let $K$ be a local field: a complete discrete valuation field of finite residue field $\mathbb{F}_q$. Let $X$ be a proper smooth variety over $K$.

**Conjecture**

- *(Serre, C5)* $\det(1 - Fr_t, (H^i_{\ell})^I_K) \in \mathbb{Z}[t]$ and is independent of $\ell \nmid q$.
- *(Serre-Tate, C8)* For each lifting $F \in \text{Gal}(\bar{K}/K)$ of $Fr$, $\det(1 - Ft, H^i_{\ell}) \in \mathbb{Z}[t]$ and is independent of $\ell \nmid q$. 
Monodromy Weight Conjecture

Let $M$ denote the monodromy filtration.

**Conjecture**

Eigenvalues of $F$ lifting $\text{Fr}$ on $\text{gr}^M_n H^i_\ell$ are of weight $i + n$. 
Monodromy Weight Conjecture

Let $M$ denote the monodromy filtration.

**Conjecture**

*Eigenvalues of $F$ lifting $Fr$ on $\text{gr}_n^M H^i_\ell$ are of weight $i + n$.***

C8 + Monodromy Weight Conjecture

$\Rightarrow \det(1 - Ft, \text{gr}_n^M H^i_\ell) \in \mathbb{Z}[t]$ and is independent of $\ell$

$\Rightarrow$ C5
Monodromy Weight Conjecture

Let $M$ denote the monodromy filtration.

Conjecture

*Eigenvalues of $F$ lifting $\text{Fr}$ on $\text{gr}_n^M H^i_\ell$ are of weight $i + n$.*

C8 + Monodromy Weight Conjecture

$\Rightarrow \det(1 - Ft, \text{gr}_n^M H^i_\ell) \in \mathbb{Z}[t]$ and is independent of $\ell$

$\Rightarrow$ C5

(Monodromy Weight Conjecture $\Rightarrow$ C6 + C7)
Let $K$ be a complete discrete valuation field of residue field $k$. Let $X$ be a proper smooth variety over $K$.

Conjecture (Serre-Tate, C4)

For each $F \in I_K$, $\det(1 - Ft, H^i_\ell) \in \mathbb{Z}[t]$ and is independent of $\ell \neq \text{char}(k)$. 
Local monodromy theorem

Let $X$ be a variety over $K$.

Theorem

- (Grothendieck) An open subgroup of $I_K$ acts on $H^i_{\ell,c}$ unipotently.
Local monodromy theorem

Let $X$ be a variety over $K$.

**Theorem**

- **(Grothendieck)** An open subgroup of $I_K$ acts on $H_{\ell,c}^i$ unipotently.
- **(Deligne, Gabber, Illusie)** There exists an open subgroup $I'$ of $I_K$, independent of $\ell$, such that for every $g \in I'$, $(g - 1)^{i+1}$ acts by 0 on $H_{\ell}^i$ and $H_{\ell,c}^i$. 
Equal characteristic case

Theorem (Deligne, Terasoma, Ito)

Monodromy Weight Conjecture holds in equal characteristic.
Let $K$ be a complete discrete valuation field of residue field $k$, both of characteristic $p > 0$. Let $X$ be a proper smooth variety over $K$.

**Theorem**

- (Lu-Z., C4) For each $F \in I_K$, $\det(1 - Ft, H^i_{\ell}) \in \mathbb{Z}[t]$ and is independent of $\ell \neq p$.

- (Deligne, Terasoma, Lu-Z., C8) Assume $k = \mathbb{F}_q$. For each lifting $F \in \text{Gal}(\bar{K}/K)$ of Fr, $\det(1 - Ft, H^i_{\ell}) \in \mathbb{Z}[t]$ and is independent of $\ell \neq p$. 
Let $K$ be a complete discrete valuation field of residue field $k$, both of characteristic $p > 0$. Let $X$ be a proper smooth variety over $K$.

**Theorem**

- *(Lu-Z., C4)* For each $F \in I_K$, $\det(1 - Ft, H_i^\ell) \in \mathbb{Z}[t]$ and is independent of $\ell \neq p$.
- *(Deligne, Terasoma, Lu-Z., C8)* Assume $k = \mathbb{F}_q$. For each lifting $F \in \text{Gal}(\overline{K}/K)$ of $\text{Fr}$, $\det(1 - Ft, H_i^\ell) \in \mathbb{Z}[t]$ and is independent of $\ell \neq p$.

**Corollary (C5)**

Assume $k = \mathbb{F}_q$. For each lifting $F \in \text{Gal}(\overline{K}/K)$ of $\text{Fr}$, $\det(1 - Ft, (H_i^\ell)^{I_K}) \in \mathbb{Z}[t]$ and is independent of $\ell \nmid q$. 
Let $X$ be a variety over a field $K$.

**Theorem**

- **(Gabber, C1')** Assume $K = \mathbb{F}_q$. For each $F \in W(\bar{K}/K)$, 
  \[ \sum_i (-1)^i \text{tr}(F, H^i_\ell) \in \mathbb{Q} \text{ and is independent of } \ell \nmid q. \]
General characteristic: alternating sums

Let $X$ be a variety over a field $K$.

**Theorem**

- **(Gabber, C1')** Assume $K = \mathbb{F}_q$. For each $F \in W(\bar{K}/K)$, \( \sum_i (-1)^i \text{tr}(F, H^i_{\ell}) \in \mathbb{Q} \) and is independent of $\ell \nmid q$.

- **(Vidal, C4')** Assume $K$ is a complete discrete valuation field of residue characteristic $p > 0$. For each $F \in I_K$, \( \sum_i (-1)^i \text{tr}(F, H^i_{\ell}) \in \mathbb{Z} \) and is independent of $\ell \neq p$.

- **(Ochiai, Z., C8')** Assume $K$ is a local field of residue field $\mathbb{F}_q$. For each $F \in W(\bar{K}/K)$, \( \sum_i (-1)^i \text{tr}(F, H^i_{\ell}) \in \mathbb{Q} \) and is independent of $\ell \nmid q$. 

Weizhe Zheng

Compatible systems along the boundary

June 15, 2018 15 / 30
Plan of the talk

1. Serre’s conjectures on $\ell$-independence
2. Compatible systems along the boundary
3. Relation with wild ramification
Let $X$ be a proper smooth variety over a field $F$ of characteristic $p > 0$. There exists a scheme $B$ of finite type over $\mathbb{F}_p$ and a Cartesian square

\[
\begin{array}{ccc}
X & \longrightarrow & X' \\
\downarrow & & \downarrow f \\
\text{Spec}(F) & \longrightarrow & B
\end{array}
\]

with $f$ proper smooth. We have

\[
H^i(X_{\overline{\mathbb{F}}}, \mathbb{Q}_\ell) \simeq (R^i f_* \mathbb{Q}_\ell)_{\overline{\mathbb{F}}}
\]

This leads us to study the system $(R^i f_* \mathbb{Q}_\ell)_\ell$ of (lisse) $\mathbb{Q}_\ell$-sheaves on $B$. 
Compatible systems

Let $\mathcal{O}_K$ be an excellent Henselian discrete valuation ring of residue field $k = \mathbf{F}_q$ (no restriction on the characteristic of the fraction field $K$). Let $X$ be a scheme of finite type over $S = \text{Spec}(\mathcal{O}_K)$. Let $K(X, \overline{\mathbf{Q}}\ell)$ be the Grothendieck group of $\mathbf{Q}\ell$-sheaves on $X$. Fix $\ell_i$, $i \in I$.

**Definition**

$(L_i) \in \prod_i K(X, \overline{\mathbf{Q}}\ell_i)$ is *compatible* if for every $x \in |X|$, and every $F \in W(\overline{x}/x)$, $\text{tr}(F, (L_i)\overline{x}) \in \mathbf{Q}$ and is independent of $i$. Here $|X| := |X_K| \cup |X_k|$ denotes the set of locally closed points of $X$. 

Weizhe Zheng

Compatible systems along the boundary

June 15, 2018
Compatible systems

Let $\mathcal{O}_K$ be an excellent Henselian discrete valuation ring of residue field $k = \mathbb{F}_q$ (no restriction on the characteristic of the fraction field $K$). Let $X$ be a scheme of finite type over $S = \text{Spec}(\mathcal{O}_K)$. Let $K(X, \overline{\mathbb{Q}_\ell})$ be the Grothendieck group of $\mathbb{Q}_\ell$-sheaves on $X$. Fix $\ell_i, i \in I$.

**Definition**

$(L_i) \in \prod_i K(X, \overline{\mathbb{Q}_\ell_i})$ is compatible if for every $x \in |X|$, and every $F \in W(\overline{x}/x)$, $\text{tr}(F, (L_i)_{\overline{x}}) \in \mathbb{Q}$ and is independent of $i$. Here $|X| := |X_K| \cup |X_k|$ denotes the set of locally closed points of $X$.

More general notion with fixed embeddings $Q \hookrightarrow \overline{\mathbb{Q}_\ell_i}$. 

Weizhe Zheng
Compatible systems along the boundary
June 15, 2018 18 / 30
Gabber’s theorem

Theorem (Gabber, Z.)

Over $S$, compatible systems are preserved by duality and Grothendieck’s six operations:

$$f^*, f_*, f!, f^!, \otimes, R\mathcal{H}om.$$
Local fundamental groups

Let $\bar{C}$ be a smooth curve over $\mathbb{F}_q$ and let $C \subseteq \bar{C}$ be a Zariski dense open. For $x \in \bar{C} \setminus C$, we have $\text{Spec}(K_x) = \bar{C}_x \times \bar{C} \to C$, where $\bar{C}_x$ denotes the Henselization of $\bar{C}$ at $x$. Short exact sequence:

$$1 \to I_x \to \text{Gal}(\overline{K_x}/K_x) \to \text{Gal}(\bar{x}/x) \to 1.$$
Local fundamental groups

Let \( \bar{C} \) be a smooth curve over \( \mathbb{F}_q \) and let \( C \subseteq \bar{C} \) be a Zariski dense open. For \( x \in \bar{C} \setminus C \), we have \( \text{Spec}(K_x) = \bar{C}_x \times \bar{C} \to C \), where \( \bar{C}_x \) denotes the Henselization of \( \bar{C} \) at \( x \). Short exact sequence:

\[
1 \to I_x \to \text{Gal}(\bar{K}_x/K_x) \to \text{Gal}(\bar{x}/x) \to 1.
\]

More generally, let \( \bar{X} \) be a normal scheme of finite type over \( S \) and let \( X \subseteq \bar{X} \) be a Zariski dense open. For \( x \in \bar{X} \), the open immersion \( \bar{X}_x \times \bar{X} \subseteq \bar{X}_x \) induces a surjection

\[
\pi_1(\bar{X}_x \times \bar{X}) \to \pi_1(\bar{X}_x) \simeq \text{Gal}(\bar{x}/x).
\]
Compatible systems along the boundary

Definition

\((L_i) \in \prod_i K_{\text{lisse}}(X, \overline{\mathbb{Q}_{\ell_i}})\) is compatible on \(\tilde{X}\) if for every \(x \in |\tilde{X}|\), for every \(F \in W(\tilde{X}(x) \times \tilde{X}, \tilde{a})\) (where \(\tilde{a}\) is a geometric point), \(\text{tr}(F, (L_i)_{\tilde{a}}) \in \mathbb{Q}\) and is independent of \(i\).
Definition

\((L_i) \in \prod_i K_{\text{lisse}}(X, \overline{Q_{\ell_i}})\) is compatible on \(\bar{X}\) if for every \(x \in |\bar{X}|\), for every \(F \in W(\bar{X}_x \times \bar{X}, \bar{a})\) (where \(\bar{a}\) is a geometric point), \(\text{tr}(F, (L_i)_{\bar{a}}) \in \mathbb{Q}\) and is independent of \(i\).

Question

Assume \((L_i) \in \prod_i K_{\text{lisse}}(X, \overline{Q_{\ell_i}})\) compatible on \(X\). Is \((L_i)\) compatible on \(\bar{X}\)?
Definition

\((L_i) \in \prod_i K_{\text{lis}}(X, Q_{\ell_i})\) is compatible on \(\bar{X}\) if for every \(x \in |\bar{X}|\), for every \(F \in W(\bar{X}(x) \times \bar{X}, \bar{a})\) (where \(\bar{a}\) is a geometric point), \(\text{tr}(F, (L_i)_{\bar{a}}) \in Q\) and is independent of \(i\).

Question

Assume \((L_i) \in \prod_i K_{\text{lis}}(X, Q_{\ell_i})\) compatible on \(X\). Is \((L_i)\) compatible on \(\bar{X}\)?

Yes up to stratification or modification.
Theorem (Lu-Z.)

Let $X$ be a scheme of finite type over $S$ and let $(L_i) \in \prod_{i \in I} \mathcal{K}_{lisse}(X, \overline{Q}_\ell_i)$ compatible with $I$ finite. There exists a partition $X = \bigcup_{\alpha} X_\alpha$ into locally closed subschemes such that each $X_\alpha$ admits a normal compactification $X_\alpha \subseteq \tilde{X}_\alpha$ over $S$ with $(L_i|_{X_\alpha})$ compatible on $\tilde{X}_\alpha$. 

Due to Deligne in the case where $X$ is a curve over $F_q$. 

Weizhe Zheng

Compatible systems along the boundary

June 15, 2018 22 / 30
Compatible systems along the boundary

Compatible $\Rightarrow$ Compatible along the boundary up to ...

**Theorem (Lu-Z.)**

Let $X$ be a scheme of finite type over $S$ and let $(L_i) \in \prod_{i \in I} K_{\text{lisse}}(X, \overline{Q}_{\ell_i})$ compatible with $I$ finite. There exists a partition $X = \bigcup_{\alpha} X_{\alpha}$ into locally closed subschemes such that each $X_{\alpha}$ admits a normal compactification $X_{\alpha} \subseteq \bar{X}_{\alpha}$ over $S$ with $(L_i|_{X_{\alpha}})$ compatible on $\bar{X}_{\alpha}$.

**Theorem (Lu-Z.)**

Let $\bar{X}$ be a reduced scheme separated of finite type over $S$ and let $X \subseteq \bar{X}$ be a Zariski dense open. Let $(L_i) \in \prod_{i \in I} K_{\text{lisse}}(X, \overline{Q}_{\ell_i})$ compatible with $I$ finite. There exists a proper birational transformation $f: \bar{Y} \rightarrow \bar{X}$ such that $(L_i|_{f^{-1}(X)})$ is compatible on $\bar{Y}$.

Due to Deligne in the case where $X$ is a curve over $\mathbb{F}_q$.
Theorem (Lu-Z.)

Let $X$ be a scheme of finite type over $S$ and let $(L_i) \in \prod_{i \in I} K_{\text{lis}e}(X, \overline{Q_{\ell_i}})$ compatible with $I$ finite. There exists a partition $X = \bigcup_{\alpha} X_\alpha$ into locally closed subschemes such that each $X_\alpha$ admits a normal compactification $X_\alpha \subseteq \overline{X}_\alpha$ over $S$ with $(L_i|_{X_\alpha})$ compatible on $\overline{X}_\alpha$.

Theorem (Lu-Z.)

Let $\overline{X}$ be a reduced scheme separated of finite type over $S$ and let $X \subseteq \overline{X}$ be a Zariski dense open. Let $(L_i) \in \prod_{i \in I} K_{\text{lis}e}(X, \overline{Q_{\ell_i}})$ compatible with $I$ finite. There exists a proper birational transformation $f : \overline{Y} \to \overline{X}$ such that $(L_i|_{f^{-1}(X)})$ is compatible on $\overline{Y}$.

Due to Deligne in the case where $X$ is a curve over $\mathbb{F}_q$. 

Compatible $\Rightarrow$ Compatible along the boundary up to ...
Corollary

Let $X$ be a scheme of finite type over $S$ and let $(L_i) \in \prod_{i \in I} K(X, \overline{Q}_{\ell_i})$. Consider commutative squares

$$
\begin{array}{ccc}
\text{Spec}(L) & \longrightarrow & X \\
& \downarrow & \\
\text{Spec}(\mathcal{O}_L) & \longrightarrow & S,
\end{array}
$$

where $\mathcal{O}_L$ is a Henselian valuation ring and $L = \text{Frac}(\mathcal{O}_L)$.

1. $(L_i)_{i \in I}$ compatible $\iff$ for every square with closed point of $\text{Spec}(\mathcal{O}_L)$ quasi-finite over $S$, $\text{tr}(F, (L_i)_{\overline{L}}) \in \mathbb{Q}$ and is independent of $\ell$ for all $F \in W(\overline{L}/L)$.

2. $(L_i)_{i \in I}$ compatible $\Rightarrow$ for every square with $\mathcal{O}_L$ strictly Henselian, $\text{tr}(F, (L_i)_{\overline{L}}) \in \mathbb{Q}$ and is independent of $\ell$ for all $F \in \text{Gal}(\overline{L}/L)$. 
Serre’s conjectures in equal characteristic

Let $\mathcal{O}_L$ be a Henselian (not necessarily discrete) valuation field ring of residue field $k$ and characteristic $p > 0$. Let $L = \text{Frac}(\mathcal{O}_L)$. Let $X$ be a proper smooth variety over $L$.

**Corollary**

- (C4) For each $F \in I_L$, $\det(1 - Ft, H^i_{\ell}) \in \mathbb{Z}[t]$ and is independent of $\ell \neq p$.
- (C8) Assume $k = \mathbb{F}_q$. For each lifting $F \in \text{Gal}(\overline{L}/L)$ of $Fr$, $\det(1 - Ft, H^i_{\ell}) \in \mathbb{Z}[t]$ and is independent of $\ell \neq p$. 
Let $\mathcal{O}_L$ be a Henselian (not necessarily discrete) valuation field ring of residue field $k$ and characteristic $p > 0$. Let $L = \text{Frac}(\mathcal{O}_L)$. Let $X$ be a proper smooth variety over $L$.

**Corollary**

- (C4) For each $F \in \mathcal{I}_L$, $\det(1 - Ft, H^i_\ell) \in \mathbb{Z}[t]$ and is independent of $\ell \neq p$.
- (C8) Assume $k = \mathbb{F}_q$. For each lifting $F \in \text{Gal}(\overline{L}/L)$ of Fr, $\det(1 - Ft, H^i_\ell) \in \mathbb{Z}[t]$ and is independent of $\ell \neq p$.

The valuative criterion was inspired by Gabber’s valuative criterion for the ramified part of $\pi_1$.
Plan of the talk

1. Serre’s conjectures on $\ell$-independence

2. Compatible systems along the boundary

3. Relation with wild ramification
Ramified part of $\pi_1$

Let $\mathcal{O}_K$ be an excellent Henselian discrete valuation ring of residue characteristic $p > 0$.

Definition (Vidal)

Let $X$ be an integral normal scheme separated of finite type over $S = \text{Spec}(\mathcal{O}_K)$. Closed subsets $\pi_1^{\text{wr}}(X) \subseteq \pi_1^r(X) \subseteq \pi_1(X)$:

- For any normal compactification $X \subseteq \bar{X}$ over $S$, $\pi_1^r(X)_{\bar{X}}$ is the closure of the union of the conjugates of $\text{Im}(\pi_1(\bar{X}(\bar{x}) \times_{\bar{X}} X) \to \pi_1(X))$, where $\bar{x}$ runs through geometric points of $\bar{X}$.

- (ramified part) $\pi_1^r(X) = \bigcap_{\bar{X}} \pi_1^r(X)_{\bar{X}}$. 

Weizhe Zheng
Relation with wild ramification

Ramified part of $\pi_1$

Let $O_K$ be an excellent Henselian discrete valuation ring of residue characteristic $p > 0$.

Definition (Vidal)

Let $X$ be an integral normal scheme separated of finite type over $S = \text{Spec}(O_K)$. Closed subsets $\pi^{wr}_1(X) \subseteq \pi^r_1(X) \subseteq \pi_1(X)$:

- For any normal compactification $X \subseteq \bar{X}$ over $S$, $\pi^r_1(X)_{\bar{X}}$ is the closure of the union of the conjugates of $\text{Im}(\pi_1(\bar{X}(\bar{x}) \times_{\bar{X}} X) \to \pi_1(X))$, where $\bar{x}$ runs through geometric points of $\bar{X}$.

- (ramified part) $\pi^r_1(X) = \bigcap_{\bar{X}} \pi^r_1(X)_{\bar{X}}$.

- (wildly ramified part) $\pi^{wr}_1(X) = \pi^r_1(X) \cap \bigcup_H H$, where $H$ runs through pro-$p$-Sylows of $\pi_1(X)$. 
Gabber’s valuative criterion

Theorem (Gabber)

\[ \pi_1^r(X) \text{ is the closure of the union of the conjugates of } \text{Im}(\text{Gal}(\bar{L}/L) \rightarrow \pi_1(X)), \text{ indexed by commutative squares} \]

\[
\begin{array}{ccc}
\text{Spec}(L) & \rightarrow & X \\
\downarrow & & \downarrow \\
\text{Spec}(\mathcal{O}_L) & \rightarrow & S
\end{array}
\]

where \( \mathcal{O}_L \) is a strictly Henselian valuation ring.
Compatible wild ramification

Let $X$ be a scheme of finite type over $S$.

**Definition**

$(L_i) \in \prod_{i \in I} K(X, \overline{F_{\ell_i}})$ has compatible wild ramification if for every separated integral normal subscheme $Y$ and every $g \in \pi_1^{wr}(Y, \overline{a})$ (where $\overline{a}$ is a geometric point), $\text{tr}^{Br}(g, (L_i)_{\overline{a}}) \in \mathbb{Q}$ and is independent of $\ell$ (as long as $L_i \in K_{\text{lis}}$).

Saito-Yatagawa and Yatagawa studied a weaker condition “same wild ramification”.

Weizhe Zheng
Compatible systems along the boundary
June 15, 2018 28 / 30
Compatible wild ramification

Let $X$ be a scheme of finite type over $S$.

**Definition**

$(L_i) \in \prod_{i \in I} K(X, \overline{F_{\ell_i}})$ has **compatible wild ramification** if for every separated integral normal subscheme $Y$ and every $g \in \pi_1^{\text{wr}}(Y, \overline{a})$ (where $\overline{a}$ is a geometric point), $\text{tr}^\text{Br}(g, (L_i)_{\overline{a}}) \in \mathbb{Q}$ and is independent of $\ell$ (as long as $L_i \in K_{\text{lisse}}$).

Saito-Yatagawa and Yatagawa studied a weaker condition “same wild ramification”.

**Theorem (Deligne, Vidal, Saito-Yatagawa, Yatagawa, Guo)**

- “Compatible wild ramification” is preserved by $f^*, f_*, f_!, f^!$, $\otimes$, $R\mathcal{H}\text{om}$.
- “Same wild ramification” is preserved by $f^*, f_*, f_!, f^!$. 
Assume that the residue field of $O_K$ is finite. The decomposition map $d_\ell$ is the composition
\[ K(X, \overline{Q_\ell}) \leftarrow K(X, \overline{Z_\ell}) \rightarrow K(X, \overline{F_\ell}), \]
where both arrows are given by extension of scalars. Combining Gabber’s valuative criterion with ours, we get:

**Corollary**

\[(L_i) \in \prod_i K(X, \overline{Q_{\ell_i}}) \text{ compatible } \Rightarrow (d_{\ell_i}L_i) \in \prod_i K(X, \overline{F_{\ell_i}}) \text{ has compatible wild ramification.}\]
Thank you!