Shintani Generating Class and the p-adic Polylogarithm for Totally Real Fields

Kenichi Bannai

Keio University/RIKEN

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Joint work with Kei Hagihara, Kazuki Yamada, and Shuji Yamamoto.
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Simple Question

- Rational Field $\mathbb{Q}$

		logarithmic derivative	Generating Function
	$1-t$	$\overset{\partial}{\mapsto}$	$\dfrac{t}{1-t}$
	Cyclotomic Units		Dirichlet L-values

- Imaginary Quadratic Field

Robert's Theta Function	$\theta(t)$	$\overset{\partial}{\mapsto}$	$\dfrac{\theta'(t)}{\theta(t)}$
	Elliptic Units		Hecke L-values

- Totally Real Field

			TODAY
	?	$\overset{\partial}{\mapsto}$	$\boxed{?}$
			Hecke L-values

Generating Function

Theorem (Classical)

Let $\mathcal{G}(t) := \dfrac{t}{1-t} \in \Gamma(\mathbb{G}_m, \mathscr{O}_{\mathbb{G}_m})$, and let $\partial := t\frac{d}{dt}$. Then we have

$$\partial^k \mathcal{G}(t)\big|_{t=\xi} = \mathcal{L}(\xi, -k)$$

for *any* integer $k \geq 0$ and *any* root of unity $\xi \neq 1$.

Here, $\mathcal{L}(\xi, s)$ is the *Lerch zeta function*, a function given by

$$\mathcal{L}(\xi, s) := \sum_{n=1}^{\infty} \xi(n) n^{-s}$$

for $\mathrm{Re}(s) > 1$, where $\xi(n) := \xi^n$. This function has an analytic continuation to $s \in \mathbb{C}$.

Dirichlet L-functions

Let $c_\chi(\xi) := N^{-1} \sum_{m \in \mathbb{Z}/N\mathbb{Z}} \chi(m)\xi(-m)$.

Lemma

For $N > 1$, let $\chi\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character. Then we have

$$L(\chi, s) = \sum_{\xi \in \mu_N} c_\chi(\xi) \mathcal{L}(\xi, s).$$

If χ is primitive, then sum is over primitive N-th roots of unity in μ_N.

Proof. Follows from the Fourier transform for finite characters $\mathrm{Re}(s) > 1$

$$L(\chi, s) = \sum_{n=1}^{\infty} \chi(n) n^{-s} = \sum_{n=1}^{\infty} \sum_{\xi \in \mu_N} c_\chi(\xi)\xi(n) n^{-s} = \sum_{\xi \in \mu_N} c_\chi(\xi) \sum_{n=1}^{\infty} \xi(n) n^{-s}$$

If χ is primitive, then sum is over primitive N-th roots of unity in μ_N. $\square$

Related Results

- Our Results Based On:
  - Shintani, Shintani zeta function and its generating function, Shintani decomposition.
  - Barsky, Cassou-Nougès, Construction of the p-adic Hecke L-function for totally real fields.
  - Katz, Use of the algebraic torus $\mathbb{T}^{\mathfrak{a}}$.
- Related Results:
  - Eisenstein Cocycles, Sczech's Cocycles, Shintani Cocycles... (Charollois–Dasgupta, Charollois–Dasgupta–Greenberg, ...)
  - Topological Polylogarithms (Beilinson–Levin–Kings)
  - Our Construction – Algebraic, arising from study of polylogarithm.

Lerch Zeta Function for Totally Real Fields

Finite Hecke Character

- F: totally real field, degree $g := [F : \mathbb{Q}]$, $\mathcal{O}_F$: ring of integers.
- $F_+^\times$: group of totally positive elements.
- $\mathfrak{I}$: group of nonzero fractional ideals of F.
  $\mathfrak{I}_\mathfrak{g}$: subgroup of nonzero fractional ideals prime to $\mathfrak{g}$, for any ideal $\mathfrak{g} \subset \mathcal{O}_F$.
- $\mathrm{Cl}_F^+(\mathfrak{g}) := \mathfrak{I}_\mathfrak{g}/P_\mathfrak{g}^+$, where $P_\mathfrak{g}^+ := \{(\alpha) \mid \alpha \in F_+^\times, \alpha \equiv 1 \bmod^\times \mathfrak{g}\}$.

Definition (Finite Hecke Character)

A finite Hecke character is a homomorphism $\chi\colon \mathrm{Cl}_F^+(\mathfrak{g}) \to \mathbb{C}^\times$.

Extend χ to a function on $\mathfrak{I}$ by *zero*. The Hecke L-function is defined by

$$L(\chi, s) = \sum_{\mathfrak{a} \subset \mathcal{O}_F} \chi(\mathfrak{a}) N\mathfrak{a}^{-s},$$

which converges absolutely for $\mathrm{Re}(s) > 1$.

Finite Hecke Character

Let

- $\mathrm{Cl}_F^+(1) := \mathfrak{I}/P^+$: *narrow ray class group*, where $P^+ := \{(\alpha) \mid \alpha \in F_+^\times\}$.
- $\mathfrak{a}_+$: subset of totally positive elements in $\mathfrak{a}$, for $\mathfrak{a} \in \mathfrak{I}$.
- $\Delta := \mathcal{O}_{F+}^\times$: *set of totally positive units* in $\mathcal{O}_F$.

Then we have

$$L(\chi, s) = \sum_{\mathfrak{a} \subset \mathcal{O}_F} \chi(\mathfrak{a}) N\mathfrak{a}^{-s} = \sum_{\mathfrak{a} \in \mathrm{Cl}_F^+(1)} \sum_{\alpha \in \Delta \backslash \mathfrak{a}_+} \underline{\chi(\mathfrak{a}^{-1}\alpha)} N(\mathfrak{a}^{-1}\alpha)^{-s}.$$

For any fractional ideal $\mathfrak{a}$ prime to $\mathfrak{g}$, we let

$$\chi_{\mathfrak{a}} : (\mathfrak{a}/\mathfrak{g}\mathfrak{a})^\times \to \mathbb{C}^\times, \qquad \chi_{\mathfrak{a}}(\alpha) := \underline{\chi(\mathfrak{a}^{-1}\alpha)} \quad \forall \alpha \in \mathfrak{a}_+,$$

where $(\mathfrak{a}/\mathfrak{a}\mathfrak{g})^\times \subset \mathfrak{a}/\mathfrak{a}\mathfrak{g}$ is the set of generators of $\mathfrak{a}/\mathfrak{a}\mathfrak{g}$ as a $\mathcal{O}_F/\mathfrak{g}$-module.

Lerch Zeta Function

Definition (Lerch Zeta Function)

Let $\xi \in \mathbb{T}^{\mathfrak{a}} := \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{a}, \mathbb{C}^{\times})$ be a torsion element. Define the *Lerch zeta function for the totally real case* by

$$\mathcal{L}(\xi\Delta, s) := \sum_{\alpha \in \Delta\backslash \mathfrak{a}_+} \xi\Delta(\alpha) N(\mathfrak{a}^{-1}\alpha)^{-s},$$

where $\xi\Delta := \sum_{\varepsilon \in \Delta_\xi \backslash \Delta} \xi^{\varepsilon}$ for $\Delta_\xi := \{\varepsilon \in \Delta \mid \xi^{\varepsilon} = \xi\}$. Here, $\xi^{\varepsilon}(\alpha) := \xi(\varepsilon\alpha)$ for any $\alpha \in \mathfrak{a}$.

Lemma

Let $\chi\colon \mathrm{Cl}_F^+(\mathfrak{g}) \to \mathbb{C}^{\times}$ be a finite Hecke character. Then we have

$$L(\chi, s) = \sum_{\mathfrak{a} \in \mathrm{Cl}_F^+(1)} \sum_{\xi \in \mathbb{T}^{\mathfrak{a}}[\mathfrak{g}]/\Delta} c_\chi(\xi)\mathcal{L}(\xi\Delta, s),$$

where $c_\chi(\xi) := \sum_{\beta \in \mathfrak{a}/\mathfrak{g}\mathfrak{a}} \chi_{\mathfrak{a}}(\beta)\xi(-\beta)$ for any $\xi \in \mathbb{T}^{\mathfrak{a}}[\mathfrak{g}] := \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{a}/\mathfrak{g}\mathfrak{a}, \mathbb{C}^{\times})$.

Shintani Zeta Function and the Generating Function

Cones in $\mathbb{R}_+^g$.

Let $I := \mathrm{Hom}(F, \mathbb{R}) = \{\tau_1, \ldots, \tau_g\}$. Then we have

$$F \otimes \mathbb{R} \cong \mathbb{R}^g, \qquad \alpha \otimes 1 \mapsto (\alpha^{\tau_i}).$$

We define a cone in $\mathbb{R}_+^g \cup \{0\}$ as follows.

Definition (Cone)

We define a g-dimensional F-rational simplicial closed polyhedral cone in $\mathbb{R}_+^g \cup \{0\}$, which we simply call a *cone*, to be the set of the form

$$\sigma_{\boldsymbol{\alpha}} := \{x_1\alpha_1 + \cdots + x_g\alpha_g \mid x_1, \ldots, x_g \in \mathbb{R}_{\geq 0}\}$$

for some $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_g) \in F_+^g$ linearly independent over $\mathbb{R}$. In this case, we say that $\boldsymbol{\alpha}$ is the generator of $\sigma_{\boldsymbol{\alpha}}$.

Shintani Zeta Function

Fix a fractional ideal $\mathfrak{a} \in \mathfrak{I}$.

Definition (Shintani Zeta Function)

Let σ be a cone, and let $\xi \neq 1$ be a torsion element in $\mathbb{T}^{\mathfrak{a}}(\mathbb{C}) = \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{a}, \mathbb{C}^{\times})$. The Shintani Zeta Function is given by

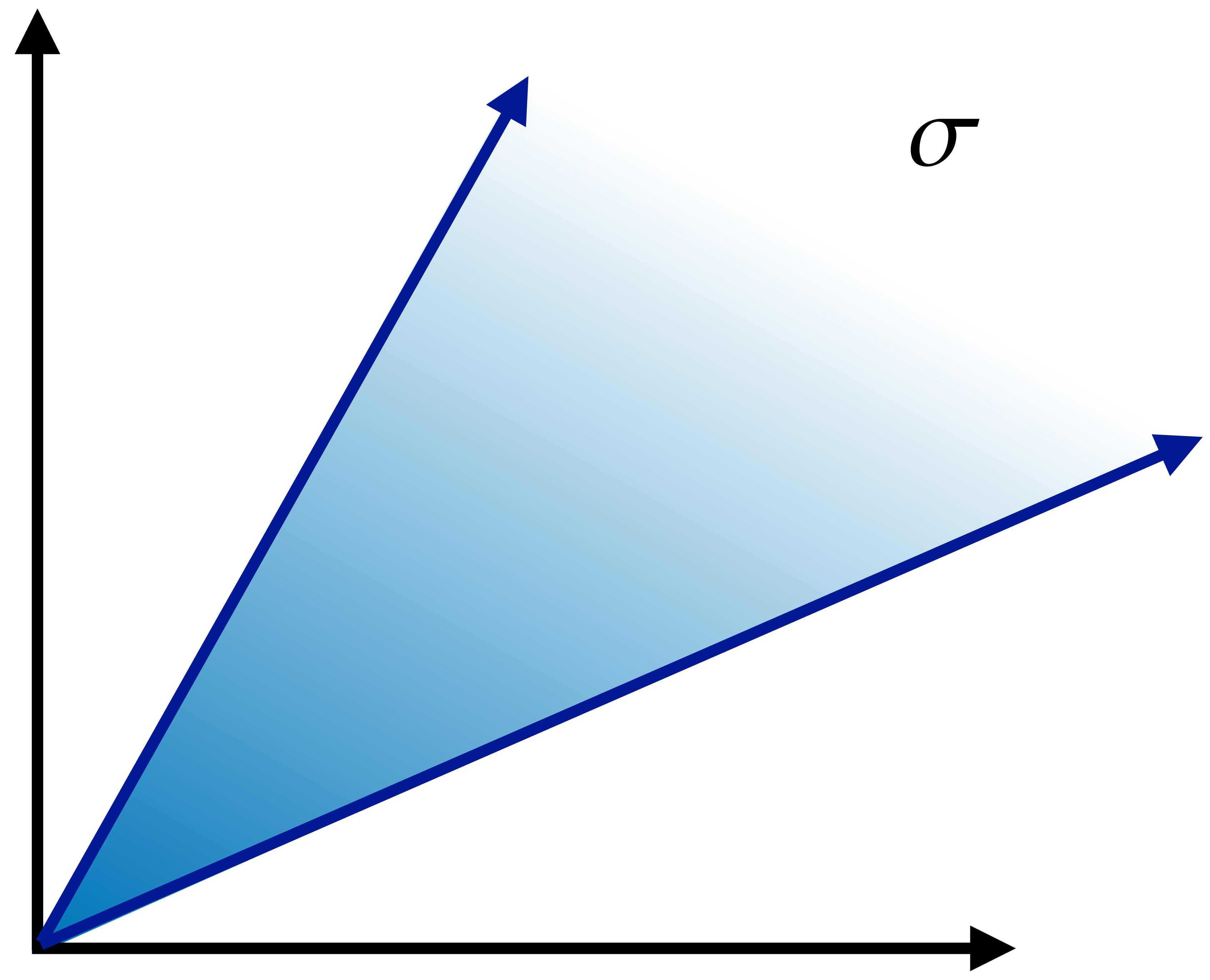
$$\zeta_{\sigma}(\xi, (s_1, \ldots, s_g)) := \sum_{\alpha \in \mathfrak{a} \cap \widehat{\sigma}} \xi(\alpha) \alpha_1^{-s_1} \cdots \alpha_g^{-s_g}.$$

This function has analytic continuation to $s_1, \ldots, s_g \in \mathbb{C}$.

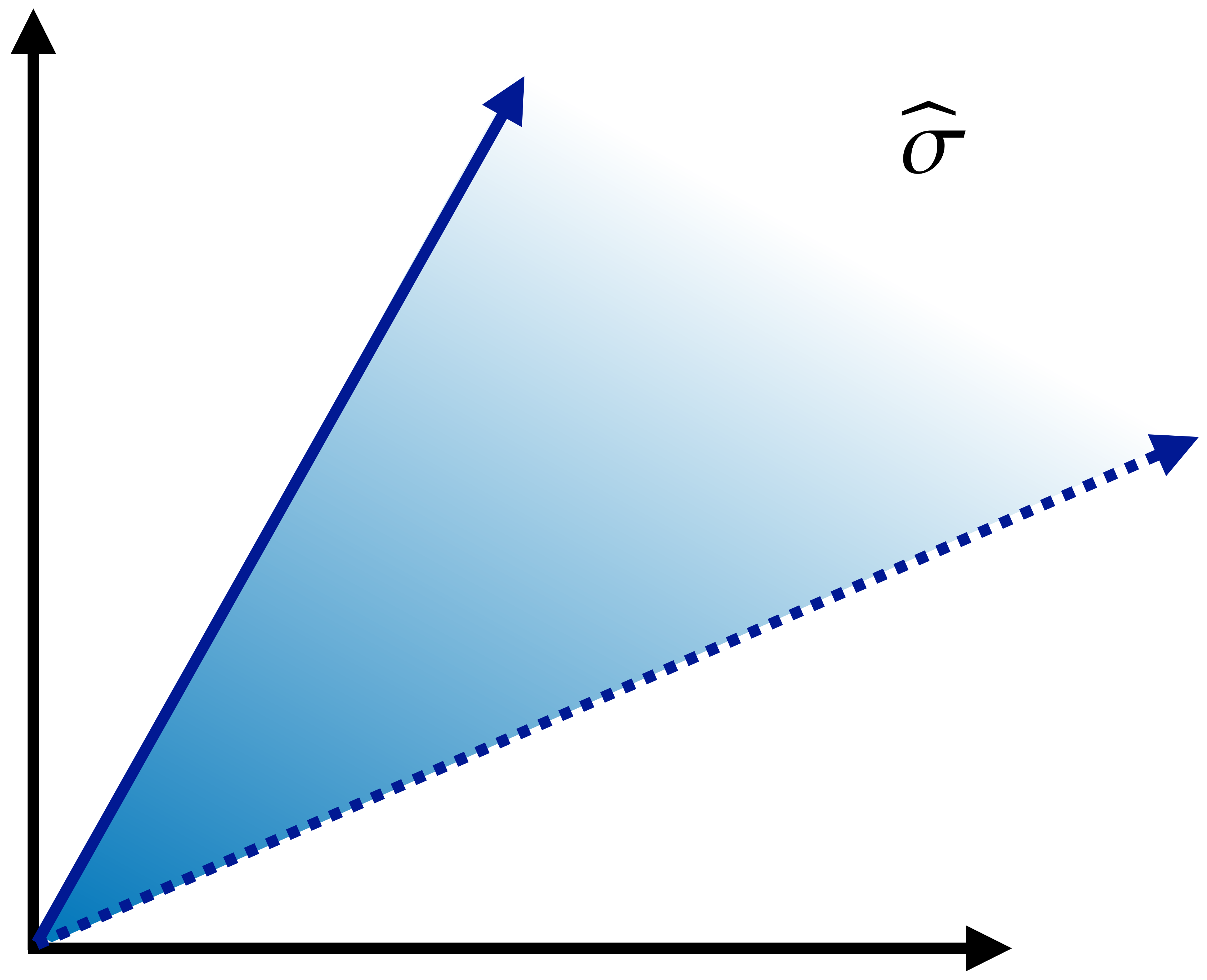
Here, $\widehat{\sigma}$ is the upper closure of σ given by

$$\widehat{\sigma} := \{u = (u_i) \in \mathbb{R}^g \mid \exists \delta > 0, 0 < \forall \delta' < \delta, (u_1, \ldots, u_{g-1}, u_g - \delta') \in \sigma\}.$$

Shintani Zeta Function



Upper Closure



$$\widehat{\sigma} := \{u = (u_i) \in \mathbb{R}^g \mid \exists \delta > 0, 0 < \forall \delta' < \delta, (u_1, \ldots, u_{g-1}, u_g - \delta') \in \sigma\}$$

Shintani Decomposition

The Shintani decomposition gives a choice of a fundamental domain of $\Delta\backslash\mathfrak{a}_+$.

Theorem (Shintani, Yamamoto [2, Proposition 5.6])

There exists a set Φ of g-dimensional cones stable under the action of Δ, such that $\Delta\backslash\Phi$ is a finite set, and

$$\mathbb{R}_+^g = \coprod_{\sigma\in\Phi} \widehat{\sigma}.$$

Using this decomposition, for any torsion $\xi \in \mathbb{T}^{\mathfrak{a}}(\mathbb{C})$ such that $\xi \neq 1$, we have the following.

$$\mathcal{L}(\xi\Delta, s) = \sum_{\alpha\in\Delta\backslash\mathfrak{a}_+} \xi\Delta(\alpha)N(\mathfrak{a}^{-1}\alpha)^{-s} = N\mathfrak{a}^s \sum_{\sigma\in\Delta\backslash\Phi}\sum_{\varepsilon\in\Delta/\Delta_\xi} \underline{\zeta_\sigma(\xi^\varepsilon, (s, \dots, s))}.$$

Shintani Zeta Function has a Generating Function

Underlying Geometry

- $F = \mathbb{Q}$

$$\mathbb{G}_m(\mathbb{C}) = \mathrm{Hom}(\mathbb{Z}, \mathbb{C}^\times)$$

$$\begin{array}{ccc} \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \xi & \mapsto & (n \mapsto \xi^n) \end{array}$$

$$\mathbb{G}_m = \mathrm{Hom}(\mathbb{Z}, \mathbb{G}_m)$$

$$= \mathrm{Spec}\,\mathbb{Z}[t, t^{-1}]$$

$$\mathcal{G}(t) = \frac{t}{1-t}$$

- F: Totally Real Field

$$\mathbb{T}^{\mathfrak{a}}(\mathbb{C}) = \mathrm{Hom}(\mathfrak{a}, \mathbb{C}^\times)$$

$$\mathbb{T}^{\mathfrak{a}} = \mathrm{Hom}(\mathfrak{a}, \mathbb{G}_m)$$

$$= \mathrm{Spec}\,\mathbb{Z}[t^\alpha \mid \alpha \in \mathfrak{a}]$$

Generating Function

We say that $\alpha \in \mathfrak{a}$ is *primitive*, if $\alpha/N \notin \mathfrak{a}$ for any integer $N > 0$. Let $\mathscr{A}_\mathfrak{a}$ be the set of primitive elements in $\mathfrak{a}$. For any $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_g) \in \mathscr{A}_\mathfrak{a}^g$ and cone $\sigma := \sigma_{\boldsymbol{\alpha}}$, let

$$\mathcal{G}^{\mathfrak{a}}_{\sigma_{\boldsymbol{\alpha}}}(t) := \frac{\sum_{\alpha \in \mathfrak{a} \cap \widehat{P}_{\boldsymbol{\alpha}}} t^\alpha}{(1 - t^{\alpha_1}) \cdots (1 - t^{\alpha_g})},$$

where $P_{\boldsymbol{\alpha}} := \{x_1\alpha_1 + \cdots + x_g\alpha_g \mid 0 < x_1, \dots, x_g < 1\}$ is the parallelepiped defined by $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_g)$. We let $U^{\mathfrak{a}}_\alpha := \mathbb{T}^{\mathfrak{a}} \setminus \{t^\alpha = 1\}$.

Theorem (Shintani)

For any integer $k_1, \dots, k_g \geq 0$ and torsion point ξ in $U^{\mathfrak{a}}_{\alpha_1} \cap \cdots \cap U^{\mathfrak{a}}_{\alpha_g}$, we have

$$\partial^{k_1}_{\tau_1} \cdots \partial^{k_g}_{\tau_g} \mathcal{G}^{\mathfrak{a}}_{\sigma_{\boldsymbol{\alpha}}}(t)\Big|_{t=\xi} = \zeta_{\sigma_{\boldsymbol{\alpha}}}(\xi, (-k_1, \dots, -k_g)),$$

where ∂_τ is the differential satisfying $\partial_\tau(t^\alpha) = \alpha^\tau t^\alpha$ for any $\tau \colon F \hookrightarrow \mathbb{R}$.

Generating Function

Since for a Shimura decomposition Φ, we have

$$\mathcal{L}(\xi\Delta, s) = \sum_{\alpha \in \Delta \backslash \mathfrak{a}_+} \xi\Delta(\alpha) N(\mathfrak{a}^{-1}\alpha)^{-s} = N\mathfrak{a}^s \sum_{\sigma \in \Delta \backslash \Phi} \sum_{\varepsilon \in \Delta / \Delta_\xi} \zeta_\sigma(\xi^\varepsilon, (s, \ldots, s)).$$

If we let

$$\mathcal{G}_\Phi^{\mathfrak{a}}(t) := \sum_{\sigma \in \Delta \backslash \Phi} \sum_{\varepsilon \in \Delta / \Delta_\xi} \mathcal{G}_\sigma^{\mathfrak{a}}(t^\varepsilon),$$

then for $\partial := \prod_{i=1}^{g} \partial_{\tau_i}$, we have $\partial^k \mathcal{G}_\Phi^{\mathfrak{a}}(t)\big|_{t=\xi} = \mathcal{L}(\xi\Delta, -k)$ for any integer $k \geq 0$, if $\mathcal{G}_\Phi^{\mathfrak{a}}(t)$ is defined at ξ. However,

- $\mathcal{G}_\Phi^{\mathfrak{a}}(t)$ depends on ξ.
- $\mathcal{G}_\Phi^{\mathfrak{a}}(t)$ depends on the choice of the Shintani decomposition Φ.

How can we create a canonical generating function?

Shintani Generating Class

Action of a Group

Any $x \in F_+^\times$ defines an isomorphism $\mathfrak{a} \xrightarrow{\cong} x\mathfrak{a}$ of $\mathcal{O}_F$-modules. This gives an isomorphism

$$\langle x \rangle : \mathbb{T}^{x\mathfrak{a}} \xrightarrow{\cong} \mathbb{T}^{\mathfrak{a}},$$

which on $\mathbb{C}$-valued points is given by

$$\begin{aligned} \langle x \rangle : \mathbb{T}^{x\mathfrak{a}}(\mathbb{C}) &\xrightarrow{\cong} \mathbb{T}^{\mathfrak{a}}(\mathbb{C}), \\ \xi : x\mathfrak{a} \to \mathbb{C}^\times \qquad &\mapsto \qquad \xi^x : \mathfrak{a} \to \mathbb{C}^\times \\ &\qquad\quad \xi^x(\alpha) = \xi(x\alpha) \quad \forall \alpha \in \mathfrak{a}. \end{aligned}$$

More generally, if we let $\mathbb{T} := \coprod_{\mathfrak{a} \in \mathfrak{I}} \mathbb{T}^{\mathfrak{a}}$, then $x \in F_+^\times$ induces

$$\langle x \rangle : \mathbb{T} \xrightarrow{\cong} \mathbb{T},$$

hence an action of $F_+^\times$ on $\mathbb{T}$.

Equivariant Sheaf and Cohomology

Let $\mathbb{T} := \coprod_{\mathfrak{a}\in\mathfrak{I}} \mathbb{T}^{\mathfrak{a}}$, and $U := \coprod_{\mathfrak{a}\in\mathfrak{I}} U^{\mathfrak{a}}$, where $U^{\mathfrak{a}} := \mathbb{T}^{\mathfrak{a}} \setminus \{1\}$. Then U also has an action of $F_+^\times$ induced from the action on $\mathbb{T}$.

Equivariant Sheaf

We define a $F_+^\times$-*equivariant sheaf on* U to be a family of $\mathcal{O}_{U^{\mathfrak{a}}}$-modules $(\mathscr{F}_{\mathfrak{a}})_{\mathfrak{a}\in\mathfrak{I}}$ with isomorphisms $\iota_{x,\mathfrak{a}} : \langle x\rangle^* \mathscr{F}_{\mathfrak{a}} \cong \mathscr{F}_{x\mathfrak{a}}$ for any $x \in F_+^\times$, compatible with the composition.

We define the equivariant cohomology of U with coefficients in $\mathscr{F}$ by

$$H^m(U/F_+^\times, \mathscr{F}) := R^m\Gamma(U/F_+^\times, \mathscr{F}),$$

where $R^m\Gamma(U/F_+^\times, \mathscr{F})$ is the m-th right derived functor of $\Gamma(U, -)^{F_+^\times}$.

We next define an explicit complex to calculate this cohomology

Čech Cocycle

We let $U^{\mathfrak{a}}_{\alpha} := \mathbb{T}^{\mathfrak{a}} \setminus \{t^{\alpha} = 1\}$ for any $\alpha \in \mathscr{A}_{\mathfrak{a}}$. Then $\mathfrak{U} := \{U^{\mathfrak{a}}_{\alpha}\}_{\mathfrak{a} \in \mathfrak{I}, \alpha \in \mathscr{A}_{\mathfrak{a}}}$ is an affine open covering of $U := \coprod U^{\mathfrak{a}}$. The group $F^{\times}_{+}$ naturally acts on the sets $\coprod_{\mathfrak{a} \in \mathfrak{I}} \mathscr{A}_{\mathfrak{a}}$ and $\mathfrak{U} = \{U^{\mathfrak{a}}_{\alpha}\}$.

Definition (Equivariant Čech Complex)

Let $\mathscr{F} = (\mathscr{F}_{\mathfrak{a}})$ be a $F^{\times}_{+}$-equivariant sheaf on U. Define the complex $C^{\bullet}(\mathfrak{U}/F^{\times}_{+}, \mathscr{F})$ by

$$C^{q}(\mathfrak{U}/F^{\times}_{+}, \mathscr{F}) := \left(\prod_{\mathfrak{a} \in \mathfrak{I}} \prod^{\mathrm{alt}}_{\alpha \in \mathscr{A}_{\mathfrak{a}}^{q+1}} \Gamma\big(U^{\mathfrak{a}}_{\alpha_0} \cap \cdots \cap U^{\mathfrak{a}}_{\alpha_q}, \mathscr{F}_{\mathfrak{a}}\big) \right)^{F^{\times}_{+}}$$

for any integer $q \geq 0$, with the usual Čech differential.

Then we have

$$H^{m}(U/F^{\times}_{+}, \mathscr{F}) = H^{m}(C^{\bullet}(\mathfrak{U}/F^{\times}_{+}, \mathscr{F})).$$

Shintani Generating Class

We fix a numbering $I = \{\tau_1, \ldots, \tau_g\}$. For any $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_g) \in \mathscr{A}_{\mathfrak{a}}^g$, let $\mathrm{sgn}(\boldsymbol{\alpha})$ be the sign of the determinant of the matrix $(\alpha_i^{\tau_j})$. Then we may see that

$$(\mathrm{sgn}(\boldsymbol{\alpha})\mathcal{G}^{\mathfrak{a}}_{\sigma_{\boldsymbol{\alpha}}}(t)) \in \left(\prod_{\mathfrak{a} \in \mathfrak{I}} \prod^{\mathrm{alt}}_{\boldsymbol{\alpha} \in \mathscr{A}_{\mathfrak{a}}^g} \Gamma\big(U^{\mathfrak{a}}_{\alpha_1} \cap \cdots \cap U^{\mathfrak{a}}_{\alpha_g}, \mathscr{O}_{\mathbb{T}^{\mathfrak{a}}}\big)\right)^{F_+^\times}.$$

Theorem (B.–Hagihara–Yamada–Yamamoto)

The functions $(\mathrm{sgn}(\boldsymbol{\alpha})\mathcal{G}_{\sigma_{\boldsymbol{\alpha}}}(t))$ define a cocycle, and form a canonical class

$$\mathcal{G}(t) = (\mathrm{sgn}(\boldsymbol{\alpha})\mathcal{G}^{\mathfrak{a}}_{\sigma_{\boldsymbol{\alpha}}}(t)) \in H^{g-1}(U/F_+^\times, \mathscr{O}_{\mathbb{T}}),$$

which we call the *Shintani Generating Class*.

The generating functions paste together to form a single canonical class!

Main Theorem

The differential $\partial := \prod_{i=1}^{g} \partial_\tau$ induces an homomorphism

$$\partial \colon H^{g-1}(U/F_+^\times, \mathscr{O}_{\mathbb{T}}) \longrightarrow H^{g-1}(U/F_+^\times, \mathscr{O}_{\mathbb{T}}).$$

∂ is given by $\partial(t^\alpha) = N(\alpha)t^\alpha$ for any $\alpha \in \mathfrak{a}$ on $U^{\mathfrak{a}}$.

Theorem (B.–Hagihara–Yamada–Yamamoto)

For any integer $k \geq 0$ and any torsion point $\xi \neq 1$ in $\mathbb{T}^{\mathfrak{a}}(\overline{\mathbb{Q}})$, we have

$$\begin{tikzcd}
 & H^{g-1}(U/F_+^\times, \mathscr{O}_{\mathbb{T}}) \arrow[d, "i_\xi^*"] & \ni & \partial^k \mathcal{G}(t) \arrow[d, mapsto] \\
\Delta \cong \mathbb{Z}^{g-1} & H^{g-1}(\xi\Delta/\Delta, \mathscr{O}_{\xi\Delta}) = \mathbb{Q}(\xi) & \ni & \partial^k \mathcal{G}(t)\big|_{t=\xi} = \mathcal{L}(\xi\Delta, -k),
\end{tikzcd}$$

where $i_\xi \colon \xi\Delta \longrightarrow U$ is equivariant with respect to the action of Δ.

Case g=1 is exactly the Theorem (Classical)

p-adic Polylogarithm

p-adic Polylogarithm

We fix embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. We let K be a finite extension of $\mathbb{Q}_p$ containing the Galois closure of F. Let $A := \mathcal{O}_K[t^\alpha \mid \alpha \in \mathfrak{a}]$, $\widehat{\mathbb{T}}^{\mathfrak{a}}_K := \mathrm{Sp}(A\widehat{\otimes}K)$ be the affinoid space associated to $(A\widehat{\otimes}K)$, and $\widehat{U}^{\mathfrak{a}}_{\alpha K} := \mathrm{Sp}(A[(t^\alpha - 1)^{-1}]\widehat{\otimes}K) = \widehat{\mathbb{T}}^{\mathfrak{a}}_K \backslash]t^\alpha = 1[$. We let

$$\widehat{U}^{\mathfrak{a}}_K := \bigcup_{\alpha \in \mathscr{A}_{\mathfrak{a}}} \widehat{U}^{\mathfrak{a}}_{\alpha K} \subset \widehat{\mathbb{T}}^{\mathfrak{a}}_K, \qquad \widehat{U}_K := \coprod_{\mathfrak{a} \in \mathfrak{I}} \widehat{U}^{\mathfrak{a}}_K \subset \widehat{\mathbb{T}}_K.$$

Proposition (B.–Hagihara–Yamada–Yamamoto)

For $\mathfrak{a} \in \mathfrak{I}$, $k \in \mathbb{Z}$, and cone σ, we have

$$\mathrm{Li}^{\mathfrak{a},(p)}_{k,\sigma}(t) := \sum_{\substack{\alpha \in \widehat{\sigma} \cap \mathfrak{a} \\ \alpha \in (\mathfrak{a} \otimes \mathbb{Z}_p)^\times}} t^\alpha N(\alpha)^{-k} \in \Gamma\big(\widehat{U}^{\mathfrak{a}}_{\alpha K}, \mathcal{O}_{\widehat{U}^{\mathfrak{a}}_{\alpha K}}\big),$$

where $(\mathfrak{a} \otimes \mathbb{Z}_p)^\times$ is the set of generators of the $(\mathcal{O}_F \otimes \mathbb{Z}_p)$-module $\mathfrak{a} \otimes \mathbb{Z}_p$.

p-adic Polylogarithm

We fix a numbering $I = \{\tau_1, \dots, \tau_g\}$. For any $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_g) \in \mathscr{A}_{\mathfrak{a}}^g$, let $\mathrm{sgn}(\boldsymbol{\alpha})$ be the sign of the determinant of the matrix $(\alpha_i^{\tau_j})$. Then we may see that

$$(\mathrm{sgn}(\boldsymbol{\alpha})\,\mathrm{Li}_{k,\sigma}^{\mathfrak{a},(p)}(t)) \in \left(\prod_{\mathfrak{a}\in\mathfrak{I}} \prod_{\alpha\in\mathscr{A}_{\mathfrak{a}}^g}^{\mathrm{alt}} \Gamma\left(\widehat{U}^{\mathfrak{a}}_{\alpha_1 K} \cap \cdots \cap \widehat{U}^{\mathfrak{a}}_{\alpha_g K}, \mathscr{O}_{\widehat{\mathbb{T}}^{\mathfrak{a}}_K}\right)\right)^{F_+^\times}.$$

Theorem (B.–Hagihara–Yamada–Yamamoto)

The functions $(\mathrm{sgn}(\boldsymbol{\alpha})\,\mathrm{Li}_{k,\sigma}^{\mathfrak{a},(p)}(t))$ define a cocycle, and form a canonical class

$$\mathrm{Li}_k^{(p)}(t) = (\mathrm{sgn}(\boldsymbol{\alpha})\,\mathrm{Li}_{k,\sigma}^{\mathfrak{a},(p)}(t)) \in H^{g-1}\left(\widehat{U}_K/F_+^\times, \mathscr{O}_{\widehat{\mathbb{T}}_K}\right).$$

We will prove it's relation to special values of p-adic L-functions

p-adic Polylogarithm

Theorem (B.–Hagihara–Yamada–Yamamoto)

The functions $(\operatorname{sgn}(\alpha)\,\mathrm{Li}^{\mathfrak{a},(p)}_{k,\sigma}(t))$ define a cocycle, and form a canonical class

$$\mathrm{Li}^{(p)}_k(t) = (\operatorname{sgn}(\alpha)\,\mathrm{Li}^{\mathfrak{a},(p)}_{k,\sigma}(t)) \in H^{g-1}\big(\widehat{U}_K/F_+^\times, \mathscr{O}_{\widehat{\mathbb{T}}_K}\big).$$

One can evaluate the polylogarithm at any torsion point ξ in $\widehat{U}^{\mathfrak{a}}_K$.

$$\Delta \cong \mathbb{Z}^{g-1}$$

$$\begin{tikzcd}
H^{g-1}(\widehat{U}_K/F_+^\times, \mathscr{O}_{\mathbb{T}}) \arrow[d, "i_\xi^*"] & \ni & \mathrm{Li}^{(p)}_k(t) \arrow[d, mapsto] \\
H^{g-1}(\xi\Delta/\Delta, \mathscr{O}_{\xi\Delta}) = K(\xi) & \ni & \mathrm{Li}^{(p)}_k(t)\big|_{t=\xi}
\end{tikzcd}$$

where $i_\xi : \xi\Delta \longrightarrow \widehat{U}_K$ is equivariant with respect to the action of Δ.

p-adic L-functions

Theorem (Barsky, Cassou-Nougès, Deligne-Ribet)

Let $\mathfrak{g} \neq (1)$ be a nonzero fractional ideal of F, and let $\chi \colon \mathrm{Cl}_F^+(\mathfrak{g}) \to \mathbb{C}^\times$ be a finite primitive Hecke character. There exists an analytic function $L_p(\chi, s)$ for $s \in \mathbb{Z}_p$ satisfying

$$L_p(\chi, -k) = \left(\prod_{\mathfrak{p} \mid (p)} \left(1 - \chi\omega_p^{-k-1}(\mathfrak{p}) N\mathfrak{p}^k\right)\right) L\left(\chi\omega_p^{-k-1}, -k\right)$$

for any integer $k \geq 0$, where ω_p denotes the composition of the norm map with the Teichmüller character.

Action of $\mathsf{Cl}_F^+(\mathfrak{g})$ on $\mathscr{T}_0[\mathfrak{g}]$.

For any integral ideal $\mathfrak{b} \subset \mathcal{O}_F$, the natural inclusion $\mathfrak{a}\mathfrak{b} \subset \mathfrak{a}$ for $\mathfrak{a} \in \mathfrak{I}$ induces a map $\rho(\mathfrak{b})\colon \mathbb{T}^{\mathfrak{a}} \to \mathbb{T}^{\mathfrak{a}\mathfrak{b}}$. This induces a map

$$\rho(\mathfrak{b})\colon \mathbb{T} \to \mathbb{T},$$

which is compatible with the action of $F_+^\times$. Let

$$\mathscr{T}_0[\mathfrak{g}] := \Big(\coprod_{\mathfrak{a}\in\mathfrak{I}} \mathbb{T}_0^{\mathfrak{a}}[\mathfrak{g}]\Big)/F_+^\times,$$

where $\mathbb{T}_0^{\mathfrak{a}}[\mathfrak{g}]$ is the set of primitive $\mathfrak{g}$-torsion points in $\mathbb{T}^{\mathfrak{a}}(\overline{\mathbb{Q}})$. Then ρ gives an action of $\mathsf{Cl}_F^+(\mathfrak{g})$ on $\mathscr{T}_0[\mathfrak{g}]$ which is simply transitive. For any $\xi \in \mathbb{T}^{\mathfrak{a}}[\mathfrak{g}]$ and integral ideal $\mathfrak{b}$ prime to $\mathfrak{g}$, we let $\xi^{\mathfrak{b}} := \rho(\mathfrak{b})(\xi) \in \mathbb{T}^{\mathfrak{a}\mathfrak{b}}[\mathfrak{g}]$.

Result concerning p-adic L-functions

Theorem (B.–Hagihara–Yamada–Yamamoto)

Suppose $\mathfrak{g}$ does not divide any power of (p), and let ξ be an arbitrary primitive $\mathfrak{g}$-torsion point in $\mathbb{T}^{\mathfrak{a}}(\overline{\mathbb{Q}})$. Then for any integer $k \in \mathbb{Z}$, we have

$$L_p\left(\chi\omega_p^{1-k}, k\right) = \frac{g(\chi,\xi)}{N\mathfrak{g}} \sum_{\mathfrak{b}\in \mathrm{Cl}_F^+(\mathfrak{g})} \chi(\mathfrak{b})^{-1} \operatorname{Li}_k^{(p)}(t)\Big|_{t=\xi^{\mathfrak{b}}},$$

where $g(\chi,\xi) := N\mathfrak{g}c_\chi(\xi) = \sum_{\beta\in\mathfrak{a}/\mathfrak{a}\mathfrak{g}} \chi_{\mathfrak{a}}(\beta)\xi(-\beta)$.

Conclusion

- We newly defined the Lerch zeta function for totally real fields.
- We constructed the Shintani generating class as a canonical class in cohomology, and proved that it generates *all* non-positive special values of Lerch zeta functions for *all* nontrivial finite characters.
- We constructed the p-adic polylogarithm and gave its relation to the special values of p-adic Hecke L-functions for totally real fields.

Conjectures/Questions

- We conjecture that the specialization to torsion points of the equivariant plectic polylogarithm for $\mathbb{T}$ should be related to positive values of our Lerch zeta function.
- We conjecture that the syntomic relalization of the equivariant plectic polylogarithm for $\mathbb{T}$ should be expressed using our p-adic polylogarithm $\mathrm{Li}_k^{(p)}(t)$.
- The stack $\mathscr{T} := \mathbb{T}/F_+^\times$ seems to contain important arithmetic information, somewhat similar to information possessed by elliptic curves with complex multiplication in the imaginary quadratic case. What does this mean?