Non-abelian Cohomology and Diophantine Geometry

Minhyong Kim

Bures-sur-Yvette, April, 2018

 $In \mathbb{Q}_2$,

$$\sum_{n=1}^{\infty} \frac{2^n}{n^2} = ?$$

$$\sum_{n=1}^{\infty} \frac{2^n}{n^2} = 0$$

in the 2-adics. Actually, also true in \mathbb{Q}_p for all p.

$$\sum_{n=1}^{\infty} \frac{z^n}{n^2} = \int_0^z \frac{dt}{t} \frac{dt}{1-t} =: \ell_2(z)$$

Right hand side can be defined via

$$M_b^z = \begin{bmatrix} 1 & \int_b^z dt/t & \int_0^z (dt/t)(dt/(1-t)) \\ 0 & 1 & \int_b^z dt/(1-t) \\ 0 & 0 & 1 \end{bmatrix}$$

where M_b^z is the holonomy of the rank 3 unipotent connection on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ given by the connection form

$$-\begin{bmatrix} 0 & dt/t & 0 \\ 0 & 0 & dt/(1-t) \\ 0 & 0 & 0 \end{bmatrix}$$

Locally, we are solving the equations

$$d\ell_1 = dt/(1-t); \ d\ell_2 = \ell_1 dt/t.$$

Given a unipotent connection (V, ∇) and two points

$$x, y \in \mathbb{P}^1 \setminus \{0, 1, \infty\}(\mathbb{Q}_p)$$

(possibly tangental), there is a canonical isomorphism

$$M_{x}^{y}(V,\nabla):V(]\bar{x}[)^{\nabla=0}\xrightarrow{\simeq}V(]\bar{y}[)^{\nabla=0}$$

determined by the property that it's compatible with Frobenius pull-backs. This is the holonomy matrix above.

More generally, the k-logarithm

$$\ell_k(z):=\int_0^z (dt/t)(dt/t)\cdots(dt/t)(dt/(1-t))$$

is defined as the upper right hand corner of the holonomy matrix arising from the (k+1) imes (k+1) connection form

$$-\begin{bmatrix} 0 & dt/t & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & dt/t & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & dt/t & \dots & 0 & 0 \\ & \vdots & & \dots & dt/t & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & dt/(1-t) \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

Coleman derived the following functional equations:

$$\ell_k(z) + (-1)^k \ell_k(z^{-1}) = rac{-1}{k!} \log^k(z);$$

 $D_2(z) = -D_2(z^{-1});$
 $D_2(z) = -D_2(1-z);$

where

$$D_2(z) = \ell_2(z) + (1/2)\log(z)\log(1-z).$$

(The upper right hand corner of the log of the holonomy matrix.)

From this, we get

$$D_2(-1) = -D_2(1/(-1)) = 0,$$

 $\quad \text{and} \quad$

$$D_2(2) = -D_2(1-2) = 0.$$

But

$$D_2(2) = \ell_2(2) + (1/2)\log(2)\log(-1) = \ell_2(2).$$

Also,

$$D_2(1/2) = -D_2(2) = 0.$$

Note:

$$\{-1, 2, 1/2\}$$

are exactly the 2-integral points of $\mathbb{P}^1\setminus\{0,1,\infty\}$, and one can give a global proof of the vanishing.

Use, in some sense, the arithmetic geometry of

 $\overline{\mathsf{Spec}(\mathbb{Z})} \setminus \{2, p, \infty\}.$

[Ishai Dan-Cohen and Stefan Wewers]

Let

$$D_4(z) = \zeta(3)\ell_4(z) + (8/7)[\log^3 2/24 + \ell_4(1/2)/\log 2]\log(z)\ell_3(z)$$
$$+[(4/21)(\log^3 2/24 + \ell_4(1/2)/\log 2) + \zeta(3)/24]\log^3(z)\log(1-z).$$

$$= \zeta(3)\ell_4(z) + A\log(z)\ell_3(z) + B\log^3(z)\log(1-z).$$

Then

$$[\mathbb{P}^1 \setminus \{0, 1, \infty\}](\mathbb{Z}[1/2]) \subset \{D_2(z) = 0, D_4(z) = 0\}$$

and numerical computations for $p \leq 29$ indicate equality.

The inclusions above are examples of *non-abelian explicit reciprocity laws*.

Remark: The extra equation is definitely necessary in general, since, for example, $\sqrt{5}\in\mathbb{Z}_{11},$ and

$$D_2(\frac{-1\pm\sqrt{5}}{2}) = D_2(\frac{3\pm\sqrt{5}}{2}) = D_2(\frac{1\pm\sqrt{5}}{2}) = 0.$$

Diophantine Geometry: Main Local-to-Global Problem

Given number field F and X/F smooth variety (with an integral model), locate

$$X(F) \subset X(\mathbb{A}_F) = \prod_{\nu}' X(F_{\nu})$$

The question is

How do the global points sit inside the local points?

In fact, there is a classical answer for $X = \mathbb{G}_m$, in which case

$$X(F)=F^*, \quad X(F_v)=F_v^*.$$

Problem becomes that of locating

$$F^* \subset \mathbb{A}_F^{\times}.$$

Diophantine Geometry: Abelian Class Field Theory

We have the Artin reciprocity map

$$\operatorname{rec} = \prod_{v} \operatorname{rec}_{v} : \mathbb{A}_{F}^{\times} \longrightarrow G_{F}^{ab},$$

and the reciprocity law, which says that the composed map

$$F^* \hookrightarrow \mathbb{A}_F^{\times} \xrightarrow{rec} G_F^{ab}$$

is zero.

That is, the reciprocity map gives a defining equation for

 $\mathbb{G}_m(F) \subset \mathbb{G}_m(\mathbb{A}_F).$

We would like to generalize this to other equations by way of a *non-abelian reciprocity law*.

Start with a rather general variety \boldsymbol{X} for which we would like to understand

X(F)

via

$$X(F) \hookrightarrow X(\mathbb{A}_F) \xrightarrow{\operatorname{rec}^{NA}} \text{ some target with base-point 0}$$

in such way that

$$rec^{NA} = 0.$$

becomes an equation for X(F).

Notation:

F: number field. $G_F = \text{Gal}(\overline{F}/F).$ $G_v = \text{Gal}(\overline{F}_v/F_v)$ for a place v of F. S: finite set of places of F. \mathbb{A}_F : finite Adeles of F. \mathbb{A}_F^S : finite S-integral adeles of F.

 $G_S = \text{Gal}(F^S/F)$, where F^S is the maximal extension of F unramified outside S.

 $\prod_{S} : \text{ product over non-Archimedean places in } S.$ $\prod^{S} H^{1}(G_{v}, A): \text{ product over non-Archimedean places in } S \text{ and } `unramified cohomology' outside of } S.$

X: a smooth variety over F.

Fix base-point $b \in X(F)$ (sometimes tangential).

$$\Delta = \pi_1(\bar{X}, b)^{(2)},$$

pro-finite prime-to-2, étale fundamental group of

$$\bar{X} = X imes_{\operatorname{\mathsf{Spec}}(F)} \operatorname{\mathsf{Spec}}(\bar{F})$$

with base-point b.

 $\Delta^{[n]}$,

lower central series with $\Delta^{[1]} = \Delta$.

$$\Delta_n = \Delta / \Delta^{[n+1]}.$$

 $T_n = \Delta^{[n]} / \Delta^{[n+1]}.$

Denote by Δ^M , $(\Delta_n)^M$, T_n^M pro-M quotients for various finite sets of prime M.

[Coh] For each n and M sufficiently large, T_n^M is torsion-free. This implies

$$H^1(G_F^S, T_n^M) \xrightarrow{\mathsf{loc}} \prod' H^1(G_v, T_n^M)$$

is injective.

Assuming [Coh], we get a non-abelian class field theory with coefficients in the nilpotent completion of X.

This consists of a filtration

$$\begin{aligned} X(\mathbb{A}_F) &= X(\mathbb{A}_F)_1 \supset X(\mathbb{A}_F)_1^2 \supset X(\mathbb{A}_F)_2 \supset X(\mathbb{A}_F)_2^3 \\ &\supset X(\mathbb{A}_F)_3 \supset X(\mathbb{A}_F)_3^4 \supset \cdots \end{aligned}$$

and a sequence of maps

$$rec_{n}: X(\mathbb{A}_{F})_{n} \longrightarrow \mathfrak{G}_{n}(X)$$
$$rec_{n}^{n+1}: X(\mathbb{A}_{F})_{n}^{n+1} \longrightarrow \mathfrak{G}_{n}^{n+1}(X)$$

to a sequence $\mathfrak{G}_n(X), \mathfrak{G}_n^{n+1}(X)$ of profinite abelian groups in such a way that

$$X(\mathbb{A}_F)_n^{n+1} = rec_n^{-1}(0)$$

and

$$X(\mathbb{A}_F)_{n+1} = (rec_n^{n+1})^{-1}(0).$$

The $\mathfrak{G}_n(X)$ are defined as

 $\mathfrak{G}_n(X) :=$

Hom[$H^1(G_F, D(T_n)), \mathbb{Q}/\mathbb{Z}$]

where

$$D(T_n) = \varinjlim_m \operatorname{Hom}(T_n, \mu_m).$$

$$\mathfrak{G}_n^{n+1}(X) := \varprojlim_M [\varinjlim_S s \amalg_S^2(T_{n+1}^M)]$$

where

$$s \amalg_{S}^{2}(\mathcal{T}_{n+1}^{M}) = \operatorname{Ker}[H^{2}(G_{F}^{S}, \mathcal{T}_{n+1}^{M}) \longrightarrow \prod' H^{2}(G_{v}, \mathcal{T}_{n+1}^{M})].$$

When
$$X = \mathbb{G}_m$$
, then $\mathfrak{G}_n(X) = 0$ for $n \ge 2$, $\mathfrak{G}_n^{n+1}(X) = 0$

for all *n*, and

$$\mathfrak{G}_{1} = \operatorname{Hom}[H^{1}(G_{F}, D(\hat{\mathbb{Z}}(1)^{(2)})), \mathbb{Q}/\mathbb{Z}]$$

= $\operatorname{Hom}[H^{1}(G_{F}, [\mathbb{Q}/\mathbb{Z}]^{(2)}), \mathbb{Q}/\mathbb{Z}] = [G^{(2)}]_{F}^{ab}.$

In this case, rec_1 reduces to the prime-to-2 part of the usual reciprocity map.

The reciprocity maps are defined using the local period maps

$$j^{\nu}: X(F_{\nu}) \longrightarrow H^{1}(G_{\nu}, \Delta);$$
$$x \mapsto [\pi_{1}^{(2)}(\bar{X}; b, x)].$$

Because the homotopy classes of étale paths

$$\pi_1^{(2)}(\bar{X};b,x)$$

form a torsor for Δ with compatible action of G_{ν} , we get a corresponding class in non-abelian cohomology of G_{ν} with coefficients in Δ .

These assemble to a map

$$j^{loc}: X(\mathbb{A}_F) \longrightarrow \prod H^1(G_v, \Delta),$$

which comes in levels

$$j_n^{loc}: X(\mathbb{A}_F) \longrightarrow \prod H^1(G_v, \Delta_n).$$

Also have pro-M versions

$$j_n^{loc}: X(\mathbb{A}_F) \longrightarrow \prod' H^1(G_v, \Delta_n^M)$$

and integral versions

$$j_n^{loc}: X(\mathbb{A}_F^S) \longrightarrow \prod^S H^1(G_v, \Delta_n^M).$$

To indicate the definition of the reciprocity maps, will just define pro-M versions on $X(\mathbb{A}_F^S)$ and assume that

$$H^1(G_F^S, T_n^M) \xrightarrow{\mathsf{loc}_S} \prod_S H^1(G_v, T_n^M)$$

are injective.

In general, one needs first to work with a pro-M quotient for a finite set of primes M and $S \supset M$. Then take a limit over S and M.

The first reciprocity map is just defined using

$$x \in X(\mathbb{A}_F) \mapsto d_1(j_1^{loc}(x)),$$

where

$$D_1:\prod_{S}H^1(G_{v},\Delta_1^{M})\longrightarrow \prod_{S}H^1(G_{v},D(\Delta_1^{M}))^{\vee}\xrightarrow{\mathsf{loc}^*}H^1(G_{F}^{S},D(\Delta_1^{M}))^{\vee},$$

is obtained from Tate duality and the dual of localization.

To define the higher reciprocity maps, we use the exact sequences

$$0 \longrightarrow H^{1}(G_{F}^{S}, T_{n+1}^{M}) \longrightarrow H^{1}(G_{F}^{S}, \Delta_{n+1}^{M}) \xrightarrow{p_{n+1}^{n+1}} H^{1}(G_{F}^{S}, \Delta_{n})$$
$$\xrightarrow{\delta_{n+1}} H^{2}(G_{F}^{S}, T_{n+1}^{M})$$

for non-abelian cohomology and Poitou-Tate duality stating that

$$H^{1}(G_{F}^{S}, T_{n+1}^{M}) \longrightarrow \prod_{S} H^{1}(G_{v}, T_{n}^{M}) \xrightarrow{D_{n+1}} H^{1}(G_{S}, D(T_{n+1}^{M}))^{\vee}$$

is exact.

We proceed as follows:

$$\operatorname{rec}_1^2(x) = \delta_2 \circ \operatorname{loc}^{-1}(j_1(x)) \in \operatorname{III}_S^2(T_2^M)$$

 and

$$rec_2(x) = D_2(loc((p_1^2)^{-1}(loc^{-1}(j_1(x)))) - j_2(x)) \in H^1(G_S, D(T_2^M))^{\vee}.$$

$$egin{aligned} &k_2(x) := \log [[p_1^2]^{-1}(\log^{-1}(j_1(x)))] - j_2(x) \ &\mapsto D_2(k_2(x)) \in H^1(G_F^S, D(T_2^M))^ee \end{aligned}$$

In general,

$$\operatorname{rec}_n^{n+1}(x) = \delta_{n+1} \circ \operatorname{loc}^{-1}(j_n(x)) \in \operatorname{III}^2_{\mathcal{S}}(T^M_{n+1})$$

 ${\sf and}$

$$rec_{n+1}(x)$$

= $D((loc(p_n^{n+1})^{-1}(j_n(x))) - j_{n+1}(x)) \in H^1(G_S, D(T_2^M))^{\vee}.$

Put

$$X(\mathbb{A}_F)_{\infty} = \cap_{n=1}^{\infty} X(\mathbb{A}_F)_n.$$

Theorem (Non-abelian reciprocity)

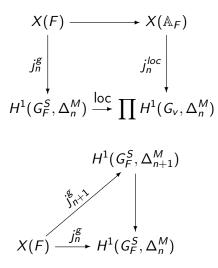
 $X(F) \subset X(\mathbb{A}_F)_{\infty}.$

Remark: When $F = \mathbb{Q}$ and p is a prime of good reduction, suppose there is a finite set T of places such that

$$H^1(G_F^S, \Delta_n^p) \longrightarrow \prod_{v \in T} H^1(G_v, \Delta_n^p)$$

is injective. Then the reciprocity law implies finiteness of X(F).

Non-Abelian Reciprocity: idea of proof



Non-Abelian Reciprocity: Idea of proof

If $x \in X(\mathbb{A}_F)$ comes from a global point $x^g \in X(F)$, then there will be a class

$$j_n^g(x^g) \in H^1(G_F^S, \Delta_n^M)$$

for every n corresponding to the global torsor

$$\pi_1^{et,M}(\bar{X};b,x^g).$$

That is,
$$j_n^g(x^g) = \log^{-1}(j_n^{loc}(x))$$
,
 $\delta_{n+1}(j_n^g(x^g)) = 0$

and

$$\log[(p_n^{n+1})^{-1}(\log^{-1}(j_n(x)))] - j_{n+1}(x) = \log(j_{n+1}^g) - j_{n+1}(x) = 0$$

for every n.

A non-abelian conjecture of Birch and Swinnerton-Dyer type Let

$$Pr_{v}: X(\mathbb{A}_{F}) \longrightarrow X(F_{v})$$

be the projection to the v-adic component of the adeles. Define

$$X(F_{v})_{n} := Pr_{v}(X(\mathbb{A}_{F})_{n})$$

and

$$X(F_{\nu})_n^{n+1} := Pr_{\nu}(X(\mathbb{A}_F)_n^{n+1}).$$

Thus,

$$X(F_{\nu}) = X(F_{\nu})_1 \supset X(F_{\nu})_1^2 \supset X(F_{\nu})_2 \supset \cdots \supset X(F_{\nu})_{\infty} \supset X(F).$$

Conjecture: Let X/\mathbb{Q} be a projective smooth curve of genus at least 2. Then for any prime *p* of good reduction, we have

$$X(\mathbb{Q}_p)_{\infty} = X(\mathbb{Q}).$$

A non-abelian conjecture of Birch and Swinnerton-Dyer type

Can consider more generally S-integral points on affine hyperbolic X as well where we get an induced filtration

$$X(\mathbb{A}_F^S) \supset X(\mathbb{A}_F^S)_1^2 \supset X(\mathbb{A}_F^S)_2 \supset X(\mathbb{A}_F^S)_2^3 \supset \cdots$$

By projecting to $X(O_{F_v})$ for $v \notin S$, get a flitration

$$X(\mathcal{O}_{F_{v}}) \supset X(\mathcal{O}_{F_{v}})^{2}_{5,1} \supset X(\mathcal{O}_{F_{v}})_{5,2} \supset X(\mathcal{O}_{F_{v}})^{3}_{5,2} \supset \cdots$$

and

$$X(O_{F_{v}})_{S,\infty} = \cap_{n} X(O_{F_{v}})_{S,n}.$$

A non-abelian conjecture of Birch and Swinnerton-Dyer type

Conjecture: Let X/\mathbb{Q} be an affine smooth curve with non-abelian fundamental group and S a finite set of primes. Then for any prime $p \notin S$ of good reduction, we have

$$X(\mathbb{Z}[1/S]) = X(\mathbb{Z}_p)_{S,\infty}.$$

These give us conjectural methods to 'compute'

$$X(\mathbb{Q}) \subset X(\mathbb{Q}_p)$$

or

$$X(\mathbb{Z}[1/S]) \subset X(\mathbb{Z}_p).$$

A non-abelian conjecture of Birch and Swinnerton-Dyer type

Whenever we have an element

$$k_n \in H^1(G_T, \operatorname{Hom}(T_n^M, \mathbb{Q}_p(1))),$$

we get a function

$$X(\mathbb{A}_{\mathbb{Q}})_n \xrightarrow{rec_n} H^1(G_T, D(T_n^M))^{\vee} \xrightarrow{k_n} \mathbb{Q}_p$$

that kills $X(\mathbb{Q}) \subset X(\mathbb{A}_{\mathbb{Q}})_n$.

Need an explicit reciprocity law that describes the image

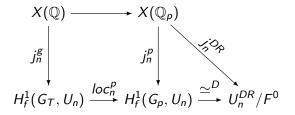
 $X(\mathbb{Q}_p)_n.$

A non-abelian conjecture of Birch and Swinnerton-Dyer type

Computations all rely on the theory of

U(X, b),

the \mathbb{Q}_p -pro-unipotent fundamental group of \bar{X} with Galois action, and the diagram



A non-abelian conjecture of Birch and Swinnerton-Dyer type

The key point is that the map

$$X(\mathbb{Q}_p) \xrightarrow{j^{DR}} U^{DR}/F^0$$

can be computed explicitly using iterated integrals, and

$$X(\mathbb{Q}) \subset X(\mathbb{Q}_p)_n \subset [j_n^{DR}]^{-1}[\operatorname{Im}(D \circ \operatorname{loc}_n^p)].$$

[Jennifer Balakrishnan, Ishai Dan-Cohen, Stefan Wewers, M.K.] and [Dan-Cohen, Wewers]

Let $X = \mathbf{P}^1 \setminus \{0, 1, \infty\}$. Then $X(\mathbb{Z}[1/2]) = \{2, -1, 1/2\}$.

$$X(\mathbb{Z}_p)_{\{2\},2} \subset \bigcup_{n,m} \{z \mid \log(z) = n \log(2), \log(1-z) = m \log(2)\}.$$

 $X(\mathbb{Z}_p)_{\{2\},3}$

 $\subset [\cup_{m,n} \{z \mid \log(z) = n \log(2), \log(1-z) = m \log(2)\}] \cap \{D_2(z) = 0\}.$

Probably,

$$X(\mathbb{Z}_p)_{\{2\},4} = X(\mathbb{Z}_p)_{\{2\},3}.$$

Also,

$$X(\mathbb{Z}_p)_{\{2\},5}$$

 $\subset [\cup_{m,n} \{z \mid \log(z) = n \log(2), \log(1-z) = m \log(2)\}]$
 $\cap \{D_2(z) = 0\} \cap \{D_4(z) = 0\}.$

Numerically, this appears to be equal to $\{2, -1, 1/2\}$.

[Balakrishnan, Dan-Cohen, Wewers, K.]

Let $X = E \setminus O$ where E is a semi-stable elliptic curve of rank 0 and $|III(E)(p)| < \infty$.

$$\log(z) = \int_b^z (dx/y).$$

(*b* is a tangential base-point.) Then

$$X(\mathbb{Z}_p)_2 = \{z \in X(\mathbb{Z}_p) \mid \log(z) = 0\} = E(\mathbb{Z}_p)[tor] \setminus O.$$

Now examine the inclusion

$$X(\mathbb{Z}) \subset X(\mathbb{Z}_p)_3$$

Let

$$D_2(z) = \int_b^z (dx/y)(xdx/y).$$

Let T be the set of primes of bad reduction. For each $I \in T$, let

 $N_l = \operatorname{ord}_l(\Delta_{\mathcal{E}}),$

where $\Delta_{\mathcal{E}}$ is the minimal discriminant. Define a set

$$W_{I} := \{ (n(N_{I} - n)/2N_{I}) \log I \mid 0 \le n < N_{I} \},\$$

and for each $w = (w_l)_{l \in S} \in W := \prod_{l \in S} W_l$, define

$$\|w\|=\sum_{I\in S}w_I.$$

Theorem Suppose E has rank zero and that $\operatorname{III}_E[p^{\infty}] < \infty$. With assumptions as above

$$X(\mathbb{Z}_p)_3 \subset \cup_{w \in W} \Psi(w),$$

where

$$\Psi(w) := \{ z \in \mathcal{X}(\mathbb{Z}_p) \mid \log(z) = 0, \ D_2(z) = \|w\| \}.$$

Of course,

$$X(\mathbb{Z}) \subset \mathcal{X}(\mathbb{Z}_p)_3,$$

but depending on the reduction of E, the latter could be made up of a large number of $\Psi(w)$, creating potential for some discrepancy.

In fact, so far, we have checked

$$X(\mathbb{Z}) = X(\mathbb{Z}_p)_3$$

for the prime p = 5 and 256 semi-stable elliptic curves of rank zero.

Cremona label	number of w -values
1122m1	128
1122m2	384
1122m4	84
1254a2	140
1302d2	96
1506a2	112
1806h1	120
2442h1	78
2442h2	84
2706d2	120
2982j1	160
2982j2	140
3054b1	108

Hence, for example, for the curve $1122m^2$,

$$y^2 + xy = x^3 - 41608x - 90515392$$

there are potentially 384 of the $\Psi(w)$'s that make up $X(\mathbb{Z}_p)_3$.

Of these, all but 4 end up being empty, while the points in those $\Psi(w)$ consist exactly of the integral points

(752, -17800), (752, 17048), (2864, -154024), (2864, 151160).

[Jennifer Balakrishnan, Netan Dogra, Stefan Mueller-Stach, Jan Tuitman, Jan Vonk]

$$X_s^+(N) = X(N)/C_s^+(N),$$

where X(N) is the compactification of the moduli space of pairs

$$(E,\phi:E[N]\simeq (\mathbb{Z}/N)^2),$$

and $C_s^+(N) \subset GL_2(\mathbb{Z}/N)$ is the normaliser of a split Cartan subgroup.

Bilu-Parent-Rebolledo had shown that $X_s^+(p)(\mathbb{Q})$ consists entirely of cusps and CM points for all primes p > 7, $p \neq 13$. They called p = 13 the 'cursed level'.

Theorem (BDMTV)

$$X_{s}^{+}(13)(\mathbb{Q}) = X_{s}^{+}(13)(\mathbb{Q}_{17})_{3}.$$

This set consists of 7 rational points, which are the cusp and 6 CM points.

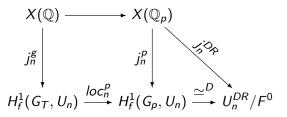
This concludes an important chapter of a question of Serre:

Find an absolute constant A such that

$$G \longrightarrow \operatorname{Aut}(E[p])$$

is surjective for all non-CM elliptic curves E/\mathbb{Q} and primes p > A.

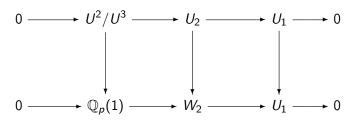
Careful computation of the lower horizontal map, which is algebraic:



The right vertical map is analytic and expressed in terms of iterated integrals.

Defining equation for $Im(Ioc_n^p)$ pulls back to analytic defining equation for rational points.

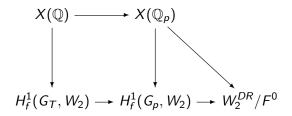
In fact, in this case, there is a pushout:



induced by a polarisation

$$U^2/U^3 \simeq \wedge^2 U_1/\mathbb{Q}_p(1) \longrightarrow \mathbb{Q}_p(1)$$

orthogonal to the Weil pairing. Recall that $U_1 \simeq V_p = T_p J_X \otimes \mathbb{Q}_p$.



In fact, the target can be identified with a space of *mixed* extensions:

$$E \supset E^1 \supset E^2$$
,

such that

$$E_2 \simeq \mathbb{Q}_p(1), \ E^1/E^2 \simeq V_p, \ E/E^1 \simeq \mathbb{Q}_p.$$

Thus, they are mixtures of

$$0 \longrightarrow \mathbb{Q}_{\rho}(1) \longrightarrow E^1 \longrightarrow V_{\rho} \longrightarrow 0$$

and

$$0 \longrightarrow V_p \longrightarrow E/E^2 \longrightarrow \mathbb{Q}_p \longrightarrow 0,$$

coming up in Nekovar's theory of height pairings

$$H^1_f(V) \times H^1_f(V) \longrightarrow \mathbb{Q}_p$$

We give a general idea of how this works with a simpler example:

 $X = E \setminus 0$

where E/\mathbb{Q} is an elliptic curve of rank 1 with square-free minimal discriminant.

We have

$$h: E(\mathbb{Q}) \longrightarrow \mathbb{Q}_p,$$

the *p*-adic quadratic height.

Thus, if $y \in E(\mathbb{Q})$ is non-torsion, then

$$c_E := h(y)/\log^2(y)$$

is independent of y.

But log is an analytic function on $E(\mathbb{Q}_p)$, while *h* has a decomposition

$$h=h_p+\sum_{\nu\neq p}h_{\nu},$$

with

$$h_p(z) = \int_b^z \alpha \beta + C_E,$$

where

$$C_E = (a_1^2 + 4a_2)/12 - Eis_2(E, \alpha)/12,$$

 α is an integral invariant differential, and $\beta = x\alpha$. But if z is integral, then

$$h(z)=h_p(z).$$

Thus, the equation

$$h(z)/\log^2(z) = c_E = h(y)/\log^2(y)$$

becomes

$$\int_b^z \alpha\beta + (C_E - c_E) \log^2(z) = 0,$$

a defining equation for integral points.

The case of $X_s^+(13)$ is a substantially more complicated version of this argument using relation between the functions

$$h(z), \log_i(z)\log_j(z)$$

for $1 \leq i \leq j \leq 3$.

Some speculations on rational points and critical points

Actually, interested in

$$Im(H^1(G_T, U)) \cap \prod_{v \in T} H^1_f(G_v, U) \subset \prod_{v \in T} H^1(G_v, U),$$

where

$$H^1_f(G_v, U) \subset H^1(G_v, U)$$

is a subvariety defined by some integral or Hodge-theoretic conditions.

In order to apply symplectic techniques, replace U by

$$T^*(1)U := (LieU)^*(1) \rtimes U.$$

Rational points and critical points

Then

$$\prod_{v\in T} H^1(G_v, T^*(1)U)$$

is a symplectic variety and

$$Im(H^1(G_T, T^*(1)U)), \quad \prod_{v \in T} H^1_f(G_v, T^*(1)U)$$

are Lagrangian subvarieties.

Thus, the derived intersection

$$\mathit{Im}(\mathit{H}^{1}(\mathit{G}_{\mathcal{T}},\mathit{T}^{*}(1)\mathit{U}))\cap\prod_{v\in\mathcal{T}}\mathit{H}^{1}_{\mathit{f}}(\mathit{G}_{v},\mathit{T}^{*}(1)\mathit{U})$$

has a [-1]-shifted symplectic structure. Should be the critical set of a function.