Towards a modulo p Langlands correspondence for GL_2

Christophe Breuil
C.N.R.S. & I.H.É.S.
Le Bois-Marie
35 route de Chartres
91440 Bures-sur-Yvette
France
breuil@ihes.fr

Vytautas Paškūnas
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
D-33501 Bielefeld
Germany
paskunas@math.uni-bielefeld.de

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1 Introduction

Fix a prime number p, an integer $f \geq 1$ and let F be an unramified extension of $\mathbb{Q}_p$ of degree f. Let:

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

be a continuous representation. Assuming ρ is "generic", the main aim of this paper is (i) to associate to ρ a (usually infinite) family of smooth admissible representations π of $\mathrm{GL}_2(F)$ over $\overline{\mathbb{F}}_p$ with fixed central character (matching $\det(\rho)$ via local class field theory) and (ii) to prove that these representations are all irreducible and supersingular ([4]) when ρ is irreducible. In the case $f = 1$, i.e. $F = \mathbb{Q}_p$, one can naturally refine this process into a correspondence and associate to ρ a single smooth admissible representation $\pi(\rho)$ (see [6], [17], [14] and the last section). However, when $f > 1$, this is not possible anymore as the family becomes much too big. It is then not clear so far how to formulate a correct "modulo p local Langlands correspondence" and we content ourselves here with the construction and study of the family of representations π associated to ρ.

During the genesis of this paper, the authors have experienced a succession of good and bad surprises (most of the time bad!). In particular, the theory has revealed itself infinitely more complicated than expected at first. Its origin is a conference which was held in February 2006 at the American Institute of Mathematics in Palo Alto. During the open sessions, discussions involving several mathematicians resulted in the construction for $f = 2$ and for each irreducible ρ as above arising from a global Galois representation of (at least) one new supersingular representation π of $\mathrm{GL}_2(F)$ via global (and slightly heuristic) arguments ([10], [16]). This representation π was such that:

$$\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_F)}\, \pi = \oplus_{\sigma \in \mathcal{D}(\rho)} \sigma \qquad (1)$$

where $\mathcal{D}(\rho)$ is the set of weights σ associated to $\rho|_{\mathrm{inertia}}$ ([11], [15]) and where $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_F)}\, \pi$ denotes the socle of π seen as a $\mathrm{GL}_2(\mathcal{O}_F)$-representation. Recall that a weight is an irreducible representation of $\mathrm{GL}_2(\mathcal{O}_F)$ over $\overline{\mathbb{F}}_p$ and that the $\mathrm{GL}_2(\mathcal{O}_F)$-socle is the maximal semi-simple $\mathrm{GL}_2(\mathcal{O}_F)$-subrepresentation. Let us emphasize that none of the smooth admissible representations of $\mathrm{GL}_2(F)$ that were known at the time ([25]) satisfied condition (1).

After the conference, the two authors tried to construct representations π satisfying (1) via purely local means and, more generally, embarked on the project of trying to classify all smooth irreducible admissible representations of $\mathrm{GL}_2(F)$ over $\overline{\mathbb{F}}_p$. The first good surprise was that, using a generalization of the main construction of [25], it was indeed possible to construct for $f = 2$ new supersingular representations of $\mathrm{GL}_2(F)$ having property (1) for ρ irreducible and "generic". Unfortunately soon after, came the first bad surprise: there was an infinity of such representations! Such a phenomena could not happen for $f = 1$. A little later, we realized the situation was even worse: not only were there infinitely many representations π satisfying (1), but a host of other supersingular representations also existed for $f = 2$ with arbitrary $\mathrm{GL}_2(\mathcal{O}_F)$-socles having nothing to do with that of (1) for any ρ. The naïve hope for a simple 1-1 local Langlands correspondence as for the case $f = 1$ had gone away...

However, we still hoped for a simple classification of all admissible irreducible representations of $\mathrm{GL}_2(F)$. All of the above new irreducible representations for $f = 2$ were constructed via a general process (available for any local field F with finite residue field) involving a finite group theoretic structure called an irreducible "basic 0-diagram" (see below or §9). These irreducible basic 0-diagrams turn out to be much more numerous when $f > 1$ (just as happens with representations), so our natural hope was: may-be there are just as many irreducible basic 0-diagrams as irreducible admissible representations of $\mathrm{GL}_2(F)$ over $\overline{\mathbb{F}}_p$, as happens for $f = 1$? More work soon convinced us that for F unramified over $\mathbb{Q}_p$ and distinct from $\mathbb{Q}_p$ this was not the case: one single irreducible basic 0-diagram could lead to an infinite family of supersingular representations. Besides, a reducible basic 0-diagram could also lead to irreducible representations. To have a glimpse at how complicated the situation can be for $f = 2$ (compared to $f = 1$), the interested reader should take a look at §10.

Since a full classification of smooth irreducible admissible representations seemed too complicated, we decided to focus only on those representations satisfying (1) and at least try to associate to ρ a reasonable (although usually infinite) family of such representations. Then came a good surprise. We had noticed that extending our construction from $f = 2$ to $f = 3$ seemed to involve reducible basic 0-diagrams even in the case where ρ is irreducible and hence (following our previous hope) seemed to lead to reducible representations of $\mathrm{GL}_2(F)$. Just after we realized this hope was erroneous, we discovered that the representations of $\mathrm{GL}_2(F)$ we could associate to ρ irreducible for $f = 3$ were indeed irreducible even though the basic 0-diagrams were not. This phenomena comforted us with the hope that we were constructing interesting representations in that case. Finally, to extend these results from

$f = 2, 3$ to arbitrary f (including the cases where ρ is reducible), we found that the condition we needed on the basic 0-diagrams was a certain multiplicity one assumption together with a maximality condition (see below or §13).

Let us now explain with more details the main results of this paper.

Let $I \subset \mathrm{GL}_2(\mathcal{O}_F)$ be the Iwahori subgroup of upper triangular matrices modulo p, $I_1 \subset I$ its maximal pro-p subgroup, $K_1 \subset I_1$ the first congruence subgroup of $\mathrm{GL}_2(\mathcal{O}_F)$ and $\mathfrak{K}_1 \subset \mathrm{GL}_2(F)$ the normalizer of I in $\mathrm{GL}_2(F)$. If $\chi : I \to \overline{\mathbb{F}}_p$ is a smooth character, let $\chi^s := \chi(\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)^{-1})$ be the conjugate character. Our main idea to construct representations π of $\mathrm{GL}_2(F)$ over $\overline{\mathbb{F}}_p$ is to first construct what we ultimately hope to be the "right" triple $(\pi^{K_1}, \pi^{I_1}, \mathrm{can})$. Here π^{K_1} is seen as a representation of $\mathrm{GL}_2(\mathcal{O}_F)/K_1 = \mathrm{GL}_2(\mathbb{F}_q)$ where $\mathbb{F}_q$ is the residue field of $\mathcal{O}_F$, π^{I_1} is seen as a representation of $\mathfrak{K}_1$ and can is the canonical injection.

Theorem 1.1. *Fix a Galois representation ρ as above and assume ρ is generic (Definition 11.7).*

(i) There exists a unique finite dimensional representation $D_0(\rho)$ of $\mathrm{GL}_2(\mathbb{F}_q)$ over $\overline{\mathbb{F}}_p$ such that:

  (a) $\mathrm{soc}_{\mathrm{GL}_2(\mathbb{F}_q)} D_0(\rho) \simeq \oplus_{\sigma \in \mathcal{D}(\rho)} \sigma$

  (b) each irreducible σ in $\mathcal{D}(\rho)$ only occurs once as a Jordan-Hölder factor of $D_0(\rho)$ (hence in the socle)

  (c) $D_0(\rho)$ is maximal for properties (a) and (b).

(ii) Each Jordan-Hölder factor of $D_0(\rho)$ only occurs once in $D_0(\rho)$.

(iii) As an I-representation, one has:

$$D_0(\rho)^{I_1} \simeq \bigoplus_{\substack{\text{certain } (\chi, \chi^s) \\ \chi \neq \chi^s}} \chi \oplus \chi^s$$

(in particular $D_0(\rho)^{I_1}$ is stable under $\chi \mapsto \chi^s$).

(iv) Assume ρ is tamely ramified, that is either split or irreducible, then:

$$\dim_{\overline{\mathbb{F}}_p} D_0(\rho)^{I_1} = 3^f \pm 1$$

with $+$ in the reducible case and $-$ in the irreducible case.

(v) Assume ρ is reducible non-split and let $d \in \{0, \cdots, f-1\}$ such that $|\mathcal{D}(\rho)| = 2^d$, then:
$$\dim_{\overline{\mathbb{F}}_p} D_0(\rho)^{I_1} = 2^{f-d}3^d.$$

Let us point out the following important comment concerning the above theorem. First, (i) is a general fact that works for any set of distinct weights (not just the sets $\mathcal{D}(\rho)$), see §13. But (ii) seems quite specific to the combinatorics of the weights of $\mathcal{D}(\rho)$, see §12. In particular, when ρ is reducible non-split, we rely in (ii) on a property of $\mathcal{D}(\rho)$ which we call being of Galois type (see §11). Moreover, to construct and study the representation $D_0(\rho)$, in particular to prove (iv) and (v) above, we need a fine knowledge of the injective envelope of a weight that we couldn't find in the literature (where the results were not strong enough and only available for $\mathrm{SL}_2(\mathbb{F}_q)$). We were therefore forced to provide our own proofs (see §§2 to 4). For instance, let $\sigma := \mathrm{Sym}^{r_0}\overline{\mathbb{F}}_p^2 \otimes_{\overline{\mathbb{F}}_p} (\mathrm{Sym}^{r_1}\overline{\mathbb{F}}_p^2)^{\mathrm{Fr}} \otimes \cdots \otimes_{\overline{\mathbb{F}}_p} (\mathrm{Sym}^{r_{f-1}}\overline{\mathbb{F}}_p^2)^{\mathrm{Fr}^{f-1}}$ be a weight with all r_i in $\{0, \cdots, p-2\}$ (see below for notations) and let V_σ be the maximal representation of $\mathrm{GL}_2(\mathbb{F}_q)$ with socle σ such that σ occurs only once in V_σ. We completely determine the structure of the representation V_σ including its socle and cosocle filtrations (see Proposition 3.6 and Theorem 4.7). We also completely determine the structure of the $\mathrm{GL}_2(\mathbb{F}_q)$ representation $D_0(\rho)$ in Theorem 1.1 when ρ is tamely ramified: see §13 and §14 for details.

Let us call a basic 0-diagram any triple $D := (D_0, D_1, r)$ where D_0 is a smooth representation of $\mathrm{GL}_2(\mathcal{O}_F)F^\times$ over $\overline{\mathbb{F}}_p$ such that $p \in F^\times$ acts trivially, D_1 a smooth representation of $\mathfrak{K}_1$ over $\overline{\mathbb{F}}_p$ and $r : D_1 \hookrightarrow D_0$ an injection inducing an $IF^\times$-equivariant isomorphism $D_1 \xrightarrow{\sim} D_0^{I_1}$. For instance $(\pi^{K_1}, \pi^{I_1}, \mathrm{can})$ for π a smooth representation of $\mathrm{GL}_2(F)$ over $\overline{\mathbb{F}}_p$ is such a diagram. Let us say that a basic 0-diagram is irreducible if it doesn't contain any non-zero strict basic subdiagram (in the obvious sense).

Theorem 1.2. *Let $D = (D_0, D_1, r)$ be a basic 0-diagram such that D_0 is finite dimensional and K_1 acts trivially on D_0.*

(i) There exists at least one smooth admissible representation π of $\mathrm{GL}_2(F)$ over $\overline{\mathbb{F}}_p$ such that:

  (a) $\mathrm{soc}_K\, \pi = \mathrm{soc}_K\, D_0$

  (b) $(\pi^{K_1}, \pi^{I_1}, \mathrm{can})$ *contains D*

  (c) π is generated by D_0.

(ii) Assume D is irreducible. Then any π satisfying (a), (b), (c) of (i) is irreducible.

This theorem has to be thought of as an existence theorem only, as unicity in (i) is wrong in general. Moreover, it has nothing to do with F unramified over $\mathbb{Q}_p$ and works for any local field F with finite residue field, see §9 (it is a special case of Theorem 9.8 by taking π to be the G-subrepresentation generated by D_0). The idea is to build π inside the injective envelope $\mathrm{Inj}\, D_0$ of the $\mathrm{GL}_2(\mathcal{O}_F)$-representation D_0 in the category of smooth representations of $\mathrm{GL}_2(\mathcal{O}_F)$ over $\overline{\mathbb{F}}_p$. Roughly speaking, the main point is to prove one can non-canonically extend the action of I on $\mathrm{Inj}\, D_0$ to an action of $\mathfrak{K}_1$ such that $(\mathrm{Inj}\, D_0, \mathrm{Inj}\, D_0, \mathrm{id})$ contains D (up to isomorphism), which is possible as injective envelopes are very flexible. Then the two compatible actions of $\mathrm{GL}_2(\mathcal{O}_F)$ and $\mathfrak{K}_1$ on the same vector space $\mathrm{Inj}\, D_0$ glue to give an action of $\mathrm{GL}_2(F)$ and we define π as the subspace generated by D_0. The whole process is highly non-canonical both because the action of $\mathrm{GL}_2(\mathcal{O}_F)$ on $\mathrm{Inj}\, D_0$ is only defined up to non-unique isomorphism and because the extension to an action of $\mathfrak{K}_1$ involves choices. Note also that the converse to (ii) is wrong in general: reducible basic 0-diagrams (in the above sense) can lead to π as in (i) being irreducible (we provide ample examples in the sequel). However, one can prove under certain mild conditions that any admissible irreducible π gives rise to an irreducible basic "e-diagram" for some $e \geq 0$, see Theorem 9.13. Unfortunately, basic e-diagrams when $e > 0$ are much more difficult to handle than basic 0-diagrams.

Let us now go back to the setting of the first theorem and assume that p acts trivially on $\det(\rho)$ (via the local reciprocity map) which is always possible up to twist. One can use (iii) of Theorem 1.1 to extend the action of I on $D_0(\rho)^{I_1}$ to an action of $\mathfrak{K}_1$. Moreover, multiplicity 1 in (ii) implies that this extension is unique up to isomorphism and we denote by $D_1(\rho)$ the resulting representation of $\mathfrak{K}_1$. The idea is then to use $D_0(\rho)$ and $D_1(\rho)$ to associate a basic 0-diagram to ρ but one needs to choose an $IF^\times$-equivariant injection $r : D_1(\rho) \hookrightarrow D_0(\rho)$. Up to isomorphisms of commutative diagrams, it turns out there are infinitely many such injections as soon as $f > 1$. Denote by $D(\rho, r) := (D_0(\rho), D_1(\rho), r)$ any such basic 0-diagram. Most of the time, $D(\rho, r)$ is not irreducible, but one can prove the following structure theorem:

Theorem 1.3. *Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous generic representation such that p acts trivially on $\det(\rho)$ and let $D(\rho, r)$ be one of the basic 0-diagrams associated to ρ.*

(i) Assume ρ is indecomposable, then $D(\rho, r)$ cannot be written as the direct sum of two non-zero basic 0-diagrams.

(ii) Assume ρ is reducible split, then we have:

$$D(\rho, r) \cong \bigoplus_{\ell=0}^{f} D(\rho, r_\ell)$$

where $D(\rho, r_\ell)$ is a non-zero basic 0-diagram that cannot be written as the direct sum of two non-zero basic 0-diagrams.

In fact, $D(\rho, r_\ell) = (D_{0,\ell}(\rho), D_{1,\ell}(\rho), r_\ell)$ where $D_{0,\ell}(\rho)$ and $D_{1,\ell}(\rho)$ only depend on ρ and not on r. When ℓ varies, the $\mathrm{GL}_2(\mathbb{F}_q)$-representations $D_{0,\ell}(\rho)$ do not have the same flavour. For instance $D_{0,0}(\rho)$ and $D_{0,f}(\rho)$ have an irreducible socle but this is not the case for the other $D_{0,\ell}(\rho)$. Note that the basic 0-diagrams $D(\rho, r_\ell)$ are usually not irreducible.

Applying (i) of Theorem 1.2 to the diagrams $D(\rho, r)$, one gets:

Theorem 1.4. *Keep the setting of Theorem 1.3.*

(i) There exists a smooth admissible representation π of $\mathrm{GL}_2(F)$ such that:
(a) $\mathrm{soc}_K\, \pi = \bigoplus_{\sigma \in \mathcal{D}(\rho)} \sigma$
(b) $(\pi^{K_1}, \pi^{I_1}, \mathrm{can})$ contains $D(\rho, r)$
(c) π is generated by $D_0(\rho)$.

(ii) If $D(\rho, r)$ and $D(\rho, r')$ are two non-isomorphic basic 0-diagrams associated to ρ, and π, π' satisfy (a), (b), (c) respectively for $D(\rho, r)$ and $D(\rho, r')$, then π and π' are non-isomorphic.

The proof of (ii) crucially relies on property (i) of Theorem 1.1 defining $D_0(\rho)$. We also have an exactly similar theorem replacing everywhere π by π_ℓ and $D(\rho, r)$ by $D(\rho, r_\ell)$ for $0 \leq \ell \leq f$ (see Theorem 19.9).

For a given generic ρ, the family of all π satisfying (i) of Theorem 1.4 for all $D(\rho, r)$ is the family of admissible representations of $\mathrm{GL}_2(F)$ we associate to ρ in this paper. One big task, which we only start here, is to better understand this family in order eventually to rule out some of the representations it contains. For instance, some of the representations π in this family are such that π^{I_1} contains strictly $D_1(\rho)$ ([22]). One could thus only consider those π in the family satisfying $(\pi^{K_1}, \pi^{I_1}, \mathrm{can}) \simeq D(\rho, r)$. L. Dembélé's appendix in [9] suggests that there should at least exist some π with $\pi^{I_1} = D_1(\rho)$. In another direction, [8] suggests a refinement based on the condition that a certain (φ, Γ)-module associated to a π in this family should be exactly the (φ, Γ)-module of the tensor induction of ρ from F to $\mathbb{Q}_p$. However, those two refinements are probably still not enough to select, e.g., a finite subset of representations in the above infinite family of representations associated to ρ (if $F \neq \mathbb{Q}_p$).

Theorem 1.3 suggests that the right $\mathrm{GL}_2(F)$-representations associated to ρ should also somehow satisfy the same properties as $D(\rho, r)$, i.e. should be indecomposable (resp. semi-simple) if and only if ρ is. We at least have the following irreducibility result:

Theorem 1.5. *Keep the setting of Theorem 1.3.*

(i) Assume ρ is irreducible. Then any π as in (i) of Theorem 1.4 is irreducible and is a supersingular representation.

(ii) Assume ρ is reducible split. Then any π_ℓ as in (i) of Theorem 1.4 for $D(\rho, r_\ell)$ is irreducible. Moreover, π_ℓ is a principal series if $\ell \in \{0, f\}$ and is a supersingular representation otherwise.

For instance, when ρ is semi-simple split, (ii) of Theorem 1.3 together with (ii) of Theorem 1.5 imply that the representations $\oplus_{\ell=0}^{f} \pi_\ell$ belong to the family associated to ρ and are semi-simple. The proof of Theorem 1.5 is too technical to be described here (in particular it can't follow from (ii) of Theorem 1.2 as the basic 0-diagrams involved are not irreducible). It relies on controlling $\mathrm{GL}_2(\mathcal{O}_F)$-extensions between weights in certain quotients of the compact induction $\text{c-Ind}_{\mathrm{GL}_2(\mathcal{O})F^\times}^{\mathrm{GL}_2(F)} \sigma$ (see Lemma 18.4). This uses computations with Witt vectors and we suspect that the argument here breaks down when F is ramified over $\mathbb{Q}_p$. The existence of many non-split such extensions, together with Theorem 1.3, is responsible for the irreducibility of the above representations π or π_ℓ (see §19). Note that the proof requires one to distinguish which $\mathrm{GL}_2(\mathcal{O}_F)$-extensions between two weights are actually $\mathrm{GL}_2(\mathbb{F}_q)$-extensions, which is done in detail in §5.

For some time, we hoped that, given a basic 0-diagram $D(\rho, r)$ as in Theorem 1.3, there was a unique smooth representation π of $\mathrm{GL}_2(F)$ generated by its K_1-invariants vectors and such that $(\pi^{K_1}, \pi^{I_1}, \mathrm{can}) \simeq D(\rho, r)$ (and we even dared state this as a conjecture!). However, due to recent work of Y. Hu ([22]), this statement is false when $F \neq \mathbb{Q}_p$ (i.e. such a representation, if it exists, is in general not unique). Hu went on in [21] to associate to π what he calls a canonical diagram (which is not a basic 0-diagram in general) that contains $D(\rho, r)$ (more precisely that contains $(\langle \mathrm{GL}_2(\mathcal{O}_F) \cdot D_1(\rho)\rangle, D_1(\rho), r)$) and that uniquely determines π. However, the explicit computation of this canonical diagram when $F \neq \mathbb{Q}_p$ is still unknown.

For $F = \mathbb{Q}_p$, everything works well: our basic 0-diagram determines π and we have the following result:

Theorem 1.6. *Assume $F = \mathbb{Q}_p$ and keep the setting of Theorem 1.3.*

(i) There exists a unique (up to isomorphism) smooth representation $\pi(\rho, r)$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ which is generated by its K_1-invariant vectors and such that:

$$(\pi(\rho, r)^{K_1}, \pi(\rho, r)^{I_1}, \mathrm{can}) \cong D(\rho, r).$$

(ii) If ρ is irreducible, this representation is irreducible.

(iii) If ρ is semi-simple, then $\pi(\rho, r) \simeq \pi(\rho, r)_0 \oplus \pi(\rho, r)_1$ where $\pi(\rho, r)_\ell$ ($\ell \in \{0, 1\}$) is a smooth irreducible admissible principal series of $\mathrm{GL}_2(\mathbb{Q}_p)$ such that $(\pi(\rho, r)_\ell^{K_1}, \pi(\rho, r)_\ell^{I_1}, \mathrm{can}) \cong D(\rho, r_\ell)$.

(iv) If ρ is indecomposable, then $\pi(\rho, r)^{\mathrm{ss}} \simeq \pi(\rho^{\mathrm{ss}}, r^{\mathrm{ss}})_0 \oplus \pi(\rho^{\mathrm{ss}}, r^{\mathrm{ss}})_1$ where ρ^{ss} is the semi-simplification of ρ and $\pi(\rho^{\mathrm{ss}}, r^{\mathrm{ss}})_\ell$ ($\ell \in \{0, 1\}$) is a smooth irreducible principal series as in (iii). Moreover, the $\mathrm{GL}_2(\mathbb{Q}_p)$-socle of $\pi(\rho, r)$ is $\pi(\rho^{\mathrm{ss}}, r^{\mathrm{ss}})_0$.

Cases of this theorem were already known for a long time thanks to [6], [14] or [17] but we provide here an essentially complete proof (§20). The main novelty concerns the case (iv) (ρ reducible non-split) where we correspondingly have a non-trivial extension between two principal series. The existence of such extensions was already known (see [14] and [17]) but it was not known that, when ρ is generic, they are completely determined by their K_1-invariants. Our proof relies on the computation of the first derived functor $H^1(I_1, \pi)$ of the functor $\pi \mapsto \pi^{I_1}$ as a module over the Hecke algebra of I_1 when π is a principal series (see Theorem 7.16). As the proof is not substantially longer, we give it for all extensions F of $\mathbb{Q}_p$ (even ramified) although we only use it for $F = \mathbb{Q}_p$. An interesting fact is that this computation shows the appearance of supersingular Hecke-modules ([33]) in $H^1(I_1, \pi)$ if and only if F is not $\mathbb{Q}_p$, which seems to be consistent with the appearance of the $D(\rho, r_\ell)$ for $1 \leq \ell \leq f - 1$ in (ii) of Theorem 1.3 when $f > 1$. We are confident that Theorem 7.16 will find other applications in the future. Our method for computing extensions extends to supersingular representations of $\mathrm{GL}_2(\mathbb{Q}_p)$, (unlike [14] and [17]), see [27]. Let us add that, for $f = 1$, the canonical diagram of [21] is exactly $(\langle \mathrm{GL}_2(\mathcal{O}_F) \cdot D_1(\rho) \rangle, D_1(\rho), r)$.

An exegesis to this paper, much easier to read but lacking many technical proofs, is available in [7]. The reader will have realized that this paper contains more questions than answers. Apart from understanding the family of representations π we associate to a given ρ (that is, the representations constructed in Theorem 1.4) it seems to us that the most important problem is to prove that the representations of $\mathrm{GL}_2(F)$ that appear as subobject in the étale cohomology modulo p of towers of p-power level Shimura varieties (over totally real fields at places above p where the real field is unramified) belong to such families. The papers [18] and [9] give first important steps in that direction. If so, then comes the question of the cohomological meaning of the various "parameters" r of the family $(D(\rho, r))_r$: for instance can several distinct r "occur" (via some π as in Theorem 1.4) on various cohomology groups for a given ρ? The paper [8] (see also [9, §8]) singles out a few parameters r, but most of them remain mysterious. One can also wonder how one can build the right $\mathrm{GL}_2(F)$-representations associated to ρ when ρ is not generic (for example trivial). Also, can one extend the constructions via (φ, Γ)-modules of the case $F = \mathbb{Q}_p$ ([14], [5])? Despite partial results ([8], [34]), it is still not known how to (or if one can) produce functorially a finite dimensional (φ, Γ)-module from a finite length admissible π. The case where F is ramified over $\mathbb{Q}_p$ is largely open although a set of weights analogous to $\mathcal{D}(\rho)$ exists thanks to [28] (see however [19] and [29]). And there still simply remains the open problem to classify all supersingular representations of $\mathrm{GL}_2(F)$ over $\overline{\mathbb{F}}_p$ when $f > 1$. Note that the non-supersingular representations of $\mathrm{GL}_n(F)$ over $\overline{\mathbb{F}}_p$ are now well understood thanks to [20] up to understanding the supersingular representations of $\mathrm{GL}_m(F)$ for $m < n$. Finally, let us point out that the p-adic theory is not expected to be simpler (see [26]).

Let us now quickly describe the organization of the paper.

In §§2 to 4, we provide the necessary results or references on the representation theory of the group $\mathrm{GL}_2(\mathbb{F}_q)$ over $\overline{\mathbb{F}}_p$. In §5, we study $\mathrm{GL}_2(\mathcal{O}_F)$-extensions between two weights, in particular we explain which such extensions are $\mathrm{GL}_2(\mathbb{F}_q)$-extensions. In §6, we give preliminary material on the Hecke modules for I_1 associated to principal series of $\mathrm{GL}_2(F)$. This material is used in §7 to compute the Hecke module $H^1(I_1, \pi)$ when π is a principal series. This computation is then used in §8 to construct extensions between principal series for $F = \mathbb{Q}_p$ and to show that, in that case, a principal series has no non-trivial extension with a supersingular representation. In §9, we develop a general theory of diagrams for any local field F with finite residue field and we prove Theorem 1.2. In §10, we give the most obvious examples of basic 0-diagrams, in particular we list all irreducible such diagrams for $F = \mathbb{Q}_p$, and we show that the situation gets more complicated when $F \neq \mathbb{Q}_p$. In §11, we define generic Galois representations ρ and recall the set of weights $\mathcal{D}(\rho)$ associated to them in [11]. In §12, we prove a combinatorial unicity Lemma involving these weights which is used in §13 to prove (ii) of Theorem 1.1. In §13, we also prove (i) and (iii) of Theorem 1.1 and define the basic 0-diagrams $(D_0(\rho), D_1(\rho), r)$. In §14, we study more closely the $\mathrm{GL}_2(\mathbb{F}_q)$-representation $D_0(\rho)$ and prove (iv) and (v) of Theorem 1.1. In §15, we prove Theorem 1.3. In §16, we give explicitly the diagrams $(D_0(\rho), D_1(\rho), r)$ when $f = 1$ and $f = 2$. In §17, for each non-trivial weight σ we define and study a $\mathrm{GL}_2(\mathcal{O}_F)$-subrepresentation $R(\sigma)$ of $\text{c-Ind}_{\mathrm{GL}_2(\mathcal{O}_F)F^\times}^{\mathrm{GL}_2(F)} \sigma$. We use these results in §18 to prove that $R(\sigma)$ contains many non-split extensions between weights. In §19, we prove Theorem 1.4 and use the results of §§17 and 18 to prove Theorem 1.5. Finally, in §20, we prove Theorem 1.6.

Let us now fix the main notations of the text.

Throughout the paper, we denote by $\mathcal{O}_F$ a complete discrete valuation ring with fraction field F, residue field $\mathbb{F}_q = \mathbb{F}_{p^f}$, and maximal ideal $\mathfrak{p}_F$. We fix a uniformizer ϖ of $\mathcal{O}_F$ which is p when F is unramified over $\mathbb{Q}_p$ (so $\mathfrak{p}_F = \varpi\mathcal{O}_F$). We also fix once and for all an embedding $\mathbb{F}_{p^f} \hookrightarrow \overline{\mathbb{F}}_p$.

We let $\Gamma' := \mathrm{SL}_2(\mathbb{F}_q)$, $\Gamma := \mathrm{GL}_2(\mathbb{F}_q)$, $B \subset \Gamma$ the subgroup of upper triangular matrices, $U \subset B$ the subgroup of upper unipotent matrices, H the subgroup of diagonal matrices, $K := \mathrm{GL}_2(\mathcal{O}_F)$ and $I \subset K$ the subgroup of matrices that are sent to B via the reduction map $K \twoheadrightarrow \Gamma$. Recall that we have a group isomorphism $H \xrightarrow{\sim} B/U$ and a bijection $K/I \xrightarrow{\sim} \Gamma/B$. We also denote by U^s the subgroup of lower unipotent matrices. For $m \geq 1$, we define the following subgroups of K:

$$I_m := \begin{pmatrix} 1+\mathfrak{p}_F^m & \mathfrak{p}_F^{m-1} \\ \mathfrak{p}_F^m & 1+\mathfrak{p}_F^m \end{pmatrix} \quad \text{and} \quad K_m := \begin{pmatrix} 1+\mathfrak{p}_F^m & \mathfrak{p}_F^m \\ \mathfrak{p}_F^m & 1+\mathfrak{p}_F^m \end{pmatrix}.$$

For instance, $I_1 \subset I$ is the subgroup of matrices that are sent to $U \subset B$ via $K \twoheadrightarrow \Gamma$. We set $G := \mathrm{GL}_2(F)$, $Z := F^\times$ the center of G, $\mathfrak{K}_0 := KZ$ and $\mathfrak{K}_1$ the normalizer of I in G. We let $s := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $n_s := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\Pi := \left(\begin{smallmatrix} 0 & 1 \\ \varpi & 0 \end{smallmatrix}\right)$. We let $P \subset G$ (resp. $P^s := sPs^{-1} \subset G$) be the subgroup of upper (resp. lower) triangular matrices and $T := P \cap P^s$ the diagonal matrices. Recall that we have $\mathfrak{K}_1 = IZ \amalg IZ\Pi$.

All representations are over $\overline{\mathbb{F}}_p$-vector spaces. We denote by Rep_Γ (resp. Rep_G, Rep_K, Rep_I, Rep_{K_m}, etc.) the category of finite dimensional (resp.

smooth) representations of Γ (resp. G, K, I, K_m, etc.) over $\overline{\mathbb{F}}_p$. If $S \in \mathrm{Rep}_\Gamma$ (resp. Rep_G, Rep_K, etc.) and $E \subseteq S$ is any subset, we denote by $\langle \Gamma \cdot E\rangle$ (resp. $\langle G \cdot E\rangle$, $\langle K \cdot E\rangle$, etc.) the subrepresentation of S generated by E under the action of the group. If $S \in \mathrm{Rep}_\Gamma$, we denote by $\mathrm{inj}\, S$ the injective envelope of S in Rep_Γ ([31, §14], [25, §4]) and if $S \in \mathrm{Rep}_K$ (resp. Rep_I), we denote by $\mathrm{Inj}\, S$ the injective envelope of S in Rep_K (resp. Rep_I) ([25, §6.2]). If $\chi : F^\times \to \overline{\mathbb{F}}_p^\times$ is a smooth character, we denote by $\mathrm{Rep}_{G,\chi}$ (resp. $\mathrm{Rep}_{K,\chi}$, $\mathrm{Rep}_{I,\chi}$) those smooth representations which have central character χ. If $S \in \mathrm{Rep}_\Gamma$ is non-zero and indecomposable, we denote by $(S_i)_{0\leq i}$ (resp. $(S^i)_{0\leq i}$) the graded pieces of its socle filtration (resp. of its cosocle filtration, also called its radical filtration) with $S_0 \neq 0$ (resp. $S^0 \neq 0$) (see e.g. [1, §I.1]). We denote by $\mathrm{soc}_\Gamma\, S = S_0$ its socle and by $\mathrm{cosoc}_\Gamma\, S$ its cosocle, that is, its maximal semi-simple quotient. If $S \in \mathrm{Rep}_K$, we denote by $\mathrm{soc}_K\, S$ its K-socle. If $G' \subseteq G$ is a closed subgroup and R a smooth representation of G' on an $\overline{\mathbb{F}}_p$-vector space, we denote by $\text{c-Ind}_{G'}^G R$ the $\overline{\mathbb{F}}_p$-vector space of functions $f : G \to R$ such that $f(g'g) = g' \cdot f(g)$ ($g' \in G'$, $g \in G$) and such that the support of f is compact modulo G'. The group G acts on $\text{c-Ind}_{G'}^G R$ by right translation on functions. If $\chi : B \to \overline{\mathbb{F}}_p^\times$ is a smooth character, we denote by $\mathrm{Ind}_B^\Gamma \chi$ the $\overline{\mathbb{F}}_p$-vector space of functions $f : \Gamma \to \overline{\mathbb{F}}_p$ such that $f(b\gamma) = \chi(b)f(\gamma)$ ($b \in B$, $\gamma \in \Gamma$) with left action of Γ by right translation on functions. Likewise with $\mathrm{Ind}_I^K \chi$.

If $\chi : H \to \overline{\mathbb{F}}_p^\times$ is a character, we denote by χ^s the character $\chi^s(h) := \chi(shs)$ where $h \in H$. We denote by $\alpha : H \to \overline{\mathbb{F}}_p^\times$ the character:

$$\alpha : \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \mapsto \lambda\mu^{-1}$$

that we also see as a character of B or I via $I \twoheadrightarrow B \twoheadrightarrow H$. If σ is an irreducible representation of Γ over $\overline{\mathbb{F}}_p$ and $\chi : I \to \overline{\mathbb{F}}_p^\times$ the character giving the action of I on σ^{I_1}, we denote by $\sigma^{[s]}$ the unique irreducible representation of Γ over $\overline{\mathbb{F}}_p$ which is distinct from σ and such that I acts on $(\sigma^{[s]})^{I_1}$ via χ^s (we can't use σ^s which denotes the conjugation of σ by s). For $x \in \overline{\mathbb{F}}_p^\times$, we let $\delta_x : G \to \overline{\mathbb{F}}_p^\times$, $g \mapsto x^{\mathrm{val}(\det(g))}$ where val is the valuation normalized by $\mathrm{val}(\varpi) := 1$.

We call a weight an irreducible representation of K (hence of Γ) on $\overline{\mathbb{F}}_p$. A weight can be written:

$$\mathrm{Sym}^{r_0}\overline{\mathbb{F}}_p^2 \otimes_{\overline{\mathbb{F}}_p} (\mathrm{Sym}^{r_1}\overline{\mathbb{F}}_p^2)^{\mathrm{Fr}} \otimes \cdots \otimes_{\overline{\mathbb{F}}_p} (\mathrm{Sym}^{r_{f-1}}\overline{\mathbb{F}}_p^2)^{\mathrm{Fr}^{f-1}} \otimes_{\overline{\mathbb{F}}_p} \eta \qquad (2)$$

where the r_i are integers between 0 and $p-1$, η is a smooth character $\mathcal{O}_F^\times \to \overline{\mathbb{F}}_p^\times$, Γ acts on the first Sym via the fixed embedding $\mathbb{F}_{p^f} \hookrightarrow \overline{\mathbb{F}}_p$ and on the others via twists by powers of the Frobenius Fr where $\mathrm{Fr}(x) := x^p$ ($x \in \mathbb{F}_{p^f}$). Throughout the text, we often denote by $(r_0, \cdots, r_{f-1}) \otimes \eta$ the representation (2) (although sometimes $(r_0, \cdots, r_{f-1})$ just means the corresponding f-tuple, the context avoiding any possible confusion). For instance if $\sigma = (r_0, \cdots, r_{f-1})$ then $\sigma^{[s]} = (p-1-r_0, \cdots, p-1-r_{f-1}) \otimes \det^{\sum_{i=0}^{f-1} p^i r_i}$.

We normalize the local reciprocity map so that it sends a geometric Frobenius to a uniformizer. Using the fixed embedding $\mathbb{F}_{p^f} \hookrightarrow \overline{\mathbb{F}}_p$, we define:

$$\omega : \mathcal{O}_F \twoheadrightarrow \mathcal{O}_F/\varpi\mathcal{O}_F \simeq \mathbb{F}_{p^f} \hookrightarrow \overline{\mathbb{F}}_p.$$

When $F = \mathbb{Q}_{p^f}$ is unramified, we define:

$$\omega_f : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^f}) \to \overline{\mathbb{F}}_p^\times \tag{3}$$

via the local reciprocity map as the unique character which is the reduction modulo p on $\mathbb{Z}_{p^f}^\times$ and which sends p to 1.

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2 Representation theory of Γ over $\overline{\mathbb{F}}_p$ I

In this section, we study the structure of the principal series of Γ over $\overline{\mathbb{F}}_p$.

Fix $\sigma := (r_0, \cdots, r_{f-1}) \otimes \eta$ a weight. Then B acts on σ^U by a character:

$$\chi : \begin{pmatrix} a & * \\ 0 & d \end{pmatrix} \mapsto a^r \eta(ad)$$

where $r := \sum_{i=0}^{f-1} p^i r_i$. Recall the space $\mathrm{Ind}_B^\Gamma \chi$ (resp. $\mathrm{Ind}_B^\Gamma \chi^s$) has dimension $q+1$ and is isomorphic to $\mathrm{Ind}_I^K \chi$ (resp. $\mathrm{Ind}_I^K \chi^s$) in an obvious way. We now recall results on the structure of $\mathrm{Ind}_B^\Gamma \chi^s$ and $\mathrm{Ind}_B^\Gamma \chi$, mainly from [3]. First, we give its Jordan-Hölder components.

Let $(x_0, \cdots, x_{f-1})$ be f variables. We define a set $\mathcal{P}(x_0, \cdots, x_{f-1})$ of f-tuples $\lambda := (\lambda_0(x_0), \cdots, \lambda_{f-1}(x_{f-1}))$ where $\lambda_i(x_i) \in \mathbb{Z} \pm x_i$ as follows. If $f = 1$, $\lambda_0(x_0) \in \{x_0, p-1-x_0\}$. If $f > 1$, then:

(i) $\lambda_i(x_i) \in \{x_i, x_i - 1, p-2-x_i, p-1-x_i\}$ for $i \in \{0, \cdots, f-1\}$

(ii) if $\lambda_i(x_i) \in \{x_i, x_i - 1\}$, then $\lambda_{i+1}(x_{i+1}) \in \{x_{i+1}, p-2-x_{i+1}\}$

(iii) if $\lambda_i(x_i) \in \{p-2-x_i, p-1-x_i\}$, then $\lambda_{i+1}(x_{i+1}) \in \{p-1-x_{i+1}, x_{i+1}-1\}$

with the conventions $x_f = x_0$ and $\lambda_f(x_f) = \lambda_0(x_0)$.

For $\lambda \in \mathcal{P}(x_0, \cdots, x_{f-1})$, define:

$$\begin{aligned} e(\lambda) &:= \frac{1}{2}\Big(\sum_{i=0}^{f-1} p^i(x_i - \lambda_i(x_i))\Big) \text{ if } \lambda_{f-1}(x_{f-1}) \in \{x_{f-1}, x_{f-1} - 1\} \\ e(\lambda) &:= \frac{1}{2}\Big(p^f - 1 + \sum_{i=0}^{f-1} p^i(x_i - \lambda_i(x_i))\Big) \text{ otherwise.} \end{aligned}$$

The following straightforward lemma is left to the reader.

Lemma 2.1. *One has* $e(\lambda) \in \mathbb{Z} \oplus \bigoplus_{i=0}^{f-1} \mathbb{Z}x_i$.

Lemma 2.2. *The irreducible subquotients of* $\mathrm{Ind}_B^\Gamma \chi$ *or* $\mathrm{Ind}_B^\Gamma \chi^s$ *are exactly the (all distinct) weights:*

$$(\lambda_0(r_0), \cdots, \lambda_{f-1}(r_{f-1})) \otimes \det{}^{e(\lambda)(r_0, \cdots, r_{f-1})}\eta$$

for $\lambda \in \mathcal{P}(x_0, \cdots, x_{f-1})$ *forgetting the weights such that* $\lambda_i(r_i) < 0$ *for some* i.

Proof. See [15, Prop. 1]. □

For $\lambda \in \mathcal{P}(x_0, \cdots, x_{f-1})$, we define:

$$J(\lambda) := \{i \in \{0, \cdots, f-1\}, \lambda_i(x_i) \in \{p-2-x_i, p-1-x_i\}\}$$

and set $\ell(\lambda) := |J(\lambda)|$. If $\lambda, \lambda' \in \mathcal{P}(x_0, \cdots, x_{f-1})$, we write $\lambda' \leq \lambda$ if $J(\lambda') \subseteq J(\lambda)$. If τ is an irreducible subquotient of $\mathrm{Ind}_B^\Gamma \chi^s$ and $\lambda \in \mathcal{P}(x_0, \cdots, x_{f-1})$ its associated f-tuple by Lemma 2.2, we set $\ell(\tau) := \ell(\lambda)$. We also write $\tau' \leq \tau$ if the corresponding f-tuples λ', λ satisfy $\lambda' \leq \lambda$. The following lemma is well known.

Lemma 2.3. *Assume $\chi = \chi^s$, then:*

$$\mathrm{Ind}_B^\Gamma \chi = \mathrm{Ind}_B^\Gamma \chi^s \simeq (0, \cdots, 0) \otimes \eta \oplus (p-1, \cdots, p-1) \otimes \eta.$$

The following theorem is easily derived from the results of [3]. It can also be derived from Theorem 4.7 below.

Theorem 2.4. *Assume $\chi \neq \chi^s$.*

(i) The socle and cosocle filtrations (see e.g. [1, §I.1]) on $\mathrm{Ind}_B^\Gamma \chi^s$ are the same, with graded pieces:

$$(\mathrm{Ind}_B^\Gamma \chi^s)_i = \bigoplus_{\ell(\tau)=i} \tau$$

for $0 \leq i \leq f$.

(ii) We have $(\mathrm{Ind}_B^\Gamma \chi)_i = (\mathrm{Ind}_B^\Gamma \chi^s)_{f-i}$.

(iii) Let τ be an irreducible subquotient of $\mathrm{Ind}_B^\Gamma \chi^s$ and $U(\tau)$ the unique subrepresentation with cosocle τ. Then the socle and cosocle filtrations on $U(\tau)$ are the same, with graded pieces:

$$(U(\tau))_i = \bigoplus_{\substack{\ell(\tau')=i \\ \tau' \leq \tau}} \tau'$$

for $0 \leq i \leq \ell(\tau)$.

(iv) Let τ be an irreducible subquotient of $\mathrm{Ind}_B^\Gamma \chi^s$ and $Q(\tau)$ the unique quotient with socle τ. Then the socle and cosocle filtrations on $Q(\tau)$ are the same, with graded pieces:

$$(Q(\tau))_i = \bigoplus_{\substack{\ell(\tau')=i+\ell(\tau) \\ \tau \leq \tau'}} \tau'$$

for $0 \leq i \leq f - \ell(\tau)$.

Let $\phi \in \mathrm{Ind}_B^\Gamma \chi^s$ with support in B such that $\phi(u) = 1$ for all $u \in U$. In particular, ϕ is U-invariant and H acts on ϕ via the character χ^s (we recall that $\chi^s : B \to \overline{\mathbb{F}}_p^\times$ is the character $\left(\begin{smallmatrix} a & * \\ 0 & d \end{smallmatrix}\right) \mapsto d^r \eta(ad)$). For $0 \leq j \leq q-1$, set:

$$f_j := \sum_{\lambda \in \mathbb{F}_q} \lambda^j \begin{pmatrix} \lambda & 1 \\ 1 & 0 \end{pmatrix} \phi$$

with the convention $0^0 = 1$ and $0^{q-1} = 0$. If $\chi \neq \chi^s$, or equivalently $0 < r < q-1$, f_0 is the "other" U-invariant element in $\mathrm{Ind}_B^\Gamma \chi^s$. The following two easy lemmas are left to the reader.

Lemma 2.5. *(i) The group H acts on f_j via the character $\chi\alpha^{-j} = \chi^s\alpha^{r-j}$. (ii) The set $\{f_j, 0 \le j \le q-1, \phi\}$ is a basis of $\mathrm{Ind}_B^\Gamma \chi^s$.*

Lemma 2.6. *Assume $\chi = \chi^s$. Then $f_0 + \eta(-1)\phi$ is an H-eigenvector and a basis of $(0, \cdots, 0) \otimes \eta$, while $\{f_j, 0 \le j \le q-2, f_{q-1} + \eta(-1)\phi\}$ is a basis of H-eigenvectors for $(p-1, \cdots, p-1) \otimes \eta$.*

We now describe the analogous result for $\chi \ne \chi^s$. Note that σ is then the socle of $\mathrm{Ind}_B^\Gamma \chi^s$ by Theorem 2.4.

Lemma 2.7. *Assume $\chi \ne \chi^s$. With the notations of Lemma 2.2, let $\tau := (\lambda_0(r_0), \cdots, \lambda_{f-1}(r_{f-1})) \otimes \det^{e(\lambda)(r_0,\cdots,r_{f-1})}\eta$ be an irreducible subquotient of $\mathrm{Ind}_B^\Gamma \chi^s$.*

(i) Assume $\tau = \sigma$. Then the following H-eigenvectors of $\mathrm{Ind}_B^\Gamma \chi^s$:

$$f_{\sum_{i=0}^{f-1} p^i d_i},\ 0 \le d_i \le r_i \text{ not all } d_i = r_i;\ f_r + \eta(-1)(-1)^r\phi$$

form a basis of H-eigenvectors of σ inside $\mathrm{Ind}_B^\Gamma \chi^s$.

(ii) Assume $\tau \ne \sigma$. Then the following H-eigenvectors of $\mathrm{Ind}_B^\Gamma \chi^s$:

$$f_{\sum_{i=0}^{f-1} p^i d_i},\ 0 \le d_i \le \lambda_i(r_i) \text{ if } i \notin J(\lambda),\ p-1-\lambda_i(r_i) \le d_i \le p-1 \text{ if } i \in J(\lambda)$$

map to a basis of H-eigenvectors of τ in any quotient of $\mathrm{Ind}_B^\Gamma \chi^s$ where τ is a subrepresentation.

Proof. Using the equality for $\lambda \ne 0$:

$$\begin{pmatrix}\delta & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\lambda & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}\lambda^{-1}+\delta & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\lambda & 0\\ 0 & -\lambda^{-1}\end{pmatrix}\begin{pmatrix}1 & \lambda^{-1}\\ 0 & 1\end{pmatrix}$$

and the fact that ϕ is U-invariant and an H-eigenvector of eigencharacter χ^s, we get for $\delta \in \mathbb{F}_q$:

$$\begin{aligned}\begin{pmatrix}\delta & 1\\ 1 & 0\end{pmatrix} f_0 &= \phi + \eta(-1)\sum_{\lambda\in\mathbb{F}_q^\times}(-\lambda^{-1})^r \begin{pmatrix}\lambda^{-1}+\delta & 1\\ 1 & 0\end{pmatrix}\phi\\ &= \phi + \eta(-1)(-1)^r\sum_{\lambda\in\mathbb{F}_q}(\lambda-\delta)^r\begin{pmatrix}\lambda & 1\\ 1 & 0\end{pmatrix}\phi \qquad (4)\end{aligned}$$

and for $1 \le j \le q-1$:

$$\begin{aligned}\begin{pmatrix}\delta & 1\\ 1 & 0\end{pmatrix} f_j &= \eta(-1)\sum_{\lambda\in\mathbb{F}_q^\times}(-\lambda^{-1})^r\lambda^j \begin{pmatrix}\lambda^{-1}+\delta & 1\\ 1 & 0\end{pmatrix}\phi\\ &= \eta(-1)(-1)^r\sum_{\lambda\in\mathbb{F}_q}(\lambda-\delta)^{q-1+r-j}\begin{pmatrix}\lambda & 1\\ 1 & 0\end{pmatrix}\phi. \qquad (5)\end{aligned}$$

We prove (i). As $(\mathrm{Ind}_B^\Gamma \chi^s)^U$ is generated by f_0 and ϕ and as ϕ generates $\mathrm{Ind}_B^\Gamma \chi^s$, we have that f_0 generates σ^U. As f_0 is U-invariant, it is enough to compute $\left(\begin{smallmatrix} \delta & 1 \\ 1 & 0 \end{smallmatrix}\right) f_0$. By (4), we have:

$$\begin{pmatrix} \delta & 1 \\ 1 & 0 \end{pmatrix} f_0 = (\phi + \eta(-1)(-1)^r f_r) + \eta(-1)(-1)^r \sum_{\substack{0\le d_i\le r_i \\ d\neq r}} \binom{r}{d} (-\delta)^{r-d} f_d$$

where $d := \sum_{i=0}^{f-1} p^i d_i$. This implies (i) by varying δ in $\mathbb{F}_q$. We prove (ii). Let $U(\tau)_0 := \mathrm{Ker}(U(\tau) \twoheadrightarrow \tau)$ (see (iii) of Theorem 2.4 for $U(\tau)$), it is enough to prove that the image of the elements (ii) in the quotient $(\mathrm{Ind}_B^\Gamma \chi^s)/U(\tau)_0$ form a basis of τ. Using the equality:

$$\begin{pmatrix} 1 & \delta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \lambda+\delta & 1 \\ 1 & 0 \end{pmatrix},$$

one easily checks that $f_{\sum_{i\in J(\lambda)} p^i(p-1-\lambda_i(r_i))}$ is U-invariant in $(\mathrm{Ind}_B^\Gamma \chi^s)/U(\tau)_0$. By (i) of Lemma 2.5, B acts on it by $\chi\alpha^{-\sum_{i\in J(\lambda)} p^i(p-1-\lambda_i(r_i))}$ which is also the action of B on τ^U, hence it generates τ in $(\mathrm{Ind}_B^\Gamma \chi^s)/U(\tau)_0$. Now a calculation shows that for $\lambda \in \mathbb{F}_q$:

$$\lambda^{q-1+r-\sum_{i\in J(\lambda)} p^i(p-1-\lambda_i(r_i))} = \lambda^{\sum_{i\in J(\lambda)} p^i(p-1)+\sum_{i\notin J(\lambda)} p^i\lambda_i(r_i)},$$

hence, by (5), we have as previously in $\mathrm{Ind}_B^\Gamma \chi^s$:

$$\begin{pmatrix} \delta & 1 \\ 1 & 0 \end{pmatrix} f_{\sum_{i\in J(\lambda)} p^i(p-1-\lambda_i(r_i))} = \eta(-1)(-1)^r \sum_{\substack{0\le d_i\le \lambda_i(r_i) \text{ if } i\notin J(\lambda) \\ 0\le d_i\le p-1 \text{ if } i\in J(\lambda)}} \binom{c}{d} (-\delta)^{c-d} f_d$$

where $c := \sum_{i\in J(\lambda)} p^i(p-1) + \sum_{i\notin J(\lambda)} p^i\lambda_i(r_i)$ and $d = \sum_{i=0}^{f-1} p^i d_i$. But in $(\mathrm{Ind}_B^\Gamma \chi^s)/U(\tau)_0$ and arguing by induction on $\ell(\tau) = |J(\lambda)|$ starting from (i) (where $\ell(\tau) = 0$), we have in particular:

$$\begin{aligned} f_r + \eta(-1)(-1)^r\phi &= 0 \\ f_d &= 0 \end{aligned} \tag{6}$$

for $d = \sum_{i=0}^{f-1} p^i d_i$ with $0 \le d_i \le \lambda_i(r_i)$ if $i \notin J(\lambda)$, $0 \le d_i \le p-1$ if $i \in J(\lambda)$ with at least one $i \in J(\lambda)$ such that $d_i < p-1-\lambda_i(r_i)$ (one checks using (iii) of Theorem 2.4 and (ii) above with the induction that these elements are in $U(\tau)_0$). By (6), the only f_d remaining in $(\mathrm{Ind}_B^\Gamma \chi^s)/U(\tau)_0$ are exactly those in (ii). We get the result by varying δ in $\mathbb{F}_q$. □

3 Representation theory of Γ over $\overline{\mathbb{F}}_p$ II

In this section, we study the structure of the injective indecomposable representations of Γ over $\overline{\mathbb{F}}_p$ and prove some useful technical results.

We start with the description of the irreducible components of these injective envelopes (without the multiplicities). Let $(x_0, \cdots, x_{f-1})$ be f variables. We define a set $\mathcal{I}(x_0, \cdots, x_{f-1})$ of f-tuples $\lambda := (\lambda_0(x_0), \cdots, \lambda_{f-1}(x_{f-1}))$ where $\lambda_i(x_i) \in \mathbb{Z} \pm x_i$ as follows. If $f = 1$, $\lambda_0(x_0) \in \{x_0, p-1-x_0, p-3-x_0\}$. If $f > 1$, then:

(i) $\lambda_i(x_i) \in \{x_i, x_i - 1, x_i + 1, p - 2 - x_i, p - 3 - x_i, p - 1 - x_i\}$ for $i \in \{0, \cdots, f-1\}$

(ii) if $\lambda_i(x_i) \in \{x_i, x_i - 1, x_i + 1\}$, then $\lambda_{i+1}(x_{i+1}) \in \{x_{i+1}, p - 2 - x_{i+1}\}$

(iii) if $\lambda_i(x_i) \in \{p-2-x_i, p-3-x_i, p-1-x_i\}$, then $\lambda_{i+1}(x_{i+1}) \in \{x_{i+1} - 1, x_{i+1} + 1, p - 3 - x_{i+1}, p - 1 - x_{i+1}\}$

with the conventions $x_f = x_0$ and $\lambda_f(x_f) = \lambda_0(x_0)$.

For $\lambda \in \mathcal{I}(x_0, \cdots, x_{f-1})$, define:

$$e(\lambda) := \frac{1}{2}\Big(\sum_{i=0}^{f-1} p^i(x_i - \lambda_i(x_i))\Big) \text{ if } \lambda_{f-1}(x_{f-1}) \in \{x_{f-1}, x_{f-1} - 1, x_{f-1} + 1\}$$

$$e(\lambda) := \frac{1}{2}\Big(p^f - 1 + \sum_{i=0}^{f-1} p^i(x_i - \lambda_i(x_i))\Big) \text{ otherwise.}$$

The following straightforward lemma is left to the reader.

Lemma 3.1. *One has* $e(\lambda) \in \mathbb{Z} \oplus \bigoplus_{i=0}^{f-1} \mathbb{Z}x_i$.

Let $\sigma := (r_0, \cdots, r_{f-1}) \otimes \eta$ be a weight. The following lemma makes explicit the weights which are subquotients of $\operatorname{inj}\sigma$ (counted without multiplicities).

Lemma 3.2. *(i) Assume* $(r_0, \cdots, r_{f-1}) \neq (0, \cdots, 0)$ *and* $(r_0, \cdots, r_{f-1}) \neq (p-1, \cdots, p-1)$. *The irreducible subquotients of* $\operatorname{inj}\sigma$ *(without multiplicities) are exactly the (all distinct) weights:*

$$(\lambda_0(r_0), \cdots, \lambda_{f-1}(r_{f-1})) \otimes \det{}^{e(\lambda)(r_0, \cdots, r_{f-1})}\eta$$

for $\lambda \in \mathcal{I}(x_0, \cdots, x_{f-1})$ *forgetting the weights such that* $\lambda_i(r_i) < 0$ *or* $\lambda_i(r_i) > p-1$ *for some* i.

(ii) Assume $(r_0,\cdots,r_{f-1})=(0,\cdots,0)$*. The irreducible subquotients of* $\mathrm{inj}\,\sigma$ *(without multiplicities) are exactly the (all distinct) weights:*

$$(\lambda_0(r_0),\cdots,\lambda_{f-1}(r_{f-1}))\otimes\det{}^{e(\lambda)(r_0,\cdots,r_{f-1})}\eta$$

for $\lambda\in\mathcal{I}(x_0,\cdots,x_{f-1})$ *forgetting the weights such that* $\lambda_i(r_i)<0$ *for some* i *and forgetting the weight* $(p-1,\cdots,p-1)\otimes\eta$*.*

(iii) Assume $(r_0,\cdots,r_{f-1})=(p-1,\cdots,p-1)$*. Then we have* $\mathrm{inj}\,\sigma=\sigma=(r_0,\cdots,r_{f-1})\otimes\eta$*.*

Proof. See [32] or [2] for $\mathrm{SL}_2(\mathbb{F}_{p^f})$ from which the case $\mathrm{GL}_2(\mathbb{F}_{p^f})$ is easily derived. It can also be derived from Proposition 3.7 and Theorem 4.7 below (see proof of Corollary 4.11). □

For $r\geq 0$, recall we can identify $\mathrm{Sym}^r\overline{\mathbb{Q}}_p^2$ with $\oplus_{i=0}^r\overline{\mathbb{Q}}_p x^{r-i}y^i$ (see [4] or [6] or [25]). Let $V_{r,\overline{\mathbb{Z}}_p}$ be the $\overline{\mathbb{Z}}_p$-lattice in $\mathrm{Sym}^r\overline{\mathbb{Q}}_p^2$ spanned by $\binom{r}{i}x^{r-i}y^i$, $0\leq i\leq r$ and set:

$$V_r:=V_{r,\overline{\mathbb{Z}}_p}\otimes_{\overline{\mathbb{Z}}_p}\overline{\mathbb{F}}_p$$

with the convention $V_r:=0$ if $r<0$. For convenience, we introduce the following notation:

Definition 3.3. *Given an* f*-tuple of integers* $\mathbf{r}:=(r_0,\cdots,r_{f-1})$ *with* $0\leq r_i$*, we define:*

$$V_{\mathbf{r}}:=\bigotimes_{i=0}^{f-1}V_{r_i}^{\mathrm{Fr}^i}.$$

One can easily verify that $V_{r,\overline{\mathbb{Z}}_p}$ and hence V_r is stable under the action of K and that K_1 acts trivially on V_r. Moreover, if $0\leq n\leq p-1$ then $\binom{n}{i}$ is a unit in $\overline{\mathbb{Z}}_p$ and hence $V_r\cong\mathrm{Sym}^r\overline{\mathbb{F}}_p^2$. This isomorphism doesn't hold in general. Given an irreducible representation σ of Γ, there exists a unique pair $(\mathbf{r},a)$ where $\mathbf{r}$ is an f-tuple as above with $0\leq r_i\leq p-1$ for all i and where $0\leq a<q-1$ such that $\sigma\cong V_{\mathbf{r}}\otimes\det^a=(r_0,\cdots,r_{f-1})\otimes\det^a$. By expressing $a=\sum_{i=0}^{f-1}p^ia_i$ with $0\leq a_i\leq p-1$, one may reformulate this as follows: given an irreducible representation σ of Γ, there exist unique f-tuples $\mathbf{r}$ (as above) and $(a_0,\cdots,a_{f-1})$ with $0\leq a_i\leq p-1$ and not all of a_i equal $p-1$ such that $\sigma\cong\otimes_{i=0}^{f-1}(V_{r_i}\otimes\det^{a_i})^{\mathrm{Fr}^i}$ (see (2)).

We now recall more precise results on the structure of injective envelopes in Rep_Γ, following [25] (which is based on [23]). If $r=p-1$ we set $R_{p-1}:=V_{p-1}$. For $0\leq r<p-1$, R_r is an (explicit) Γ-invariant subspace of $V_{p-r-1}\otimes$

V_{p-1} defined in [25, Def. 4.2.10] (we won't really need its precise definition here). Let $\mathbf{r}$ be an f-tuple such that $0 \le r_i \le p-1$ for all i, then $R_{\mathbf{r}} := \otimes_{i=0}^{f-1} R_{r_i}^{\mathrm{Fr}^i}$ is an injective object in Rep_Γ. Moreover, if $\mathbf{r} \neq \mathbf{0}$ then $R_{\mathbf{r}}$ is an injective envelope of $V_{\mathbf{r}} \otimes \det^{-\sum_{i=0}^{f-1} p^i r_i}$ and $R_{\mathbf{0}} \cong \operatorname{inj} V_{\mathbf{0}} \oplus V_{\mathbf{p-1}}$ ([25, Cor. 4.2.22 and Cor. 4.2.31]).

Lemma 3.4. *For $0 \le r < p-1$, there exists an exact sequence of Γ-representations:*

$$0 \longrightarrow V_{2p-2-r} \longrightarrow R_r \longrightarrow V_r \otimes \det^{p-1-r} \longrightarrow 0.$$

Proof. This follows directly from [25, Lem. 4.2.9 and Def. 4.2.10]. $\square$

Lemma 3.5. *For $0 \le r < p-1$, there exists an exact sequence of Γ-representations:*

$$0 \longrightarrow V_r \otimes \det^{p-1-r} \longrightarrow V_{2p-2-r} \longrightarrow V_{p-r-2} \otimes V_1^{\mathrm{Fr}} \longrightarrow 0.$$

Proof. The injection is given by [25, Prop. 4.2.13]. We denote the quotient by Q. If $0 \le r < p-1$, then it follows from [32, Prop. 2] that $Q \cong V_{p-r-2} \otimes V_1^{\mathrm{Fr}}$ as a representation of Γ'. Without loss of generality we may assume that q is arbitrarily large, in particular that $V_{p-r-2} \otimes V_1^{\mathrm{Fr}}$ is irreducible. Then the image of x^{2p-r-2} spans Q^U. And since:

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} x^{2p-r-2} = \lambda^{2p-r-2} x^{2p-r-2} \quad \lambda, \mu \in \mathbb{F}_q^\times,$$

we obtain $Q \cong V_{p-r-2} \otimes V_1^{\mathrm{Fr}}$ as Γ-representation. $\square$

Given $\mathbf{r}$ an f-tuple of integers as in Definition 3.3 with $0 \le r_i \le 2p-2$, we define $\mathbf{2p-2-r} := (2p-2-r_0, \cdots, 2p-2-r_{f-1})$.

Proposition 3.6. *Let $\mathbf{r}$ be an f-tuple with $0 \le r_i \le p-1$ for all i and let τ be a representation of Γ such that $\sigma := \mathrm{soc}_\Gamma \tau$ is isomorphic to $V_{\mathbf{r}} \otimes \det^{-\sum_{i=0}^{f-1} p^i r_i} = (r_0, \cdots, r_{f-1}) \otimes \det^{-\sum_{i=0}^{f-1} p^i r_i}$ and σ occurs in τ with multiplicity 1. Then there exists a Γ-equivariant injection $\tau \hookrightarrow V_{\mathbf{2p-2-r}}$.*

Proof. Since $R_{\mathbf{r}}$ contains (and is isomorphic to if $\mathbf{r} \neq \mathbf{0}$) an injective envelope of $\mathrm{soc}_\Gamma \tau$, there exists a Γ-equivariant embedding $\tau \hookrightarrow R_{\mathbf{r}}$. For $0 \le i \le f-1$ set:

$$W_i := \bigotimes_{j=0}^{i-1} R_{r_j}^{\mathrm{Fr}^j} \otimes \bigotimes_{j=i+1}^{f-1} V_{2p-2-r_j}^{\mathrm{Fr}^j}.$$

Suppose that $r_i \neq p-1$ so that $R_{r_i} \neq V_{2p-2-r_i}$. Twisting and tensoring the exact sequence of Lemma 3.4, we obtain an exact sequence:

$$0 \longrightarrow V_{2p-2-r_i}^{\mathrm{Fr}^i} \otimes W_i \longrightarrow R_{r_i}^{\mathrm{Fr}^i} \otimes W_i \longrightarrow (V_r \otimes \det{}^{p-1-r})^{\mathrm{Fr}^i} \otimes W_i \longrightarrow 0. \quad (7)$$

It follows from Lemmas 3.4 and 3.5 that we can embed $(V_r \otimes \det^{p-1-r})^{\mathrm{Fr}^i} \otimes W_i$ into $R_{\mathbf{r}}$. Hence, if $\mathbf{r} \neq \mathbf{0}$ then $\mathrm{soc}_\Gamma((V_r \otimes \det^{p-1-r})^{\mathrm{Fr}^i} \otimes W_i) \cong \sigma$. Since σ occurs in τ with multiplicity 1, we obtain an isomorphism:

$$\mathrm{Hom}_\Gamma(\tau, V_{2p-2-r_i}^{\mathrm{Fr}^i} \otimes W_i) \cong \mathrm{Hom}_\Gamma(\tau, R_{r_i}^{\mathrm{Fr}^i} \otimes W_i).$$

Applying this identity recursively we obtain that the image of τ is contained in $V_{\mathbf{2p-2-r}}$. If $\mathbf{r} = \mathbf{0}$ then $\mathrm{soc}_\Gamma((V_r \otimes \det^{p-1-r})^{\mathrm{Fr}^i} \otimes W_i) \subseteq \sigma \oplus V_{\mathbf{p-1}}$. Since $V_{\mathbf{p-1}}$ is irreducible, injective (and projective) and does not appear in $\mathrm{soc}_\Gamma \tau$, $V_{\mathbf{p-1}}$ cannot be a subquotient of τ. The same argument as above implies the assertion. □

Proposition 3.7. *Let* $\mathbf{r}$ *be an* f*-tuple with* $0 \leq r_i \leq p-1$ *for all* i*. Then* σ *is an irreducible subquotient of* $R_{\mathbf{r}}$ *if and only if it is an irreducible subquotient of* $V_{\mathbf{2p-2-r}}$*. In particular, if* $\mathbf{r} \neq 0$*,* σ *appears in* $\mathrm{inj}\, V_{\mathbf{r}} \otimes \det^{-\sum_{i=0}^{f-1} p^i r_i}$ *if and only if it appears in* $V_{\mathbf{2p-2-r}}$*.*

Proof. We keep the notation of the proof of Proposition 3.6. Lemma 3.5 implies that there exists a Γ-equivariant injection $(V_{r_i} \otimes \det^{p-1-r_i})^{\mathrm{Fr}^i} \otimes W_i \hookrightarrow V_{2p-2-r_i}^{\mathrm{Fr}^i} \otimes W_i$. It follows from (7) that σ is an irreducible subquotient of $V_{2p-2-r_{i+1}}^{\mathrm{Fr}^{i+1}} \otimes W_{i+1}$ if and only if σ is an irreducible subquotient of $V_{2p-2-r_i}^{\mathrm{Fr}^i} \otimes W_i$. This implies the assertion. □

We need to study more closely the structure of $V_{\mathbf{2p-2-r}}$ when $0 \leq r_i \leq p-2$. Let $\mathbf{r} := (r_0, \cdots, r_{f-1})$ be an f-tuple of integers with $0 \leq r_i \leq p-1$ for all i and set:

$$S_{\mathbf{r}} := \{i \in \{0, \cdots, f-1\}, r_i \neq p-1\}.$$

For a subset $J \subseteq S_{\mathbf{r}}$, we define $W_J := \otimes_{i=0}^{f-1} W_{J,i}^{\mathrm{Fr}^i}$ where $W_{J,i} := V_{r_i} \otimes \det^{p-1-r_i}$ if $i \notin J$ and $W_{J,i} := V_{p-r_i-2} \otimes V_1^{\mathrm{Fr}}$ if $i \in J$. If $f > 1$ then by shifting V_1 to a neighbouring component we obtain that $W_J \cong \otimes_{i=0}^{f-1} U_{J,i}^{\mathrm{Fr}^i}$ where:

(i) if $i \in J$ and $i-1 \in J$ then $U_{J,i} := V_{p-r_i-2} \otimes V_1$

(ii) if $i \in J$ and $i-1 \notin J$ then $U_{J,i} := V_{p-r_i-2}$

(iii) if $i \notin J$ and $i-1 \in J$ then $U_{J,i} := V_{r_i} \otimes \det^{p-1-r_i} \otimes V_1$

(iv) if $i \notin J$ and $i-1 \notin J$ then $U_{J,i} := V_{r_i} \otimes \det^{p-1-r_i}$.

Assume $0 \le r_i < p-1$, we define a filtration on V_{2p-2-r_i} by:

$$\begin{aligned} \mathrm{Fil}^0 V_{2p-2-r_i} &:= V_{2p-2-r_i} \\ \mathrm{Fil}^1 V_{2p-2-r_i} &:= V_{r_i} \otimes \det^{p-r_i-1} \\ \mathrm{Fil}^2 V_{2p-2-r_i} &:= 0. \end{aligned}$$

If $r_i = p-1$, we define a filtration on V_{p-1} by $\mathrm{Fil}^0 V_{p-1} := V_{p-1}$ and $\mathrm{Fil}^1 V_{p-1} := 0$. This induces the usual tensor product filtration on $V_{\mathbf{2p-2-r}}$. It follows from [2, §1] that:

$$\frac{\mathrm{Fil}^i V_{\mathbf{2p-2-r}}}{\mathrm{Fil}^{i+1} V_{\mathbf{2p-2-r}}} \cong \bigoplus_{\substack{J \subseteq S_{\mathbf{r}} \\ |J| = |S_{\mathbf{r}}| - i}} W_J. \tag{8}$$

Lemma 3.8. *(i) Assume $0 \le r \le p-2$, we have an isomorphism of Γ-representations $V_r \otimes V_1 \cong V_{r+1} \oplus (V_{r-1} \otimes \det)$.*
(ii) We have $V_1 \otimes V_{p-1} \cong R_{p-2}$.

Proof. If $0 \le r \le p-2$ then it follows from [2, Lem. 2.5] that there exists a Γ'-equivariant isomorphism $V_r \otimes V_1 \cong V_{r+1} \oplus V_{r-1}$. Since the order of Γ/Γ' is prime to p, $V_r \otimes V_1$ is a semi-simple representation of Γ. Hence there exist integers a, b such that $0 \le a, b < q-1$ and:

$$V_r \otimes V_1 \cong (V_{r+1} \otimes \det^a) \oplus (V_{r-1} \otimes \det^b).$$

One may verify that $x^r \otimes x$ and $x^r \otimes y - x^{r-1} y \otimes x$ are fixed by U. This implies that $a = 0$ and $b = 1$. If $r = p-1$ then R_{p-2} is a $2p$-dimensional subspace of $V_1 \otimes V_{p-1}$ by [25, Prop. 4.2.11]. Since $\dim(V_1 \otimes V_{p-1}) = 2p$, we get $V_1 \otimes V_{p-1} \cong R_{p-2}$. $\square$

Proposition 3.9. *Assume $0 \le r \le p-2$ and let $L_r := (V_r \otimes \det^{p-1-r}) \oplus (V_r \otimes \det^{p-r-1} \otimes V_1) \oplus V_{p-r-2} \oplus (V_{p-r-2} \otimes V_1)$.*

(i) L_r is isomorphic to:

$$(V_r \otimes \det^{p-1-r}) \oplus (V_{r+1} \otimes \det^{p-r-1}) \oplus (V_{r-1} \otimes \det^{p-r}) \oplus V_{p-r-2} \oplus V_{p-r-1} \oplus (V_{p-r-3} \otimes \det).$$

(ii) Suppose that $q > 3$, then L_r is multiplicity free.

Proof. The hypothesis on r ensures that $0 \le p-r-2 \le p-2$ and (i) follows from Lemma 3.8. So L_r is semi-simple. Let us assume $q > 3$. We will deduce (ii) from the fact that if $0 \le r, s \le p-1$ then:

$$V_r \otimes \det^a \cong V_s \otimes \det^b \iff r = s \text{ and } a \equiv b \ (q-1).$$

Since $r-1, r, r+1$ are distinct the representation $(V_r \otimes \det^{p-1-r}) \oplus (V_{r+1} \otimes \det^{p-1-r}) \oplus (V_{r-1} \otimes \det^{p-r})$ is multiplicity free. Similarly $V_{p-r-2} \oplus V_{p-r-1} \oplus (V_{p-r-3} \otimes \det)$ is multiplicity free. So if L_r is not multiplicity free then one of the following must hold:

$$\begin{array}{llllll} (a) & r-1 = p-r-1 & (b) & r = p-r-1 & (c) & r+1 = p-r-1 \\ (d) & r+1 = p-r-2 & (e) & r+1 = p-r-3. & & \end{array}$$

In cases (a), (c) and (e), p is even, hence $p = 2$ and $r = 0$ so that:

$$L_0 \cong (V_0 \otimes \det) \oplus (V_1 \otimes \det) \oplus V_0 \oplus V_1.$$

Since $q > 2$ we have $1 \not\equiv 0 \ (q-1)$ hence L_0 is multiplicity free. In case (b), $p \neq 2$ and $r = (p-1)/2$ hence:

$$L_r \cong (V_r \otimes \det^r) \oplus (V_{r+1} \otimes \det^r) \oplus (V_{r-1} \otimes \det^{r+1}) \oplus V_{r-1} \oplus V_r \oplus (V_{r-2} \otimes \det).$$

Since $r \not\equiv 0 \ (q-1)$ and $r+1 \not\equiv 0 \ (q-1)$ as $q > 3$, L_r is multiplicity free. In case (d), $p \neq 2$ and $r = (p-3)/2$ hence L_r is isomorphic to:

$$(V_r \otimes \det^{r+2}) \oplus (V_{r+1} \otimes \det^{r+2}) \oplus (V_{r-1} \otimes \det^{r+3}) \oplus V_{r+1} \oplus V_{r+2} \oplus (V_r \otimes \det).$$

Now $r+2 \not\equiv 1 \ (q-1)$ as this would imply $\frac{p-3}{2} \equiv p-2 \ (p-1)$ which is impossible. Also $r+2 \not\equiv 0 \ (q-1)$ as $q > 3$. Hence L_r is always multiplicity free. □

Proposition 3.10. *Let* $\mathbf{r}$ *and* $\mathbf{s}$ *be* f*-tuples such that* $0 \le r_i, s_i \le p-1$ *for all* i*. If* $p \neq 2$ *then assume that if* $r_i = p-1$ *then* $s_i = p-1$ *or* $s_i = 0$*. If* $p = 2$ *then assume that if* $r_i = p-1$ *then* $s_i = p-1$*. Then* $V_{\mathbf{s}} \otimes \det^a$ *can occur in* $V_{\mathbf{2p-2-r}}$ *with multiplicity at most* 1*.*

Proof. Note first that if $\mathbf{r} = \mathbf{p-1}$, then $V_{\mathbf{2p-2-r}} = V_{\mathbf{p-1}}$ which is irreducible, hence we can assume $\mathbf{r} \neq \mathbf{p-1}$. If $f = 1$ this is either trivial or follows from Lemma 3.5. Assume $f > 1$. We argue by induction on:

$$n(\mathbf{r}) := |\{i \in \{0, \cdots, f-1\}, r_i = p-1 \text{ and } s_i = 0\}|.$$

Assume $n(\mathbf{r}) = 0$ and suppose that $V_{\mathbf{s}} \otimes \det^a$ occurs in $V_{\mathbf{2p-2-r}}$. It follows from (8) that $V_{\mathbf{s}} \otimes \det^a$ occurs in W_J for some $J \subseteq S_{\mathbf{r}}$. We claim that if

$r_i = p-1$ then $i-1 \notin J$. If $i-1 \in J$ then $r_{i-1} \neq p-1$ by definition of $S_{\mathbf{r}}$ and $U_{J,i} = V_{p-1} \otimes V_1$. Lemma 3.8 implies that $V_1 \otimes V_{p-1} \cong R_{p-2}$ and it follows from Lemmas 3.4 and 3.5 that the irreducible subquotients of R_{p-2} are $V_{p-2} \otimes \det$ and V_1^{Fr}. Hence if $V_{\mathbf{s}} \otimes \det^a$ is a subquotient of W_J then either $s_i = p-2$ or $s_i = 0$. But both are impossible: the first because of our assumptions and the second because of $n(\mathbf{r}) = 0$. This proves the claim. Let $\mathcal{P}$ be the set of subsets of $S_{\mathbf{r}}$ such that $J \in \mathcal{P}$ if and only if for all i, $r_i = p-1$ implies that $i-1 \notin J$. If $r_i = p-1$ set $M_{r_i} := V_{p-1}$, if $i \in S_{\mathbf{r}}$ and $r_{i+1} = p-1$ set $M_{r_i} := (V_{r_i} \otimes \det^{p-1-r_i}) \oplus (V_{r_i} \otimes \det^{p-1-r_i} \otimes V_1)$ and if $i \in S_{\mathbf{r}}$ and $r_{i+1} \neq p-1$ set $M_{r_i} := L_{r_i}$. Proposition 3.9 implies that the representation $\otimes_{i=0}^{f-1} M_{r_i}^{\mathrm{Fr}^i}$ is semi-simple and multiplicity free. Hence, if $J \in \mathcal{P}$ then W_J is semi-simple and multiplicity free. Now, suppose $V_{\mathbf{s}} \otimes \det^a$ occurs at least twice in $V_{\mathbf{2p-2-r}}$. From what we have just proven, this means there exists $I, J \in \mathcal{P}$ with $I \neq J$ such that $V_{\mathbf{s}} \otimes \det^a$ is a subquotient of W_I and W_J. If $j \in I$ and $j \notin J$, it follows from the definitions that $U_{J,j} \neq U_{I,j}$. Then $V_{\mathbf{s}} \otimes \det^a$ is a subquotient of $U_{I,j}^{\mathrm{Fr}^j} \otimes (\otimes_{i \neq j} M_{r_i}^{\mathrm{Fr}^i})$ and $U_{J,j}^{\mathrm{Fr}^j} \otimes (\otimes_{i \neq j} M_{r_i}^{\mathrm{Fr}^i})$. Since $U_{I,j}^{\mathrm{Fr}^j} \neq U_{J,j}^{\mathrm{Fr}^j}$, this implies that $V_{\mathbf{s}} \otimes \det^a$ appears in $\otimes_{i=0}^{f-1} M_{r_i}^{\mathrm{Fr}^i}$ at least twice, which cannot happen as $\otimes_{i=0}^{f-1} M_{r_i}^{\mathrm{Fr}^i}$ is multiplicity free. This proves our statement for $n(\mathbf{r}) = 0$. Assume $n(\mathbf{r}) > 0$ i.e. there exists i such that $r_i = p-1$ and $s_i = 0$ and hence $p > 2$ from our assumption. Suppose that $V_{\mathbf{s}} \otimes \det^a$ is a subquotient of W_J for some $J \subseteq S_{\mathbf{r}}$. The only possibility for having $s_i = 0$ with $r_i = p-1$ is to have:

$$W_{J,i} \otimes V_1^{\mathrm{Fr}^i} = (V_{r_i} \otimes \det^{p-1-r_i})^{\mathrm{Fr}^i} \otimes V_1^{\mathrm{Fr}^i} = (V_{p-1} \otimes V_1)^{\mathrm{Fr}^i}$$

appear as a $\otimes$-factor of a subquotient of W_J and this implies either $r_{i-1} = p-1$ and $s_{i-1} = 0$ or $r_{i-1} \neq p-1$ and $i-1 \in J$. By an obvious induction and as $\mathbf{r} \neq \mathbf{p-1}$, this implies that there exists j such that $r_j \neq p-1$, $j \in J$ and $r_{j+1} = p-1$, $s_{j+1} = 0$. From the definition of W_J, we see that $V_{\mathbf{s}} \otimes \det^a$ is then a subquotient of:

$$(V_{p-2-r_j} \otimes V_1^{\mathrm{Fr}})^{\mathrm{Fr}^j} \otimes (\otimes_{k \neq j} V_{2p-2-r_k}^{\mathrm{Fr}^k}) \cong V_{p-2-r_j}^{\mathrm{Fr}^j} \otimes R_{p-2}^{\mathrm{Fr}^{j+1}} \otimes (\otimes_{k \notin \{j,j+1\}} V_{2p-2-r_k}^{\mathrm{Fr}^k})$$

where the isomorphism follows from Lemma 3.8. Every irreducible subquotient of $V_{p-2-r_j}^{\mathrm{Fr}^j} \otimes (V_{p-2} \otimes \det)^{\mathrm{Fr}^{j+1}} \otimes (\otimes_{k \notin \{j,j+1\}} V_{2p-2-r_k}^{\mathrm{Fr}^k})$ has its $(j+1)$-th digit equal to $p-2$. Since $p > 2$ and $s_{j+1} = 0$, $V_{\mathbf{s}} \otimes \det^a$ cannot be such a subquotient. Lemma 3.4 then implies that $V_{\mathbf{s}} \otimes \det^a$ occurs in $V_{\mathbf{2p-2-r}}$ with the same multiplicity as in $V_{p-2-r_j}^{\mathrm{Fr}^j} \otimes V_p^{\mathrm{Fr}^{j+1}} \otimes (\otimes_{k \notin \{j,j+1\}} V_{2p-2-r_k}^{\mathrm{Fr}^k})$. Let $\mathbf{r}'$ be the f-tuple $r'_j := p-2-r_j$, $r'_{j+1} := p-2$, $r'_k := r_k$ for $k \notin \{j, j+1\}$. Since $2p-2-(p-2) = p$ and $n(\mathbf{r}') = n(\mathbf{r}) - 1$, the induction hypothesis implies that $V_{\mathbf{s}} \otimes \det^a$ can occur in $V_{p-2-r_j}^{\mathrm{Fr}^j} \otimes V_p^{\mathrm{Fr}^{j+1}} \otimes (\otimes_{k \notin \{j,j+1\}} V_{2p-2-r_k}^{\mathrm{Fr}^k})$ with multiplicity at most 1. This finishes the proof. $\square$

Corollary 3.11. *Let* $\mathbf{r}$ *be an* f*-tuple such that* $0 \leq r_i \leq p-2$*. Then* $V_{\mathbf{2p-2-r}}$ *is multiplicity free and the graded pieces of the filtration (8) are semi-simple.*

Proof. Since $0 \leq r_i \leq p-2$, the conditions on $\mathbf{s}$ in Proposition 3.10 are empty and $\mathcal{P}$ as in the proof of Proposition 3.10 is the set of all subsets of $S_{\mathbf{r}}$. It follows from the proof of Proposition 3.10 that if $J \subseteq S_{\mathbf{r}}$ then W_J is semi-simple. □

Corollary 3.12. *Let* σ *and* τ *be two irreducible representations of* Γ *over* $\overline{\mathbb{F}}_p$*. Assume* $\sigma = (r_0, \cdots, r_{f-1}) \otimes \eta$ *with* $0 \leq r_i \leq p-2$ *for all* i *and assume there exist finite dimensional representations of* Γ *over* $\overline{\mathbb{F}}_p$ *with socle* σ *and cosocle* τ*.*

(i) Among these indecomposable representations, there is a unique one $I(\sigma, \tau)$ *such that* σ *appears with multiplicity* 1 *(hence as subobject).*

(ii) The representation $I(\sigma, \tau)$ *is multiplicity free.*

Proof. Let κ be a representation of Γ such that $\operatorname{soc}_\Gamma \kappa \cong \sigma$ and $\operatorname{cosoc}_\Gamma \kappa \cong \tau$. Since $\operatorname{soc}_\Gamma \kappa$ is irreducible there exists an injection $\kappa \hookrightarrow \operatorname{inj} \sigma$. In particular, τ is a subquotient of $\operatorname{inj} \sigma$ and hence by Proposition 3.7 τ is a subquotient of $V_{\mathbf{2p-2-r}}$. Corollary 3.11 implies that τ occurs in $V_{\mathbf{2p-2-r}}$ with multiplicity 1. Since Γ is a finite group, an injective envelope of τ is also its projective envelope (see Ex.14.1 in [31, §14]) hence $\dim_{\overline{\mathbb{F}}_p} \operatorname{Hom}_\Gamma(\operatorname{inj} \tau, V_{\mathbf{2p-2-r}}) = 1$. Choose a non-zero $\phi \in \operatorname{Hom}_\Gamma(\operatorname{inj} \tau, V_{\mathbf{2p-2-r}})$ and set $I(\sigma, \tau) := \operatorname{Im} \phi$. Since $I(\sigma, \tau)$ is a quotient of $\operatorname{inj} \tau$, $\operatorname{cosoc}_\Gamma I(\sigma, \tau) \cong \tau$ and since $I(\sigma, \tau)$ is a subrepresentation of $V_{\mathbf{2p-2-r}}$, $\operatorname{soc}_\Gamma I(\sigma, \tau) \cong \sigma$. Since $V_{\mathbf{2p-2-r}}$ is multiplicity free so is $I(\sigma, \tau)$. Let λ be a representation of Γ such that $\operatorname{soc}_\Gamma \lambda \cong \sigma$, σ occurs in λ with multiplicity 1 and $\operatorname{cosoc}_\Gamma \lambda \cong \tau$. Then λ is a quotient of $\operatorname{inj} \tau$ and Proposition 3.6 implies that λ is isomorphic to a subrepresentation of $V_{\mathbf{2p-2-r}}$. Since $\dim_{\overline{\mathbb{F}}_p} \operatorname{Hom}_\Gamma(\operatorname{inj} \tau, V_{\mathbf{2p-2-r}}) = 1$, λ is isomorphic to $I(\sigma, \tau)$. □

4 Representation theory of Γ over $\overline{\mathbb{F}}_p$ III

We prove important results on the socle filtration and B-invariants of Γ-representations with irreducible socle appearing only once.

Let $\mathbf{r} := (r_0, \cdots, r_{f-1})$ be an f-tuple of integers such that $0 \leq r_i \leq p-2$ for all i. Let Σ be the set consisting of f-tuples $\boldsymbol{\varepsilon} = (\epsilon_0, \cdots, \epsilon_{f-1})$ where $\epsilon_i \in \{-1, 0, 1\}$. For $\boldsymbol{\varepsilon} \in \Sigma$, we set $|\boldsymbol{\varepsilon}| := |\{i, \epsilon_i \neq 0\}|$. Let $\Sigma_{\mathbf{r}}$ be the subset of Σ consisting of f-tuples $\boldsymbol{\varepsilon}$ such that if $r_i = 0$ then $\epsilon_{i-1} \neq -1$ and if $r_i = p-2$ and $\epsilon_i \neq 0$ then $\epsilon_{i-1} \neq -1$ (as usual $f = 0$ and $-1 = f-1$). In particular, $\Sigma_{\mathbf{r}} = \Sigma$ if $1 \leq r_i \leq p-3$ for all i.

Definition 4.1. *Let $\boldsymbol{\varepsilon}, \boldsymbol{\delta} \in \Sigma_{\mathbf{r}}$ we write $\boldsymbol{\varepsilon} \prec \boldsymbol{\delta}$ if there exists $k \in \{0, \cdots, f-1\}$ such that the following hold:*

(i) $\epsilon_k = 0$, $\delta_k \neq 0$

(ii) $\delta_{k-1} = -\epsilon_{k-1}$

(iii) $\delta_i = \epsilon_i$ *for all* $i \notin \{k-1, k\}$

(iv) if $r_k = p-2$ *then* $\epsilon_{k-1} \neq 1$.

We write $\boldsymbol{\varepsilon} < \boldsymbol{\delta}$ if there exists a sequence $\boldsymbol{\varepsilon}_0, \cdots, \boldsymbol{\varepsilon}_j$ in $\Sigma_{\mathbf{r}}$ with $j > 0$ such that $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_0$, $\boldsymbol{\delta} = \boldsymbol{\varepsilon}_j$ and $\boldsymbol{\varepsilon}_i \prec \boldsymbol{\varepsilon}_{i+1}$ for $0 \leq i < j$. We write $\boldsymbol{\varepsilon} \leq \boldsymbol{\delta}$ if $\boldsymbol{\varepsilon} = \boldsymbol{\delta}$ or $\boldsymbol{\varepsilon} < \boldsymbol{\delta}$.

To $\boldsymbol{\varepsilon} \in \Sigma_{\mathbf{r}}$ we associate an f-tuple $\mathbf{r}(\boldsymbol{\varepsilon}) = (r(\boldsymbol{\varepsilon})_0, \cdots, r(\boldsymbol{\varepsilon})_{f-1})$ such that $0 \leq r(\boldsymbol{\varepsilon})_i \leq p-1$ and an integer $e(\boldsymbol{\varepsilon}) := \sum_{i=0}^{f-1} p^i e(\boldsymbol{\varepsilon})_i$ as follows:

(i) if $\epsilon_i \neq 0$, $r(\boldsymbol{\varepsilon})_i := p-2-r_i+\epsilon_{i-1}$

(ii) if $\epsilon_i = 0$, $r(\boldsymbol{\varepsilon})_i := r_i+\epsilon_{i-1}$

(iii) if $\epsilon_i \neq 0$, $e(\boldsymbol{\varepsilon})_i := \begin{cases} 1 & \text{if } \epsilon_{i-1} = -1 \\ 0 & \text{otherwise} \end{cases}$

(iv) if $\epsilon_i = 0$, $e(\boldsymbol{\varepsilon})_i := \begin{cases} p-r_i & \text{if } \epsilon_{i-1} = -1 \\ p-1-r_i & \text{otherwise.} \end{cases}$

Lemma 4.2. *Let $J \subseteq \{0, \cdots, f-1\}$ then:*

$$W_J \cong \bigoplus_{\substack{\boldsymbol{\varepsilon} \in \Sigma_{\mathbf{r}} \\ |\boldsymbol{\varepsilon}| = |J|}} V_{\mathbf{r}(\boldsymbol{\varepsilon})} \otimes \det{}^{e(\boldsymbol{\varepsilon})}.$$

Proof. This follows from Lemma 3.8 and the definition of W_J (see §3). □

In the next lemmas, we sometimes use some results that will be proved in §5 (the reader can check that these results do not depend on the ones we prove below using them!).

Lemma 4.3. *Let $\boldsymbol{\varepsilon}, \boldsymbol{\delta} \in \Sigma_{\mathbf{r}}$ with $|\boldsymbol{\varepsilon}| < |\boldsymbol{\delta}|$, then* $\mathrm{Ext}^1_\Gamma(V_{\mathbf{r}(\boldsymbol{\delta})} \otimes \det^{e(\boldsymbol{\delta})}, V_{\mathbf{r}(\boldsymbol{\varepsilon})} \otimes \det^{e(\boldsymbol{\varepsilon})}) \neq 0$ *if and only if* $\boldsymbol{\varepsilon} \prec \boldsymbol{\delta}$.

Proof. If $f = 1$ this follows from (i) of Corollary 5.6. Assume $f > 1$. If $\boldsymbol{\varepsilon} \prec \boldsymbol{\delta}$, a straightforward computation yields $r(\boldsymbol{\delta})_k = p-2-r(\boldsymbol{\varepsilon})_k$, $r(\boldsymbol{\delta})_{k+1} = r(\boldsymbol{\varepsilon})_{k+1} + \delta_k$, $e(\boldsymbol{\delta})_k = e(\boldsymbol{\varepsilon})_k + r(\boldsymbol{\varepsilon})_k + 1 - p$ and $e(\boldsymbol{\delta})_{k+1} = e(\boldsymbol{\varepsilon})_{k+1} + 1$ if $\delta_k = -1$, $e(\boldsymbol{\delta})_{k+1} = e(\boldsymbol{\varepsilon})_{k+1}$ if $\delta_k = 1$. By (i) of Corollary 5.6, we have $\mathrm{Ext}^1_\Gamma(V_{\mathbf{r}(\boldsymbol{\delta})} \otimes \det^{e(\boldsymbol{\delta})}, V_{\mathbf{r}(\boldsymbol{\varepsilon})} \otimes \det^{e(\boldsymbol{\varepsilon})}) \neq 0$. Conversely suppose that $\mathrm{Ext}^1_\Gamma(V_{\mathbf{r}(\boldsymbol{\delta})} \otimes \det^{e(\boldsymbol{\delta})}, V_{r(\boldsymbol{\varepsilon})} \otimes \det^{e(\boldsymbol{\varepsilon})}) \neq 0$, let $j \in \{0, \cdots, f-1\}$ be an index as in (i) of Corollary 5.6 and set $k := j - 1$. If $i \notin \{k, k+1\}$ then this corollary implies that $r(\boldsymbol{\varepsilon})_i = r(\boldsymbol{\delta})_i$, $e(\boldsymbol{\varepsilon})_i = e(\boldsymbol{\delta})_i$. We claim that this implies $\epsilon_{i-1} = \delta_{i-1}$ for $i \notin \{k, k+1\}$. If ϵ_i and δ_i are either both zero or both non-zero then $r(\boldsymbol{\varepsilon})_i = r(\boldsymbol{\delta})_i$ implies that $\epsilon_{i-1} = \delta_{i-1}$. If one of them is zero and the other non-zero then $e(\boldsymbol{\varepsilon})_i = e(\boldsymbol{\delta})_i$ implies that $r_i = p-2$ and $r(\boldsymbol{\varepsilon})_i = r(\boldsymbol{\delta})_i$ implies that either $r_i = (p-2+\delta_{i-1}-\epsilon_{i-1})/2$ or $r_i = (p-2+\epsilon_{i-1}-\delta_{i-1})/2$. If $p > 3$ this is impossible. If $p = 3$, a case by case analysis shows that this is also impossible. If $p = 2$ we get $\epsilon_{i-1} = \delta_{i-1}$. Hence $\epsilon_i = \delta_i$ for all $i \notin \{k-1, k\}$. By (i) of Corollary 5.6, we have $r(\boldsymbol{\delta})_k = p - 2 - r(\boldsymbol{\varepsilon})_k$. If $\epsilon_k \neq 0$ then $r(\boldsymbol{\delta})_k = r_k - \epsilon_{k-1}$. This together with exponent considerations imply that $\delta_k = 0$ and hence $\delta_{k-1} = -\epsilon_{k-1}$. However this contradicts $|\boldsymbol{\varepsilon}| < |\boldsymbol{\delta}|$. Hence $\epsilon_k = 0$. The same argument gives $\delta_k \neq 0$ and $\delta_{k-1} = -\epsilon_{k-1}$. Since $r(\boldsymbol{\delta})_k \geq 0$ we get that $r(\boldsymbol{\varepsilon})_k \leq p-2$ and hence if $r_k = p-2$ then $\epsilon_{k-1} \neq 1$. Putting this together gives $\boldsymbol{\varepsilon} \prec \boldsymbol{\delta}$. □

Lemma 4.4. *We have* $V_{p-1} \otimes V_1^{\mathrm{Fr}} \cong V_{2p-1}$.

Proof. We may assume that $f > 1$. The image of x^{2p-1} in V_{2p-1} (see §3 for notations) is fixed by U and we have $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right) x^{2p-1} = \lambda^{p-1+p} x^{2p-1}$ in V_{2p-1}. This implies that $V_{p-1} \otimes V_1^{\mathrm{Fr}}$ occurs as an irreducible subquotient of V_{2p-1}. However, both have dimension $2p$, hence they are isomorphic. □

Lemma 4.5. *Let r be an integer such that $0 \leq r \leq p-2$, we have:*

$$V_{2p-2-r} \otimes V_1 \cong (V_{2p-3-r} \otimes \det) \oplus V_{2p-1-r}.$$

Proof. We may assume $f > 1$ since by restricting to $\mathrm{GL}_2(\mathbb{F}_p) \subseteq \mathrm{GL}_2(\mathbb{F}_q) = \Gamma$ we get the result for $f = 1$. Tensoring with V_1 the exact sequence of Lemma 3.5 and using Lemma 3.8 gives an exact sequence:

$$\begin{aligned} 0 \to (V_{r+1} \otimes \det^{p-r-1}) \oplus (V_{r-1} \otimes \det^{p-r}) \to V_{2p-2-r} \otimes V_1 \to \\ (V_{p-r-1} \otimes V_1^{\mathrm{Fr}}) \oplus (V_{p-r-3} \otimes \det \otimes V_1^{\mathrm{Fr}}) \to 0 \end{aligned}$$

where one forgets the term involving $p-r-3$ (resp. $r-1$) if $r = p-2$ (resp. $r = 0$). It follows from (i) of Corollary 5.6 below that $\mathrm{Ext}^1_\Gamma(V_{p-r-1} \otimes V_1^{\mathrm{Fr}}, V_{r+1} \otimes \det^{p-r-1}) = 0$ and $\mathrm{Ext}^1_\Gamma(V_{p-r-3} \otimes \det \otimes V_1^{\mathrm{Fr}}, V_{r-1} \otimes \det^{p-r}) = 0$.

Moreover, $\dim_{\overline{\mathbb{F}}_p} \mathrm{Ext}^1_\Gamma(V_{p-r-3} \otimes \det \otimes V_1^{\mathrm{Fr}}, V_{r+1} \otimes \det^{p-r-1}) = 1$ if $r < p-2$ and the unique non-split extension is given by the representation $V_{2p-3-r} \otimes \det$ (Lemma 3.5). Likewise $\dim_{\overline{\mathbb{F}}_p} \mathrm{Ext}^1_\Gamma(V_{p-r-1} \otimes V_1^{\mathrm{Fr}}, V_{r-1} \otimes \det^{p-r}) = 1$ if $r > 0$ and the unique non-split extension is given by V_{2p-1-r}. Now an irreducible representation σ can occur in the socle of $V_{2p-2-r} \otimes V_1$ if and only if $\mathrm{Hom}_\Gamma(\sigma \otimes V_1^*, V_{2p-2-r}) \neq 0$. Since $V_r \otimes \det^{p-1-r}$ is the socle of V_{2p-2-r}, Lemmas 5.3 and 3.8 imply that $V_s \otimes V_1^{\mathrm{Fr}} \otimes \det^b$ cannot occur in the socle of $V_{2p-2-r} \otimes V_1$ if $0 \leq s \leq p-2$. Putting this together we obtain $V_{2p-2-r} \otimes V_1 \cong (V_{2p-3-r} \otimes \det) \oplus V_{2p-1-r}$ if $1 \leq r \leq p-3$. If $r = p-2$ the same proof gives $V_p \otimes V_1 \cong (V_{p-1} \otimes \det) \oplus V_{p+1}$. Finally if $r = 0$ we get $V_{2p-2} \otimes V_1 \cong (V_{p-1} \otimes V_1^{\mathrm{Fr}}) \oplus (V_{2p-3} \otimes \det) \cong V_{2p-1} \oplus (V_{2p-3} \otimes \det)$ where the last isomorphism is given by Lemma 4.4. □

Proposition 4.6. *Let $\boldsymbol{\varepsilon}, \boldsymbol{\delta} \in \Sigma_{\mathbf{r}}$ with $\boldsymbol{\varepsilon} \prec \boldsymbol{\delta}$ and let $E(\boldsymbol{\varepsilon}, \boldsymbol{\delta})$ be the unique non-split extension (see Lemma 4.3):*

$$0 \longrightarrow V_{\mathbf{r}(\boldsymbol{\varepsilon})} \otimes \det{}^{e(\boldsymbol{\varepsilon})} \longrightarrow E(\boldsymbol{\varepsilon}, \boldsymbol{\delta}) \longrightarrow V_{\mathbf{r}(\boldsymbol{\delta})} \otimes \det{}^{e(\boldsymbol{\delta})} \longrightarrow 0.$$

Then there exists a Γ-equivariant injection:

$$E(\boldsymbol{\varepsilon}, \boldsymbol{\delta}) \hookrightarrow \frac{\mathrm{Fil}^{f-|\boldsymbol{\delta}|} V_{\mathbf{2p-2-r}}}{\mathrm{Fil}^{f-|\boldsymbol{\delta}|+2} V_{\mathbf{2p-2-r}}}.$$

Proof. Let $J := \{i, \epsilon_i \neq 0\}$ then there exists an injection:

$$\tau := V_{2p-2-r_j}^{\mathrm{Fr}^j} \otimes (\otimes_{i \neq j} W_{J,i}^{\mathrm{Fr}^i}) \hookrightarrow \frac{\mathrm{Fil}^{f-|\boldsymbol{\delta}|} V_{\mathbf{2p-2-r}}}{\mathrm{Fil}^{f-|\boldsymbol{\delta}|+2} V_{\mathbf{2p-2-r}}}.$$

It follows from Lemma 3.8 that if $j-1 \notin J$ then $V_{2p-2-r_j}^{\mathrm{Fr}^j} \otimes (\otimes_{i \neq j} (V_{r(\boldsymbol{\varepsilon})_i} \otimes \det^{e(\boldsymbol{\varepsilon})_i})^{\mathrm{Fr}^i})$ is a summand of τ and if $j-1 \in J$ then $(V_1 \otimes V_{2p-2-r_j})^{\mathrm{Fr}^j} \otimes (\otimes_{i \neq j} (V_{r(\boldsymbol{\varepsilon})_i} \otimes \det^{e(\boldsymbol{\varepsilon})_i})^{\mathrm{Fr}^i})$ is a summand of τ. Lemmas 3.8 and 4.5 imply that $E(\boldsymbol{\varepsilon}, \boldsymbol{\delta})$ is a summand of τ, and hence we obtain an injection as in the statement. □

Recall that an ideal A of a partially ordered set $(P, \leq_P)$ is a subset such that $x \in A$ and $y \leq_P x$ imply $y \in A$.

Theorem 4.7. *Let $\mathbf{r} := (r_0, \cdots, r_{f-1})$ be an f-tuple of integers with $0 \leq r_i \leq p-2$, then $V_{\mathbf{2p-2-r}}$ is multiplicity free and the set $\Sigma_{\mathbf{r}}$ parametrizes its composition factors. For each subrepresentation τ of $V_{\mathbf{2p-2-r}}$ let $\Sigma_{\mathbf{r}}(\tau)$ be the set of composition factors of τ, then $\Sigma_{\mathbf{r}}(\tau)$ is an ideal of the partially ordered set $(\Sigma_{\mathbf{r}}, \leq)$. The mapping $\tau \mapsto \Sigma_{\mathbf{r}}(\tau)$ defines a lattice isomorphism between the lattice of subrepresentations of $V_{\mathbf{2p-2-r}}$ and the lattice of ideals, ordered by inclusion, of the partially ordered set $(\Sigma_{\mathbf{r}}, \leq)$.*

Proof. The first assertion is given by Corollary 3.11, (8) and Lemma 4.2. We identify in the sequel the irreducible subquotients of $V_{\mathbf{2p-2-r}}$ with the elements of $\Sigma_{\mathbf{r}}$. Let us define a new partial ordering $\leq'$ on $\Sigma_{\mathbf{r}}$ as follows: $\boldsymbol{\varepsilon} \leq' \boldsymbol{\delta}$ if and only if there exists a subrepresentation τ of $V_{\mathbf{2p-2-r}}$ such that $\mathrm{cosoc}_\Gamma \tau \cong \boldsymbol{\delta}$ and $\boldsymbol{\varepsilon}$ occurs as an irreducible subquotient of τ. Tautologically the lattice of ideals of $(\Sigma_{\mathbf{r}}, \leq')$ is isomorphic to the lattice of subrepresentations of $V_{\mathbf{2p-2-r}}$. Now if $\boldsymbol{\varepsilon} \leq' \boldsymbol{\delta}$ it follows from Corollary 3.11 and Lemma 4.2 that $\boldsymbol{\varepsilon}$ occurs in $\mathrm{Fil}^{f-|\boldsymbol{\delta}|-1} V_{\mathbf{2p-2-r}}$. Hence $|\boldsymbol{\varepsilon}| \leq |\boldsymbol{\delta}|$ by (8) and Lemma 4.2. Lemma 4.3 and Proposition 4.6 imply then that the partial orderings $\leq'$ and $\leq$ coincide. $\square$

Corollary 4.8. *Let τ be a subrepresentation of $V_{\mathbf{2p-2-r}}$. If $\mathbf{r} = \mathbf{0}$ assume that $V_{\mathbf{p-1}}$ is not a direct factor of τ. Then the graded pieces of the socle filtration of τ are given by:*

$$\tau_i \cong \bigoplus_{\substack{\boldsymbol{\varepsilon} \in \Sigma_{\mathbf{r}}(\tau) \\ |\boldsymbol{\varepsilon}| = i}} V_{\mathbf{r}(\boldsymbol{\varepsilon})} \otimes \det{}^{e(\boldsymbol{\varepsilon})}.$$

Proof. Let $\boldsymbol{\delta} \in \Sigma_{\mathbf{r}}$ then the number of $\boldsymbol{\varepsilon} \in \Sigma_{\mathbf{r}}$ such that $\boldsymbol{\varepsilon} \prec \boldsymbol{\delta}$ is $|\boldsymbol{\delta}| - |\{i, \delta_i \neq 0, \delta_{i-1} = 1, r_{i-1} = 0\}|$. This implies that, unless $\mathbf{r} = \mathbf{0}$ and $\boldsymbol{\delta} = \mathbf{1}$, there will exist $\boldsymbol{\varepsilon} \in \Sigma_{\mathbf{r}}$ such that $\boldsymbol{\varepsilon} \prec \boldsymbol{\delta}$. The case $\mathbf{r} = \mathbf{0}$ and $\boldsymbol{\delta} = \mathbf{1}$ corresponds to $V_{\mathbf{p-1}}$ and we have excluded it here. It follows from Definition 4.1 that $\boldsymbol{\varepsilon} \prec \boldsymbol{\delta}$ implies $|\boldsymbol{\varepsilon}| + 1 = |\boldsymbol{\delta}|$. Theorem 4.7 gives then the assertion. $\square$

Corollary 4.9. *Let τ be a subrepresentation of $V_{\mathbf{2p-2-r}}$. Assume that there exists an integer $k \in \{0, \cdots, f\}$ such that, if $V_{\mathbf{r}(\boldsymbol{\varepsilon})} \otimes \det^{e(\boldsymbol{\varepsilon})}$ occurs in $\mathrm{cosoc}_\Gamma \tau$, then $|\boldsymbol{\varepsilon}| = k$. Then the graded pieces of the cosocle filtration of τ are given by:*

$$\tau^i \cong \bigoplus_{\substack{\boldsymbol{\varepsilon} \in \Sigma_{\mathbf{r}}(\tau) \\ |\boldsymbol{\varepsilon}| = k-i}} V_{\mathbf{r}(\boldsymbol{\varepsilon})} \otimes \det{}^{e(\boldsymbol{\varepsilon})}.$$

Proof. This follows again from Theorem 4.7 together with $\boldsymbol{\varepsilon} \prec \boldsymbol{\delta} \Rightarrow |\boldsymbol{\varepsilon}| = |\boldsymbol{\delta}| - 1$. $\square$

Definition 4.10. *Let $\lambda, \lambda' \in \mathcal{I}(x_0, \cdots, x_{f-1})$ (see §3). We say λ and λ' are compatible if, whenever $\lambda_i(x_i) \in \{p - 2 - x_i - \pm 1, x_i \pm 1\}$ and $\lambda'_i(x_i) \in \{p - 2 - x_i - \pm 1, x_i \pm 1\}$ for the same i, then the signs of the ± 1 are the same in $\lambda_i(x_i)$ and $\lambda'_i(x_i)$.*

For $\lambda \in \mathcal{I}(x_0, \cdots, x_{f-1})$, set $\mathcal{S}(\lambda) := \{i \in \{0, \cdots, f-1\}, \lambda_i(x_i) = p - 2 - x_i - \pm 1, x_i \pm 1\}$ and $\ell(\lambda) := |\mathcal{S}(\lambda)|$.

Corollary 4.11. *Let σ and τ be two irreducible representations of Γ over $\overline{\mathbb{F}}_p$. Assume $\sigma = (r_0, \cdots, r_{f-1}) \otimes \eta$ with $0 \le r_i \le p-2$ for all i and let $I(\sigma,\tau)$ be the Γ-representation with socle σ and cosocle τ constructed in Corollary 3.12 (assuming it exists). Let $\lambda \in \mathcal{I}(x_0, \cdots, x_{f-1})$ such that $\tau = (\lambda_0(r_0), \cdots, \lambda_{f-1}(r_{f-1})) \otimes \det^{e(\lambda)(r_0,\cdots,r_{f-1})}\eta$ by Lemma 3.2. Then:*

$$I(\sigma,\tau)_i = \bigoplus_{\substack{\mathcal{S}(\lambda')\subseteq\mathcal{S}(\lambda)\\ \ell(\lambda')=i\\ \lambda' \text{ compatible with } \lambda}} (\lambda'_0(r_0), \cdots, \lambda'_{f-1}(r_{f-1})) \otimes \det{}^{e(\lambda')(r_0,\cdots,r_{f-1})}\eta$$

forgetting the weights such that $\lambda'_i(r_i) < 0$ for some i.

Proof. To an element $\lambda' \in \mathcal{I}(x_0, \cdots, x_{f-1})$, we associate an element $\boldsymbol{\varepsilon}' \in \Sigma$ as follows:

(i) if $\lambda'_i(x_i) \in \{p-2-x_i, p-3-x_i, p-1-x_i\}$ then $\epsilon'_{i-1} := \lambda'_i(x_i) - (p-2-x_i)$

(ii) if $\lambda'_i(x_i) \in \{x_i, x_i-1, x_i+1\}$ then $\epsilon'_{i-1} := \lambda'_i(x_i) - x_i$.

This defines a map $\mathcal{I}(x_0, \cdots, x_{f-1}) \to \Sigma$ which is bijective, the inverse map being given by $\lambda'_i(x_i) := p-2-x_i+\epsilon'_{i-1}$ if $\epsilon'_i \neq 0$ and $\lambda'_i(x_i) := x_i + \epsilon'_{i-1}$ if $\epsilon'_i = 0$. The reader can then easily check that the following properties hold:

(i) $0 \le \lambda'_i(r_i) \le p-1$ for all i if and only if $\boldsymbol{\varepsilon}' \in \Sigma_{\mathbf{r}} \subseteq \Sigma$ where $\mathbf{r} := (r_0, \cdots, r_{f-1})$

(ii) $(\lambda'_0(r_0), \cdots, \lambda'_{f-1}(r_{f-1})) \otimes \det^{e(\lambda')(r_0,\cdots,r_{f-1})} \cong V_{\mathbf{r}(\boldsymbol{\varepsilon}')} \otimes \det^{e(\boldsymbol{\varepsilon}')+\sum_{i=0}^{f-1}p^i r_i}$

(iii) if $\lambda \mapsto \boldsymbol{\varepsilon}$ and $\lambda' \mapsto \boldsymbol{\varepsilon}'$ with $\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}' \in \Sigma_{\mathbf{r}}$, we have $\boldsymbol{\varepsilon}' \le \boldsymbol{\varepsilon}$ if and only if $\mathcal{S}(\lambda') \subseteq \mathcal{S}(\lambda)$ and λ' is compatible with λ.

By Proposition 3.6 and the definition of $I(\sigma,\tau)$ (Corollary 3.12), we may embed $I(\sigma,\tau)$ into $V_{\mathbf{2p-2-r}} \otimes \det^{\sum_{i=0}^{f-1}p^i r_i}\eta$. Let $\boldsymbol{\varepsilon} \in \Sigma_{\mathbf{r}}$ correspond to λ as in the statement. From (iii), (i) and Theorem 4.7, we get that $\Sigma_{\mathbf{r}}(I(\sigma,\tau))$ corresponds to the f-tuples $\lambda' \in \mathcal{I}(x_0, \cdots, x_{f-1})$ such that $0 \le \lambda'_i(r_i) \le p-1$ for all i, $\mathcal{S}(\lambda') \subseteq \mathcal{S}(\lambda)$ and λ' is compatible with λ. The result follows then from Corollary 4.8 together with (ii) and the fact $\ell(\lambda') = |\boldsymbol{\varepsilon}'|$ if $\lambda' \mapsto \boldsymbol{\varepsilon}'$. $\square$

Let $\mathbf{r} := (r_0, \cdots, r_{f-1})$ be an f-tuple of integers with $0 \le r_i \le p-1$ for all i this time. Let Σ' be the subset of Σ consisting of f-tuples $\boldsymbol{\varepsilon} = (\epsilon_0, \cdots, \epsilon_{f-1})$ such that $\epsilon_i \in \{0,1\}$ for all i. Let $\Sigma'_{\mathbf{r}}$ be the subset of Σ' consisting of f-tuples $\boldsymbol{\varepsilon}$ such that if $r_i = p-1$ then $\epsilon_i = 0$. In particular, $\Sigma'_{\mathbf{r}} = \Sigma'$ if $0 \le r_i \le p-2$ for all i. Define $\chi : H \to \overline{\mathbb{F}}_p^\times$ by $\chi(\left(\begin{smallmatrix}\lambda & 0\\ 0 & \mu\end{smallmatrix}\right)) := \mu^{-\sum_{i=0}^{f-1}p^i r_i}$, so that B acts on the U-invariants of $V_{\mathbf{r}} \otimes \det^{-\sum_{i=0}^{f-1}p^i r_i}$ by χ. For $\boldsymbol{\varepsilon} \in \Sigma'_{\mathbf{r}}$ set $\chi_{\boldsymbol{\varepsilon}} := \chi\alpha^{\sum_{i=0}^{f-1}p^i\epsilon_i(p-1-r_i)}$.

Then $\chi_{\boldsymbol{\varepsilon}} = \chi_{\boldsymbol{\delta}}$ implies $\boldsymbol{\varepsilon} = \boldsymbol{\delta}$ or $\mathbf{r} = \mathbf{0}$ and $\{\boldsymbol{\varepsilon}, \boldsymbol{\delta}\} = \{\mathbf{0}, \mathbf{1}\}$. Moreover, [25, Lem. 4.2.33] implies that $\chi_{\boldsymbol{\varepsilon}} \neq \chi_{\boldsymbol{\varepsilon}}^s$ unless $\mathbf{r} = \mathbf{0}$, $\boldsymbol{\varepsilon} \in \{\mathbf{0}, \mathbf{1}\}$ or $\mathbf{r} = \mathbf{p-1}$ (and $\boldsymbol{\varepsilon} = \mathbf{0}$). To $\boldsymbol{\varepsilon} \in \Sigma'_{\mathbf{r}}$ we associate an f-tuple $\mathbf{r}(\boldsymbol{\varepsilon}) = (r(\boldsymbol{\varepsilon})_0, \cdots, r(\boldsymbol{\varepsilon})_{f-1})$ such that $0 \leq r(\boldsymbol{\varepsilon})_i \leq p-1$ and an integer $e(\boldsymbol{\varepsilon}) := \sum_{i=0}^{f-1} p^i e(\boldsymbol{\varepsilon})_i$ as follows:

(i) if $\mathbf{r} = \mathbf{0}$ and $\boldsymbol{\varepsilon} = \mathbf{0}$ then $\mathbf{r}(\mathbf{0}) := \mathbf{0}$ and $e(\mathbf{0}) := 0$

(ii) if $\mathbf{r} = \mathbf{0}$ and $\boldsymbol{\varepsilon} = \mathbf{1}$ then $\mathbf{r}(\mathbf{1}) := \mathbf{p-1}$ and $e(\mathbf{1}) := 0$

(iii) if $\mathbf{r} = \mathbf{p-1}$ (and $\boldsymbol{\varepsilon} = \mathbf{0}$) then $\mathbf{r}(\mathbf{0}) := \mathbf{p-1}$ and $e(\mathbf{0}) := 0$

(iv) in all other cases, $\mathbf{r}(\boldsymbol{\varepsilon})$ and $e(\boldsymbol{\varepsilon})$ are such that H acts on the U-invariants of $V_{\mathbf{r}(\boldsymbol{\varepsilon})} \otimes \det^{e(\boldsymbol{\varepsilon})}$ by $\chi_{\boldsymbol{\varepsilon}}$.

Note that $(\mathbf{r}(\boldsymbol{\varepsilon}), e(\boldsymbol{\varepsilon}))$ in (iv) is well defined since $\chi_{\boldsymbol{\varepsilon}} \neq \chi_{\boldsymbol{\varepsilon}}^s$.

Lemma 4.12. *Let $\boldsymbol{\varepsilon} \in \Sigma'_{\mathbf{r}}$. If $\mathbf{r} = \mathbf{0}$ or $\mathbf{r} = \mathbf{1}$, assume that $\boldsymbol{\varepsilon} \notin \{\mathbf{0}, \mathbf{1}\}$. Then $\mathbf{r}(\boldsymbol{\varepsilon})$ is determined by:*

$$\sum_{i=0}^{f-1} p^i r(\boldsymbol{\varepsilon})_i \equiv \sum_{i=0}^{f-1} p^i (r_i + 2\epsilon_i(p-1-r_i)) \qquad (q-1)$$

and we have $e(\boldsymbol{\varepsilon})_i = (1-\epsilon_i)(p-1-r_i)$ for all i. In particular, if $r_i = p-1$ then $r(\boldsymbol{\varepsilon})_i = p-1$ or $r(\boldsymbol{\varepsilon})_i = 0$. Moreover, if $0 \leq r_i \leq p-2$ for all i then the definition of $\mathbf{r}(\boldsymbol{\varepsilon})$ and $e(\boldsymbol{\varepsilon})$ coincides with the previous one.

Proof. This follows from:

$$\begin{aligned} \chi_{\boldsymbol{\varepsilon}}\Big(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\Big) &= \lambda^{\sum_{i=0}^{f-1} p^i (r_i + 2\epsilon_i(p-1-r_i))} \\ \chi_{\boldsymbol{\varepsilon}}\Big(\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}\Big) &= \lambda^{\sum_{i=0}^{f-1} p^i (1-\epsilon_i)(p-1-r_i)}. \end{aligned}$$

□

If τ is any representation of Γ, we denote by $\Sigma(\tau)$ the set of its irreducible subquotients and by $\Sigma(\tau) \cap \Sigma'_{\mathbf{r}}$ the subset of $\Sigma'_{\mathbf{r}}$ of $\boldsymbol{\varepsilon}$ corresponding to irreducible representations $V_{\mathbf{r}(\boldsymbol{\varepsilon})} \otimes \det^{e(\boldsymbol{\varepsilon})}$ that are also in $\Sigma(\tau)$.

Proposition 4.13. *Let τ be a representation of Γ. Suppose that $\mathrm{soc}_\Gamma \tau \cong V_{\mathbf{r}} \otimes \det^{-\sum_{i=0}^{f-1} r_i p^i}$ and that $\mathrm{soc}_\Gamma \tau$ occurs in τ with multiplicity 1. If $p = 2$ we additionally assume $\mathbf{r} = \mathbf{0}$ or $\mathbf{r} = \mathbf{p-1}$. Then we have:*

$$\tau^U \cong \bigoplus_{\boldsymbol{\varepsilon} \in \Sigma(\tau) \cap \Sigma'_{\mathbf{r}}} \chi \alpha^{\sum_{i=0}^{f-1} p^i \epsilon_i (p-1-r_i)}.$$

In particular $\dim_{\mathbb{F}_p} \tau^U = |\Sigma(\tau) \cap \Sigma'_{\mathbf{r}}|$.

Proof. By Proposition 3.6, we may embed τ into $V_{\mathbf{2p-2-r}}$ and we denote the quotient by Q. It follows from [25, Lem. 4.2.19] and [25, Lem. 4.2.20] that:

$$R_{\mathbf{r}}^U \cong V_{\mathbf{2p-2-r}}^U \cong \oplus_{\boldsymbol{\varepsilon}\in\Sigma'_{\mathbf{r}}}\chi_{\boldsymbol{\varepsilon}} = \oplus_{\boldsymbol{\varepsilon}\in\Sigma'_{\mathbf{r}}}\chi\alpha^{\sum_{i=0}^{f-1} p^i\epsilon_i(p-1-r_i)}. \tag{9}$$

If $\mathbf{r} = \mathbf{p-1}$ we necessarily have $\tau \cong V_{\mathbf{p-1}}$ and $\Sigma'_{\mathbf{r}} = \{\mathbf{0}\}$: the assertion follows trivially. Assume $\mathbf{r} \neq \mathbf{p-1}$, then the assumption on $\mathrm{soc}_\Gamma\,\tau$ implies that $V_{\mathbf{p-1}}$ can't occur in τ and we are left to prove that $\chi_{\boldsymbol{\varepsilon}}$ occurs in τ^U if and only if $\boldsymbol{\varepsilon} \in \Sigma(\tau) \cap \Sigma'_{\mathbf{r}}$. Assume $\mathbf{r} = \mathbf{0}$ and $\chi_{\boldsymbol{\varepsilon}} = \chi_{\boldsymbol{\varepsilon}}^s$, which implies $\boldsymbol{\varepsilon} \in \{\mathbf{0},\mathbf{1}\}$ and $\chi_{\boldsymbol{\varepsilon}} = 1$. We have that 1 occurs in τ^U (as $\mathrm{soc}_\Gamma(\tau)$ is the trivial representation) and that $\{\mathbf{0},\mathbf{1}\} \cap \Sigma(\tau) \cap \Sigma'_{\mathbf{0}} = \{\mathbf{0}\}$: the assertion follows in that case. Assume now $\chi_{\boldsymbol{\varepsilon}} \neq \chi_{\boldsymbol{\varepsilon}}^s$. If $\chi_{\boldsymbol{\varepsilon}}$ occurs in τ^U then $\mathrm{Hom}_\Gamma(\mathrm{Ind}_B^\Gamma\,\chi_{\boldsymbol{\varepsilon}}, \tau) \neq 0$. Since $V_{\mathbf{r}(\boldsymbol{\varepsilon})} \otimes \det^{e(\boldsymbol{\varepsilon})}$ is the cosocle of $\mathrm{Ind}_B^\Gamma\,\chi_{\boldsymbol{\varepsilon}}$, it must be a subquotient of τ. We thus have $\boldsymbol{\varepsilon} \in \Sigma(\tau) \cap \Sigma'_{\mathbf{r}}$. Conversely, assume $\boldsymbol{\varepsilon} \in \Sigma(\tau) \cap \Sigma'_{\mathbf{r}}$ i.e. $V_{\mathbf{r}(\boldsymbol{\varepsilon})} \otimes \det^{e(\boldsymbol{\varepsilon})} \in \Sigma(\tau)$. Equivalently, we have $\mathrm{Hom}_\Gamma(\mathrm{inj}\,V_{\mathbf{r}(\boldsymbol{\varepsilon})} \otimes \det^{e(\boldsymbol{\varepsilon})}, \tau) \neq 0$ (using the fact that $\mathrm{inj}\,V_{\mathbf{r}(\boldsymbol{\varepsilon})}$ has cosocle $V_{\mathbf{r}(\boldsymbol{\varepsilon})}$ and is a projective object). It follows from Proposition 3.10, Lemma 4.12 and (9) that $\dim_{\overline{\mathbb{F}}_p} \mathrm{Hom}_\Gamma(\mathrm{inj}\,V_{\mathbf{r}(\boldsymbol{\varepsilon})} \otimes \det^{e(\boldsymbol{\varepsilon})}, V_{\mathbf{2p-2-r}}) = 1$. As $\mathrm{inj}\,V_{\mathbf{r}(\boldsymbol{\varepsilon})} \otimes \det^{e(\boldsymbol{\varepsilon})}$ is projective, this implies $\mathrm{Hom}_\Gamma(\mathrm{inj}\,V_{\mathbf{r}(\boldsymbol{\varepsilon})}\otimes\det^{e(\boldsymbol{\varepsilon})}, Q) = 0$ i.e. $V_{\mathbf{r}(\boldsymbol{\varepsilon})}\otimes\det^{e(\boldsymbol{\varepsilon})}$ can't occur in Q. This implies in particular $\mathrm{Hom}_\Gamma(\mathrm{Ind}_B^\Gamma\,\chi_{\boldsymbol{\varepsilon}}, Q) = 0$ as $\mathrm{Ind}_B^\Gamma\,\chi_{\boldsymbol{\varepsilon}}$ has cosocle $V_{\mathbf{r}(\boldsymbol{\varepsilon})} \otimes \det^{e(\boldsymbol{\varepsilon})}$. We thus have an isomorphism:

$$\mathrm{Hom}_\Gamma(\mathrm{Ind}_B^\Gamma\,\chi_{\boldsymbol{\varepsilon}}, \tau) \cong \mathrm{Hom}_\Gamma(\mathrm{Ind}_B^\Gamma\,\chi_{\boldsymbol{\varepsilon}}, V_{\mathbf{2p-2-r}}).$$

But the right hand side is non-zero by (9) and we are done. □

5 Results on K-extensions

In this section, we assume F is a finite extension of $\mathbb{Q}_p$. We determine Γ-extensions between two weights and give partial results on K-extensions for $p > 2$ which are not Γ-extensions.

Proposition 5.1. *Let τ be a representation of K such that K_1 acts trivially. Then there exists an isomorphism of K-representations:*

$$H^1(K_1, \tau) \cong \bigoplus_{i=0}^{f-1} \left(\tau \otimes (V_2 \otimes \det^{-1})^{\mathrm{Fr}^i}\right) \oplus \bigoplus_{i=1}^{d} \tau,$$

where if $p \neq 2$ then $d := \dim_{\overline{\mathbb{F}}_p} \mathrm{Hom}(1 + \mathfrak{p}_F, \overline{\mathbb{F}}_p)$ and if $p = 2$ then $d := \dim_{\overline{\mathbb{F}}_p} \mathrm{Hom}(1 + \mathfrak{p}_F, \overline{\mathbb{F}}_p) - f$.

Proof. It is enough to prove the claim when τ is the trivial representation of K (as K_1 acts trivially on τ). We have $H^1(K_1,1)\simeq \mathrm{Hom}(K_1,\overline{\mathbb{F}}_p)$ (continuous group homomorphisms) with the action of K given by:

$$(g\psi)(h)=\psi(g^{-1}hg),\ g\in K,\ \psi\in \mathrm{Hom}(K_1,\overline{\mathbb{F}}_p),\ h\in K_1.$$

For $0\le i\le f-1$, define $\kappa_i^u,\kappa_i^l,\epsilon_i\in \mathrm{Hom}(K_1,\overline{\mathbb{F}}_p)$ as follows:

$$\kappa_i^u(A):=\omega(b)^{p^i},\quad \kappa_i^l(A):=\omega(c)^{p^i},\quad \epsilon_i(A):=\omega(a-d)^{p^i}$$

where $A:=\left(\begin{smallmatrix}1+\varpi a & \varpi b\\ \varpi c & 1+\varpi d\end{smallmatrix}\right)\in K_1$ (see §1 for ω). Since:

$$\begin{pmatrix}1 & -\alpha\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1 & \alpha\\ 0 & 1\end{pmatrix}=\begin{pmatrix}a-c\alpha & b+(a-d)\alpha-c\alpha^2\\ c & d+c\alpha\end{pmatrix}$$

the action of I_1 fixes κ_i^l and for all $\alpha\in\mathbb{F}_q$ we get:

$$\begin{aligned}\begin{pmatrix}1 & [\alpha]\\ 0 & 1\end{pmatrix}\kappa_i^u &= \kappa_i^u+\alpha^{p^i}\epsilon_i-\alpha^{2p^i}\kappa_i^l\\ \begin{pmatrix}1 & [\alpha]\\ 0 & 1\end{pmatrix}\epsilon_i &= \epsilon_i-2\alpha^{p^i}\kappa_i^l.\end{aligned}$$

Moreover we have:

$$\begin{pmatrix}[\lambda] & 0\\ 0 & [\mu]\end{pmatrix}\kappa_i^l=(\lambda\mu^{-1})^{p^i}\kappa_i^l,\quad \lambda,\mu\in\mathbb{F}_q^\times.$$

One may then check that the map:

$$y^2\mapsto\kappa_i^u,\quad \binom{2}{1}xy\mapsto\epsilon_i,\quad x^2\mapsto-\kappa_i^l$$

induces a K-equivariant isomorphism $(V_2\otimes\det^{-1})^{\mathrm{Fr}^i}\cong\langle\kappa_i^u,\epsilon_i,\kappa_i^l\rangle$. As $K_2\cap U\subset[K_1\cap T,K_1\cap U]$, $K_2\cap U^s\subset[K_1\cap T,K_1\cap U^s]$ (where square brackets denote the subgroup generated by the commutators) and:

$$\left\{\begin{pmatrix}x & 0\\ 0 & x^{-1}\end{pmatrix},x\in 1+\mathfrak{p}_F^2\right\}\subset\langle K_1\cap U^s,K_1\cap U\rangle,$$

we deduce that every $\psi\in\mathrm{Hom}(K_1,\overline{\mathbb{F}}_p)$ can be written as a linear combination of κ_i^u, ϵ_i, κ_i^l for $0\le i\le f-1$ and a homomorphism which factors through the determinant. If ψ factors through the determinant then K acts trivially on ψ. Let $x\in\mathbb{F}_q$ and $\alpha\in[x]+\mathfrak{p}_F$ then:

$$\epsilon_i\left(\begin{pmatrix}1+\varpi\alpha & 0\\ 0 & (1+\varpi\alpha)^{-1}\end{pmatrix}\right)=(2x)^{p^i}.$$

Hence, ϵ_i factors through det if and only if $p=2$. This implies the claim. □

Corollary 5.2. *Let σ and τ be finite dimensional representations of Γ over $\overline{\mathbb{F}}_p$. Suppose that $\mathrm{Hom}_\Gamma(\sigma,\tau)=0$ and $\mathrm{Hom}_\Gamma(\sigma,(V_2\otimes\det^{-1})^{\mathrm{Fr}^i}\otimes\tau)=0$ for $0\le i\le f-1$. Then $\mathrm{Ext}^1_\Gamma(\sigma,\tau)\cong\mathrm{Ext}^1_K(\sigma,\tau)$.*

Proof. Since for all representations π of K (over $\overline{\mathbb{F}}_p$) we have $\mathrm{Hom}_K(\sigma,\pi)\simeq\mathrm{Hom}_\Gamma(\sigma,\pi^{K_1})$, the Grothendieck spectral sequence gives an exact sequence:

$$0\longrightarrow\mathrm{Ext}^1_\Gamma(\sigma,\tau)\longrightarrow\mathrm{Ext}^1_K(\sigma,\tau)\longrightarrow\mathrm{Hom}_\Gamma(\sigma,H^1(K_1,\tau))\ .$$

The result follows then from Proposition 5.1. □

We now use notations from §3.

Lemma 5.3. *For $0\le r\le p-1$ we have $\mathrm{Hom}_{\overline{\mathbb{F}}_p}(V_r,\overline{\mathbb{F}}_p)\simeq V_r\otimes\det^{-r}$.*

Proof. Exercise. □

Proposition 5.4. *Assume $p>2$.*

(i) For $0\le r<p-2$ we have an isomorphism of Γ-representations:

$$V_2\otimes V_r\cong V_{r+2}\oplus(V_r\otimes\det)\oplus(V_{r-2}\otimes\det^2).$$

(ii) We have an isomorphism of Γ-representations (recall $V_r=0$ if $r<0$):

$$\begin{aligned}V_2\otimes V_{p-2}&\cong R_{p-2}\oplus(V_{p-4}\otimes\det^2)\\ V_2\otimes V_{p-1}&\cong R_{p-3}\oplus(V_{p-1}\otimes\det).\end{aligned}$$

Proof. The assumption $p>2$ ensures that $1<p-1$. Hence by Lemma 3.8 we have $V_1\otimes V_1\cong V_2\oplus(V_0\otimes\det)$. If $r<p-2$ then using Lemma 3.8 twice we obtain that $V_1\otimes V_1\otimes V_r$ is isomorphic to:

$$V_1\otimes(V_{r+1}\oplus(V_{r-1}\otimes\det))\cong V_{r+2}\oplus(V_r\otimes\det)\oplus(V_r\otimes\det)\oplus(V_{r-2}\otimes\det^2),$$

hence $V_2\otimes V_r\cong V_{r+2}\oplus(V_r\otimes\det)\oplus(V_{r-2}\otimes\det^2)$, which proves (i). If $r=p-2$ the same argument gives $V_2\otimes V_{p-2}\cong(V_1\otimes V_{p-1})\oplus(V_{p-4}\otimes\det^2)$ and Lemma 3.8 implies $V_1\otimes V_{p-1}\cong R_{p-2}$. If $r=p-1$ then by [25, Prop. 4.2.11], R_{p-3} is a $2p$-dimensional subspace of $V_2\otimes V_{p-1}$. The restrictions of R_{p-3} and $V_2\otimes V_{p-1}$ to $\mathrm{GL}_2(\mathbb{F}_p)$ are injective objects in $\mathrm{Rep}_{\mathrm{GL}_2(\mathbb{F}_p)}$. Hence there exists a $\mathrm{GL}_2(\mathbb{F}_p)$-equivariant isomorphism:

$$V_2\otimes V_{p-1}\cong R_{p-3}\oplus J,$$

where J is an injective object in $\mathrm{Rep}_{\mathrm{GL}_2(\mathbb{F}_p)}$. Since $\dim_{\overline{\mathbb{F}}_p}J=3p-2p=p$, we have $J\simeq V_{p-1}\otimes\det^a$ as an $\mathrm{GL}_2(\mathbb{F}_p)$ representation. Now since J is irreducible as $\mathrm{GL}_2(\mathbb{F}_p)$ representation, there exists an exact sequence of Γ-representations:

$$0 \longrightarrow R_{p-3} \longrightarrow V_2 \otimes V_{p-1} \longrightarrow V_{p-1}^{\mathrm{Fr}^i} \otimes \det{}^a \longrightarrow 0. \qquad (10)$$

for some $0 \leq i \leq f-1$ and some $0 \leq a < q-1$. By dualizing and using Lemma 5.3, we obtain an injection $V_{p-1}^{\mathrm{Fr}^i} \otimes \det^b \hookrightarrow V_2 \otimes V_{p-1}$ for some $0 \leq b < q-1$. Since $\mathrm{soc}_\Gamma R_{p-3} \simeq V_{p-3} \otimes \det^2$, we can't have $V_{p-1}^{\mathrm{Fr}^i} \otimes \det^b \hookrightarrow R_{p-3}$. Thus (10) must split and we have $b = a$. Now the element $xy \otimes x^{p-1} - x^2 \otimes x^{p-2}y$ is fixed by U in $V_2 \otimes V_{p-1}$ (with obvious notations) and it follows from [25, Prop. 4.2.13 and Lem. 4.2.14] that it does not lie in R_{p-3}. Hence, the image of $xy \otimes x^{p-1} - x^2 \otimes x^{p-2}y$ spans the U-invariants of $V_{p-1}^{\mathrm{Fr}^i} \otimes \det^a$. Since:

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} (xy \otimes x^{p-1} - x^2 \otimes x^{p-2}y) = \lambda^p \mu (xy \otimes x^{p-1} - x^2 \otimes x^{p-2}y), \quad \lambda, \mu \in \mathbb{F}_q^\times,$$

we must have $i = 0$ and $a = 1$. Hence $V_2 \otimes V_{p-1} \simeq R_{p-3} \oplus (V_{p-1} \otimes \det)$. $\square$

Corollary 5.5. *Assume $p > 2$ and let $\mathbf{r}$ and $\mathbf{s}$ be f-tuples such that $0 \leq r_j, s_j \leq p-1$ for all j.*

(i) For $0 \leq i \leq f-1$ and all integers a, b we have:

$$\dim_{\overline{\mathbb{F}}_p} \mathrm{Hom}_\Gamma(V_{\mathbf{s}} \otimes \det{}^b, V_2^{\mathrm{Fr}^i} \otimes V_{\mathbf{r}} \otimes \det{}^a) \leq 1.$$

(ii) We have $\mathrm{Hom}_\Gamma(V_{\mathbf{s}} \otimes \det^b, V_2^{\mathrm{Fr}^i} \otimes V_{\mathbf{r}} \otimes \det^a) \neq 0$ if and only if $s_j = r_j$ for all $j \neq i$ and one of the following holds:

(a) $s_i = r_i + 2$ and $b \equiv a \ (q-1)$

(b) $s_i = r_i$ and $b \equiv a + p^i \ (q-1)$

(c) $s_i = r_i - 2$ and $b \equiv a + 2p^i \ (q-1)$

(d) $f = 1$, $p = 3$, $s_0 = r_0 = p-1$ and $b \equiv a \ (q-1)$.

Proof. The result is obvious from Proposition 5.4 if $0 \leq r_i < p-2$ (note that if $r_i < 2$ or $r_i > p-3$, some cases are empty as we must have $0 \leq s_i \leq p-1$). If $r_i = p-2$, there exists an injection:

$$R_{p-2}^{\mathrm{Fr}^i} \otimes \Big(\bigotimes_{j \neq i} V_{r_j}^{\mathrm{Fr}^j} \Big) \hookrightarrow R_{p-2}^{\mathrm{Fr}^i} \otimes \Big(\bigotimes_{j \neq i} (R_{r_j} \otimes \det{}^{r_j - p+1})^{\mathrm{Fr}^j} \Big).$$

As $\mathbf{r} \neq \mathbf{0}$ (because $r_i = p-2$ and $p > 2$), we have (see §3):

$$R_{p-2}^{\mathrm{Fr}^i} \otimes \Big(\bigotimes_{j \neq i} (R_{r_j} \otimes \det{}^{r_j - p+1})^{\mathrm{Fr}^j} \Big) \cong \mathrm{inj} \Big((V_{p-2} \otimes \det)^{\mathrm{Fr}^i} \otimes \bigotimes_{j \neq i} V_{r_j}^{\mathrm{Fr}^j} \Big),$$

which implies in particular:

$$\mathrm{soc}_\Gamma\Big(R_{p-2}^{\mathrm{Fr}^i}\otimes\Big(\bigotimes_{j\neq i}V_{r_j}^{\mathrm{Fr}^j}\Big)\Big)\cong(V_{p-2}\otimes\det)^{\mathrm{Fr}^i}\otimes\Big(\bigotimes_{j\neq i}V_{r_j}^{\mathrm{Fr}^j}\Big).$$

Using Proposition 5.4, we then deduce the result in that case. If ($r_i=p-1$ and $p>3$) or ($r_i=p-1$, $p=3$ and $f>1$), the proof is analogous using:

$$\mathrm{soc}_\Gamma\Big(R_{p-3}^{\mathrm{Fr}^i}\otimes\Big(\bigotimes_{j\neq i}V_{r_j}^{\mathrm{Fr}^j}\Big)\Big)\cong(V_{p-3}\otimes\det)^{\mathrm{Fr}^i}\otimes\Big(\bigotimes_{j\neq i}V_{r_j}^{\mathrm{Fr}^j}\Big).$$

Finally, if $r_i=p-1$ and $p=q=3$, the result follows from $R_0\simeq\mathrm{inj}(V_0)\oplus V_{p-1}$. □

We finally obtain the main result of that section:

Corollary 5.6. *Let $\sigma:=(r_0,\cdots,r_{f-1})\otimes\det^a$ and $\tau:=(s_0,\cdots,s_{f-1})\otimes\det^b$ be two weights ($0\leq r_i,s_i\leq p-1$ and $0\leq a,b$).*

(i) We always have $\dim_{\overline{\mathbb{F}}_p}\mathrm{Ext}^1_\Gamma(\tau,\sigma)\leq 1$. If $f=1$, $p>2$ and $(r_0,s_0)\neq(0,p-1)$, we have $\mathrm{Ext}^1_\Gamma(\tau,\sigma)\neq 0$ if and only if $s_0=p-2-r_0\pm 1$ and $b\equiv a+r_0+1-p^{\frac{(1\pm1)}{2}}\ (p-1)$. If $f>1$, we have $\mathrm{Ext}^1_\Gamma(\tau,\sigma)\neq 0$ if and only if there exists $j\in\{0,\cdots,f-1\}$ such that $s_i=r_i$ for all $i\notin\{j-1,j\}$ (with the convention $-1=f-1$) and one of the following holds:

(a) $s_{j-1}=p-2-r_{j-1}$, $s_j=r_j-1$ and $b\equiv a+p^{j-1}(r_{j-1}+1)\ (q-1)$

(b) $s_{j-1}=p-2-r_{j-1}$, $s_j=r_j+1$ and $b\equiv a+p^{j-1}(r_{j-1}+1)-p^j\ (q-1)$.

(ii) Assume $p>2$. If $\mathrm{Ext}^1_\Gamma(\tau,\sigma)\not\cong\mathrm{Ext}^1_K(\tau,\sigma)$ then there exists $j\in\{0,\cdots,f-1\}$ such that $s_i=r_i$ for all $i\neq j$ and one of the following holds:

(a) $s_j=r_j-2$ and $b\equiv a+p^j\ (q-1)$

(b) $s_j=r_j$ and $b\equiv a\ (q-1)$

(c) $s_j=r_j+2$ and $b\equiv a-p^j\ (q-1)$

(d) $f=1$, $p=3$, $s_0=r_0=p-1$ and $b\equiv a-1\ (q-1)$.

Proof. We start with (i). The case $f=1$ is a direct consequence of Lemmas 3.4 and 3.5. Assume $f>1$. Twisting everything by $\det^{-a}$, we can assume $\sigma=V_{\mathbf{r}}$ (with obvious notations). Let τ be as in (a) or (b), then Lemma 3.8 together with (8) imply that τ occurs in the socle of:

$$\mathrm{Fil}^{|S_{\mathbf{r}}|-1}(V_{\mathbf{2p-2-r}}\otimes\det^r)/\mathrm{Fil}^{|S_{\mathbf{r}}|}(V_{\mathbf{2p-2-r}}\otimes\det^r)$$

where $r := \sum_{i=0}^{f-1} r_i p^i$. As $\mathrm{Fil}^{|S_{\mathbf{r}}|}(V_{\mathbf{2p-2-r}} \otimes \det^r) \cong V_{\mathbf{r}} \cong \sigma$, the inverse image of τ by the surjection $\mathrm{Fil}^{|S_{\mathbf{r}}|-1}(V_{\mathbf{2p-2-r}} \otimes \det^r) \twoheadrightarrow \mathrm{Fil}^{|S_{\mathbf{r}}|-1}(V_{\mathbf{2p-2-r}} \otimes \det^r)/V_{\mathbf{r}}$ gives an element in $\mathrm{Ext}^1_\Gamma(\tau, \sigma)$. Moreover, this element is non-zero as $\mathrm{soc}_\Gamma(V_{\mathbf{2p-2-r}} \otimes \det^r) = V_{\mathbf{r}}$ by Lemmas 3.4 and 3.5. Now we have:

$$\mathrm{Ext}^1_{\Gamma'}(\tau, \sigma) \cong \mathrm{Ext}^1_\Gamma(\mathrm{Ind}_{\Gamma'}^{\Gamma} \tau, \sigma) \cong \bigoplus_{c=1}^{q-1} \mathrm{Ext}^1_\Gamma(\tau \otimes \det^c, \sigma).$$

By [2, Cor. 4.5], $\mathrm{Ext}^1_{\Gamma'}(\tau, \sigma) \neq 0$ if and only if there exists j such that $s_i = r_i$ for $i \notin \{j-1, j\}$, $s_{j-1} = p-2-r_{j-1}$ and $s_j = r_j \pm 1$. In that case, $\dim_{\overline{\mathbb{F}}_p} \mathrm{Ext}^1_{\Gamma'}(\tau, \sigma) = 1$ unless $f = 2$ and $\mathbf{r} = ((p-3)/2, (p-1)/2)$ or $\mathbf{r} = ((p-1)/2, (p-3)/2)$ in which case the dimension is 2. If the dimension is 1 we are done. If $f = 2$ and $\mathbf{r} = ((p-3)/2, (p-1)/2)$ then case (a) with $j = 1$ and case (b) with $j = 0$ give the same 2-tuple $\mathbf{s}$ but a different exponent b, which implies the assertion. The other case is analogous. Finally (ii) follows from Corollary 5.2 and Corollary 5.5. □

The second part of Corollary 5.6 is presumably not optimal in the sense that there might exist σ and τ satisfying one of the conditions in (ii) with $\mathrm{Ext}^1_K(\tau, \sigma) = 0$.

Corollary 5.7. *Assume $p > 2$ and let W be a representation of K on a finite dimensional $\overline{\mathbb{F}}_p$-vector space. Assume W is multiplicity free and for any pair of distinct irreducible constituents (σ, τ) of W, none of the conditions (a) to (d) in (ii) of Corollary 5.6 are satisfied for any j. Then W is a Γ-representation.*

Proof. We argue by induction on $n(W) :=$ the number of irreducible subquotients of W. If $n(W) = 1$ then W is irreducible and so K_1 acts trivially. Suppose that $n(W) > 1$ and let σ be an irreducible quotient of W. Consider an exact sequence $0 \to W_1 \to W \to \sigma \to 0$. Since $n(W_1) = n(W) - 1$, K_1 acts trivially on W_1. Moreover, our assumptions imply:

$$\mathrm{Hom}_\Gamma(\sigma, W_1^{\mathrm{ss}}) = \mathrm{Hom}_\Gamma(\sigma, (V_2 \otimes \det^{-1})^{\mathrm{Fr}^i} \otimes W_1^{\mathrm{ss}}) = 0$$

for all i, where ss denotes semi-simplification. This implies:

$$\mathrm{Hom}_\Gamma(\sigma, W_1) = \mathrm{Hom}_\Gamma(\sigma, (V_2 \otimes \det^{-1})^{\mathrm{Fr}^i} \otimes W_1) = 0$$

for all i. By Corollary 5.2 we have $\mathrm{Ext}^1_\Gamma(\sigma, W_1) \cong \mathrm{Ext}^1_K(\sigma, W_1)$ and hence K_1 acts trivially on W. □

6 Hecke algebra

We recall certain results on the representation theory of the Hecke algebra of I_1. We follow (most of) the notations of [25, §2] and don't assume anything on F.

Let $\mathcal{H} := \mathrm{End}_G(\text{c-Ind}_{I_1}^G 1)$. The algebra $\mathcal{H}$ has an $\overline{\mathbb{F}}_p$-basis indexed by the double cosets $I_1 \backslash G / I_1$. We write T_g for the element corresponding to the double coset $I_1 g I_1$. For a character $\chi : H \to \overline{\mathbb{F}}_p^\times$, we set $e_\chi := |H|^{-1} \sum_{h \in H} \chi(h) T_h$. The elements T_{n_s}, T_Π, $T_{\Pi^{-1}}$ and the idempotents e_χ for all characters $\chi : H \to \overline{\mathbb{F}}_p^\times$ generate $\mathcal{H}$ as an algebra. For relations see [25, Lem. 2.0.12].

Let $\mathcal{I} : \mathrm{Rep}_G \to \mathrm{Mod}_{\mathcal{H}}$ be the functor:

$$\mathcal{I}(\pi) := \pi^{I_1} \cong \mathrm{Hom}_G(\text{c-Ind}_{I_1}^G 1, \pi).$$

Let $\mathcal{T} : \mathrm{Mod}_{\mathcal{H}} \to \mathrm{Rep}_G$ be the functor:

$$\mathcal{T}(M) := M \otimes_{\mathcal{H}} \text{c-Ind}_{I_1}^G 1.$$

One checks that $(\mathcal{I}, \mathcal{T})$ is a pair of adjoint functors i.e. $\mathrm{Hom}_{\mathcal{H}}(M, \mathcal{I}(\pi)) \cong \mathrm{Hom}_G(\mathcal{T}(M), \pi)$. Let $\mathrm{Rep}_G^{\varpi=1}$ be the full subcategory of Rep_G with objects G-representations on which the fixed uniformizer ϖ acts trivially. Then the functors restrict to $\mathcal{I} : \mathrm{Rep}^{\varpi=1} \to \mathrm{Mod}_{\mathcal{H}_{\varpi=1}}$ and $\mathcal{T} : \mathrm{Mod}_{\mathcal{H}_{\varpi=1}} \to \mathrm{Rep}^{\varpi=1}$ where $\mathcal{H}_{\varpi=1} := \mathcal{H}/(T_\Pi^2 - 1)$.

Let $\mathbf{r} := (r_0, \cdots, r_{f-1})$ be an f-tuple such that $0 \leq r_i \leq p-1$ for all i. We consider $V_{\mathbf{r}}$ (see Definition 3.3) as a representation of $\mathfrak{K}_0$ by lifting it to a representation of K and letting ϖ act trivially. It is shown in [4, Prop. 8] that there exists an isomorphism of algebras:

$$\mathrm{End}_G(\text{c-Ind}_{\mathfrak{K}_0}^G V_{\mathbf{r}}) \cong \overline{\mathbb{F}}_p[T]$$

for a certain $T \in \mathrm{End}_G(\text{c-Ind}_{\mathfrak{K}_0}^G V_{\mathbf{r}})$ defined in [4, §3]. Fix $\varphi \in \text{c-Ind}_{\mathfrak{K}_0}^G V_{\mathbf{r}}$ such that $\mathrm{Supp}\,\varphi = \mathfrak{K}_0$ and $\varphi(1)$ spans $V_{\mathbf{r}}^{I_1}$. Since φ generates $\text{c-Ind}_{\mathfrak{K}_0}^G V_{\mathbf{r}}$ as a G-representation, T is determined by $T\varphi$.

Lemma 6.1. *(i) If $\mathbf{r} = \mathbf{0}$ then $T\varphi = \Pi\varphi + \sum_{\lambda \in \mathbb{F}_q} \left(\begin{smallmatrix} \varpi & [\lambda] \\ 0 & 1 \end{smallmatrix}\right)\varphi$.*

(ii) If $\mathbf{r} \neq \mathbf{0}$ then $T\varphi = \sum_{\lambda \in \mathbb{F}_q} \left(\begin{smallmatrix} \varpi & [\lambda] \\ 0 & 1 \end{smallmatrix}\right)\varphi$.

Proof. In the notation of [4] this is the calculation of $T([1, e_{\vec{0}}])$. The claim follows from the formula (19) in the proof of [4, Th. 19]. $\square$

Definition 6.2. *Let $r := \sum_{i=0}^{f-1} p^i r_i$ with $r_i \in \{0, \cdots, p-1\}$ and $\mathbf{r}$ the f-tuple $(r_0, \cdots, r_{f-1})$. Let $\lambda \in \overline{\mathbb{F}}_p$ and $\eta : F^\times \to \overline{\mathbb{F}}_p^\times$ be a smooth character. We define an $\mathcal{H}$-module $M(r, \lambda)$ by the exact sequence:*

$$0 \longrightarrow (\text{c-Ind}_{\mathfrak{K}_0}^G V_{\mathbf{r}})^{I_1} \xrightarrow{T-\lambda} (\text{c-Ind}_{\mathfrak{K}_0}^G V_{\mathbf{r}})^{I_1} \longrightarrow M(r, \lambda) \longrightarrow 0.$$

We define a G-representation $\pi(r, \lambda)$ be the exact sequence:

$$0 \longrightarrow \text{c-Ind}_{\mathfrak{K}_0}^G V_{\mathbf{r}} \xrightarrow{T-\lambda} \text{c-Ind}_{\mathfrak{K}_0}^G V_{\mathbf{r}} \longrightarrow \pi(r, \lambda) \longrightarrow 0.$$

We set $\pi(r, \lambda, \eta) := \pi(r, \lambda) \otimes \eta \circ \det$ and $M(r, \lambda, \eta) := M(r, \lambda) \otimes \eta \circ \det$.

For r, η as in Definition 6.2, let $\chi : H \to \overline{\mathbb{F}}_p^\times$ be the character given by $\chi\big(\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right)\big) := \mu^r \eta([\lambda\mu])$. Set $\gamma := \{\chi, \chi^s\}$ and $e_\gamma := \sum_{\psi \in \gamma} e_\psi \in \mathcal{H}$. The idempotents e_γ are central in $\mathcal{H}$. If $|\gamma| = 1$ (i.e. if $r = 0$ or $r = q - 1$), set $Z_\gamma := T_{n_s} T_\Pi + T_\Pi T_{n_s} + T_\Pi$. Otherwise set $Z_\gamma := T_{n_s} T_\Pi + T_\Pi T_{n_s}$. The elements Z_γ are central in $e_\gamma \mathcal{H}$ and were used in [33].

Proposition 6.3. *Letting $\lambda_\eta := \lambda\eta(-\varpi^{-1})$, there exist exact sequences of $\mathcal{H}$-modules:*

(i) if $r = 0$:

$$0 \longrightarrow (1 + T_{n_s}) e_\chi \mathcal{H}_{\varpi=1} \xrightarrow{Z_\gamma - \lambda_\eta} (1 + T_{n_s}) e_\chi \mathcal{H}_{\varpi=1} \longrightarrow M(0, \lambda, \eta) \longrightarrow 0$$

(ii) if $r \neq 0$:

$$0 \longrightarrow T_{n_s} e_{\chi^s} \mathcal{H}_{\varpi=1} \xrightarrow{Z_\gamma - \lambda_\eta} T_{n_s} e_{\chi^s} \mathcal{H}_{\varpi=1} \longrightarrow M(r, \lambda, \eta) \longrightarrow 0.$$

Proof. We prove the statement in the (harder) case when $r \neq 0$ and $r \neq q - 1$. We can assume $\eta = 1$, since twisting by η has no effect on the action by T_{n_s} and $(v \otimes 1) T_\Pi = \Pi^{-1}(v \otimes 1) = (v T_\Pi) \otimes \eta(-\varpi^{-1})$. We claim that $(\text{c-Ind}_{\mathfrak{K}_0}^G V_{\mathbf{r}})^{I_1} \cong T_{n_s} e_{\chi^s} \mathcal{H}_{\varpi=1}$. It follows from [25, Rem. 3.1.6] that the image $\text{Im}(\text{c-Ind}_{IZ}^G \chi^s \xrightarrow{T_{n_s}} \text{c-Ind}_{IZ}^G \chi)$ is isomorphic to $\text{c-Ind}_{\mathfrak{K}_0}^G V_{\mathbf{r}}$, where we consider χ and χ^s as representations of IZ with ϖ acting trivially. Now $(\text{c-Ind}_{IZ}^G \chi)^{I_1} \cong e_\chi \mathcal{H}_{\varpi=1}$ as an $\mathcal{H}$-module, hence $T_{n_s} e_{\chi^s} \mathcal{H}_{\varpi=1}$ is a submodule of $(\text{c-Ind}_{\mathfrak{K}_0}^G V_{\mathbf{r}})^{I_1}$. This is an isomorphism since $T_{n_s}^2 e_\chi = 0$ and it can be deduced from [25, Lem. 2.0.15] that:

$$\operatorname{Ker}(\, e_\chi \mathcal{H}_{\varpi=1} \xrightarrow{T_{n_s}} e_{\chi^s}\mathcal{H}_{\varpi=1}\,) \cong T_{n_s} e_{\chi^s}\mathcal{H}_{\varpi=1}.$$

Let $\varphi_{\chi^s} \in \text{c-Ind}_{IZ}^G \chi^s$ be the function such that $\operatorname{Supp}\varphi = IZ$ and $\varphi_\chi(g) := \chi^s(g)$ for all $g \in IZ$, then:

$$Z_\gamma T_{n_s}\varphi_{\chi^s} = T_{n_s} T_\Pi T_{n_s}\varphi_{\chi^s} = \sum_{\mu\in\mathbb{F}_q} \begin{pmatrix} 1 & [\mu] \\ 0 & 1 \end{pmatrix} n_s^{-1}\Pi^{-1} T_{n_s}\varphi_{\chi^s}.$$

It follows from Lemma 6.1 that $Z_\gamma = T$. □

Corollary 6.4. *(i) There exists a basis $\{v_1, v_2\}$ of the underlying vector space of $M(r,\lambda,\eta)$ such that:*

$$v_1 e_{\chi^s} = v_1, \quad v_1 T_\Pi = v_2, \quad v_2 e_\chi = v_2, \quad v_2 T_\Pi = v_1$$

and such that $v_1 T_{n_s} = -v_1$ if $r = q-1$ and $v_1 T_{n_s} = 0$ otherwise.

(ii) We have $v_2(1 + T_{n_s}) = \eta(-\varpi^{-1})\lambda v_1$ if $r = 0$ and $v_2 T_{n_s} = \eta(-\varpi^{-1})\lambda v_1$ otherwise.

Proof. One may show that if $r \neq 0$ (resp. $r = 0$) the images of $T_{n_s}e_{\chi^s}$, $T_{n_s}e_{\chi^s}T_\Pi$ (resp. $(1+T_{n_s})e_\chi$, $(1+T_{n_s})e_\chi T_\Pi$) form a basis of $M(r,\lambda)$. One may then immediately verify the assertions. □

One can deduce from Corollary 6.4 that $M(r,\lambda)$ is irreducible unless $(r,\lambda) = (0,\pm 1)$ or $(r,\lambda) = (q-1,\pm 1)$ (see [33]). Moreover, there exist exact non-split sequences of $\mathcal{H}_{\varpi=1}$ modules:

$$0 \longrightarrow \mathcal{I}(\delta_{\pm 1}) \longrightarrow M(q-1,\pm 1) \longrightarrow \mathcal{I}(\operatorname{St}\otimes\delta_{\pm 1}) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{I}(\operatorname{St}\otimes\delta_{\pm 1}) \longrightarrow M(0,\pm 1) \longrightarrow \mathcal{I}(\delta_{\pm 1}) \longrightarrow 0$$

where St denotes the Steinberg representation of G over $\overline{\mathbb{F}}_p$.

Corollary 6.5. *Let M be a subquotient of $M(s,\mu,\omega^a)$ in $\operatorname{Mod}_{\mathcal{H}_{\varpi=1}}$. Assume that $\operatorname{Ext}^1_{\mathcal{H}_{\varpi=1}}(M, M(r,\lambda)) \neq 0$, then $\lambda = (-1)^a\mu$ and either $r = s$ and $a \equiv 0\ (q-1)$ or $r = q-1-s$ and $a \equiv r\ (q-1)$.*

Proof. Let $0 \to M(r,\lambda) \to E \to M \to 0$ be a non-split extension. Set $\gamma := \{\chi,\chi^s\}$. Since e_γ and $Z_\gamma e_\gamma$ are central in $\mathcal{H}_{\varpi=1}$, the maps $E \to E$, $v \mapsto v(1-e_\gamma)$, $v \mapsto v(Z_\gamma e_\gamma - \lambda)$ are maps of $\mathcal{H}_{\varpi=1}$ modules and factor through $M \to E$. Since E is not split, $M(1-e_\gamma) = 0$ and hence $s = r$ and $a \equiv 0\ (q-1)$ or $s = q-1-r$ and $a \equiv r\ (q-1)$. Since Z_γ has eigenvalue $\mu(-1)^a$ on $M(s,\mu,\omega^a)$, we obtain $\lambda = \mu(-1)^a$. □

Corollary 6.6. *Assume that $r \neq 0$, $r \neq q-1$. Set $M := M(r, \lambda)$ and $M' := M(q-1-r, \lambda(-1)^r, \omega^r)$, if $\lambda \neq 0$ then:*

$$\dim_{\overline{\mathbb{F}}_p} \mathrm{Ext}^1_{\mathcal{H}_{\varpi=1}}(M, M) = 1, \quad \dim_{\overline{\mathbb{F}}_p} \mathrm{Ext}^1_{\mathcal{H}_{\varpi=1}}(M, M') = 0.$$

If $\lambda = 0$ then $M \cong M'$ and $\dim_{\overline{\mathbb{F}}_p} \mathrm{Ext}^1_{\mathcal{H}_{\varpi=1}}(M, M) = 2$.

Proof. Consider an exact sequence of $\mathcal{H}_{\varpi=1}$-modules:

$$0 \longrightarrow M \longrightarrow T_{n_s} e_{\chi^s} \mathcal{H}_{\varpi=1}/(Z_\gamma - \lambda)^2 \longrightarrow M \longrightarrow 0\ . \tag{11}$$

If the sequence was split then $Z_\gamma - \lambda$ would kill $T_{n_s} e_{\chi^s} \mathcal{H}_{\varpi=1}/(Z_\gamma - \lambda)^2$, which is not the case. Hence, $\dim_{\overline{\mathbb{F}}_p} \mathrm{Ext}^1_{\mathcal{H}_{\varpi=1}}(M, M) \geq 1$. Set $E := e_{\chi^s} \mathcal{H}_{\varpi=1}/(Z_\gamma - \lambda)$ then we have an exact sequence of $\mathcal{H}_{\varpi=1}$-modules:

$$0 \longrightarrow e_{\chi^s} \mathcal{H}_{\varpi=1} \xrightarrow{Z_\gamma - \lambda} e_{\chi^s} \mathcal{H}_{\varpi=1} \longrightarrow E \longrightarrow 0. \tag{12}$$

Since $e_{\chi^s} \mathcal{H}_{\varpi=1}$ is a direct summand of a free module, (12) is a projective resolution of E in $\mathrm{Mod}_{\mathcal{H}_{\varpi=1}}$. In particular, for all $\mathcal{H}_{\varpi=1}$-modules N, we have $\mathrm{Ext}^i_{\mathcal{H}_{\varpi=1}}(E, N) = 0$ for $i > 1$ and an exact sequence:

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{H}_{\varpi=1}}(E, N) \longrightarrow N e_{\chi^s} \xrightarrow{Z_\gamma - \lambda} N e_{\chi^s} \longrightarrow \mathrm{Ext}^1_{\mathcal{H}_{\varpi=1}}(E, N) \longrightarrow 0. \tag{13}$$

Since $0 < r < q-1$ we have $\chi \neq \chi^s$ and so it follows from Proposition 6.3 and (12) that:

$$\dim_{\overline{\mathbb{F}}_p} \mathrm{Hom}_{\mathcal{H}_{\varpi=1}}(E, M) = \dim_{\overline{\mathbb{F}}_p} \mathrm{Hom}_{\mathcal{H}_{\varpi=1}}(E, M') = 1. \tag{14}$$

One may verify that the images of e_{χ^s}, $e_{\chi^s} T_\Pi$, $e_{\chi_s} T_{n_s}$, $e_{\chi^s} T_\Pi T_{n_s}$ form a basis of E, so $\dim_{\overline{\mathbb{F}}_p} E = 4$. If $\lambda \neq 0$ then $M \not\cong M'$ and as M and M' are irreducible and 2-dimensional we obtain $E \cong M \oplus M'$. It follows from (13) that $\dim_{\overline{\mathbb{F}}_p} \mathrm{Ext}^1_{\mathcal{H}_{\varpi=1}}(E, M) = 1$. Since $\mathrm{Ext}^1_{\mathcal{H}_{\varpi=1}}(M, M) \neq 0$, if $\lambda \neq 0$ we obtain:

$$\mathrm{Ext}^1_{\mathcal{H}_{\varpi=1}}(M', M) = 0, \quad \dim_{\overline{\mathbb{F}}_p} \mathrm{Ext}^1_{\mathcal{H}_{\varpi=1}}(M, M) = 1.$$

After replacing r with $q-r-1$ and twisting, we also obtain $\mathrm{Ext}^1_{\mathcal{H}_{\varpi=1}}(M, M') = 0$. Assume that $\lambda = 0$ then $M \cong M'$. One may check that the subspace of E spanned by the images of $e_{\chi_s} T_{n_s}$, $e_{\chi^s} T_\Pi T_{n_s}$ is an $\mathcal{H}_{\varpi=1}$-module isomorphic to M and the corresponding quotient is isomorphic to M. It follows from (14) that the exact sequence of $\mathcal{H}_{\varpi=1}$-modules:

$$0 \longrightarrow M \longrightarrow E \longrightarrow M \longrightarrow 0 \tag{15}$$

is non-split. The classes of extensions (11) and (15) are linearly independent in $\mathrm{Ext}^1_{\mathcal{H}_{\varpi=1}}(M,M)$ since one of them is killed by Z_γ and the other one by Z_γ^2. So $\dim_{\overline{\mathbb{F}}_p} \mathrm{Ext}^1_{\mathcal{H}_{\varpi=1}}(M,M) \geq 2$. From (15) we obtain an exact sequence:

$$\mathrm{Hom}_{\mathcal{H}_{\varpi=1}}(M,M) \hookrightarrow \mathrm{Ext}^1_{\mathcal{H}_{\varpi=1}}(M,M) \to \mathrm{Ext}^1_{\mathcal{H}_{\varpi=1}}(E,M).$$

Since $\dim_{\overline{\mathbb{F}}_p} \mathrm{Ext}^1_{\mathcal{H}_{\varpi=1}}(E,M) = 1$, we get $\dim_{\overline{\mathbb{F}}_p} \mathrm{Ext}^1_{\mathcal{H}_{\varpi=1}}(M,M) \leq 2$. □

Corollary 6.7. *Assume $r = 0$ or $r = q-1$. Set $M := M(r,\lambda)$ and $M' := M(q-1-r,\lambda)$, then* $\dim_{\overline{\mathbb{F}}_p} \mathrm{Ext}^1_{\mathcal{H}_{\varpi=1}}(M,M) = \dim_{\overline{\mathbb{F}}_p} \mathrm{Ext}^1_{\mathcal{H}_{\varpi=1}}(M,M') = 1$.

Proof. Since $-e_1 T_{n_s}$ is an idempotent in $\mathcal{H}$, the exact sequences in Proposition 6.3 are projective resolution for M. If $r = q-1$ then for all $\mathcal{H}_{\varpi=1}$-modules N we have $\mathrm{Ext}^i_{\mathcal{H}_{\varpi=1}}(M,N) = 0$ for $i > 1$ and an exact sequence:

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{H}_{\varpi=1}}(M,N) \longrightarrow N e_1 T_{n_s} \stackrel{Z_\gamma - \lambda}{\longrightarrow} N e_1 T_{n_s} \longrightarrow \mathrm{Ext}^1_{\mathcal{H}_{\varpi=1}}(M,N) \longrightarrow 0.$$

The assertion for $r = q-1$ follows from this exact sequence. The case $r = 0$ is analogous. □

Proposition 6.8. *Assume $\lambda \neq 0$ then we have:*

$$\mathcal{T}(M(r,\lambda,\eta)) \cong \pi(r,\lambda,\eta), \quad \mathcal{I}(\pi(r,\lambda,\eta)) \cong M(r,\lambda,\eta).$$

Proof. It is enough to consider the case $\eta = 1$. Let $\mathcal{K}$ be the kernel of the natural map $\alpha : \mathcal{T}(\mathcal{I}(\text{c-Ind}^G_{\mathfrak{K}_0} V_{\mathbf{r}})) \to \text{c-Ind}^G_{\mathfrak{K}_0} V_{\mathbf{r}}$. If $r = 0$ or $r = q-1$ then $-T_{n_s} e_1$ is an idempotent which implies that $\mathcal{K} = 0$. The first isomorphism then follows by applying $\mathcal{T}$ to the exact sequence defining $M(r,\lambda)$. Assume $r \neq 0$ and $r \neq q-1$. It follows from the proof of Proposition 6.3 that $(\text{c-Ind}^G_{\mathfrak{K}_0} V_{\mathbf{r}})^{I_1} \cong T_{n_s} e_\chi \mathcal{H}_{\varpi=1}$. Hence every $v \in \mathcal{K}$ can be written $v = T_{n_s} e_\chi \otimes f$ with $T_{n_s} e_\chi f = T_{n_s} f = 0$. Now we have:

$$Z_\gamma v = T_{n_s} T_\Pi T_{n_s} e_\chi \otimes f = T_{n_s} e_\chi \otimes (T_\Pi T_{n_s} f) = 0.$$

Since $\lambda \neq 0$, multiplication by $Z_\gamma - \lambda$ induces thus an isomorphism $\mathcal{K} \xrightarrow{\sim} \mathcal{K}$. Since α is surjective, we deduce the first isomorphism from a diagram chase. Applying $\mathcal{I}$ to the exact sequence defining $\pi(r,\lambda,\eta)$, we obtain an injection $M(r,\lambda,\eta) \hookrightarrow \mathcal{I}(\pi(r,\lambda,\eta))$. It follows from [4, Th. 30] that this map is an isomorphism. □

Corollary 6.9. *Let τ be a smooth admissible irreducible non-supersingular representation of G over $\overline{\mathbb{F}}_p$, then $\mathcal{T}\mathcal{I}(\tau) \cong \tau$.*

Proof. It follows from [4, Cor. 36] that either $\tau \cong \pi(r,\lambda,\eta)$ with $(r,\lambda) \neq (0,\pm 1)$, $(r,\lambda) \neq (q-1,\pm 1)$, or $\tau \cong \eta \circ \det$, or $\tau \cong \mathrm{St} \otimes \eta \circ \det$. The assertion follows from Proposition 6.8, [4, Th. 30] and right exactness of $\mathcal{T}$. □

Corollary 6.10. *Let $M = M(r,\lambda)$ with $\lambda \neq 0$, or $M = \mathcal{I}(1)$, or $M = \mathcal{I}(\mathrm{St})$. Then $\mathbb{L}^1\mathcal{T}(M) = 0$.*

Proof. We prove the statement when $M = M(r,\lambda)$ and $\lambda \neq 0$. Assume that $r = 0$ or $r = q-1$ then $-e_1 T_{n_s}$ is an idempotent. Hence the exact sequences in Proposition 6.3 are projective resolutions. Moreover, $\mathcal{T}(T_{n_s} e_1 \mathcal{H}_{\varpi=1}) \cong \text{c-Ind}_{\mathfrak{K}_0}^G V_{\mathbf{p-1}}$ and $\mathcal{T}((1+T_{n_s})e_1\mathcal{H}_{\varpi=1}) \cong \text{c-Ind}_{\mathfrak{K}_0}^G V_{\mathbf{0}}$. Hence, applying $\mathcal{T}$ to the exact sequence defining $M(r,\lambda)$, we obtain an exact sequence defining $\pi(r,\lambda)$ and thus $\mathbb{L}^1\mathcal{T}(M) = 0$. Applying $\mathcal{T}$ to the exact sequences:

$$0 \longrightarrow \mathcal{I}(1) \longrightarrow M(q-1,1) \longrightarrow \mathcal{I}(\mathrm{St}) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{I}(\mathrm{St}) \longrightarrow M(0,1) \longrightarrow \mathcal{I}(1) \longrightarrow 0$$

we obtain $\mathbb{L}^1\mathcal{T}(\mathcal{I}(1)) = \mathbb{L}^1\mathcal{T}(\mathcal{I}(\mathrm{St})) = 0$. Assume $r \neq 0$, $r \neq q-1$ and $\lambda \neq 0$. Let E be an $\mathcal{H}_{\varpi=1}$-module defined by the exact sequence:

$$0 \longrightarrow e_{\chi^s}\mathcal{H}_{\varpi=1} \xrightarrow{Z_\gamma - \lambda} e_{\chi^s}\mathcal{H}_{\varpi=1} \longrightarrow E \longrightarrow 0.$$

Since e_{χ^s} is an idempotent, this is a projective resolution of E. Applying $\mathcal{T}$, we obtain an exact sequence using $\mathcal{T}(e_{\chi^s}\mathcal{H}_{\varpi=1}) \cong \text{c-Ind}_{IZ}^G \chi^s$:

$$0 \longrightarrow \mathbb{L}^1\mathcal{T}(E) \longrightarrow \text{c-Ind}_{IZ}^G \chi^s \xrightarrow{Z_\gamma - \lambda} \text{c-Ind}_{IZ}^G \chi^s \longrightarrow \mathcal{T}(E) \longrightarrow 0.$$

Now $Z_\gamma - \lambda$ is an injection since it is an injection on $\mathcal{I}(\text{c-Ind}_{IZ}^G \chi^s) \cong e_{\chi^s}\mathcal{H}_{\varpi=1}$. Hence $\mathbb{L}^1\mathcal{T}(E) = 0$ and $\mathcal{T}(E) \cong \text{c-Ind}_{IZ}^G \chi^s/(Z_\gamma - \lambda)$. Set $\pi' := \pi(q-1-r, \lambda(-1)^r, \omega^r)$, $\pi := \pi(r,\lambda)$ and $M' := M(q-1-r, \lambda(-1)^r, \omega^r)$. Applying $\mathcal{T}$ to the exact sequence $0 \longrightarrow M' \longrightarrow E \longrightarrow M \longrightarrow 0$ we obtain an exact sequence:

$$0 \longrightarrow \mathbb{L}^1\mathcal{T}(M) \longrightarrow \pi' \longrightarrow \mathcal{T}(E) \longrightarrow \pi \longrightarrow 0.$$

Since $r \neq 0$ and $r \neq q-1$, π' is irreducible and thus $\mathcal{T}(E) \cong \pi$ if $\mathbb{L}^1\mathcal{T}(M) \neq 0$. But this is impossible as $\mathcal{I}(\pi)$ has dimension 2 and $\mathcal{I}(\mathcal{T}(E))$ has dimension at least 4. □

7 Computation of $\mathbb{R}^1\mathcal{I}$ for principal series

We assume F is a finite extension of $\mathbb{Q}_p$. Let $\chi : P \twoheadrightarrow T \to \overline{\mathbb{F}}_p^\times$ be a smooth character. We also denote by χ the restriction of χ to Z. We compute $\mathbb{R}^1\mathcal{I}(\pi)$ in $\mathrm{Rep}_{G,\chi}$ for $\pi = \mathrm{Ind}_P^G \chi$.

Set $Z_1 := I_1 \cap Z$. Since Z_1 is pro-p and χ is smooth, we have $\chi(Z_1) = 1$. Hence Z_1 acts trivially on all the representations in $\mathrm{Rep}_{G,\chi}$. Forgetting the $\mathcal{H}$-module structure gives an isomorphism of vector spaces $\mathbb{R}^1\mathcal{I}(\pi) \cong H^1(I_1/Z_1, \pi)$.

Lemma 7.1. *For a cocycle $f \in Z^1(I_1/Z_1, \mathrm{Ind}_P^G \chi)$ define functions ψ^u and ψ^l as follows:*

$$\psi^u(u) := [f(u)](1), \quad u \in I_1 \cap P, \quad \psi^l(u) := [f(u)](n_s), \quad u \in I_1 \cap P^s.$$

Then the map $f \mapsto (\psi^u, \psi^l)$ induces an isomorphism:

$$H^1(I_1/Z_1, \mathrm{Ind}_P^G \chi) \xrightarrow{\sim} \mathrm{Hom}((I_1 \cap P)/Z_1, \overline{\mathbb{F}}_p) \oplus \mathrm{Hom}((I_1 \cap P^s)/Z_1, \overline{\mathbb{F}}_p). \quad (16)$$

Proof. Since $G = PI_1 \amalg Pn_sI_1$, we have an isomorphism:

$$\mathrm{Ind}_P^G \chi|_{I_1} \cong \mathrm{Ind}_{I_1 \cap P}^{I_1} 1 \oplus \mathrm{Ind}_{I_1 \cap P^s}^{I_1} 1.$$

As Z_1 acts trivially on both sides, we may rewrite this as $\mathrm{Ind}_{P/Z_1}^{G/Z_1} \chi|_{I_1/Z_1} \cong \mathrm{Ind}_{(I_1\cap P)/Z_1}^{I_1/Z_1} 1 \oplus \mathrm{Ind}_{(I_1\cap P^s)/Z_1}^{I_1/Z_1} 1$. It follows from [30, §2.5] that:

$$H^1(I_1/Z_1, \mathrm{Ind}_P^G \chi) \cong H^1((I_1 \cap P)/Z_1, 1) \oplus H^1((I_1 \cap P^s)/Z_1, 1)$$

which implies the assertion. □

Fix $\psi^u \in \mathrm{Hom}((I_1 \cap P)/Z_1, \overline{\mathbb{F}}_p)$ and $\psi^l \in \mathrm{Hom}((I_1 \cap P^s)/Z_1, \overline{\mathbb{F}}_p)$ (the superscripts u and l stand for "upper" and "lower"). We consider the pair (ψ^u, ψ^l) as an element of $\mathbb{R}^1\mathcal{I}(\mathrm{Ind}_P^G \chi)$ via (16).

Lemma 7.2. *There exists a locally constant function $\varphi : G \to \overline{\mathbb{F}}_p$ satisfying the following equalities:*

(i) $\varphi(pgu) - \varphi(pg) = \chi(p)(\varphi(gu) - \varphi(g)), \quad p \in P,\ g \in G,\ u \in I_1$

(ii) $\varphi(u) - \varphi(1) = \psi^u(u), \quad u \in I_1 \cap P$

(iii) $\varphi(n_s u) - \varphi(n_s) = \psi^l(u), \quad u \in I_1 \cap P^s$

(iv) $\varphi(zg) = \chi(z)\varphi(g), \quad z \in Z,\ g \in G.$

Proof. Consider the exact sequence of G-representations:

$$0 \longrightarrow \operatorname{Ind}_P^G \chi \xrightarrow{\ \iota\ } \operatorname{Ind}_Z^G \chi \longrightarrow Q \longrightarrow 0$$

where ι is the natural inclusion. For all τ in $\operatorname{Rep}_{G,\chi}$, we have:

$$\operatorname{Hom}_G(\tau, \operatorname{Ind}_Z^G \chi) \cong \operatorname{Hom}_Z(\tau, \chi) \cong \operatorname{Hom}_{\overline{\mathbb{F}}_p}(\tau, \overline{\mathbb{F}}_p)$$

and hence $\operatorname{Ind}_Z^G \chi$ is an injective object in $\operatorname{Rep}_{G,\chi}$. Applying $\mathcal{I}$ we obtain an exact sequence of $\mathcal{H}$-modules:

$$0 \longrightarrow \mathcal{I}(\operatorname{Ind}_P^G \chi) \xrightarrow{\ \iota\ } \mathcal{I}(\operatorname{Ind}_Z^G \chi) \longrightarrow \mathcal{I}(Q) \longrightarrow \mathbb{R}^1\mathcal{I}(\operatorname{Ind}_P^G \chi) \longrightarrow 0.$$

Let $\overline{\varphi}$ be a preimage of (ψ^u, ψ^l) in $\mathcal{I}(Q)$ and φ be a preimage of $\overline{\varphi}$ in $\operatorname{Ind}_Z^G \chi$. Then φ satisfies (iv). Since $\overline{\varphi}$ is fixed by I_1, for all $u \in I_1$ we get $(u-1)\varphi \in \operatorname{Ind}_P^G \chi$ and hence φ satisfies (i). Moreover, (ψ^u, ψ^l) is the class of the cocycle $u \mapsto (u-1)\varphi$. Lemma 7.1 implies then that φ satisfies (ii) and (iii). □

Proposition 7.3. *Let $n \in G$ and, for each coset $c \in I_1/(I_1 \cap n^{-1}I_1 n)$, fix a representative $\overline{c} \in I_1$. With the notations of Lemma 7.2, set $\varphi T_n := \sum_c \overline{c} n^{-1}\varphi$ and:*

$$\begin{aligned}
\theta^u(u) &:= [\varphi T_n](u) - [\varphi T_n](1) = \sum_c (\varphi(u\overline{c}n^{-1}) - \varphi(\overline{c}n^{-1})) \\
\theta^l(u) &:= [\varphi T_n](n_s u) - [\varphi T_n](n_s) = \sum_c (\varphi(n_s u\overline{c}n^{-1}) - \varphi(n_s\overline{c}n^{-1}))
\end{aligned}$$

where u is respectively in $I_1 \cap P$ and $I_1 \cap P^s$, and where the sum is taken over all cosets $c \in I_1/(I_1 \cap n^{-1}I_1 n)$. Then the action of $\mathcal{H}$ on $\mathbb{R}^1\mathcal{I}(\operatorname{Ind}_P^G \chi)$ is given by $(\psi^u, \psi^l)T_n = (\theta^u, \theta^l)$.

Proof. It follows from [25, Cor. 2.0.7] that the action of $\mathcal{H}$ on $\mathcal{I}(Q)$ is given by $\overline{\varphi}T_n = \sum_c \overline{c}n^{-1}\overline{\varphi}$. As φT_n is a preimage of $\overline{\varphi}T_n$ in $\operatorname{Ind}_Z^G \chi$, $(\psi^u, \psi^l)T_n$ is the class of the cocycle $u \mapsto (u-1)(\varphi T_n)$. The assertion follows from Lemma 7.1. □

We now fix an integer r such that $0 < r \leq q-1$ and $\lambda \in \overline{\mathbb{F}}_p^\times$. Let $\chi : T \to \overline{\mathbb{F}}_p^\times$ be the character:

$$\chi\Big(\begin{pmatrix} \varpi^n a & 0 \\ 0 & \varpi^m d \end{pmatrix}\Big) := \lambda^{m-n}\omega(d)^r \quad (a, d \in \mathcal{O}_F^\times,\ m, n \in \mathbb{Z}). \tag{17}$$

It follows from [4, Th. 30] that $\mathcal{I}(\mathrm{Ind}_P^G \chi) \cong M(r, \lambda)$.

Fix $\psi^u \in \mathrm{Hom}((I_1 \cap P)/Z_1, \overline{\mathbb{F}}_p)$, $\psi^l \in \mathrm{Hom}((I_1 \cap P^s)/Z_1, \overline{\mathbb{F}}_p)$ and let φ be a function as in Lemma 7.2. We may choose coset representatives so that $\varphi T_{n_s} = \sum_{\alpha \in \mathbb{F}_q} \left(\begin{smallmatrix} 1 & [\alpha] \\ 0 & 1 \end{smallmatrix}\right) n_s^{-1} \varphi$, $\varphi T_\Pi = \Pi^{-1}\varphi$ and so that, for all $\xi : H \to \overline{\mathbb{F}}_p^\times$:

$$\varphi e_\xi = |H|^{-1} \sum_{\lambda, \mu \in \mathbb{F}_q^\times} \xi\left(\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \mu^{-1} \end{pmatrix} \right) \begin{pmatrix} [\lambda] & 0 \\ 0 & [\mu] \end{pmatrix} \varphi.$$

We consider (ψ^u, ψ^l) as an element of $\mathbb{R}^1\mathcal{I}(\mathrm{Ind}_P^G \chi)$ via (16). We are going to determine $(\psi^u, \psi^l)T_{n_s}$, $(\psi^u, \psi^l)T_\Pi$ and $(\psi^u, \psi^l)e_\xi$ using Proposition 7.3.

Lemma 7.4. *Let $\mu \in \mathbb{F}_q^\times$ then* $\left(\begin{smallmatrix} 1 & 0 \\ -[\mu] & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & [\mu]^{-1} \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} -[\mu]^{-1} & 1 \\ 0 & -[\mu] \end{smallmatrix}\right) n_s$.

Lemma 7.5. *Let $c \in \varpi\mathcal{O}_F$ then:*

$$[\varphi T_{n_s}]\left(n_s \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \right) - [\varphi T_{n_s}](n_s) = \sum_{\mu \in \mathbb{F}_q^\times} \mu^r \psi^l\left(\begin{pmatrix} 1 & 0 \\ [\mu^2]c & 1 \end{pmatrix} \right).$$

Proof. Note that the left hand side is equal to $\sum_{\mu \in \mathbb{F}_q} \left(\varphi\left(\left(\begin{smallmatrix} 1 & -c \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ -[\mu] & 1 \end{smallmatrix}\right)\right) - \varphi\left(\left(\begin{smallmatrix} 1 & 0 \\ -[\mu] & 1 \end{smallmatrix}\right)\right)\right)$. Since $\left(\begin{smallmatrix} 1 & \mathfrak{p}_F \\ 0 & 1 \end{smallmatrix}\right)$ is contained in the derived subgroup of $I_1 \cap P$, the term corresponding to $\mu = 0$ is:

$$\varphi\left(\begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix} \right) - \varphi(1) = \psi^u\left(\begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix} \right) = 0.$$

If $\mu \in \mathbb{F}_q^\times$, using (i) of Lemma 7.2 with $p := \left(\begin{smallmatrix} 1 & -c \\ 0 & 1 \end{smallmatrix}\right)$, $g := \left(\begin{smallmatrix} 1 & 0 \\ -[\mu] & 1 \end{smallmatrix}\right)$, $u := \left(\begin{smallmatrix} 1 & [\mu]^{-1} \\ 0 & 1 \end{smallmatrix}\right)$ and Lemma 7.4 we get:

$$\begin{aligned}
\varphi\left(\left(\begin{smallmatrix} 1 & -c \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ -[\mu] & 1 \end{smallmatrix}\right)\right) - \varphi\left(\left(\begin{smallmatrix} 1 & 0 \\ -[\mu] & 1 \end{smallmatrix}\right)\right) &= \varphi\left(\left(\begin{smallmatrix} 1 & -c \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} -[\mu]^{-1} & 1 \\ 0 & -[\mu] \end{smallmatrix}\right) n_s\right) - \varphi\left(\left(\begin{smallmatrix} -[\mu]^{-1} & 1 \\ 0 & -[\mu] \end{smallmatrix}\right) n_s\right) \\
&= \varphi\left(\left(\begin{smallmatrix} -[\mu]^{-1} & 1 \\ 0 & -[\mu] \end{smallmatrix}\right) n_s \left(\begin{smallmatrix} 1 & 0 \\ [\mu^2]c & 1 \end{smallmatrix}\right)\right) - \varphi\left(\left(\begin{smallmatrix} -[\mu]^{-1} & 1 \\ 0 & -[\mu] \end{smallmatrix}\right) n_s\right) \\
&= \chi\left(\left(\begin{smallmatrix} -[\mu]^{-1} & 0 \\ 0 & -[\mu] \end{smallmatrix}\right)\right) \psi^l\left(\left(\begin{smallmatrix} 1 & 0 \\ [\mu^2]c & 1 \end{smallmatrix}\right)\right).
\end{aligned}$$

Summing over $\mu \in \mathbb{F}_q^\times$ we get the claim. □

Lemma 7.6. *Let $a, d \in 1 + \varpi\mathcal{O}_F$ then:*

$$[\varphi T_{n_s}]\left(n_s \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \right) - [\varphi T_{n_s}](n_s) = \psi^u\left(\begin{pmatrix} d & 0 \\ 0 & a \end{pmatrix} \right) + \sum_{\mu \in \mathbb{F}_q^\times} \mu^r \psi^l\left(\begin{pmatrix} 1 & 0 \\ [\mu](1 - da^{-1}) & 1 \end{pmatrix} \right) + \left(\sum_{\mu \in \mathbb{F}_q^\times} \mu^r\right) \psi^l\left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \right).$$

Lemma 7.7. *Let $b \in \mathcal{O}_F$ then:*

$$[\varphi T_{n_s}]\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) - [\varphi T_{n_s}](1) = (-1)^r \psi^l\left(\begin{pmatrix} 1 & 0 \\ -qb & 1 \end{pmatrix}\right).$$

Proof. Let $\bar{b}$ be the image of b in $\mathcal{O}_F/\varpi\mathcal{O}_F$ then:

$$\sum_{\mu\in\mathbb{F}_q} \Big(\varphi\big(\left(\begin{smallmatrix} 1 & [\mu]+b \\ 0 & 1 \end{smallmatrix}\right)n_s^{-1}\big) - \varphi\big(\left(\begin{smallmatrix} 1 & [\mu] \\ 0 & 1 \end{smallmatrix}\right)n_s^{-1}\big)\Big) = \sum_{\mu\in\mathbb{F}_q} \Big(\varphi\big(\left(\begin{smallmatrix} 1 & [\mu] \\ 0 & 1 \end{smallmatrix}\right)n_s^{-1}\left(\begin{smallmatrix} 1 & 0 \\ [\mu]-b-[\mu-\bar{b}] & 1 \end{smallmatrix}\right)\big)$$
$$- \varphi\big(\left(\begin{smallmatrix} 1 & [\mu] \\ 0 & 1 \end{smallmatrix}\right)n_s^{-1}\big)\Big)$$

and the right hand side is equal to:

$$\chi(n_s^{-2})\psi^l\big(\left(\begin{smallmatrix} 1 & 0 \\ \sum_{\mu\in\mathbb{F}_q}([\mu]-b-[\mu-\bar{b}]) & 1 \end{smallmatrix}\right)\big) = (-1)^r\psi^l\big(\left(\begin{smallmatrix} 1 & 0 \\ -qb & 1 \end{smallmatrix}\right)\big).$$

□

A similar argument gives:

Lemma 7.8. *Let $a, d \in 1 + \varpi\mathcal{O}_F$ then:*

$$[\varphi T_{n_s}]\left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right) - [\varphi T_{n_s}](1) = \psi^l\left(\begin{pmatrix} 1 & 0 \\ (\sum_{\mu\in\mathbb{F}_q}[\mu])(1-ad^{-1}) & 1 \end{pmatrix}\right).$$

The following two lemmas can be easily obtained by using (i) of Lemma 7.2 and observing that $\chi(\Pi^{-1}n_s^{-1}) = \lambda$ and $\chi(n_s\Pi^{-1}) = \lambda^{-1}$:

Lemma 7.9. *Let $a, d \in 1 + \varpi\mathcal{O}_F$ and $c, b \in \mathcal{O}_F$ then:*

$$\begin{aligned} [\varphi T_\Pi]\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) - [\varphi T_\Pi](1) &= \lambda\psi^l\left(\begin{pmatrix} d & 0 \\ \varpi b & a \end{pmatrix}\right) \\ [\varphi T_\Pi]\left(n_s\begin{pmatrix} a & 0 \\ \varpi c & d \end{pmatrix}\right) - [\varphi T_\Pi](n_s) &= \lambda^{-1}\psi^u\left(\begin{pmatrix} d & c \\ 0 & a \end{pmatrix}\right). \end{aligned}$$

Lemma 7.10. *Let $a, d \in 1 + \varpi\mathcal{O}_F$, $c \in \varpi\mathcal{O}_F$, $b \in \mathcal{O}_F$ and let $\xi : H \to \overline{\mathbb{F}}_p^\times$ be a character then:*

$$[\varphi e_\xi]\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) - [\varphi e_\xi](1) = \sum_{\lambda,\mu\in\mathbb{F}_q^\times} \mu^{-r}\xi\left(\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}\right)\psi^u\left(\begin{pmatrix} a & b[\lambda\mu^{-1}] \\ 0 & d \end{pmatrix}\right)$$

$$[\varphi e_\xi]\left(n_s\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}\right) - [\varphi e_\xi](n_s) = \sum_{\lambda,\mu\in\mathbb{F}_q^\times} \lambda^{-r}\xi\left(\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}\right)\psi^l\left(\begin{pmatrix} a & 0 \\ c[\lambda^{-1}\mu] & d \end{pmatrix}\right).$$

Definition 7.11. *(i) For $0 \le i < f$, define $\varepsilon_i \in \mathrm{Hom}((I_1 \cap T)/Z_1, \overline{\mathbb{F}}_p)$ by:*

$$\varepsilon_i\big(\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)\big) := \omega((1 - da^{-1})/\varpi)^{p^i}$$

and $\kappa_i^u \in \mathrm{Hom}((I_1 \cap P)/Z_1, \overline{\mathbb{F}}_p)$, $\kappa_i^l \in \mathrm{Hom}((I_1 \cap P^s)/Z_1, \overline{\mathbb{F}}_p)$ by:

$$\kappa_i^u\big(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\big) := \omega(b)^{p^i}, \quad \kappa_i^l\big(\left(\begin{smallmatrix} a & 0 \\ \varpi c & d \end{smallmatrix}\right)\big) := \omega(c)^{p^i}.$$

(ii) For $\delta \in \mathrm{Hom}((I_1 \cap T)/Z_1, \overline{\mathbb{F}}_p)$, define $\delta^u \in \mathrm{Hom}((I_1 \cap P)/Z_1, \overline{\mathbb{F}}_p)$ and $\delta^l \in \mathrm{Hom}((I_1 \cap P^s)/Z_1, \overline{\mathbb{F}}_p)$ by:

$$\delta^u\big(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\big) := \delta\big(\left(\begin{smallmatrix} d & 0 \\ 0 & a \end{smallmatrix}\right)\big), \quad \delta^l\big(\left(\begin{smallmatrix} a & 0 \\ \varpi c & d \end{smallmatrix}\right)\big) := \delta\big(\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)\big).$$

Lemma 7.12. *There exist $\delta_1, \delta_2 \in \mathrm{Hom}((I_1 \cap T)/Z_1, \overline{\mathbb{F}}_p)$ and $x_i^u, x_i^l \in \overline{\mathbb{F}}_p$ for $0 \le i < f$ such that:*

$$\psi^u = \delta_1^u + \sum_{i=0}^{f-1} x_i^u \kappa_i^u, \quad \psi^l = \delta_2^l + \sum_{i=0}^{f-1} x_i^l \kappa_i^l.$$

Proof. By restricting ψ^u to $T \cap I_1$ and twisting by s, we obtain $\delta_1 \in \mathrm{Hom}((I_1 \cap T)/Z_1, \overline{\mathbb{F}}_p)$ such that $\psi^u - \delta_1^u$ is trivial on $I_1 \cap T$. Since:

$$\left(\begin{smallmatrix} 1+\varpi & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} (1+\varpi)^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -\beta \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & \varpi\beta \\ 0 & 1 \end{smallmatrix}\right),$$

the restriction of $\psi^u - \delta_1^u$ to $\left(\begin{smallmatrix} 1 & \mathfrak{p}_F \\ 0 & 1 \end{smallmatrix}\right)$ is trivial. Hence $\psi^u - \delta_1^u$ factors through $K_1 \cap P$. Now $\dim_{\overline{\mathbb{F}}_p} \mathrm{Hom}((I_1 \cap P)/(K_1 \cap P), \overline{\mathbb{F}}_p) = \dim_{\overline{\mathbb{F}}_p} \mathrm{Hom}(\mathbb{F}_q, \overline{\mathbb{F}}_p) = f$ and the κ_i^u are linearly independent. This proves the claim for ψ^u. The claim for ψ^l follows after conjugating by Π. $\square$

We will consider κ_i^u, κ_i^l, δ^u, δ^l as elements of $\mathbb{R}^1\mathcal{I}(\mathrm{Ind}_P^G \chi)$ by extending them by zero to elements of $\mathrm{Hom}((I_1 \cap P)/Z_1, \overline{\mathbb{F}}_p) \oplus \mathrm{Hom}((I_1 \cap P^s)/Z_1, \overline{\mathbb{F}}_p)$ and then using (16) and Shapiro's lemma.

Proposition 7.13. *Let $\delta \in \mathrm{Hom}((I_1 \cap T)/Z_1, \overline{\mathbb{F}}_p)$ then:*

$$\delta^u e_\chi = \delta^u, \quad \delta^u T_\Pi = \lambda^{-1}\delta^l, \quad \delta^u T_{n_s} = \delta^l, \quad \delta^l e_{\chi^s} = \delta^l, \quad \delta^l T_\Pi = \lambda\delta^u.$$

If $r = q-1$, $\delta^l T_{n_s} = -\delta^l$ and if $r \neq q-1$, $\delta^l T_{n_s} = 0$. In particular, $\langle \delta^u, \delta^l \rangle_{\overline{\mathbb{F}}_p}$ is stable under the action of $\mathcal{H}$ and is isomorphic to $M(r, \lambda)$ as an $\mathcal{H}$-module.

Proof. This follows from Lemmas 7.10, 7.9, 7.5, 7.6, 7.7 and 7.8. $\square$

Proposition 7.14. *(i) We have $\kappa_i^u e_{\chi\alpha^{-p^i}} = \kappa_i^u$, $\kappa_i^l e_{\chi^s\alpha^{p^i}} = \kappa_i^l$, $\kappa_i^u T_\Pi = \lambda^{-1}\kappa_i^l$, $\kappa_i^l T_\Pi = \lambda\kappa_i^u$ and $\kappa_i^u T_{n_s} = 0$.*

(ii) We have:

$$\kappa_i^l T_{n_s} = \Big(\sum_{\mu\in\mathbb{F}_q^\times} \mu^{2p^i+r}\Big)\kappa_i^l + \Big(\sum_{\mu\in\mathbb{F}_q^\times} \mu^{p^i+r}\Big)\varepsilon_i^l + \Big(\sum_{\mu\in\mathbb{F}_q^\times} \mu\Big)\varepsilon_i^u + (-1)^{r+1}\epsilon_F\kappa_i^u$$

where $\epsilon_F = 1$ if $F = \mathbb{Q}_p$ and $\epsilon_F = 0$ otherwise.

Proof. This follows from Lemmas 7.10, 7.9, 7.5, 7.6, 7.7, 7.8 (note that $\mathrm{val}(q) = [F : \mathbb{Q}_p]$). □

Proposition 7.15. *(i) The subspace $S := \langle \varepsilon_i^u, \varepsilon_i^l, \kappa_i^u, \kappa_i^l\rangle_{\overline{\mathbb{F}}_p}$ of $\mathbb{R}^1\mathcal{I}(\mathrm{Ind}_P^G \chi)$ is stable under the action of $\mathcal{H}$.*

(ii) Let $s^{(i)}$ be an integer such that $0 \leq s^{(i)} < q-1$ and $s^{(i)} \equiv -r-2p^i\ (q-1)$, then there exists an exact sequence of $\mathcal{H}$-modules:

$$0 \longrightarrow M(r,\lambda) \longrightarrow S \longrightarrow M(s^{(i)}, \epsilon_F\lambda^{-1}, \omega^{r+p^i}) \longrightarrow 0,$$

where $\epsilon_F = 1$ if $F = \mathbb{Q}_p$ and $\epsilon_F = 0$, otherwise. This sequence is non-split if and only if ($F = \mathbb{Q}_p$, $p > 2$, $r = p-2$ and $\lambda = \pm 1$) or ($F = \mathbb{Q}_2$ and $\lambda = 1$) (and hence $r = 1$ and $s = 0$).

Proof. (i) follows from Lemmas 7.10, 7.9, 7.5, 7.6, 7.7 and 7.8. Proposition 7.13 implies that the subspace $\langle \varepsilon_i^u, \varepsilon_i^l\rangle_{\overline{\mathbb{F}}_p}$ of S is stable under the action of $\mathcal{H}$ and is isomorphic to $M(r,\lambda)$. We denote the corresponding quotient by Q. Let v_1 be the image of κ_i^u and v_2 be the image of $\lambda^{-1}\kappa_i^l$ in Q. Proposition 7.14 implies $v_1 e_{\chi\alpha^{-p^i}} = v_1$, $v_2 e_{\chi^s\alpha^{p^i}} = v_2$, $v_1 T_\Pi = v_2$, $v_2 T_\Pi = v_1$, $v_1 T_{n_s} = 0$ and:

$$v_2\Big(T_{n_s} - \sum_{\mu\in\mathbb{F}_q^\times} \mu^{2p^i+r}\Big) = \lambda^{-1}(-1)^{r+1}\epsilon_F v_1.$$

Now $\chi\alpha^{-p^i}\big(\big(\begin{smallmatrix}\lambda & 0\\ 0 & \mu\end{smallmatrix}\big)\big) = \lambda^{-r-2p^i}(\lambda\mu)^{r+p^i}$ hence $\sum_{\mu\in\mathbb{F}_q^\times}\mu^{2p^i+r} \neq 0$ if and only if $(\chi\alpha^{-p^i})^s = \chi\alpha^{-p^i}$. These relations and Corollary 6.4 imply $\langle v_1, v_2\rangle_{\overline{\mathbb{F}}_p} \cong M(s^{(i)}, \epsilon_F\lambda^{-1}, \omega^{r+p^i})$. Suppose that the sequence does not split. Then Corollary 6.5 implies that $\lambda = (-1)^{r+1}\lambda^{-1}\epsilon_F$, and hence $F = \mathbb{Q}_p$ so that $i = 0$. Assume $p > 2$, then $r + 1 \not\equiv r\ (p-1)$ and hence Corollary 6.5 implies $r = s = p-2$ and $\lambda = \pm 1$. Let $\gamma := \{\chi\alpha^{-1}, \chi^s\alpha\}$, since $r = p-2$ we have $Z_\gamma = T_{n_s}T_\Pi + T_\Pi T_{n_s}$. If the sequence was split then $Z_\gamma - \lambda$ would kill S. However $\kappa^u(Z_\gamma - \lambda) = \lambda^{-1}\kappa^l T_{n_s} - \lambda\kappa^u = (\sum_{\mu\in\mathbb{F}_p^\times}\mu^{p-1})\varepsilon^l \neq 0$, so the sequence does not split. Assume now $p = 2$. Since $r > 0$ we get $r = 1$ and hence $s = 0$, $\lambda = 1$ and $Z_\gamma = T_{n_s}T_\Pi + T_\Pi T_{n_s} + T_\Pi$. The same argument shows that $\kappa^u(Z_\gamma - 1) \neq 0$ and hence the sequence can not split. □

We now sum up the results of this section.

Theorem 7.16. *Let $\lambda \in \overline{\mathbb{F}}_p^\times$ and $r \in \{1, \cdots, q-1\}$. For $0 \leq i \leq f-1$, let $s^{(i)} \in \{0, \cdots, q-2\}$ such that $s^{(i)} \equiv -r - 2p^i \ (q-1)$ and let $d := \dim_{\overline{\mathbb{F}}_p} \mathrm{Hom}(1+\mathfrak{p}_F, \overline{\mathbb{F}}_p)$.*

(i) Assume $F \neq \mathbb{Q}_p$ then:

$$\mathbb{R}^1\mathcal{I}(\pi(r,\lambda)) \cong M(r,\lambda)^{\oplus d} \oplus \bigoplus_{i=0}^{f-1} M(s^{(i)}, 0, \omega^{r+p^i}).$$

(ii) Assume $F = \mathbb{Q}_p$ and, if $\lambda = \pm 1$, assume furthermore $p > 2$ and $r \neq p-2$. Then:

$$\mathbb{R}^1\mathcal{I}(\pi(r,\lambda)) \cong M(r,\lambda)^{\oplus d} \oplus M(s^{(0)}, \lambda^{-1}, \omega^{r+1}).$$

(iii) Assume $F = \mathbb{Q}_p$, $\lambda = \pm 1$, $p > 2$ and $r = p-2$ then $\mathbb{R}^1\mathcal{I}(\pi(p-2, \pm 1))$ is the unique non-split extension of $\mathcal{H}_{\varpi=1}$-modules:

$$0 \longrightarrow M(p-2, \pm 1) \longrightarrow \mathbb{R}^1\mathcal{I}(\pi(p-2, \pm 1)) \longrightarrow M(p-2, \pm 1) \longrightarrow 0.$$

(iv) Assume $F = \mathbb{Q}_2$ and $\lambda = 1$. Let E be the unique non-split extension of $\mathcal{H}_{\varpi=1}$-modules:

$$0 \longrightarrow M(1,1) \longrightarrow E \longrightarrow M(0,1) \longrightarrow 0,$$

then $\mathbb{R}^1\mathcal{I}(\pi(1,1)) \cong M(1,1) \oplus E$.

Proof. This is a reformulation of Propositions 7.13, 7.14 and 7.15. In (iii) and (iv), the uniqueness of the extension is given by Corollary 6.7. Note that for $F = \mathbb{Q}_p$, $d = 1$ if $p > 2$ and $d = 2$ if $p = 2$. $\square$

8 Extensions of principal series

We keep the notations of sections 6 and 7 and still assume F is a finite extension of $\mathbb{Q}_p$. We fix a smooth character $\chi : F^\times \to \overline{\mathbb{F}}_p^\times$ and study groups $\mathrm{Ext}^1_{G,\chi}(\tau, \pi)$ of G-extensions with central character χ.

Theorem 8.1. *Let $r \in \{1, \cdots, q-1\}$, $\lambda \in \overline{\mathbb{F}}_p^\times$, $\pi := \pi(r,\lambda)$ and $M := M(r,\lambda)$ (see Definition 6.2). Let τ be a smooth admissible irreducible non-supersingular representation of G over $\overline{\mathbb{F}}_p$ with central character χ with χ as in (17). Then there exists a short exact sequence:*

$$0 \to \mathrm{Ext}^1_{\mathcal{H}_{\varpi=1}}(\mathcal{I}(\tau), M) \to \mathrm{Ext}^1_{G,\chi}(\tau, \pi) \to \mathrm{Hom}_{\mathcal{H}}(\mathcal{I}(\tau), \mathbb{R}^1\mathcal{I}(\pi)) \to 0 \quad (18)$$

where $\mathrm{Ext}^1_{G,\chi}(\tau,\pi)$ *denotes the* $\overline{\mathbb{F}}_p$*-vector space of* G*-extensions with central character* χ*.*

Proof. Let E be the class of an exact sequence in $\mathrm{Rep}_{G,\chi}$:

$$0 \longrightarrow \pi \longrightarrow \epsilon \longrightarrow \tau \longrightarrow 0\ .$$

Taking I_1-invariants we obtain an exact sequence of $\mathcal{H}$-modules:

$$0 \longrightarrow \mathcal{I}(\pi) \longrightarrow \mathcal{I}(\epsilon) \longrightarrow \mathcal{I}(\tau) \xrightarrow{\phi_E} \mathbb{R}^1\mathcal{I}(\pi)\ .$$

Hence we obtain a map $\mathrm{Ext}^1_{G,\chi}(\tau,\pi) \to \mathrm{Hom}_{\mathcal{H}}(\mathcal{I}(\tau), \mathbb{R}^1\mathcal{I}(\pi))$, $E \mapsto \phi_E$. We claim that this map is surjective. Let $\phi \in \mathrm{Hom}_{\mathcal{H}}(\mathcal{I}(\tau), \mathbb{R}^1\mathcal{I}(\pi))$ be non-zero. By Corollary 9.11 below (note that if $p = 2$, $\pi|_{\mathfrak{K}_1}$ has property (S) below by Corollary 9.3), there exists an exact sequence:

$$0 \to \pi \to \Omega \to Q \to 0,$$

where Ω is a smooth admissible representation of G over $\overline{\mathbb{F}}_p$ such that $\Omega|_K$ is an injective envelope of $\mathrm{soc}_K\,\pi$ in $\mathrm{Rep}_{K,\chi}$. Since $\Omega|_{I_1}$ is an injective object in $\mathrm{Rep}_{I_1,\chi}$, by taking I_1-invariants we obtain an exact sequence:

$$0 \to M \to \mathcal{I}(\Omega) \to \mathcal{I}(Q) \to \mathbb{R}^1\mathcal{I}(\pi) \to 0.$$

By examining the construction of Ω we observe that $\mathcal{I}(\Omega) \cong M \oplus S$, where S is a direct sum of supersingular modules (compare [25, Prop. 6.4.5]). Since τ is irreducible and non-supersingular, $\mathcal{I}(\tau)$ is irreducible and hence ϕ is an injection. Since τ is non-supersingular, Corollary 6.5 implies that there exists no non-split extensions between a supersingular module and $\mathcal{I}(\tau)$. In particular $\mathrm{Ext}^1_{\mathcal{H}_{\varpi=1}}(\mathcal{I}(\tau), \mathcal{I}(\Omega)/M) = 0$. From the exact sequence:

$$0 \to \mathrm{Hom}_{\mathcal{H}}(\mathcal{I}(\tau), \mathcal{I}(\Omega)/M) \to \mathrm{Hom}_{\mathcal{H}}(\mathcal{I}(\tau), \mathcal{I}(Q)) \to \mathrm{Hom}_{\mathcal{H}}(\mathcal{I}(\tau), \mathbb{R}^1\mathcal{I}(\pi)) \to 0$$

we thus get $\psi \in \mathrm{Hom}_{\mathcal{H}}(\mathcal{I}(\tau), \mathcal{I}(Q))$ such that the composition $\mathcal{I}(\tau) \xrightarrow{\psi} \mathcal{I}(Q) \to \mathbb{R}^1\mathcal{I}(\pi)$ is ϕ. Now let E be the image of ψ under:

$$\mathrm{Hom}_{\mathcal{H}}(\mathcal{I}(\tau), \mathcal{I}(Q)) \cong \mathrm{Hom}_G(\tau, Q) \xrightarrow{\partial} \mathrm{Ext}^1_{G,\chi}(\tau,\pi)$$

(use the adjunction property of $\mathcal{I}$ and $\mathcal{T}$ (see §6) together with $\mathcal{T}\mathcal{I}(\tau) = \tau$ for the first isomorphism), then one checks that $\phi_E = \phi$. Let us now prove exactness in the middle of (18). Suppose that E is such that $\phi_E = 0$, then we obtain an extension of $\mathcal{H}_{\varpi=1}$-modules:

$$0 \longrightarrow \mathcal{I}(\pi) = M \longrightarrow \mathcal{I}(\epsilon) \longrightarrow \mathcal{I}(\tau) \longrightarrow 0\ .$$

If this extension is split, Corollary 6.9 implies that $E = 0$. Corollaries 6.10, 6.9 and Proposition 6.8 imply that, after applying $\mathcal{T}$ to this exact sequence, we obtain an exact sequence of G-representations:

$$0 \longrightarrow \pi \longrightarrow \mathcal{T}\mathcal{I}(\epsilon) \longrightarrow \tau \longrightarrow 0\,.$$

This implies that the natural map $\mathcal{T}\mathcal{I}(\epsilon) \to \epsilon$ is an isomorphism, hence we get back E and thus exactness in the middle of (18). Let us finally check injectivity on the left of (18). Suppose that we have an exact sequence of $\mathcal{H}_{\varpi=1}$-modules:

$$0 \longrightarrow M \longrightarrow N \longrightarrow \mathcal{I}(\tau) \longrightarrow 0\,.$$

Corollary 6.10 implies that, after applying $\mathcal{T}$, we obtain an exact sequence of G-representations:

$$0 \longrightarrow \pi \longrightarrow \mathcal{T}(N) \longrightarrow \tau \longrightarrow 0\,.$$

Let E be the class of this extension in $\mathrm{Ext}^1_{G,\chi}(\tau, \pi)$. After applying $\mathcal{I}$ we obtain a diagram of $\mathcal{H}$-modules with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & \mathcal{I}(\tau) & \longrightarrow & 0 \\
 & & \downarrow{=} & & \downarrow & & \downarrow{=} & & \downarrow \\
0 & \longrightarrow & M & \longrightarrow & \mathcal{I}\mathcal{T}(N) & \longrightarrow & \mathcal{I}(\tau) & \xrightarrow{\phi_E} & \mathbb{R}^1\mathcal{I}(\pi).
\end{array}$$

Hence $\phi_E = 0$ and $N \cong \mathcal{I}\mathcal{T}(N)$. This implies that $\mathcal{T}$ induces an injection $\mathrm{Ext}^1_{\mathcal{H}_{\varpi=1}}(\mathcal{I}(\tau), M) \hookrightarrow \mathrm{Ext}^1_{G,\chi}(\tau, \pi)$. □

Corollary 8.2. *Let $\pi := \pi(r, \lambda)$ and $M := M(r, \lambda)$ as in Definition 6.2 with $r \in \{1, \cdots, q-1\}$ and $\lambda \in \overline{\mathbb{F}}_p^\times$. Let χ as in (17).*

(i) There exists an injection $\mathrm{Hom}(F^\times, \overline{\mathbb{F}}_p) \hookrightarrow \mathrm{Ext}^1_{G,\chi}(\pi, \pi)$.

(ii) If $(\lambda, F) \neq (\pm 1, \mathbb{Q}_2)$ and $(\lambda, F, r) \neq (\pm 1, \mathbb{Q}_3, 2)$ then this injection is an isomorphism and the subspace $\mathrm{Ext}^1_{\mathcal{H}_{\varpi=1}}(M, M)$ via (18) corresponds to the unramified homomorphisms.

Proof. Let $\delta \in \mathrm{Hom}(F^\times, \overline{\mathbb{F}}_p)$. We lift δ to a homomorphism of P via $P \twoheadrightarrow F^\times$, $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \mapsto ad^{-1}$. Let ϵ_δ be the extension corresponding to δ via $\mathrm{Ext}^1_P(\chi, \chi) \cong \mathrm{Hom}(P, \overline{\mathbb{F}}_p)$. By inducing to G we obtain an exact sequence:

$$0 \longrightarrow \mathrm{Ind}_P^G\,\chi \longrightarrow \mathrm{Ind}_P^G\,\epsilon_\delta \longrightarrow \mathrm{Ind}_P^G\,\chi \longrightarrow 0.$$

By evaluating functions at identity, we see that this sequence splits if and only if $\delta = 0$. This proves (i) as $\pi \simeq \operatorname{Ind}_P^G \chi$. Let v_1, v_2 be the basis of the underlying vector space of ϵ_δ such that for all $g \in P$, we have $gv_1 = \chi(g)v_1 + \chi(g)\delta(g)v_2$ and $gv_2 = \chi(g)v_2$. Denote by U (resp. U^s) in this proof (and only in this proof) the unipotent subgroup of P (resp. P^s). Let $\varphi_1 \in \operatorname{Ind}_P^G \epsilon_\delta$ be the function with $\operatorname{Supp} \varphi_1 = PI$ and $\varphi_1(u) = v_1$ for all $u \in I_1 \cap U^s$, and let $\varphi_2 := \sum_{\lambda \in \mathbb{F}_q} \left(\begin{smallmatrix} 1 & [\lambda] \\ 0 & 1 \end{smallmatrix}\right) n_s^{-1} \varphi_1$. The images of φ_1 and φ_2 form a basis of $(\operatorname{Ind}_P^G \chi)^{I_1}$. Moreover, $[(u-1)\varphi_1](n_s) = 0$ for all $u \in I_1 \cap P^s$ and $[(tu-1)\varphi_1](1) = \delta(t)v_2$ for all $t \in T \cap I_1$ and $u \in I_1 \cap U$. Seeing $\delta|_{1+\mathfrak{p}_F}$ as an element of $\operatorname{Hom}((I_1 \cap T)/Z_1, \overline{\mathbb{F}}_p)$, we let $\delta|^u_{1+\mathfrak{p}_F}$ and $\delta|^l_{1+\mathfrak{p}_F}$ be as in (ii) of Definition 7.11. We thus get that φ_1 maps to $\delta|^u_{1+\mathfrak{p}_F}$ and φ_2 maps to $\delta|^l_{1+\mathfrak{p}_F}$ in $\mathbb{R}^1\mathcal{I}(\pi)$ via (16). The assumptions on λ, F and r imply that if $\phi \in \operatorname{Hom}_{\mathcal{H}}(M, \mathbb{R}^1\mathcal{I}(\pi))$ is non-zero then ϕ is an injection and the proof of Theorem 8.1 goes through even in the case when $\pi = \tau$ is not irreducible. Moreover, $\phi(M) = \langle \delta^u, \delta^l \rangle_{\overline{\mathbb{F}}_p}$ for some $\delta \in \operatorname{Hom}(1 + \mathfrak{p}_F, \overline{\mathbb{F}}_p)$. So we obtain an exact sequence:

$$0 \longrightarrow \operatorname{Hom}^{\mathrm{un}}(F^\times, \overline{\mathbb{F}}_p) \longrightarrow \operatorname{Hom}(F^\times, \overline{\mathbb{F}}_p) \longrightarrow \operatorname{Hom}_{\mathcal{H}}(M, \mathbb{R}^1\mathcal{I}(\pi)) \longrightarrow 0$$

(where $\operatorname{Hom}^{\mathrm{un}}$ means homomorphisms which are trivial on $\mathcal{O}_F^\times$). Hence we obtain an injection $\operatorname{Hom}^{\mathrm{un}}(F^\times, \overline{\mathbb{F}}_p) \hookrightarrow \operatorname{Ext}^1_{\mathcal{H}_{\varpi=1}}(M, M)$. Since both spaces are one dimensional, this is an isomorphism. $\square$

Corollary 8.3. *Let $r \in \{1, \cdots, q-1\}$, $\lambda \in \overline{\mathbb{F}}_p^\times$ and χ as in (17). Assume $F = \mathbb{Q}_p$, $(p, \lambda) \neq (2, 1)$ and $(p, \lambda, r) \neq (3, \pm 1, 2)$. Let $s \in \{0, \cdots, p-2\}$ such that $s \equiv -r-2$ $(p-1)$. Let:*

$$d := \dim_{\overline{\mathbb{F}}_p} \operatorname{Ext}^1_{G,\chi}(\pi(s, \lambda^{-1}, \omega^{r+1}), \pi(r, \lambda)).$$

If $r = p-2$ and $\lambda = \pm 1$ then $d = 2$. Otherwise, $d = 1$.

Proof. Set $M' := M(s, \lambda^{-1}, \omega^{r+1})$ and $M := M(r, \lambda)$. If $\operatorname{Hom}_{\mathcal{H}}(M', M) \neq 0$ then $\omega^{r+1} = 1$ and hence $r+1 \equiv 0 \pmod{p-1}$, $\lambda = \lambda^{-1}$, $s \equiv r \pmod{p-1}$. This can only happen if $(p, \lambda, r) = (3, \pm 1, 2)$, $(p, \lambda) = (2, 1)$ or $p > 2$ and $(r, \lambda) = (p-2, \pm 1)$. We have excluded the first two cases. In the third case $\pi(r, \lambda) \cong \pi(s, \lambda^{-1}, \omega^{r+1}) \cong \pi(p-2, \pm 1)$ and hence it is dealt with in Corollary 8.2, so we obtain $d = \dim_{\overline{\mathbb{F}}_p} \operatorname{Hom}(\mathbb{Q}_p^\times, \overline{\mathbb{F}}_p) = 2$. If $\operatorname{Ext}^1_{\mathcal{H}_{\varpi=1}}(M', M) \neq 0$ then Corollary 6.5 implies that either we are in the cases considered before or $r+1 \equiv r$ $(p-1)$ and $\lambda = (-1)^r \lambda^{-1}$. This implies $(p, \lambda) = (2, 1)$. The assertion follows from Theorems 7.16 and 8.1. $\square$

Corollary 8.4. *Let $r \in \{1, \cdots, q-1\}$, $\lambda \in \overline{\mathbb{F}}_p^\times$ and χ as in (17). Assume $F = \mathbb{Q}_p$, then $\operatorname{Ext}^1_{G,\chi}(\pi(s, 0, \eta), \pi(r, \lambda)) = 0$ for all s and η.*

Proof. Set $\pi := \pi(r,\lambda)$, $\pi' := \pi(s,0,\eta)$, $M := M(r,\lambda)$ and $M' := M(s,0,\eta)$. It follows from [6] that $\mathcal{I}(\pi') \cong M'$ and [25, Cor. 6.1.8] implies that, for all τ in Rep_G, we have $\mathrm{Hom}_{\mathcal{H}}(M', \mathcal{I}(\tau)) \cong \mathrm{Hom}_G(\pi', \tau)$, hence $\pi' \cong \mathcal{T}(M')$. Theorem 7.16 implies that $\mathrm{Hom}_{\mathcal{H}}(M', \mathbb{R}^1\mathcal{I}(\pi)) = 0$. Corollary 6.5 implies that $\mathrm{Ext}^1_{\mathcal{H}_{\varpi=1}}(M', M) = 0$. The proof of theorem 8.1 then goes through to show that $\mathrm{Ext}^1_{G,\chi}(\pi', \pi) = 0$. □

9 General theory of diagrams and representations of GL_2

We define basic diagrams and use them to construct smooth admissible representations of G over $\overline{\mathbb{F}}_p$, generalizing the constructions of [25]. We don't assume anything on F.

We start with a few lemmas. Let $[H] := \{\begin{pmatrix} [\lambda] & 0 \\ 0 & [\mu] \end{pmatrix}, \lambda, \mu \in \mathbb{F}_q^\times\}$, $\mathcal{G}$ the subgroup of $\mathfrak{K}_1$ generated by $[H]$ and Π and set $\overline{\mathcal{G}} := \mathcal{G}/\varpi^{\mathbb{Z}}$.

Definition 9.1. *Let τ be a smooth representation of $\mathfrak{K}_1$ such that ϖ acts trivially. We say that τ has property (S) if $\tau^{I_1} \hookrightarrow \tau$ has a $\mathcal{G}$-equivariant section.*

Proposition 9.2. *Let τ be a smooth admissible representation of $\mathfrak{K}_1$ such that ϖ acts trivially. If $p \neq 2$ then τ has property (S). If $p = 2$ then assume that for every character $\chi : H \to \overline{\mathbb{F}}_p^\times$ such that $\chi = \chi^s$ there exists a subset $\mathcal{S}$ of $\tau^{I_1} e_\chi$ such that $(\Pi \cdot \mathcal{S}) \cap \mathcal{S} = \emptyset$ and $(\Pi \cdot \mathcal{S}) \cup \mathcal{S}$ is a basis of $\tau^{I_1} e_\chi$. Then τ has property (S).*

Proof. Since ϖ acts trivially the action of $\mathcal{G}$ on τ^{I_1} factors through $\overline{\mathcal{G}}$. We claim that the assumption implies that τ^{I_1} is an injective representation of $\overline{\mathcal{G}}$. The order of $\overline{\mathcal{G}}$ is equal to $2(q-1)^2$, hence if $p \neq 2$ the claim holds trivially. If $p = 2$ and $\chi \neq \chi^s$ then $\mathrm{Ind}_{[H]}^{\overline{\mathcal{G}}} \chi$ is irreducible, and injective since the order of $[H]$ is prime to 2. Hence, $\tau^{I_1}(e_\chi + e_{\chi^s})$ is injective. If $\chi = \chi^s$, the assumption on τ implies that $\tau^{I_1} e_\chi \cong \oplus \mathrm{Ind}_{[H]}^{\overline{\mathcal{G}}} \chi$. Since $\mathrm{Ind}_{[H]}^{\overline{\mathcal{G}}} \chi$ is injective we obtain the claim. But the claim implies that there exists a splitting. □

Corollary 9.3. *Let π be a smooth admissible representation of G such that ϖ acts trivially. Assume that $p = 2$, and that for every character $\chi : H \to \overline{\mathbb{F}}_p^\times$ such that $\chi = \chi^s$, there exists a filtration of $\mathcal{H}$-modules of $\pi^{I_1} e_\chi$ such that the graded pieces are isomorphic to either $M(0,\lambda,\eta)$ or $M(q-1,\lambda,\eta)$ for some λ and η. Then $\pi|_{\mathfrak{K}_1}$ has property (S).*

Proof. This follows from the fact that the underlying vector space of $M(r,\lambda,\eta)$ is two dimensional with basis $\{v, \Pi v\}$. $\square$

Corollary 9.4. *Let π be a smooth admissible representation of G such that ϖ acts trivially. Assume that $p=2$ and that $\operatorname{soc}_G \pi$ consists of supersingular representations, then $\pi|_{\mathfrak{K}_1}$ has property (S).*

Proof. It follows from Corollary 6.5 that there are no extensions between non-supersingular and supersingular $\mathcal{H}$-modules. This implies that if a non-supersingular module is a subquotient of π^{I_1}, then there exists an irreducible non-supersingular $\mathcal{H}$-submodule of π^{I_1}. However, Proposition 6.8 would imply that an irreducible non-supersingular representation occurs in $\operatorname{soc}_G \pi$. Hence, every irreducible subquotient of π^{I_1} is supersingular. Let $\chi : H \to \overline{\mathbb{F}}_p^{\times}$ be a character such that $\chi = \chi^s$ and choose any filtration of $\pi^{I_1} e_{\chi}$ such that the graded pieces are irreducible $\mathcal{H}$-modules. It follows by above that the graded pieces are isomorphic to $M(0,0,\eta)$ for some η. Hence, $\pi|_{\mathfrak{K}_1}$ has property (S). $\square$

Lemma 9.5. *Let τ be a smooth admissible representation of $\mathfrak{K}_1$ such that the fixed uniformizer ϖ acts trivially on τ and assume that τ has property (S). Let $\iota : \tau|_I \hookrightarrow \operatorname{Inj}(\tau|_I)$ be an injective envelope of $\tau|_I$ in Rep_I, then there exists an action of $\mathfrak{K}_1$ on $\operatorname{Inj}(\tau|_I)$ such that ι is $\mathfrak{K}_1$-equivariant. If τ satisfies the conditions of Proposition 9.2 then this action is unique up to isomorphism.*

Proof. Let $s : \tau \to \tau^{I_1}$ be a $\mathcal{G}$-equivariant section. Define $\iota_s : \tau \to \operatorname{Ind}_{\mathcal{G}}^{\mathfrak{K}_1} \tau^{I_1}$, $v \mapsto [g \mapsto s(gv)]$. This is a $\mathfrak{K}_1$-equivariant injection since it induces an injection of τ^{I_1}. Now $(\operatorname{Ind}_{\mathcal{G}}^{\mathfrak{K}_1} \tau^{I_1})|_I \cong \operatorname{Ind}_{[H]}^{I} \tau^{I_1}$ is an injective envelope of τ^{I_1} (and hence of τ) in Rep_I. Hence there exists an I-equivariant isomorphism $\psi : \operatorname{Inj}(\tau|_I) \cong \operatorname{Ind}_{[H]}^{I} \tau^{I_1}$ such that $\iota_s = \psi \circ \iota$. We may use ψ to define an action of $\mathfrak{K}_1$ on $\operatorname{Inj}(\tau|_I)$ such that ι is $\mathfrak{K}_1$-equivariant. If τ satisfies the conditions of Proposition 9.2 then τ^{I_1} is an injective representation of $\overline{\mathcal{G}}$, and hence $\operatorname{Ind}_{\mathcal{G}}^{\mathfrak{K}_1} \tau^{I_1}$ is an injective envelope of τ in the category of $\mathfrak{K}_1$ representations on which ϖ acts trivially. This implies the assertion. $\square$

Lemma 9.6. *Let $\sigma = \oplus_{i=1}^m \sigma_i$ where $(\sigma_i)_{1\le i\le m}$ are irreducible representations of K and recall $\sigma \hookrightarrow \operatorname{Inj}\sigma$ is an injective envelope of σ in Rep_K. Let $e \in \operatorname{End}_I(\operatorname{Inj}\sigma)$ be an idempotent and suppose that there exists an action of $\mathfrak{K}_1$ on $e(\operatorname{Inj}\sigma)$ extending the action of I with ϖ acting trivially. Then there exists an action of $\mathfrak{K}_1$ on $(1-e)(\operatorname{Inj}\sigma)$ with ϖ acting trivially.*

Proof. Set $V := e((\operatorname{Inj}\sigma)^{I_1})$ and $W := (1-e)((\operatorname{Inj}\sigma)^{I_1})$. Denote by V_χ and W_χ the χ-isotypic subspaces for the action of I, where $\chi : H \to \overline{\mathbb{F}}_p^{\times}$. We have:

$$V = \bigoplus_{\chi} V_\chi, \quad W = \bigoplus_{\chi} W_\chi,$$

where the sum is taken over all the characters χ. The action of Π on $e(\operatorname{Inj}\sigma)$ induces an isomorphism $V_\chi \cong V_{\chi^s}$ and hence $\dim_{\overline{\mathbb{F}}_p} V_\chi = \dim_{\overline{\mathbb{F}}_p} V_{\chi^s}$ for all χ. It follows from [25, Lem. 6.4.1, Lem. 4.2.19 and Lem. 4.2.20] that for every σ_i and every χ we have:

$$\dim_{\overline{\mathbb{F}}_p}(\operatorname{Inj}\sigma_i)^{I_1}_\chi = \dim_{\overline{\mathbb{F}}_p}(\operatorname{inj}\sigma_i)^{U}_\chi = \dim_{\overline{\mathbb{F}}_p}(\operatorname{inj}\sigma_i)^{U}_{\chi^s} = \dim_{\overline{\mathbb{F}}_p}(\operatorname{Inj}\sigma_i)^{I_1}_{\chi^s}.$$

Since $\operatorname{Inj}\sigma \cong \oplus_{i=1}^m \operatorname{Inj}\sigma_i$, we obtain $\dim_{\overline{\mathbb{F}}_p}(\operatorname{Inj}\sigma)^{I_1}_\chi = \dim_{\overline{\mathbb{F}}_p}(\operatorname{Inj}\sigma)^{I_1}_{\chi^s}$ and hence $\dim_{\overline{\mathbb{F}}_p} W_\chi = \dim_{\overline{\mathbb{F}}_p} W_{\chi^s}$ for all χ. For every ordered pair (χ, χ^s) such that $\chi \neq \chi^s$, choose an isomorphism of vector spaces $\phi_{\chi,\chi^s} : W_\chi \to W_{\chi^s}$ so that $\phi_{\chi,\chi^s} = \phi^{-1}_{\chi^s,\chi}$. If $\chi = \chi^s$ then $W_\chi = W_{\chi^s}$ and we set $\phi_{\chi,\chi^s} := \mathrm{id}_{W_\chi}$. Define $\phi \in \operatorname{End}_{\overline{\mathbb{F}}_p}(W)$ by:

$$\phi(w_\chi) := \phi_{\chi,\chi^s}(w_\chi), \quad \forall w_\chi \in W_\chi, \quad \forall \chi.$$

Then $\phi^2 = \mathrm{id}_W$ and $\phi u \phi^{-1} w = \Pi u \Pi^{-1} w$, $u \in I$, $w \in W$. Hence by sending Π to ϕ we obtain an action of $\mathfrak{K}_1$ on W. Since $\operatorname{Inj}\sigma$ is an injective object in Rep_K, $(\operatorname{Inj}\sigma)|_I$ is an injective object in Rep_I, and thus $(1-e)(\operatorname{Inj}\sigma)$ is an injective object in Rep_I. Since $W = (1-e)(\operatorname{Inj}\sigma)^{I_1}$, we have that $W \hookrightarrow (1-e)(\operatorname{Inj}\sigma)$ is an injective envelope of W in Rep_I. Since I_1 acts trivially on W, W has property (S) and Lemma 9.5 implies there exists an action of $\mathfrak{K}_1$ on $(1-e)(\operatorname{Inj}\sigma)$ extending the action of I and such that ϖ acts trivially. ∎

Definition 9.7. *A diagram D is a triple (D_0, D_1, r) where D_0 is a smooth representation of $\mathfrak{K}_0$, D_1 is a smooth representation of $\mathfrak{K}_1$ and $r : D_1 \to D_0$ is an IZ-equivariant morphism.*

This definition is taken from [25, §5.5]. Diagrams equipped with obvious morphisms form an abelian category $\mathcal{D}$, which is equivalent to the category of G-equivariant coefficient systems on the Bruhat-Tits tree X for $\mathrm{PGL}_2(F)$ ([25, Th. 5.5.4]). Given $D = (D_0, D_1, r) \in \mathcal{D}$, we will write $H_0(X, D)$, or more simply $H_0(D)$, for the 0-th homology of the coefficient system corresponding to D. Explicitly, one has an exact sequence:

$$\text{c-Ind}_{\mathfrak{K}_1}^G (D_1 \otimes \delta_{-1}) \xrightarrow{\partial} \text{c-Ind}_{\mathfrak{K}_0}^G D_0 \longrightarrow H_0(D) \longrightarrow 0$$

where ∂ is the composition of the following obvious maps:

$$\text{c-Ind}_{\mathfrak{K}_1}^G (D_1 \otimes \delta_{-1}) \hookrightarrow \text{c-Ind}_{IZ}^G D_1 \xrightarrow{r} \text{c-Ind}_{IZ}^G D_0 \twoheadrightarrow \text{c-Ind}_{KZ}^G D_0.$$

In particular, H_0 is a functor from $\mathcal{D}$ to Rep_G. This functor has a section, namely the constant functor $\mathcal{K} : \pi \mapsto (\pi|_{\mathfrak{K}_0}, \pi|_{\mathfrak{K}_1}, \mathrm{id})$.

Theorem 9.8. *Let $D = (D_0, D_1, r)$ be a diagram such that D_0 is admissible, r is an injection and ϖ acts trivially on D_0. If $p = 2$, assume D_1 has property (S). Let $\sigma := \mathrm{soc}_K D_0$. Then there exists an injection of diagrams:*

$$\iota : D \hookrightarrow \mathcal{K}(\Omega),$$

where Ω is a smooth representation of G such that $\Omega|_K \cong \mathrm{Inj}\,\sigma$.

Proof. Since D_0 is admissible, $D_0^{K_1}$ is finite dimensional and hence $\sigma \cong \oplus_{i=1}^m \sigma_i$ with σ_i irreducible. Since $D_0|_K$ is an essential extension of σ, there exists an injection $\iota_0 : D_0|_K \hookrightarrow \mathrm{Inj}\,\sigma$ making the diagram of K-representations:

$$\begin{array}{ccc} \sigma & \longrightarrow & \mathrm{Inj}\,\sigma \\ & \searrow & \uparrow{\scriptstyle \iota_0} \\ & & D_0|_K \end{array}$$

commute. Put an action of $\mathfrak{K}_0$ on $\mathrm{Inj}\,\sigma$ by making ϖ act trivially and denote this representation Ω_0. Then $\iota_0 : D_0 \hookrightarrow \Omega_0$ is $\mathfrak{K}_0$-equivariant. Set $\iota_1 := \iota_0 \circ r$. Since r and ι_0 are injections, we obtain an injection $\iota_1 : D_1|_I \hookrightarrow \Omega_0|_I$. Since $\Omega_0|_I$ is an injective object in Rep_I and ι_1 is an injection, there exists an idempotent $e \in \mathrm{End}_I(\Omega_0)$ such that $e \circ \iota_1 = \iota_1$, and $\iota_1 : D_1|_I \hookrightarrow e(\Omega_0)$ is an injective envelope of $D_1|_I$ in Rep_I. Since $\mathfrak{K}_1$ acts on D_1 with ϖ acting trivially, Lemma 9.5 implies there exists an action of $\mathfrak{K}_1$ on $e(\Omega_0)$ such that $\iota_1 : D_1 \to e(\Omega_0)$ is $\mathfrak{K}_1$-equivariant. Moreover, Lemma 9.6 implies there exists an action of $\mathfrak{K}_1$ on $(1-e)(\Omega_0)$ extending the action of I. This defines an action of $\mathfrak{K}_1$ on $e(\Omega_0) \oplus (1-e)(\Omega_0)$: we denote this representation by Ω_1. We obtain an injection of diagrams:

$$\iota = (\iota_0, \iota_1) : (D_0, D_1, r) \hookrightarrow (\Omega_0, \Omega_1, \mathrm{id}).$$

It then follows from [25, §5] that there exists a representation Ω of G, unique up to isomorphism, such that $(\Omega_0, \Omega_1, \mathrm{id}) \cong \mathcal{K}(\Omega)$. □

Lemma 9.9. *Let $D = (D_0, D_1, r)$ be a diagram and set:*

$$\mathcal{F}_0 := \{f \in \text{c-Ind}_{\mathfrak{K}_0}^G D_0, \mathrm{Supp}(f) \subseteq \mathfrak{K}_0\}.$$

Let Ω be a smooth representation of G and suppose that we are given an injection of diagrams $\iota : D \hookrightarrow \mathcal{K}(\Omega)$. Then the composition:

$$\mathcal{F}_0 \longrightarrow \text{c-Ind}_{\mathfrak{K}_0}^G D_0 \longrightarrow H_0(D) \xrightarrow{H_0(\iota)} \Omega$$

is an injection.

Proof. Let $\phi : \mathcal{F}_0 \to \Omega$ denote the above composition. Evaluation at 1 induces an isomorphism $\mathcal{F}_0 \cong D_0$. It follows from the proof of [25, Prop. 5.4.3] that the diagram $(\operatorname{Ker}\phi, 0, 0)$ is contained in $\operatorname{Ker}\iota$. Since ι is an injection, so is ϕ. □

Proposition 9.10. *Let $D = (D_0, D_1, r)$ be a diagram and suppose we are given an injection of diagrams $\iota : D \hookrightarrow \mathcal{K}(\Omega)$, where Ω is a smooth representation of G such that $\operatorname{soc}_K \Omega \cong \operatorname{soc}_K D_0$. Let π be the image of $H_0(\iota) : H_0(D) \to \Omega$. Then Ω is an essential extension of π. In particular, if π is irreducible then π is the G-socle of Ω.*

Proof. Lemma 9.9 implies $\operatorname{soc}_K D_0 \subseteq \operatorname{soc}_K \pi \subseteq \operatorname{soc}_K \Omega$, where we have identified D_0 with the image of $\mathcal{F}_0$ in Ω. Since $\operatorname{soc}_K D_0 \cong \operatorname{soc}_K \Omega$, we obtain $\operatorname{soc}_K \pi = \operatorname{soc}_K \Omega$. So $\Omega|_K$ is an essential extension of $\pi|_K$, which implies the first part. Suppose now that π is an irreducible representation of G. If $\pi' \subseteq \Omega$ is a non-zero G-invariant subspace, we thus have $\pi' \cap \pi \neq 0$, and hence $\pi \subseteq \pi'$ as π is irreducible. So π is the unique irreducible G-invariant subspace of Ω. □

Corollary 9.11. *Let π be an admissible representation of G such that ϖ acts trivially on π and $\sigma := \operatorname{soc}_K \pi$. If $p = 2$, assume $\pi|_{\mathfrak{K}_1}$ has property (S). Then there exists an injection $\pi \hookrightarrow \Omega$ where Ω is a representation of G such that $\Omega|_K \cong \operatorname{Inj}\sigma$.*

Proof. We apply Theorem 9.8 to the diagram $\mathcal{K}(\pi)$ and obtain a representation Ω of G such that $\Omega|_K \cong \operatorname{Inj}\sigma$ together with an injection of diagrams $\iota : \mathcal{K}(\pi) \hookrightarrow \mathcal{K}(\Omega)$. Applying H_0 on both sides gives a map $H_0(\iota) : \pi \to \Omega$. Lemma 9.9 implies that $H_0(\iota)$ is injective. □

We say that a diagram $D = (D_0, D_1, r)$ is basic if ϖ acts trivially and there exists an $m \geq 1$ such that r induces an isomorphism $r : D_1 \cong D_0^{I_m}$ (see §1 for I_m). We say that a basic diagram D is 0-irreducible (or just irreducible) if it does not contain any proper non-zero basic subdiagrams. For $e \geq 1$, we say that a basic diagram D is e-irreducible if D does not contain $(\chi \circ \det, \chi \circ \det, \mathrm{id})$ where $\chi : F^\times \to \overline{\mathbb{F}}_p^\times$ is a smooth character, $r : D_1 \cong D_0^{I_{e+1}}$ and for every basic proper subdiagram $D' \subseteq D$, we have $r : D_1' \cong (D_0')^{I_{e'+1}}$ with $e' < e$. Note that, if D is 0-irreducible, then $D = (D_0^{K_1}, D_1^{I_1}, r)$.

Theorem 9.12. *Let D be a basic e-irreducible diagram with $e \geq 0$ and suppose that we are given an injection of diagrams $\iota : D \hookrightarrow \mathcal{K}(\Omega)$ where Ω is a smooth representation of G such that $\operatorname{soc}_K \Omega \cong \operatorname{soc}_K D_0$. Then the image of $H_0(\iota) : H_0(D) \to \Omega$ is an irreducible representation of G.*

Proof. Let π be the image of $H_0(D) \to \Omega$. By Lemma 9.9, we have injections $D \hookrightarrow \mathcal{K}(\pi) \hookrightarrow \mathcal{K}(\Omega)$ and identify D with its image in $\mathcal{K}(\pi)$. Suppose that π' is a non-zero G-equivariant subspace of π and set $\mathcal{K}(\pi') \cap D := (D_0 \cap \pi', D_1 \cap \pi', \mathrm{can})$ (can stands for the canonical injection). Since $(D_0 \cap \pi')^{I_{e+1}} = D_0^{I_{e+1}} \cap \pi' = D_1 \cap \pi'$, we obtain that $\mathcal{K}(\pi') \cap D$ is basic. Lemma 9.9 implies that $\mathrm{soc}_K D_0 = \mathrm{soc}_K \pi = \mathrm{soc}_K \Omega$. Hence, $D_0 \cap \pi' \neq 0$ and hence $\mathcal{K}(\pi') \cap D \neq 0$. Assume $D \neq \mathcal{K}(\pi') \cap D$, then there exists $e' < e$ such that $D_1 \cap \pi' = (D_0 \cap \pi')^{I_{e'+1}}$. Since $sI_{e+1}s \subset I_{e'+1}$ we have $s(D_0 \cap \pi')^{I_{e'+1}} \subseteq (D_0 \cap \pi')^{I_{e+1}}$. Since $D_1 \cap \pi' = (D_0 \cap \pi')^{I_{e+1}}$, we obtain $s(D_1 \cap \pi') \subseteq D_1 \cap \pi'$ and hence that $D_1 \cap \pi'$ is a K-invariant subspace of $D_0 \cap \pi'$. It follows from [25, §5] that there exists a representation ρ of G such that $(D_1 \cap \pi', D_1 \cap \pi', \mathrm{id}) \cong \mathcal{K}(\rho)$. Since I_{e+1} acts trivially on D_1, it will be contained in the kernel of ρ, and hence $\mathrm{Ker}\,\rho$ will contain $\mathrm{SL}_2(F)$. Hence there exists a smooth character $\chi : F^\times \to \overline{\mathbb{F}}_p^\times$ such that $(\chi \circ \det, \chi \circ \det, \mathrm{id})$ is a subdiagram of D. This implies $D = (\chi \circ \det, \chi \circ \det, \mathrm{id})$ and hence $\pi = \chi \circ \det$ is irreducible. Otherwise $D \subseteq \mathcal{K}(\pi') \subseteq \mathcal{K}(\Omega)$. Taking H_0, we obtain that the image of $H_0(D) \to \Omega$ is contained in π' and hence $\pi = \pi'$. ☐

Set $t := \Pi s = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi \end{smallmatrix}\right)$. For π in Rep_G and $m \geq 0$, we define a diagram:

$$\mathcal{S}^m(\pi) := (\mathcal{S}^m(\pi)_0, \mathcal{S}^m(\pi)_1, \mathrm{can}),$$

where $\mathcal{S}^m(\pi)$ is the smallest subdiagram of $\mathcal{K}(\pi)$ such that the following hold:

(i) $\mathrm{soc}_K \pi \subseteq \mathcal{S}^m(\pi)_0$

(ii) $\mathcal{S}^m(\pi)_1 \cong \mathcal{S}(\pi)_0^{I_{m+1}}$

(and where can stands for the canonical injection). Given two subdiagrams $D = (D_0, D_1, \mathrm{can})$ and $D' = (D'_0, D'_1, \mathrm{can})$ of $\mathcal{K}(\pi)$ satisfying (i) and (ii) for a given m, we may consider $D \cap D' := (D_0 \cap D'_0, D_1 \cap D'_1, \mathrm{can})$. We have that $\mathrm{soc}_K \pi \subseteq D_0 \cap D'_0$ and $(D_0 \cap D'_0)^{I_{m+1}} = D_0^{I_{m+1}} \cap (D'_0)^{I_{m+1}} = D_1 \cap D'_1$. Hence $\mathcal{S}^m(\pi)$ is well defined. We note that $\mathcal{S}^m(\pi) \subseteq (\pi^{K_{m+1}}, \pi^{I_{m+1}}, \mathrm{can})$.

Theorem 9.13. *Let π be an admissible representation of G which is G-generated by $(\mathrm{soc}_K \pi)^{I_1}$.*

(i) If $p = 2$, assume $\pi|_{\mathfrak{K}_1}$ has property (S). If there exists $e \geq 0$ such that $\mathcal{S}^e(\pi)$ is e-irreducible, then π is irreducible.

(ii) Assume π is irreducible and either $\mathrm{soc}_K \pi$ is multiplicity free or π can be realized over a finite field. Then there exists $e \geq 0$ such that $\mathcal{S}^e(\pi)$ is e-irreducible.

Proof. We prove (i). Corollary 9.11 gives an injection $\pi \hookrightarrow \Omega$ where Ω is a representation of G such that $\Omega|_K \cong \operatorname{Inj}\sigma$ with $\sigma := \operatorname{soc}_K \pi$ (and $\operatorname{Inj}\sigma$ an injective envelope of σ in Rep_K). We obtain an injection of diagrams $\mathcal{S}^m(\pi) \hookrightarrow \mathcal{K}(\Omega)$ for all $m \geq 1$. We have $\operatorname{soc}_K(\mathcal{S}^m(\Pi)_0) = \operatorname{soc}_K \pi = \operatorname{soc}_K \Omega$, so if $\mathcal{S}^m(\pi)$ is irreducible, then Proposition 9.12 implies that the image of $H_0(X, \mathcal{S}^m(\pi)) \to \Omega$ is irreducible. Since π is generated by $(\operatorname{soc}_K \pi)^{I_1}$, π is contained in this image and is thus also irreducible. We prove (ii). Suppose π is an irreducible representation of G and $\operatorname{soc}_K \pi := \oplus_{i=1}^n \sigma_i$ is multiplicity free. Since $G = \amalg_{m \geq 0} K t^m K Z$ and π is irreducible, for each i there exists m_i such that:

$$\operatorname{soc}_K \left(\sum_{j=0}^{m_i} \langle K \cdot t^j \sigma_i \rangle \right) = \operatorname{soc}_K \pi. \tag{19}$$

Let $m > \max_i(m_i)$, $m \geq 1$ and suppose $\mathcal{S}^m(\pi)$ is not m-irreducible. Then it contains some basic m-irreducible subdiagram $D = (D_0, D_1, \mathrm{can})$. Since all the σ_i are distinct, we have $\sigma_i \subseteq D_0$ for some i. Since $I_2 \subset K_1$ and $m \geq 1$, we have $\sigma_i \subseteq D_1$, hence $t\sigma_i = \Pi\sigma_i \subseteq D_1$ and thus $\langle K \cdot t\sigma_i \rangle \subset D_0$. Since $st^j\sigma_i$ is fixed by I_{j+2}, by repeating the argument for all j we get:

$$\sum_{j=0}^{m-1} \langle K \cdot t^j \sigma_i \rangle \subseteq D_0.$$

As $m - 1 \geq m_i$, we have $\operatorname{soc}_K D_0 = \operatorname{soc}_K \pi$ by (19), which implies that $D = \mathcal{S}^m(\pi)$ is m-irreducible. Suppose π is an irreducible representation of G which can be realized over a finite field. Let $\sigma_1, \dots \sigma_n$ be distinct irreducible summands of $\operatorname{soc}_K \pi$. Since we are working over a finite field, the set $\operatorname{Hom}_K(\sigma_i, \pi)$ is finite. For each i and each $\phi \in \operatorname{Hom}_K(\sigma_i, \pi)$, there exists $m_{i,\phi}$ such that:

$$\operatorname{soc}_K \left(\sum_{j=0}^{m_{i,\phi}} \langle K \cdot t^j \phi(\sigma_i) \rangle \right) = \operatorname{soc}_K \pi.$$

Let m be an integer such that $m > \max_{i,\phi}(m_{i,\phi})$, then the previous proof goes through to show that $\mathcal{S}^m(\pi)$ is m-irreducible. □

In the rest of the paper, we call basic 0-diagram any basic diagram such that $r : D_1 \xrightarrow{\sim} D_0^{I_1} \hookrightarrow D_0$ (in particular D_1 then always has property (S)).

10 Examples of diagrams

We give a few simple examples of basic 0-diagrams, in particular we list all irreducible basic 0-diagrams for $f = 1$.

We denote by $\mathrm{st} := (p-1, \cdots, p-1)$ the Steinberg representation for Γ. Consider the following list of basic 0-diagrams $D = (D_0, D_1, r)$ where we make $\left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)$ act trivially everywhere:

(i) $(D_0, D_1, r) := (1, 1, \mathrm{id})$

(ii) $(D_0, D_1, r) := (\mathrm{st}, \mathrm{st}^{I_1}, \mathrm{can})$ where Π acts on st^{I_1} by identity and can is the canonical injection

(iii) $(D_0, D_1, r) := (1 \oplus \mathrm{st}, 1 \oplus \mathrm{st}^{I_1}, \mathrm{can})$ where Π acts on $1 \oplus \mathrm{st}^{I_1} = \overline{\mathbb{F}}_p v_0 \oplus \overline{\mathbb{F}}_p v_{q-1}$ with $v_0 \in 1$ and $v_{q-1} \in \mathrm{st}$ by ($\lambda \in \overline{\mathbb{F}}_p \setminus \{-1, 0, 1\}$):

$$\begin{aligned} \Pi v_0 &:= v_{q-1} + \lambda v_0 \\ \Pi v_{q-1} &:= (1 - \lambda^2) v_0 - \lambda v_{q-1} \end{aligned}$$

(iv) $(D_0, D_1, r) := (\mathrm{Ind}_B^\Gamma \chi, (\mathrm{Ind}_B^\Gamma \chi)^{I_1}, \mathrm{can})$ where $\chi \neq \chi^s$ and Π acts on $(\mathrm{Ind}_B^\Gamma \chi)^{I_1} = \overline{\mathbb{F}}_p f_0 \oplus \overline{\mathbb{F}}_p \phi$ (with the notations of §2) by ($\lambda \in \overline{\mathbb{F}}_p^\times$):

$$\begin{aligned} \Pi f_0 &:= \lambda \phi \\ \Pi \phi &:= \lambda^{-1} f_0 \end{aligned}$$

(v) $(D_0, D_1, r) := (\sigma \oplus \sigma^{[s]}, \sigma^{I_1} \oplus {\sigma^{[s]}}^{I_1}, \mathrm{can})$ where σ is any weight and Π acts on $\sigma^{I_1} \oplus {\sigma^{[s]}}^{I_1} = \overline{\mathbb{F}}_p v_\sigma \oplus \overline{\mathbb{F}}_p v_{\sigma^{[s]}}$ (with obvious notations) by:

$$\begin{aligned} \Pi v_\sigma &:= v_{\sigma^{[s]}} \\ \Pi v_{\sigma^{[s]}} &:= v_\sigma. \end{aligned}$$

All of the above basic 0-diagrams are irreducible. Moreover, one checks that the diagrams in (iii) and (iv) are all distinct when λ varies and the diagrams in (v) are all distinct when $\{\sigma, \sigma^{[s]}\}$ varies. Note that, after a base change on (v_0, v_{q-1}), (iii) is like (iv) but for $\chi = \chi^s$ (however, we chose to separate the two cases). The diagrams (v) are studied in [25].

From the results of [4], [6] and [25] (see also [33] and [24]), we deduce the following:

Theorem 10.1. *Assume $F = \mathbb{Q}_p$. The functor $D \mapsto H_0(D)$ (see §9) induces a bijection between the set of isomorphism classes of irreducible basic* 0*-diagrams and the set of isomorphism classes of irreducible smooth admissible representations of* $\mathrm{GL}_2(\mathbb{Q}_p)$ *over* $\overline{\mathbb{F}}_p$*. The inverse bijection is given by* $\pi \mapsto (\langle K \cdot \pi^{I_1} \rangle, \pi^{I_1}, \mathrm{can})$ *where* can *stands for the canonical injection.*

The above list exhausts all irreducible basic 0-diagrams up to twist when $f = 1$. When moreover $F = \mathbb{Q}_p$, the above bijection $D \mapsto H_0(D)$ gives:

(i) $H_0(D) = 1$

(ii) $H_0(D) = \mathrm{St} \otimes \delta_{-1}$ (St is the Steinberg representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over $\overline{\mathbb{F}}_p$)

(iii) $H_0(D)$ exhaust the unramified irreducible principal series of $\mathrm{GL}_2(\mathbb{Q}_p)$ over $\overline{\mathbb{F}}_p$ up to twist

(iv) $H_0(D)$ exhaust the ramified irreducible principal series of $\mathrm{GL}_2(\mathbb{Q}_p)$ over $\overline{\mathbb{F}}_p$ up to twist

(v) $H_0(D)$ exhaust the supersingular representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ over $\overline{\mathbb{F}}_p$.

Note that, for π irreducible admissible and $F = \mathbb{Q}_p$, $p > 3$, one has $(\langle K \cdot \pi^{I_1} \rangle, \pi^{I_1}, \mathrm{can}) \simeq (\pi^{K_1}, \pi^{I_1}, \mathrm{can})$ if and only if π is not supersingular.

Needless to say the above theorem completely breaks down when $F \neq \mathbb{Q}_p$. For instance, if $F = \mathbb{Q}_{p^f}$ and $f > 1$:

(i) there are many more irreducible basic 0-diagrams D than the ones of the list above

(ii) $H_0(D)$ can have infinitely many distinct quotients

(iii) these quotients can have a bigger K-socle than the one of D_0

(iv) π can be irreducible even though $(\langle K \cdot \pi^{I_1} \rangle, \pi^{I_1}, \mathrm{can})$ is not, etc.

Let us finish this section with a fancy series of examples of reducible basic 0-diagrams for $f = 2$ leading to irreducible admissible π when $F = \mathbb{Q}_{p^2}$.

Assume $f = 2$ and let $\sigma := (1, p-2) \otimes \det^{p-1}$ and χ the action of I on σ^{I_1}. Let τ be the following unique Γ-extension with Γ-socle 1:

$$0 \to 1 \to \tau \to \sigma \oplus \sigma^{[s]} \to 0.$$

For a positive integer n, set:

$$D_0(n) := \mathrm{st} \oplus \tau \oplus \underbrace{\mathrm{Ind}_B^\Gamma \chi^s \oplus \cdots \oplus \mathrm{Ind}_B^\Gamma \chi^s}_{n \text{ times}}$$

and recall that σ is the socle of $\mathrm{Ind}_B^\Gamma \chi^s$ and $\sigma^{[s]}$ its cosocle. Number the $\mathrm{Ind}_B^\Gamma \chi^s$ from 1 to n and let $v_i := (f_0)_i$, $v_i^s := \phi_i$ using the notations of §2, so that $(\mathrm{Ind}_B^\Gamma \chi^s)_i^{I_1} = \overline{\mathbb{F}}_p v_i \oplus \overline{\mathbb{F}}_p v_i^s$, $1 \leq i \leq n$. Let w be a basis of $1 \subset \tau$ and v_0 (resp. v_0^s) an H-eigenvector in τ^{I_1} which is sent to σ^{I_1} (resp. ${\sigma^{[s]}}^{I_1}$) under $\tau \twoheadrightarrow \sigma \oplus \sigma^{[s]}$, so that we have $\tau^{I_1} = \overline{\mathbb{F}}_p w \oplus \overline{\mathbb{F}}_p v_0 \oplus \overline{\mathbb{F}}_p v_0^s$. Finally, let w' be a basis of st^{I_1}. Hence we have:

$$D_0(n)^{I_1} = \overline{\mathbb{F}}_p w' \oplus (\overline{\mathbb{F}}_p w \oplus \overline{\mathbb{F}}_p v_0 \oplus \overline{\mathbb{F}}_p v_0^s) \oplus \Big(\bigoplus_{i=1}^{n} (\overline{\mathbb{F}}_p v_i \oplus \overline{\mathbb{F}}_p v_i^s)\Big).$$

Define an action of Π on $D_0(n)^{I_1}$ as follows:

$$\begin{aligned} \Pi w &:= w' \\ \Pi w' &:= w \\ \Pi v_i^s &:= v_{i+1},\ 0 \leq i \leq n-1 \\ \Pi v_{i+1} &:= v_i^s,\ 0 \leq i \leq n-1 \\ \Pi v_n^s &:= v_0 \\ \Pi v_0 &:= v_n^s \end{aligned}$$

and call $D_1(n)$ the resulting $\mathfrak{K}_1$-representation. Denote by $D(n)$ the basic 0-diagram $(D_0(n), D_1(n), \mathrm{can})$ where can is the canonical injection $D_1(n) \hookrightarrow D_0(n)$: the notation is actually bad since the isomorphism class of $D(n)$ depends on the choice of the vectors v_i, v_i^s. Using techniques analogous to that of §18, one can prove the following proposition (we don't give details here, as we don't use it in the paper):

Proposition 10.2. *Assume $F = \mathbb{Q}_{p^2}$. For any injection of diagrams $\iota : D(n) \hookrightarrow \mathcal{K}(\Omega)$ where Ω is a smooth representation of G such that $\mathrm{soc}_K \Omega \cong \mathrm{soc}_K D_0(n)$, the image of $H_0(\iota) : H_0(D(n)) \to \Omega$ is an irreducible admissible representation of $\mathrm{GL}_2(\mathbb{Q}_{p^2})$ with the same K-socle as $D_0(n)$.*

(The point being that, although $D(n)$ is reducible, the techniques of §9 produce irreducible representations from it). Using Theorem 9.8, this implies that each $H_0(D(n)))$ has at least one irreducible quotient with the same K-socle as $D_0(n)$. Now, let D be as in example (v) above with $\sigma = 1$ (and thus $\sigma^{[s]} = \mathrm{st}$ by definition, see §1). One can easily check that $H_0(D) = (\text{c-Ind}_{\mathfrak{K}_0}^G 1)/(T)$. As $D \hookrightarrow D(n)$, we see that $H_0(D)$ has infinitely many non-isomorphic irreducible admissible quotients with growing K-socle.

11 Generic Diamond weights

From now on and until the end of the paper, we assume $F = \mathbb{Q}_{p^f}$, although many of the forthcoming results actually only depend on the residue field of F. Following [11] and [12], we briefly recall the list of weights associated to a generic Galois representation ρ.

We first consider the case where ρ is reducible and split.

Let $(x_0, \cdots, x_{f-1})$ be f variables. We define a set $\mathcal{RD}(x_0, \cdots, x_{f-1})$ of f-tuples $\lambda := (\lambda_0(x_0), \cdots, \lambda_{f-1}(x_{f-1}))$ where $\lambda_i(x_i) \in \mathbb{Z} \pm x_i$ as follows. If $f = 1$, $\lambda_0(x_0) \in \{x_0, p-3-x_0\}$. If $f > 1$, then:

(i) $\lambda_i(x_i) \in \{x_i, x_i+1, p-2-x_i, p-3-x_i\}$ for $i \in \{0, \cdots, f-1\}$

(ii) if $\lambda_i(x_i) \in \{x_i, x_i+1\}$, then $\lambda_{i+1}(x_{i+1}) \in \{x_{i+1}, p-2-x_{i+1}\}$

(iii) if $\lambda_i(x_i) \in \{p-2-x_i, p-3-x_i\}$, then $\lambda_{i+1}(x_{i+1}) \in \{p-3-x_{i+1}, x_{i+1}+1\}$

with the conventions $x_f = x_0$ and $\lambda_f(x_f) = \lambda_0(x_0)$. An element of the set $\mathcal{RD}(x_0, \cdots, x_{f-1})$ is called a formal reducible Diamond weight.

For $\lambda \in \mathcal{RD}(x_0, \cdots, x_{f-1})$, define:

$$\begin{aligned} e(\lambda) &:= \frac{1}{2}\Big(\sum_{i=0}^{f-1} p^i(x_i - \lambda_i(x_i))\Big) \text{ if } \lambda_{f-1}(x_{f-1}) \in \{x_{f-1}, x_{f-1}+1\} \\ e(\lambda) &:= \frac{1}{2}\Big(p^f - 1 + \sum_{i=0}^{f-1} p^i(x_i - \lambda_i(x_i))\Big) \text{ otherwise.} \end{aligned}$$

The following straightforward lemma is left to the reader.

Lemma 11.1. *One has $e(\lambda) \in \mathbb{Z} \oplus \bigoplus_{i=0}^{f-1} \mathbb{Z}x_i$.*

Lemma 11.2. *Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^f}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous representation such that its restriction to inertia is:*

$$\begin{pmatrix} \omega_f^{r_0+1+p(r_1+1)+\cdots+p^{f-1}(r_{f-1}+1)} & 0 \\ 0 & 1 \end{pmatrix} \otimes \eta$$

with $-1 \le r_i \le p-2$. Let us assume $0 \le r_i \le p-3$ for all i and not all r_i equal to 0 or equal to $p-3$. Then the weights associated to ρ in [11] are exactly the (all distinct) weights:

$$(\lambda_0(r_0), \cdots, \lambda_{f-1}(r_{f-1})) \otimes \det{}^{e(\lambda)(r_0,\cdots,r_{f-1})}\eta$$

for $\lambda = (\lambda_0(x_0), \cdots, \lambda_{f-1}(x_{f-1})) \in \mathcal{RD}(x_0, \cdots, x_{f-1})$.

Proof. See [15, Prop. 1.1 and Prop. 1.3] and [15, §3]. □

We now consider the case where ρ is irreducible.

Let $(x_0, \cdots, x_{f-1})$ be f variables. We define a set $\mathcal{ID}(x_0, \cdots, x_{f-1})$ of f-tuples $\lambda := (\lambda_0(x_0), \cdots, \lambda_{f-1}(x_{f-1}))$ where $\lambda_i(x_i) \in \mathbb{Z} \pm x_i$ as follows. If $f = 1$, $\lambda_0(x_0) \in \{x_0, p-1-x_0\}$. If $f > 1$, then:

(i) $\lambda_0(x_0) \in \{x_0, x_0 - 1, p-2-x_0, p-1-x_0\}$ and $\lambda_i(x_i) \in \{x_i, x_i+1, p-2-x_i, p-3-x_i\}$ if $i > 0$

(ii) if $i > 0$ and $\lambda_i(x_i) \in \{x_i, x_i+1\}$ (resp. $\lambda_0(x_0) \in \{x_0, x_0-1\}$), then $\lambda_{i+1}(x_{i+1}) \in \{x_{i+1}, p-2-x_{i+1}\}$

(iii) if $0 < i < f-1$ and $\lambda_i(x_i) \in \{p-2-x_i, p-3-x_i\}$, then $\lambda_{i+1}(x_{i+1}) \in \{p-3-x_{i+1}, x_{i+1}+1\}$

(iv) if $\lambda_0(x_0) \in \{p-1-x_0, p-2-x_0\}$, then $\lambda_1(x_1) \in \{p-3-x_1, x_1+1\}$

(v) if $\lambda_{f-1}(x_{f-1}) \in \{p-2-x_{f-1}, p-3-x_{f-1}\}$, then $\lambda_0(x_0) \in \{p-1-x_0, x_0-1\}$

with the conventions $x_f = x_0$ and $\lambda_f(x_f) = \lambda_0(x_0)$. An element of the set $\mathcal{ID}(x_0, \cdots, x_{f-1})$ is called a formal irreducible Diamond weight.

For $\lambda \in \mathcal{ID}(x_0, \cdots, x_{f-1})$, define if $f > 1$:

$$\begin{aligned} e(\lambda) &:= \frac{1}{2}\Big(\sum_{i=0}^{f-1} p^i(x_i - \lambda_i(x_i))\Big) \text{ if } \lambda_{f-1}(x_{f-1}) \in \{x_{f-1}, x_{f-1}+1\} \\ e(\lambda) &:= \frac{1}{2}\Big(p^f - 1 + \sum_{i=0}^{f-1} p^i(x_i - \lambda_i(x_i))\Big) \text{ otherwise,} \end{aligned}$$

and, if $f = 1$, $e(\lambda) := 0$ if $\lambda_0(x_0) = x_0$, $e(\lambda) := x_0$ if $\lambda_0(x_0) = p-1-x_0$.

The following straightforward lemma is left to the reader.

Lemma 11.3. *One has* $e(\lambda) \in \mathbb{Z} \oplus \bigoplus_{i=0}^{f-1} \mathbb{Z}x_i$.

Lemma 11.4. *Let* $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^f}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ *be a continuous representation such that its restriction to inertia is:*

$$\begin{pmatrix} \omega_{2f}^{r_0+1+p(r_1+1)+\cdots+p^{f-1}(r_{f-1}+1)} & 0 \\ 0 & \omega_{2f}^{p^f(r_0+1)+p^{f+1}(r_1+1)+\cdots+p^{2f-1}(r_{f-1}+1)} \end{pmatrix} \otimes \eta$$

with $0 \leq r_0 \leq p-1$ *and* $-1 \leq r_i \leq p-2$ *for* $i > 0$*, and where* ω_{2f} *is defined as in (3) from one of the two embeddings* $\mathbb{F}_{p^{2f}}^\times \hookrightarrow \overline{\mathbb{F}}_p^\times$ *giving back the fixed embedding* $\mathbb{F}_{p^f}^\times \hookrightarrow \overline{\mathbb{F}}_p^\times$ *by restriction to* $\mathbb{F}_{p^f}^\times$*. Let us assume* $1 \leq r_0 \leq p-2$ *and* $0 \leq r_i \leq p-3$ *for* $i > 0$*. Then the weights associated to* ρ *in [11] are exactly the (all distinct) weights:*

$$(\lambda_0(r_0), \cdots, \lambda_{f-1}(r_{f-1})) \otimes \det{}^{e(\lambda)(r_0,\cdots,r_{f-1})}\eta$$

for $\lambda = (\lambda_0(x_0), \cdots, \lambda_{f-1}(x_{f-1})) \in \mathcal{ID}(x_0, \cdots, x_{f-1})$*.*

Proof. See [15, Prop. 1.1 and Prop. 1.3] and [15, §3]. $\square$

The set $\mathcal{RD}(x_0, \cdots, x_{f-1})$ (resp. $\mathcal{ID}(x_0, \cdots, x_{f-1})$) can be naturally identified with the set of subsets $\mathcal{S}$ of $\{0, \cdots, f-1\}$ as follows: set $i \in \mathcal{S}$ if and only if $\lambda_i(x_i) \in \{p-3-x_i, x_i+1\}$ (resp. if $i > 0$ set $i \in \mathcal{S}$ if and only if $\lambda_i(x_i) \in \{p-3-x_i, x_i+1\}$ and set $0 \in \mathcal{S}$ if and only if $\lambda_0(x_0) \in \{p-1-x_0, x_0-1\}$). One checks that, given $\mathcal{S}$, there is only one possibility for $(\lambda_i(x_i))_i \in \mathcal{RD}(x_0, \cdots, x_{f-1})$ (resp. $\in \mathcal{ID}(x_0, \cdots, x_{f-1})$). By Lemmas 11.2 and 11.4, when ρ is tamely ramified (and generic) we can thus identify $\mathcal{D}(\rho)$ with the subsets of $\{0, \cdots, f-1\}$. If $\sigma \in \mathcal{D}(\rho)$, $\lambda \in \mathcal{RD}(x_0, \cdots, x_{f-1})$ (resp. $\mathcal{ID}(x_0, \cdots, x_{f-1})$) and $\mathcal{S} \subseteq \{0, \cdots, f-1\}$ correspond to σ, we set $\ell(\sigma) = \ell(\lambda) := |\mathcal{S}|$.

For $\lambda, \lambda' \in \mathcal{RD}(x_0, \cdots, x_{f-1})$ (resp. $\mathcal{ID}(x_0, \cdots, x_{f-1})$) corresponding to $\mathcal{S}, \mathcal{S}' \subseteq \{0, \cdots, f-1\}$ respectively, we define $\lambda \cap \lambda' \in \mathcal{RD}(x_0, \cdots, x_{f-1})$ (resp. $\mathcal{ID}(x_0, \cdots, x_{f-1})$) as the element corresponding to $\mathcal{S} \cap \mathcal{S}'$ and $\lambda \cup \lambda' \in \mathcal{RD}(x_0, \cdots, x_{f-1})$ (resp. $\mathcal{ID}(x_0, \cdots, x_{f-1})$) as the element corresponding to $\mathcal{S} \cup \mathcal{S}'$. We also define a partial order on the elements of $\mathcal{RD}(x_0, \cdots, x_{f-1})$ (resp. $\mathcal{ID}(x_0, \cdots, x_{f-1})$) by declaring that $\lambda' \leq \lambda$ if and only if $\mathcal{S}' \subseteq \mathcal{S}$ or equivalently $\lambda \cup \lambda' = \lambda$ or equivalently $\lambda \cap \lambda' = \lambda'$. If ρ is a continuous generic tame Galois representation, $\sigma, \sigma' \in \mathcal{D}(\rho)$, and σ, σ' correspond respectively to λ, λ', we let $\sigma \cap \sigma'$ (resp. $\sigma \cup \sigma'$) be the unique weight in $\mathcal{D}(\rho)$ which corresponds to $\lambda \cap \lambda'$ (resp. $\lambda \cup \lambda'$). We also write $\sigma \leq \sigma'$ if $\lambda \leq \lambda'$.

We now consider the case where ρ is reducible but not split.

Definition 11.5. *A non-empty subset* $\mathcal{D}(x_0, \cdots, x_{f-1})$ *of* $\mathcal{RD}(x_0, \cdots, x_{f-1})$ *is said to be of Galois type if it satisfies the following properties:*

(i) if $\lambda \in \mathcal{D}(x_0, \cdots, x_{f-1})$*, then all* $\lambda' \in \mathcal{RD}(x_0, \cdots, x_{f-1})$ *such that* $\lambda' \leq \lambda$ *are in* $\mathcal{D}(x_0, \cdots, x_{f-1})$

(ii) if $\lambda, \lambda' \in \mathcal{D}(x_0, \cdots, x_{f-1})$*, then* $\lambda \cup \lambda' \in \mathcal{D}(x_0, \cdots, x_{f-1})$*.*

Note that, if $\mathcal{D}(x_0, \cdots, x_{f-1})$ is a subset of Galois type, then it follows from (ii) of Definition 11.5 that $\mathcal{D}(x_0, \cdots, x_{f-1})$ has a unique maximal element for $\leq$. If this element corresponds to $\mathcal{S} \subseteq \{0, \cdots, f-1\}$, one checks that $|\mathcal{D}(x_0, \cdots, x_{f-1})| = 2^d$ where $d := |\mathcal{S}|$. Definition 11.5 comes from the following conjecture:

Theorem 11.6. *Let $\{r_0, \cdots, r_{f-1}\}$ be such that $0 \leq r_i \leq p-3$ for all i and not all r_i equal to 0 or equal to $p-3$.*

(i) Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^f}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous representation such that its restriction to inertia is:

$$\begin{pmatrix} \omega_f^{r_0+1+p(r_1+1)+\cdots+p^{f-1}(r_{f-1}+1)} & * \\ 0 & 1 \end{pmatrix} \otimes \eta \tag{20}$$

with $* \neq 0$. Then there exists a unique subset $\mathcal{D}(x_0, \cdots, x_{f-1}) \subsetneq \mathcal{RD}(x_0, \cdots, x_{f-1})$ of Galois type such that the weights associated to ρ in [11] are exactly the (all distinct) weights:

$$(\lambda_0(r_0), \cdots, \lambda_{f-1}(r_{f-1})) \otimes \det{}^{e(\lambda)(r_0,\cdots,r_{f-1})}\eta$$

for $\lambda = (\lambda_0(x_0), \cdots, \lambda_{f-1}(x_{f-1})) \in \mathcal{D}(x_0, \cdots, x_{f-1})$.

(ii) Let $\mathcal{D}(x_0, \cdots, x_{f-1}) \subsetneq \mathcal{RD}(x_0, \cdots, x_{f-1})$ be a subset of Galois type. Then there exists at least one representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^f}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ as in (20) with $* \neq 0$ such that the weights associated to ρ in [11] are exactly the (all distinct) weights:

$$(\lambda_0(r_0), \cdots, \lambda_{f-1}(r_{f-1})) \otimes \det{}^{e(\lambda)(r_0,\cdots,r_{f-1})}\eta$$

for $\lambda = (\lambda_0(x_0), \cdots, \lambda_{f-1}(x_{f-1})) \in \mathcal{D}(x_0, \cdots, x_{f-1})$.

Proof. (i) and (ii) follow from [13, Thm. 1.1] or [9, Prop. A.3]. □

From now on, we only consider those ρ which satisfy the conditions in Lemma 11.2 or in Lemma 11.4 or in (i) of Theorem 11.6, and we give them a name:

Definition 11.7. *Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^f}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous representation, we say ρ is generic if one of the following holds:*

(i) the restriction of ρ to inertia is isomorphic to:

$$\begin{pmatrix} \omega_f^{r_0+1+p(r_1+1)+\cdots+p^{f-1}(r_{f-1}+1)} & * \\ 0 & 1 \end{pmatrix} \otimes \eta$$

for some character η and some r_i with $0 \leq r_i \leq p-3$ and $(r_0, \cdots, r_{f-1}) \notin \{(0, \cdots, 0), (p-3, \cdots, p-3)\}$

(ii) the restriction of ρ to inertia is isomorphic to:

$$\begin{pmatrix} \omega_{2f}^{r_0+1+p(r_1+1)+\cdots+p^{f-1}(r_{f-1}+1)} & 0 \\ 0 & \omega_{2f}^{p^f(r_0+1)+p^{f+1}(r_1+1)+\cdots+p^{2f-1}(r_{f-1}+1)} \end{pmatrix} \otimes \eta$$

for some character η and some r_i with $1 \leq r_0 \leq p-2$ and $0 \leq r_i \leq p-3$ for $i > 0$.

One can check this definition doesn't depend on the choice of the embedding $\mathbb{F}_q \hookrightarrow \overline{\mathbb{F}}_p$. Note that there is no such ρ for $p = 2$ and that for $p = 3$ the only possibility is ρ irreducible with $r_0 = 1$ and $r_i = 0$ for $i > 0$ (notation of Lemma 11.4).

In the rest of the paper, we denote by $\mathcal{D}(\rho)$ the set of weights associated to a generic ρ and simply call them Diamond weights.

12 The unicity Lemma

Fix ρ generic, we define the "distance" from a weight to a Diamond weight associated to ρ, and prove that there is attached to any weight a unique Diamond weight such that this distance is minimal.

Recall from §11 that $\mathcal{RD}(x_0, \cdots, x_{f-1})$ (resp. $\mathcal{ID}(x_0, \cdots, x_{f-1})$) denotes the set of formal reducible (resp. irreducible) Diamond weights which can be identified with the set of subsets $\mathcal{S} \subseteq \{0, \cdots, f-1\}$.

Definition 12.1. *Let $\lambda \in \mathcal{RD}(x_0, \cdots, x_{f-1})$ (resp. $\lambda \in \mathcal{ID}(x_0, \cdots, x_{f-1})$) corresponding to $\mathcal{S} \subseteq \{0, \cdots, f-1\}$.*

(i) Let $\mathcal{S}' \subseteq \{0, \cdots, f-1\}$. We say $\mu \in \mathcal{RD}(x_0, \cdots, x_{f-1})$ (resp. $\mu \in \mathcal{ID}(x_0, \cdots, x_{f-1})$) is the negative of λ within $\mathcal{S}'$ if μ corresponds to $(\mathcal{S} \setminus \mathcal{S} \cap \mathcal{S}') \amalg (\mathcal{S}' \setminus \mathcal{S} \cap \mathcal{S}')$.

(ii) If μ is as in (i) for $\mathcal{S}' = \{0, \cdots, f-1\}$, we simply say μ is the negative of λ (in which case μ corresponds to $\{0, \cdots, f-1\} \setminus \mathcal{S}$).

For instance, if $f = 5$, the negative of $\lambda := (x_0+1, x_1, p-2-x_2, x_3+1, p-2-x_4) \in \mathcal{RD}(x_0, \cdots, x_4)$ within $\{4, 5\} = \{4, 0\}$ is $(x_0, x_1, p-2-x_2, p-3-x_3, x_4+1)$ whereas its negative is $(p-2-x_0, p-3-x_1, x_2+1, p-2-x_3, x_4+1)$.

Lemma 12.2. *Let $\lambda, \lambda', \lambda'' \in \mathcal{RD}(x_0, \cdots, x_{f-1})$. Assume that λ' is the negative of λ within $\mathcal{S}'$, λ'' is the negative of λ within $\mathcal{S}''$ and $\mathcal{S}'' \subseteq \mathcal{S}'$. Then $\lambda'' \leq \lambda \cup \lambda'$ (see §11 for $\leq$ and $\cup$).*

Proof. Let $\mathcal{S}$ (resp. $\mathcal{T}'$, resp. $\mathcal{T}''$) correspond to λ (resp. $\lambda \cup \lambda'$, resp. $\lambda \cap \lambda''$), then we have $\mathcal{T}' = \mathcal{S} \cup \mathcal{S}'$ and $\mathcal{T}'' = \mathcal{S} \cup \mathcal{S}''$. Thus $\mathcal{T}'' \subseteq \mathcal{T}'$, hence $\lambda \cup \lambda'' \leq \lambda \cup \lambda'$. As $\lambda'' \leq \lambda \cup \lambda''$, we have $\lambda'' \leq \lambda \cup \lambda'$ by transitivity. □

Definition 12.3. *A sequence $\lambda = (\lambda_0(x_0), \cdots, \lambda_{f-1}(x_{f-1}))$ where $\lambda_i(x_i) \in \mathbb{Z} \pm x_i$ is called a weak formal reducible (resp. irreducible) Diamond weight if, for any i, one has $\lambda_i(x_i) \in \{x_i, x_i + 1, p - 2 - x_i, p - 3 - x_i\}$ (resp. $\lambda_i(x_i) \in \{x_i, x_i + 1, p - 2 - x_i, p - 3 - x_i\}$ for $i > 0$ and $\lambda_0(x_0) \in \{x_0, x_0 - 1, p - 2 - x_0, p - 1 - x_0\}$).*

That is, we don't require any condition on $\lambda_{i+1}(x_{i+1})$ with respect to $\lambda_i(x_i)$.

Lemma 12.4. *Let λ be a formal reducible (resp. irreducible) Diamond weight and $i \in \{0, \cdots, f-1\}$, $j \in \{1, \cdots, f-1\}$.*

(i) The sequence:

$$\big(\lambda_0(x_0), \cdots, \lambda_{i-1}(x_{i-1}), p - 2 - \lambda_i(x_i), p - 2 - \lambda_{i+1}(x_{i+1}) - \pm 1, \cdots, \\ p - 2 - \lambda_{i+j-1}(x_{i+j-1}) - \pm 1, \lambda_{i+j}(x_{i+j}) \pm 1, \\ \lambda_{i+j+1}(x_{i+j+1}), \cdots, \lambda_{f-1}(x_{f-1})\big) \quad (21)$$

(with the convention $i + \delta = i + \delta - f$ if $i + \delta \geq f$) is a weak formal reducible (resp. irreducible) Diamond weight if and only if it is a formal reducible (resp. irreducible) Diamond weight. If so, (21) is then the negative of λ within $\{i + 1, \cdots, i + j\}$.

(ii) The sequence:

$$\big(p - 2 - \lambda_0(x_0) - \pm 1, \cdots, p - 2 - \lambda_{f-1}(x_{f-1}) - \pm 1\big) \quad (22)$$

is a weak formal reducible (resp. irreducible) Diamond weight if and only if it is a formal reducible (resp. irreducible) Diamond weight. If so, (21) is then the negative of λ.

Proof. (i) This is easy combinatorics, so let us briefly prove only the case $\lambda \in \mathcal{RD}(x_0, \cdots, x_{f-1})$. Assume (21) is a weak formal reducible Diamond weight. Say that an index $\delta \in \{1, \cdots, j\}$ is of type $+$ if $p - 2 - \lambda_{i+\delta}(x_{i+\delta}) - (+1)$ or $\lambda_{i+\delta}(x_{i+\delta}) + 1$ occurs in (21) and of type $-$ if $p - 2 - \lambda_{i+\delta}(x_{i+\delta}) - (-1)$ or $\lambda_{i+\delta}(x_{i+\delta}) - 1$ occurs. Then we necessarily have $\lambda_{i+\delta}(x_{i+\delta}) \in \{p - 3 - x_{i+\delta}, x_{i+\delta}\}$ if δ is of type $+$ and $\lambda_{i+\delta}(x_{i+\delta}) \in \{p - 2 - x_{i+\delta}, x_{i+\delta} + 1\}$ if δ is of type $-$ (the other possibilities don't satisfy the conditions of Definition 12.3). Moreover, as $\lambda \in \mathcal{RD}(x_0, \cdots, x_{f-1})$, it turns out there are only 4 possibilities for the sequence $\lambda_i(x_i), \cdots, \lambda_{i+j}(x_{i+j})$, according to whether $\lambda_i(x_i) = x_i$, $x_i + 1$, $p-2-x_i$ or $p-3-x_i$, if we want (21) to be a weak formal reducible Diamond weight, that is, if we want $\lambda_{i+\delta}(x_{i+\delta}) \in \{p-3-x_{i+\delta}, x_{i+\delta}\}$ for δ of type $+$ and $\lambda_{i+\delta}(x_{i+\delta}) \in \{p-2-x_{i+\delta}, x_{i+\delta}+1\}$ for δ of type $-$. For instance, if $\lambda_i(x_i) = p-2-x_i$, then $\lambda_{i+\delta}(x_{i+\delta})$ must be $x_{i+\delta}+1$ for the first index of type $-$, $p-2-x_{i+\delta}$ for the second, $x_{i+\delta}+1$ for the third etc. Also, $\lambda_{i+\delta}(x_{i+\delta})$ must be $p-3-x_{i+\delta}$ for all indices of type $+$ strictly before the first index of type $-$, $x_{i+\delta}$ for all indices of type $+$ strictly between the first and second of type $-$, etc. One can check that in all cases, the resulting weak formal reducible Diamond weight (21) is always obtained from the original λ as follows: in λ replace $\lambda_i(x_i)$ by $p-2-\lambda_i(x_i)$, $\lambda_{i+j}(x_{i+j})$ by $\lambda_{i+j}(x_{i+j})+1$ if $\lambda_{i+j}(x_{i+j}) \in \{x_{i+j}, p-3-x_{i+j}\}$, by $\lambda_{i+j}(x_{i+j})-1$ if $\lambda_{i+j}(x_{i+j}) \in \{x_{i+j}+1, p-2-x_{i+j}\}$, and for $\delta \in \{1, \cdots, j-1\}$ each $p-2-x_{i+\delta}$ by $x_{i+\delta}+1$, each $p-3-x_{i+\delta}$ by $x_{i+\delta}$, each $x_{i+\delta}+1$ by $p-2-x_{i+\delta}$ and each $x_{i+\delta}$ by $p-3-x_{i+\delta}$. This weight is exactly the negative of λ within $\{i+1, \cdots, i+j\}$, and is a fortiori in $\mathcal{RD}(x_0, \cdots, x_{f-1})$. The proof of (ii) is analogous and left to the reader. $\square$

One defines the support of a sequence $p-2-\cdot, p-2-\cdot-\pm 1, \cdots, p-2-\cdot-\pm 1, \cdot \pm 1$ as in (21) to be the subset of indices $\{i+1, \cdots, i+j\}$ (with as usual the identification $i+\delta = i+\delta-f$ if $i+\delta \geq f$). One defines the support of the sequence (22) to be $\{0, \cdots, f-1\}$.

Let $\mu, \mu' \in \mathcal{I}(y_0, \cdots, y_{f-1})$ (see §3) and assume μ and μ' are compatible (Definition 4.10). We define $\mu \cap \mu' := \big((\mu \cap \mu')_0(y_0), \cdots, (\mu \cap \mu')_{f-1}(y_{f-1})\big)$ as follows:

(i) if $\mu_i(y_i) = y_i$, $(\mu \cap \mu')_i(y_i) := y_i$

(ii) if $\mu_i(y_i) \in \{p-1-y_i, p-3-y_i\}$, $(\mu \cap \mu')_i(y_i) := \mu'_i(y_i)$

(iii) if $\mu_i(y_i) \in \{y_i-1, y_i+1\}$, $(\mu\cap\mu')_i(y_i) := \begin{cases} \mu_i(y_i) \text{ if } \mu'_i(y_i) \neq \begin{cases} p-2-y_i \\ y_i \end{cases} \\ y_i \quad \text{if } \mu'_i(y_i) = \begin{cases} p-2-y_i \\ y_i \end{cases} \end{cases}$

(iv) if $\mu_i(y_i) = p-2-y_i$, $(\mu \cap \mu')_i(y_i) := \begin{cases} \mu_i(y_i) \text{ if } \mu'_i(y_i) \neq \begin{cases} y_i \pm 1 \\ y_i \end{cases} \\ y_i \quad \text{if } \mu'_i(y_i) = \begin{cases} y_i \pm 1 \\ y_i. \end{cases} \end{cases}$

We have $\mu \cap \mu' = \mu' \cap \mu$. Intuitively, we just take the "intersection" of the sequences $p-2-\cdot, p-2-\cdot-\pm 1, \cdots, p-2-\cdot-\pm 1, \cdot \pm 1$ on both μ and μ'. Recall from §4 that $\mathcal{S}(\mu) := \{i \in \{0, \cdots, f-1\}, \mu_i(x_i) = p-2-x_i-\pm 1, x_i \pm 1\}$ for $\mu \in \mathcal{I}(y_0, \cdots, y_{f-1})$. We leave the (straightforward) proof of the next lemma to the reader:

Lemma 12.5. *We have $\mu \cap \mu' \in \mathcal{I}(y_0, \cdots, y_{f-1})$ and $\mathcal{S}(\mu \cap \mu') = \mathcal{S}(\mu) \cap \mathcal{S}(\mu')$.*

We denote by $\mathcal{D}(x_0, \cdots, x_{f-1}) \subseteq \mathcal{RD}(x_0, \cdots, x_{f-1})$ an arbitrary subset of Galois type (see Definition 11.5).

Lemma 12.6. *Let $\mu, \mu' \in \mathcal{I}(y_0, \cdots, y_{f-1})$ and assume there is $x_i \in \mathbb{Z} \pm y_i$ such that both μ and μ' are in $\mathcal{RD}(x_0, \cdots, x_{f-1})$ (resp. $\mathcal{D}(x_0, \cdots, x_{f-1})$, resp. $\mathcal{ID}(x_0, \cdots, x_{f-1})$). Then μ, μ' are compatible and $\mu \cap \mu' \in \mathcal{RD}(x_0, \cdots, x_{f-1})$ (resp. $\mathcal{D}(x_0, \cdots, x_{f-1})$, resp. $\mathcal{ID}(x_0, \cdots, x_{f-1})$).*

Proof. Beware that $\mu \cap \mu'$ is computed in $\mathcal{I}(y_0, \cdots, y_{f-1})$. First, it is easy to check that μ, μ' are compatible. For instance, assume $\mu_i(y_i) = p-1-y_i$ and $\mu'_i(y_i) = y_i+1$, then $\mu_i(y_i) = p-\mu'_i(y_i)$. But $\mu_i(y_i), \mu'_i(y_i) \in \{x_i, x_i+1, p-3-x_i, p-2-x_i\}$ (resp. if $i > 0$ and $\mu_0(y_0), \mu'_0(y_0) \in \{x_0-1, x_0, p-2-x_0, p-1-x_0\}$ in the case $\mathcal{ID}$) and it is thus impossible to have $\mu_i(y_i) = p - \mu'_i(y_i)$. The other cases are similar. From the very definition of $\mathcal{I}(y_0, \cdots, y_{f-1})$ in §3, one sees one can pass from μ to $\mu \cap \mu'$ by applying successively to μ several sequences as in (21) such that the successive sets of indices that are affected are disjoint, or one sequence as in (22). Denote by $\mathcal{S}''$ the union of the supports of these sequences. From the definition of $\mu \cap \mu'$, one has either $(\mu \cap \mu')_i(y_i) = \mu_i(y_i)$ or $(\mu \cap \mu')_i(y_i) = \mu'_i(y_i)$ or $(\mu \cap \mu')_i(y_i) = p-2-\mu_i(y_i)$. In particular $\mu \cap \mu'$ is always a weak formal reducible (resp. irreducible) weight. Lemma 12.4 then gives that $\mu \cap \mu' \in \mathcal{I}(y_0, \cdots, y_{f-1})$ is an element of $\mathcal{RD}(x_0, \cdots, x_{f-1})$ (resp. $\mathcal{ID}(x_0, \cdots, x_{f-1})$) and is the negative of μ (seen as an element of $\mathcal{RD}(x_0, \cdots, x_{f-1})$ (resp. $\mathcal{ID}(x_0, \cdots, x_{f-1})$)) within $\mathcal{S}''$. This proves the cases $\mathcal{RD}(x_0, \cdots, x_{f-1})$ and $\mathcal{ID}(x_0, \cdots, x_{f-1})$. For the case $\mathcal{D}(x_0, \cdots, x_{f-1})$, note first that $\mu \cup \mu' \in \mathcal{D}(x_0, \cdots, x_{f-1})$ as $\mathcal{D}(x_0, \cdots, x_{f-1})$ is of Galois type (here, μ and μ' are considered as elements of $\mathcal{D}(x_0, \cdots, x_{f-1})$ and $\mu \cup \mu'$ is computed in $\mathcal{RD}(x_0, \cdots, x_{f-1})$, see §11). As before, one can pass from μ to μ' by applying successively to μ several sequences as in (21), or one sequence as in (22). As μ, μ' are compatible, one can take these sequences such that the successive sets of indices that are affected are disjoint, so that these sequences are uniquely determined. By Lemma 12.4, this implies μ' is the negative of μ (seen as an element of $\mathcal{RD}(x_0, \cdots, x_{f-1})$) within the support $\mathcal{S}'$ of these sequences. But the previous support $\mathcal{S}''$ is always included in $\mathcal{S}'$ by construction. Lemma 12.2 then yields $\mu \cap \mu' \leq \mu \cup \mu'$, which implies $\mu \cap \mu' \in \mathcal{D}(x_0, \cdots, x_{f-1})$ by (i) of Definition 11.5 applied to $\mu \cup \mu'$ (beware that $\mu \cap \mu'$ is computed in $\mathcal{I}(y_0, \cdots, y_{f-1})$ whereas $\mu \cup \mu'$ is computed in $\mathcal{RD}(x_0, \cdots, x_{f-1})$). This finishes the proof. □

Let σ and τ be two weights and assume $\sigma = (r_0, \cdots, r_{f-1}) \otimes \eta$ with $0 \leq r_i \leq p-2$ for all i. Assume there exist indecomposable Γ-representations with socle σ and cosocle τ. By Corollary 3.12, there is a unique such representation, call it $I(\sigma, \tau)$, such that σ appears in $I(\sigma, \tau)$ with multiplicity 1. Moreover, all of the Jordan-Hölder factors of $I(\sigma, \tau)$ are then distinct. If there is no such representation, set $I(\sigma, \tau) := 0$. For any $I(\sigma, \tau)$, set $\ell(\sigma, \tau) \in \mathbb{Z}_{>0} \cup \{+\infty\}$ to be $+\infty$ if $I(\sigma, \tau) = 0$ and otherwise the smallest integer such that $I(\sigma, \tau)_{\ell(\sigma,\tau)} = 0$.

The following lemma will be used in §14.

Lemma 12.7. *Let $\mu, \mu' \in \mathcal{I}(y_0, \cdots, y_{f-1})$ and assume μ and μ' are compatible. Let $\sigma = (r_0, \cdots, r_{f-1}) \otimes \eta$ with $0 \leq r_i \leq p-2$ for all i. Let τ, τ' and τ'' be irreducible subquotients of* $\operatorname{inj} \sigma$ *corresponding to μ, μ' and $\mu \cap \mu'$ respectively via Lemma 3.2. Then κ is an irreducible subquotient of $I(\sigma, \tau)$ and $I(\sigma, \tau')$ if and only if κ is an irreducible subquotient $I(\sigma, \tau'')$.*

Proof. Let κ be an irreducible subquotient of $\operatorname{inj} \sigma$ and let $\lambda \in \mathcal{I}(y_0, \cdots, y_{f-1})$ correspond to κ via Lemma 3.2. Corollary 4.11 implies that κ is a subquotient of $I(\sigma, \tau)$ and $I(\sigma, \tau')$ if and only if λ is compatible with μ and μ' and $\mathcal{S}(\lambda) \subseteq \mathcal{S}(\mu) \cap \mathcal{S}(\mu')$. It is immediate from Definition 4.10 and Lemma 12.5 that this is equivalent to λ is compatible with $\mu \cap \mu'$ and $\mathcal{S}(\lambda) \subseteq \mathcal{S}(\mu \cap \mu')$. Again by Corollary 4.11 this is equivalent to κ is an irreducible subquotient of $I(\sigma, \tau'')$. □

Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^f}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous generic representation as in Definition 11.7 and recall that $\mathcal{D}(\rho)$ is the set of weights associated to ρ (see §11). It is straightforward to check from the definitions of §11 that any $\sigma \in \mathcal{D}(\rho)$ is such that $\sigma = (s_0, \cdots, s_{f-1}) \otimes \eta$ with $0 \leq s_i \leq p-2$ for all i and not all s_i equal to 0. For a weight τ, define:

$$\ell(\rho, \tau) := \min\{\ell(\sigma, \tau),\ \sigma \in \mathcal{D}(\rho)\} \in \mathbb{Z}_{>0} \cup \{+\infty\}.$$

We can now prove the main result of this section:

Lemma 12.8. *Let τ be any weight such that $\ell(\rho, \tau) < +\infty$.*

(i) There is a unique $\sigma \in \mathcal{D}(\rho)$ such that $\ell(\sigma, \tau) = \ell(\rho, \tau)$.

(ii) Let $\sigma' \in \mathcal{D}(\rho)$ such that $I(\sigma', \tau) \neq 0$. If $\sigma' = \sigma$ with σ as in (i), then $I(\sigma', \tau)$ has no other weight of $\mathcal{D}(\rho)$ distinct from σ' in its Jordan-Hölder factors. If $\sigma' \neq \sigma$ with σ as in (i), then $I(\sigma', \tau)$ contains σ in its Jordan-Hölder factors.

Proof. We start with (i). Write $\tau = (s_0, \cdots, s_{f-1}) \otimes \eta$ and assume there are two distinct weights $\sigma, \sigma' \in \mathcal{D}(\rho)$ such that $\ell(\sigma, \tau) = \ell(\sigma', \tau) = \ell(\rho, \tau)$. By Lemma 3.2, there are distinct $\mu, \mu' \in \mathcal{I}(y_0, \cdots, y_{f-1})$ uniquely determined such that:

$$\begin{aligned}\sigma &= (\mu_0(s_0), \cdots, \mu_{f-1}(s_{f-1})) \otimes \det{}^{e(\mu)(s_0,\cdots,s_{f-1})}\eta \\ \sigma' &= (\mu'_0(s_0), \cdots, \mu'_{f-1}(s_{f-1})) \otimes \det{}^{e(\mu')(s_0,\cdots,s_{f-1})}\eta.\end{aligned}$$

From §11, there are distinct $\lambda, \lambda' \in \mathcal{RD}(x_0, \cdots, x_{f-1})$ or $\mathcal{D}(x_0, \cdots, x_{f-1})$ or $\mathcal{ID}(x_0, \cdots, x_{f-1})$ (according to ρ, $\mathcal{D}(x_0, \cdots, x_{f-1})$ being of Galois type if ρ is reducible non-split) uniquely determined such that:

$$\begin{aligned}\sigma &= (\lambda_0(r_0), \cdots, \lambda_{f-1}(r_{f-1})) \otimes \det{}^{e(\lambda)(r_0,\cdots,r_{f-1})} \\ \sigma' &= (\lambda'_0(r_0), \cdots, \lambda'_{f-1}(r_{f-1})) \otimes \det{}^{e(\lambda')(r_0,\cdots,r_{f-1})}.\end{aligned}$$

We claim that we have the identities for each $i \in \{0, \cdots, f-1\}$:

$$\mu_i^{-1}(\lambda_i(x_i)) = \mu_i'^{-1}(\lambda'_i(x_i)). \tag{23}$$

Indeed, we have the equalities for all i:

$$\mu_i^{-1}(\lambda_i(r_i)) = \mu_i'^{-1}(\lambda'_i(r_i)) \tag{24}$$

and:

$$\det{}^{e(\mu)(s_0,\cdots,s_{f-1})-e(\lambda)(r_0,\cdots,r_{f-1})} = \det{}^{e(\mu')(s_0,\cdots,s_{f-1})-e(\lambda')(r_0,\cdots,r_{f-1})}$$

which, by an easy calculation, amounts to:

$$\det{}^{-e(\mu^{-1})(\lambda_0(r_0),\cdots,\lambda_{f-1}(r_{f-1}))-e(\lambda)(r_0,\cdots,r_{f-1})} = \det{}^{-e(\mu'^{-1})(\lambda'_0(r_0),\cdots,\lambda'_{f-1}(r_{f-1}))-e(\lambda')(r_0,\cdots,r_{f-1})}$$

which is again equivalent to:

$$\det{}^{-e(\mu^{-1}\circ\lambda)(r_0,\cdots,r_{f-1})} = \det{}^{-e(\mu'^{-1}\circ\lambda')(r_0,\cdots,r_{f-1})}. \tag{25}$$

Here, $\mu^{-1} := (\mu_i^{-1}(y_i))$ (where $(\mu_i^{-1}\circ\mu_i)(y_i) = y_i$) and $\mu^{-1}\circ\lambda := (\mu_i^{-1}(\lambda_i(y_i)))$. We now leave as an exercise to the reader to check that the equalities (23) are equivalent to the equalities (24) and (25) (this is analogous to proving that all weights in $\mathrm{Ind}_B^\Gamma \chi$ or all Diamond weights are distinct, cf. e.g. Lemma 2.2). We can apply Lemma 12.6 to μ and μ' with $x_i = \lambda_i^{-1}(\mu_i(y_i)) \in \mathbb{Z} \pm y_i$ to deduce that $\mu \cap \mu' \in \mathcal{RD}(x_0, \cdots, x_{f-1})$ or $\mathcal{D}(x_0, \cdots, x_{f-1})$ or $\mathcal{ID}(x_0, \cdots, x_{f-1})$. In particular, one has:

$$\sigma'' := ((\mu \cap \mu')_0(s_0), \cdots, (\mu \cap \mu')_{f-1}(s_{f-1})) \otimes \det{}^{e(\mu\cap\mu')(s_0,\cdots,s_{f-1})}\eta \in \mathcal{D}(\rho). \tag{26}$$

But as μ and μ' are distinct and as $\ell(\sigma,\tau)=\ell(\sigma',\tau)$, Corollary 4.11 together with the second part of Lemma 12.5 imply:

$$\ell(\sigma'',\tau)<\ell(\sigma,\tau)$$

which is impossible as $\ell(\sigma,\tau)$ is minimal. This proves (i). Let us prove (ii). If $\sigma'=\sigma$, there can't be any other Diamond weight in $I(\sigma',\tau)$ as this would contradict the minimality of $\ell(\sigma,\tau)$. Assume $\sigma'\neq\sigma$, and let μ', μ as for (i). Then σ'' defined as in (26) is again in $\mathcal{D}(\rho)$ (the equality $\ell(\sigma,\tau)=\ell(\sigma',\tau)$ was not used here). Moreover, we have that μ' and $\mu\cap\mu'$ are compatible by construction and that $\mathcal{S}(\mu\cap\mu')\subseteq\mathcal{S}(\mu')$ by Lemma 12.5. By Corollary 4.11 (used "backwards"), we get that σ'' is a Jordan-Hölder factor in $I(\sigma',\tau)$. The same argument with μ and σ yields that σ'' is also a Jordan-Hölder factor in $I(\sigma,\tau)$, hence $\sigma''=\sigma\neq\sigma'$ by (i). This finishes the proof. □

Remark 12.9. If $\mathcal{D}'(x_0,\cdots,x_{f-1})\subseteq\mathcal{ID}(x_0,\cdots,x_{f-1})$ is a non-empty subset satisfying conditions (i) and (ii) of Definition 11.5, it is easily checked that the proof of Lemma 12.6 goes through with $\mathcal{D}'(x_0,\cdots,x_{f-1})$ instead of $\mathcal{ID}(x_0,\cdots,x_{f-1})$. Fix a ρ generic, let ρ^{ss} be its semi-simplification and let $\mathcal{D}'(\rho^{\mathrm{ss}})\subseteq\mathcal{D}(\rho^{\mathrm{ss}})$ be any subset of weights coming from a non-empty subset $\mathcal{D}'(x_0,\cdots,x_{f-1})$ of $\mathcal{RD}(x_0,\cdots,x_{f-1})$ (resp. $\mathcal{ID}(x_0,\cdots,x_{f-1})$) satisfying (i) and (ii) of Definition 11.5, using the above extension of Lemma 12.6 in the proof of Lemma 12.8, we see that Lemma 12.8 holds with $\mathcal{D}'(\rho^{\mathrm{ss}})$ instead of $\mathcal{D}(\rho)$. In particular, the function $\sigma'\mapsto\ell(\sigma',\tau)$ reaches its minimum on $\mathcal{D}'(\rho^{\mathrm{ss}})$ for a unique weight σ of $\mathcal{D}'(\rho^{\mathrm{ss}})$.

13 Generic Diamond diagrams

We associate to each generic ρ as in Definition 11.7 a "family" of basic 0-diagrams as in §9. When $f>1$, this family is always infinite.

We start with a general proposition:

Proposition 13.1. *Let $\mathcal{D}$ be a finite set of distinct weights. Then there exists a unique (up to isomorphism) finite dimensional smooth representation D_0 of Γ over $\overline{\mathbb{F}}_p$ such that:*

(i) $\mathrm{soc}_\Gamma D_0=\bigoplus_{\sigma\in\mathcal{D}}\sigma$

(ii) any weight of $\mathcal{D}$ appears at most once (as a subquotient) in D_0

(iii) D_0 is maximal with respect to properties (i), (ii).

Moreover, one has an isomorphism of Γ*-representations:*

$$D_0 \cong \bigoplus_{\sigma \in \mathcal{D}} D_{0,\sigma}$$

where $\mathrm{soc}_\Gamma D_{0,\sigma} \cong \sigma$.

Proof. Note first that condition (iii) means that, if D_0' is any finite dimensional representation of Γ over $\overline{\mathbb{F}}_p$ that strictly contains D_0 as a subrepresentation, then (ii) is not satisfied for D_0'. Set $\tau := \oplus_{\sigma\in\mathcal{D}}\sigma$ and let τ' be a representation of Γ satisfying (i). Then τ' satisfies (ii) if and only if $\mathrm{Hom}_\Gamma(\tau'/\tau, \mathrm{inj}\,\sigma) = 0$ for all $\sigma \in \mathcal{D}$. Since $\mathrm{inj}\,\sigma$ is an injective object in Rep_Γ, we have an exact sequence of Γ-representations:

$$0 \to \mathrm{Hom}_\Gamma(\tau'/\tau, \mathrm{inj}\,\sigma) \to \mathrm{Hom}_\Gamma(\tau', \mathrm{inj}\,\sigma) \to \mathrm{Hom}_\Gamma(\tau, \mathrm{inj}\,\sigma) \to 0$$

and hence τ' satisfies (ii) if and only if:

$$\dim_{\overline{\mathbb{F}}_p} \mathrm{Hom}_\Gamma(\tau', \mathrm{inj}\,\sigma) = 1, \quad \forall \sigma \in \mathcal{D}.$$

We fix an injective envelope $\mathrm{inj}\,\tau$ of τ in Rep_Γ. Let τ_1 and τ_2 be two Γ-invariant subspaces of $\mathrm{inj}\,\tau$ containing τ and satisfying (ii). Since $\mathrm{inj}\,\sigma$ is injective the sequence:

$$0 \to \mathrm{Hom}_\Gamma(\tau_1 + \tau_2, \mathrm{inj}\,\sigma) \to \mathrm{Hom}_\Gamma(\tau_1 \oplus \tau_2, \mathrm{inj}\,\sigma) \to \mathrm{Hom}_\Gamma(\tau_1 \cap \tau_2, \mathrm{inj}\,\sigma) \to 0$$

is exact. Since $\tau_1 + \tau_2$ and $\tau_1 \cap \tau_2$ both contain τ as a subobject, we get that $\mathrm{Hom}_\Gamma(\sigma, \tau_1 + \tau_2)$ and $\mathrm{Hom}_\Gamma(\sigma, \tau_1 \cap \tau_2)$ are non-zero. Hence the terms on the left and right are non-zero as $\mathrm{inj}\,\sigma$ is an injective object. Moreover, since the term in the middle has dimension 2, we obtain $\dim_{\overline{\mathbb{F}}_p} \mathrm{Hom}_\Gamma(\tau_1 + \tau_2, \mathrm{inj}\,\sigma) = 1$. Hence there exists a maximal subspace $\tau_{\max}$ of $\mathrm{inj}\,\tau$ satisfying (i) and (ii). Since any representation τ' of Γ with $\mathrm{soc}_\Gamma \tau' \cong \tau$ can be embedded in $\mathrm{inj}\,\tau$, we obtain the first part of the proposition. Since τ is multiplicity free for $\sigma \in \mathcal{D}$, there exists a unique idempotent $e_\sigma \in \mathrm{End}_\Gamma(\mathrm{inj}\,\tau)$ such that $e_\sigma(\mathrm{inj}\,\tau)$ is an injective envelope of σ. Since $\oplus_{\sigma\in\mathcal{D}} e_\sigma(\tau_{\max})$ satisfies (i) and (ii), the natural injection:

$$\tau_{\max} \hookrightarrow \oplus_{\sigma\in\mathcal{D}} e_\sigma(\tau_{\max})$$

has to be an isomorphism. ∎

We leave the proof of the following immediate corollary to the reader:

Corollary 13.2. *Let* $\mathcal{D}$ *and* D_0 *be as above, then* $\mathrm{End}_\Gamma(D_0) \cong \overline{\mathbb{F}}_p^{|\mathcal{D}|}$.

Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^f}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous generic representation as in Definition 11.7 and $\mathcal{D}(\rho)$ its set of Diamond weights (all distinct, see §11). We denote by $D_0(\rho)$ the unique representation of Proposition 13.1 with $\mathcal{D} = \mathcal{D}(\rho)$. We now start studying the Γ-representation $D_0(\rho)$.

If $\ell(\rho, \tau) < +\infty$, set:

$$I(\rho, \tau) := I(\sigma, \tau)$$

where $\sigma \in \mathcal{D}(\rho)$ is the unique Diamond weight of Lemma 12.8.

Lemma 13.3. *We have* $\mathrm{Hom}_\Gamma(I(\rho, \tau), I(\rho, \tau')) = 0$ *or* $\overline{\mathbb{F}}_p$.

Proof. Let $f : I(\rho, \tau) \to I(\rho, \tau')$ be non-zero (if it exists) and denote by σ (resp. σ') the Γ-socle of $I(\rho, \tau)$ (resp. $I(\rho, \tau')$). We first prove that f is injective. Otherwise, we have $f(\sigma) = 0$ and the Γ-socle of $I(\rho, \tau')$ must contain a Jordan-Hölder factor of $I(\rho, \tau)$ that is different from σ. But by (ii) of Lemma 12.8 this Jordan-Hölder factor can't be in $\mathcal{D}(\rho)$, hence can't either be in the socle of $I(\rho, \tau')$ by definition of $I(\rho, \tau')$. Thus f is injective and $f(\sigma) = \sigma = \sigma'$. Let $f' : I(\rho, \tau) \to I(\rho, \tau')$ be any Γ-equivariant map. If f' is non-zero, we again have $f'(\sigma) = \sigma$. As σ is irreducible, there is $\lambda \in \overline{\mathbb{F}}_p$ such that $f - \lambda f'$ is zero on σ. But $f - \lambda f' \in \mathrm{Hom}_\Gamma(I(\rho, \tau), I(\rho, \tau'))$ and the same proof as for f gives then $f = \lambda f'$. □

In particular, $I(\rho, \tau)$ is well defined up to scalar automorphism. Note that any subrepresentation of $I(\rho, \tau)$ with an irreducible cosocle τ' is automatically isomorphic to $I(\rho, \tau')$ (this follows from the definition of the representations $I(\sigma, \tau)$ in §12). We define a partial order on the representations $I(\rho, \tau)$ as follows:

$$I(\rho, \tau) \leq I(\rho, \tau') \Longleftrightarrow \mathrm{Hom}_\Gamma(I(\rho, \tau), I(\rho, \tau')) = \overline{\mathbb{F}}_p.$$

For each τ such that $I(\rho, \tau) \neq 0$, fix an embedding $\iota_\tau : \sigma \hookrightarrow I(\rho, \tau)$. If $\mathrm{Hom}_\Gamma(I(\rho, \tau), I(\rho, \tau')) \neq 0$, let $\phi_{\tau,\tau'} : I(\rho, \tau) \hookrightarrow I(\rho, \tau')$ be the unique embedding such that $\iota_{\tau'} = \phi_{\tau,\tau'} \circ \iota_\tau$.

Proposition 13.4. *With the previous notations, we have:*

$$D_0(\rho) = \oplus_{\sigma \in \mathcal{D}(\rho)} D_{0,\sigma}(\rho) \tag{27}$$

with:

$$D_{0,\sigma}(\rho) = \varinjlim_{\leq} I(\rho, \tau),$$

the inductive limit being taken with respect to the above maps $\phi_{\tau,\tau'} : I(\rho, \tau) \to I(\rho, \tau')$.

Proof. The first part is contained in Proposition 13.1, we are thus left to proving $D_{0,\sigma}(\rho) = \varinjlim I(\rho, \tau)$. Note that the inductive limit is not direct. The representation $\varinjlim I(\rho, \tau)$ has σ as socle and doesn't contain any other weight of $\mathcal{D}(\rho)$ by (ii) of Lemma 12.8. From the proof of Proposition 13.1, we thus have $\varinjlim I(\rho, \tau) \subseteq D_0(\rho)$, that is $\varinjlim I(\rho, \tau) \subseteq D_{0,\sigma}(\rho)$. Let τ be any irreducible subquotient of $D_{0,\sigma}(\rho)$ and $D_{0,\sigma}(\rho, \tau) \subseteq D_{0,\sigma}(\rho)$ a subrepresentation with cosocle τ. By Corollary 3.12, we have $D_{0,\sigma}(\rho, \tau) \simeq I(\sigma, \tau)$ and by (ii) of Lemma 12.8 (together with (ii) of Proposition 13.1), we have $I(\sigma, \tau) = I(\rho, \tau)$. This implies the surjectivity of $\varinjlim I(\rho, \tau) \to D_{0,\sigma}(\rho)$. □

Note that $D_0(\rho)$ only depends on the restriction of ρ to inertia.

Corollary 13.5. *The Γ-representation $D_0(\rho)$ is multiplicity free.*

Proof. This follows from Proposition 13.4, Corollary 3.12 and the definition of $I(\rho, \tau)$. □

So we see that, although $D_0(\rho)$ is defined so that it only satisfies multiplicity 1 for its socle, it indeed satisfies multiplicity 1 for all of its factors.

Corollary 13.6. *There exists a unique partition of the B-eigencharacters of $D_0(\rho)^U$ in pairs of eigencharacters $\{\chi, \chi^s\}$ with $\chi \neq \chi^s$.*

Proof. Unicity is clear from Corollary 13.5. Let $v \in D_0(\rho)^U$ such that B acts on v via some character χ. We have:

$$\operatorname{Ind}_B^\Gamma \chi \twoheadrightarrow \langle \Gamma \cdot v \rangle \hookrightarrow D_0(\rho).$$

Hence, there is a quotient of $\operatorname{Ind}_B^\Gamma \chi$ that injects into $D_0(\rho)$, which implies (looking at socles) that a weight σ' of $\mathcal{D}(\rho)$ must appear in $\operatorname{Ind}_B^\Gamma \chi$. This rules out $\chi = \chi^s$ as, from the assumption ρ generic, we know that $\mathcal{D}(\rho)$ doesn't contain a character nor a twist of $(p-1, \cdots, p-1)$. The weight σ' also appears in $\operatorname{Ind}_B^\Gamma \chi^s$ by Lemma 2.2, which implies that a quotient of $\operatorname{Ind}_B^\Gamma \chi^s$ is isomorphic to $I(\sigma', \sigma_{\chi^s}) \neq 0$ where σ_{χ^s} is the cosocle of $\operatorname{Ind}_B^\Gamma \chi^s$. By (ii) of Lemma 12.8, $I(\rho, \sigma_{\chi^s})$ is a non-zero quotient of $I(\sigma', \sigma_{\chi^s})$, and hence also of $\operatorname{Ind}_B^\Gamma \chi^s$. As $I(\rho, \sigma_{\chi^s}) \subseteq D_0(\rho)$ by Proposition 13.4, χ^s is an eigencharacter of B acting on $I(\rho, \sigma_{\chi^s})^U \subseteq D_0(\rho)^U$. □

We now fix $\varpi = p$ and recall from §9 that a basic 0-diagram (D_0, D_1, r) satisfies $r : D_1 \simeq D_0^{I_1} \subseteq D_0$. We also recall that the scalar matrix $\left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)$ acts now trivially everywhere.

Definition 13.7. *A family of basic* 0*-diagrams is a pair* $(D_0, \{\ \})$ *where* D_0 *is a smooth finite dimensional representation of* $\mathfrak{K}_0$ *which is trivial on* K_1 *and* $\{\ \}$ *is a partition of a basis of eigencharacters of* IZ *acting on* $D_0^{I_1}$ *in pairs of eigencharacters* $\{\chi, \chi^s\}$*.*

To a family of basic 0-diagrams as in Definition 13.7, one can attach a genuine "family" of basic 0-diagrams $((D_0, D_1, r))_r$ by making Π act on $D_1 := D_0^{I_1}$ as $\Pi v_\chi = v_{\chi^s}$, $\Pi v_{\chi^s} = v_\chi$ where v_χ, v_{χ^s} are eigenvectors corresponding to a pair of eigencharacters $\{\chi, \chi^s\}$ in the partition and by letting $r : D_1 \hookrightarrow D_0$ be an arbitrary IZ-equivariant injection. Usually, there are infinitely many such injections up to isomorphism.

We can finally sum up the main results of this section. We still denote by $D_0(\rho)$ the $\mathfrak{K}_0$-representation deduced from the K-representation $D_0(\rho)$ by making p act trivially.

Theorem 13.8. *Let* $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^f}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ *be a continuous generic representation such that* p *acts trivially on its determinant. Then there exists a unique family of basic* 0*-diagrams* $(D_0(\rho), \{\ \})$ *such that:*

(i) $\mathrm{soc}_K\, D_0(\rho) = \bigoplus_{\sigma \in \mathcal{D}(\rho)} \sigma$

(ii) any weight of $\mathcal{D}(\rho)$ *appears at most once (as a subquotient) in* $D_0(\rho)$

(iii) $D_0(\rho)$ *is maximal with respect to properties (i), (ii).*

Moreover, $D_0(\rho)$ *is then multiplicity free.*

Note that the family $(D_0(\rho), \{\ \})$ only depends on the restriction of ρ to inertia.

Remark 13.9. In case ρ is not generic, there is still defined a set of Diamond weights $\mathcal{D}(\rho)$ (see [11]) and one can still define a Γ-representation $D_0(\rho)$ as in Proposition 13.1 with $\mathcal{D} = \mathcal{D}(\rho)$. However, in general, $D_0(\rho)$ is not multiplicity free.

14 The representations $D_0(\rho)$ and $D_1(\rho)$

For ρ generic we compute the dimension of $D_1(\rho)$. When ρ is moreover tame, we explicitly determine the Jordan-Hölder factors of $D_0(\rho)$.

We start with several lemmas.

Lemma 14.1. *Let ρ be generic. For $\chi : H \to \overline{\mathbb{F}}_p^\times$ let $m_\chi \in \mathbb{Z}_{\geq 0}$ such that $(\oplus_{\sigma\in\mathcal{D}(\rho)} \operatorname{inj}\sigma)^{I_1} \cong \oplus_\chi m_\chi \chi$. Then $D_0(\rho)^{I_1} \cong \bigoplus_{m_\chi>0} \chi$.*

Proof. If χ occurs in $(\operatorname{inj}\sigma)^{I_1}$ for some $\sigma \in \mathcal{D}(\rho)$ then $\chi \neq \chi^s$ as ρ is generic. Since $D_0(\rho)$ is multiplicity free by Theorem 13.8, every χ can occur in $D_0(\rho)^{I_1}$ with multiplicity at most 1. Let χ occur in $(\operatorname{inj}\sigma)^{I_1}$ for some $\sigma \in \mathcal{D}(\rho)$. Then σ is a subquotient of $\operatorname{Ind}_I^K \chi$ and there is a unique quotient τ of $\operatorname{Ind}_I^K \chi$ with socle σ (as $\operatorname{Ind}_I^K \chi$ is multiplicity free). If there exists a Jordan-Hölder factor $\sigma_1 \neq \sigma$ of τ such that $\sigma_1 \in \mathcal{D}(\rho)$, let τ_1 be the unique quotient of τ with socle σ_1. Starting again, we obtain like this a non-zero quotient τ_m of $\operatorname{Ind}_I^K \chi$ such that the socle σ_m of τ_m lies in $\mathcal{D}(\rho)$ and no other Jordan-Hölder factor of τ_m does. By maximality of $D_0(\rho)$ (see (iii) in Theorem 13.8), we have an injection $\tau_m \hookrightarrow D_0(\rho)$. Hence $\operatorname{Hom}_K(\operatorname{Ind}_I^K \chi, D_0(\rho)) \neq 0$ and so χ occurs in $D_0(\rho)^{I_1}$. □

Recall from §4 that Σ denotes the set of f-tuples $\boldsymbol{\varepsilon} := (\epsilon_0, \cdots, \epsilon_{f-1})$ with $\epsilon_i \in \{-1, 0, 1\}$ and $\Sigma' \subset \Sigma$ the subset of f-tuples $\boldsymbol{\varepsilon}$ with $\epsilon_i \in \{0, 1\}$. If $\mathbf{s} := (s_0, \cdots, s_{f-1})$ is an f-tuple of integers with $0 \leq s_i \leq p-2$, $\eta : F^\times \to \overline{\mathbb{F}}_p^\times$ is a smooth character and $\sigma := (s_0, \cdots, s_{f-1}) \otimes \eta$, recall that Σ' parametrizes in a natural way the characters of I acting on $(\operatorname{inj}\sigma)^{I_1}$ (see Proposition 4.13 and twist). If $\boldsymbol{\varepsilon} \in \Sigma'$, denote by $\sigma(\boldsymbol{\varepsilon})$ the unique twist of $V_{\mathbf{s}(\boldsymbol{\varepsilon})} \otimes \det^{e(\boldsymbol{\varepsilon})}$ which occurs as a subquotient of $\operatorname{inj}\sigma$ (see §4).

Lemma 14.2. *Let $\sigma := (s_0, \cdots, s_{f-1}) \otimes \eta$ be an irreducible representation of Γ with $0 \leq s_i \leq p-2$. Let $j \in \{0, \cdots, f-1\}$, $\sigma' := (s_0, \cdots, s_{j-1}, p-2-s_j, s_{j+1}+1, s_{j+2}, \cdots, s_{f-1}) \otimes \eta\det^{p^j(s_j+1)-p^{j+1}}$ and assume $s_{j+1}+1 \leq p-2$. Let $\boldsymbol{\varepsilon} \in \Sigma'$ then there exists $\boldsymbol{\varepsilon}' \in \Sigma'$ such that $\sigma(\boldsymbol{\varepsilon}) \cong \sigma'(\boldsymbol{\varepsilon}')$ if and only if one of the following holds:*

(i) $\epsilon_j = 1$ *and* $\epsilon_{j+1} = 0$

(ii) $\epsilon_j = 0$ *and* $\epsilon_{j+1} = 1$.

Moreover, if the above holds, then $\boldsymbol{\varepsilon}'$ is uniquely determined as follows: $\epsilon'_k = \epsilon_k$ for $k \notin \{j, j+1\}$, in case (i) $\epsilon'_j = 0$ and $\epsilon'_{j+1} = 0$ and in case (ii) $\epsilon'_j = 1$ and $\epsilon'_{j+1} = 1$.

Proof. If such an $\boldsymbol{\varepsilon}'$ exists then it is uniquely determined since all the representations in $\{\sigma'(\boldsymbol{\varepsilon}'), \boldsymbol{\varepsilon}' \in \Sigma'\}$ are distinct. If $\boldsymbol{\varepsilon}$ satisfies (i) or (ii) then one may check that, if $\boldsymbol{\varepsilon}'$ is as above, then $\sigma(\boldsymbol{\varepsilon}) \cong \sigma'(\boldsymbol{\varepsilon}')$. Conversely, if $\boldsymbol{\varepsilon}' \in \Sigma'$ is such that $(\epsilon'_j, \epsilon'_{j+1}) \neq (0, 0)$ and $(\epsilon'_j, \epsilon'_{j+1}) \neq (1, 1)$ then one of the digits of the f-tuple $\mathbf{s}'(\boldsymbol{\varepsilon}')$ corresponding to $\sigma'(\boldsymbol{\varepsilon}')$ will be either $s_k + 2$ or $p - 3 - s_k$, $k \in \{j, j+1\}$, which implies (with considerations of determinant) that there exists no $\boldsymbol{\varepsilon} \in \Sigma'$ such that $\sigma(\boldsymbol{\varepsilon}) \cong \sigma'(\boldsymbol{\varepsilon}')$. □

Remark 14.3. Switching σ and σ' in the above proof, we obtain an analogous result with the weights $(s_0, \cdots, s_{f-1}) \otimes \eta$ and $(s_0, \cdots, s_{j-1}, p-2-s_j, s_{j+1}-1, s_{j+2}, \cdots, s_{f-1}) \otimes \eta \det^{p^j(s_j+1)}$.

We denote by σ_J the weight in $\mathcal{D}(\rho^{\mathrm{ss}})$ corresponding to a subset J of $\{0, \cdots, f-1\}$ (see §11) and by $\mathbf{s}(J)$ the f-tuple of integers such that σ_J is a twist of $V_{\mathbf{s}(J)}$ (see §3).

Lemma 14.4. *Fix a subset $\mathcal{S} \subseteq \{0, \cdots, f-1\}$ and let $J \subseteq \mathcal{S}$.*

(i) If $J \neq \{0, \cdots, f-1\}$ then there are $2^{f-|J|}$ characters of I which occur in $(\operatorname{inj} \sigma_J)^{I_1}$ and do not occur in $(\oplus_{j \in J} \operatorname{inj} \sigma_{J \setminus \{j\}})^{I_1}$.

(ii) If $J = \{0, \cdots, f-1\}$ and ρ is reducible then there are 2 characters of I satisfying the same condition. If $J = \{0, \cdots, f-1\}$ and ρ is irreducible then there are no characters of I satisfying the same condition.

Proof. Write $\mathbf{s}(J) = (s_0, \cdots, s_{f-1})$ and note that one has $0 \leq s_i \leq p-2$ for all i as ρ is generic. If $J = \emptyset$ then every character occuring in $(\operatorname{inj} \sigma_J)^{I_1}$ satisfies the (empty) condition, and hence there 2^f of them. Suppose that $J \neq \emptyset$ and let $j \in J$. Assume first ρ is reducible. If $j+1 \notin J$ then the f-tuple corresponding to $\sigma_{J \setminus \{j\}}$ is $\mathbf{s}(J \setminus \{j\}) = (s_0, \cdots, s_{j-2}, p-2-s_{j-1}, s_j - 1, s_{j+1}, \cdots, s_{f-1})$. If $j+1 \in J$ then the f-tuple corresponding to $\sigma_{J \setminus \{j\}}$ is $\mathbf{s}(J \setminus \{j\}) = (s_0, \cdots, s_{j-2}, p-2-s_{j-1}, s_j+1, s_{j+1}, \cdots, s_{f-1})$. By Proposition 4.13, it is enough to count the $\boldsymbol{\varepsilon} \in \Sigma'$ such that for all $j \in J$ and all $\boldsymbol{\varepsilon}' \in \Sigma'$, one has $\sigma_J(\boldsymbol{\varepsilon}) \not\cong \sigma_{J \setminus \{j\}}(\boldsymbol{\varepsilon}')$. If $J \neq \{0, \cdots, f-1\}$, it follows from Lemma 14.2 and Remark 14.3 that such $\boldsymbol{\varepsilon}$ can be described as follows: for every k and j such that $k \notin J$, $\{k+1, \cdots, j\} \subseteq J$ and $j+1 \notin J$, either $\epsilon_k = \cdots = \epsilon_{j-1} = 0$ and $\epsilon_j = 1$ or $\epsilon_k = \cdots = \epsilon_{j-1} = 1$ and $\epsilon_j = 0$. There are $2^{f-|J|}$ such $\boldsymbol{\varepsilon} \in \Sigma'$. If $J = \{0, \cdots, f-1\}$ then it follows from Lemma 14.2 that the only $\boldsymbol{\varepsilon} \in \Sigma'$ satisfying the above condition are $(0, \cdots, 0)$ and $(1, \cdots, 1)$. Hence we obtain 2 characters. Assume now that ρ is irreducible. If $j \neq 0$ then the f-tuple $\mathbf{s}(J \setminus \{j\})$ is the same as in the reducible case. If $1 \in J$ then $\mathbf{s}(J \setminus \{0\}) = (s_0 - 1, s_1, \cdots, s_{f-2}, p-2-s_{f-1})$ and if $1 \notin J$ then $\mathbf{s}(J \setminus \{0\}) = (s_0 + 1, s_1, \cdots, s_{f-2}, p-2-s_{f-1})$. If $J \neq \{0, \cdots, f-1\}$ then, as in the reducible case, the "new" $\boldsymbol{\varepsilon} \in \Sigma'$ can be described as follows: for every k and j such that $k \notin J$, $\{k+1, \cdots, j\} \subseteq J$, $j+1 \notin J$ and $0 \notin \{k+1, \cdots, j\}$ we have as before either $\epsilon_k = \cdots = \epsilon_{j-1} = 0$ and $\epsilon_j = 1$ or $\epsilon_k = \cdots = \epsilon_{j-1} = 1$ and $\epsilon_j = 0$; for every k and j such that $k \notin J$, $\{k+1, \cdots, j\} \subseteq J$, $j+1 \notin J$ and $0 \in \{k+1, \cdots, j\}$ we have:

(i) if $1 \in J$ then either $\epsilon_k = \cdots = \epsilon_{f-1} = 0$, $\epsilon_0 = \cdots = \epsilon_{j-1} = 1$ and $\epsilon_j = 0$ or $\epsilon_k = \cdots = \epsilon_{f-1} = 1$, $\epsilon_0 = \cdots = \epsilon_{j-1} = 0$ and $\epsilon_j = 1$

(ii) if $1 \notin J$ then either $\epsilon_k = \cdots = \epsilon_{f-1} = \epsilon_0 = 0$ or $\epsilon_k = \cdots = \epsilon_{f-1} = \epsilon_0 = 1$.

Again we get $2^{f-|J|}$ new characters. Assume ρ is irreducible and $J = \{0, \cdots, f-1\}$. Suppose that we have $\boldsymbol{\varepsilon} \in \Sigma'$ which is new. If $\epsilon_{f-1} = 1$ then by Lemma 14.2 $\epsilon_0 = 0$. By applying Lemma 14.2 repeatedly we obtain $\epsilon_1 = \cdots = \epsilon_{f-1} = 0$, which is a contradiction to $\epsilon_{f-1} = 1$. A similar argument also gives a contradiction when $\epsilon_{f-1} = 0$. Hence there are no new characters. $\square$

Lemma 14.5. *Fix a subset $\mathcal{S} \subseteq \{0, \cdots, f-1\}$ and let $I, J \subseteq \mathcal{S}$. Suppose that $I(\sigma_I, \sigma_J) \neq 0$. Then $\sigma_{I\cap J}$ is a Jordan-Hölder factor of $I(\sigma_I, \sigma_J)$.*

Proof. Let $\mathcal{D}'(\rho^{\mathrm{ss}}) := \{\sigma_{I'}, I' \subseteq I\}$ and let $\mathcal{D}'(x_0, \ldots, x_{f-1})$ be the corresponding subset of $\mathcal{RD}(x_0, \ldots, x_{f-1})$ (resp. $\mathcal{ID}(x_0, \ldots, x_{f-1})$). In particular $\mathcal{D}'(x_0, \ldots, x_{f-1})$ satisfies (i) and (ii) of Definition 11.5. Remark 12.9 applied to $\tau = \sigma_J$ and $\mathcal{D}'(\rho^{\mathrm{ss}})$ then shows that there exists a unique $I' \subseteq I$ such that $\ell(\sigma_{I'}, \sigma_J)$ is minimal and non-zero. By (ii) of Lemma 12.8 as extended in Remark 12.9, $\sigma_{I'}$ is a subquotient of $I(\sigma_I, \sigma_J)$. So it is enough to show $I' = I \cap J$. As $\sigma_{I\cap J}$ is a subquotient of $I(\sigma_\emptyset, \sigma_J)$ (as is easily checked using Corollary 4.11), we have $I(\sigma_{I\cap J}, \sigma_J) \neq 0$. Hence (ii) of Lemma 12.8 as extended in Remark 12.9 implies again that $\sigma_{I'}$ is a subquotient of $I(\sigma_{I\cap J}, \sigma_J)$. Using Corollary 4.11 and the fact that $I \cap J \subseteq J$, one checks that this implies $I \cap J \subseteq I' \subseteq J$. As $I' \subseteq I$ we have $I' \subseteq I \cap J$ and thus $I' = I \cap J$. $\square$

Lemma 14.6. *Fix a subset $\mathcal{S} \subseteq \{0, \cdots, f-1\}$ and let J, J' be distinct subsets of $\mathcal{S}$ such that $|J'| \leq |J|$. Assume that χ occurs in $(\mathrm{inj}\, \sigma_J)^{I_1}$ and in $(\mathrm{inj}\, \sigma_{J'})^{I_1}$. Then there exists $j \in J$ such that χ occurs in $(\mathrm{inj}\, \sigma_{J\setminus\{j\}})^{I_1}$.*

Proof. The assumption implies that σ_J and $\sigma_{J'}$ are subquotients of $\mathrm{Ind}_I^K \chi$. Let τ_χ (resp. τ_{χ^s}) be the cosocle (resp. socle) of $\mathrm{Ind}_I^K \chi$. Since $\mathrm{Ind}_I^K \chi$ is multiplicity free, $I(\tau_{\chi^s}, \sigma_J)$ and $I(\tau_{\chi^s}, \sigma_{J'})$ are submodules of $\mathrm{Ind}_I^K \chi$. It follows from Lemmas 12.7 and 12.6 that there exists $I \subseteq \mathcal{S}$ such that σ_I is a subquotient of $I(\tau_{\chi^s}, \sigma_J)$ and $I(\tau_{\chi^s}, \sigma_{J'})$. Lemma 14.5 implies that $\sigma_{I\cap J}$ is a subquotient of $I(\tau_{\chi^s}, \sigma_J)$. Suppose that $I \cap J \neq J$ and let $j \in J \setminus (I \cap J)$. Then $I(\sigma_{J\setminus\{j\}}, \sigma_J)$ is a quotient of $I(\sigma_{I\cap J}, \sigma_J)$ and hence $\sigma_{J\setminus\{j\}}$ is a subquotient of $I(\tau_{\chi^s}, \sigma_J) \subseteq \mathrm{Ind}_I^K \chi$. Hence χ occurs in $(\mathrm{inj}\, \sigma_{J\setminus\{j\}})^{I_1}$. Suppose $J \subseteq I$. (i) of Theorem 2.4 implies that $\mathrm{Ind}_I^K \chi$ and $\mathrm{Ind}_I^K \chi^s$ have the same irreducible subquotients. By repeating the same argument with χ^s instead of χ, we obtain $I' \subseteq \mathcal{S}$ such that $\sigma_{I'}$ is a subquotient of $I(\tau_\chi, \sigma_J)$ and $I(\tau_\chi, \sigma_{J'})$. It follows from (ii) of Theorem 2.4 that $I(\sigma_I, \sigma_{I'})$ is a subquotient of $\mathrm{Ind}_I^K \chi$ which contains both σ_J and $\sigma_{J'}$. It follows from the proof of Lemma 14.5 that $I(\rho_I, \sigma_{I'}) \cong I(\sigma_{I\cap I'}, \sigma_{I'})$. Hence $I(\sigma_I, \sigma_{I\cap I'})$ is a subobject of $I(\sigma_I, \sigma_{I'})$

and contains σ_J and $\sigma_{I\cap J'}$ as subquotients. Since $I(\sigma_I,\sigma_{I\cap I'})$, $I(\sigma_I,\sigma_J)$ and $I(\sigma_I,\sigma_{I\cap J'})$ are all subquotients of $I(\sigma_{\mathcal{S}},\sigma_\emptyset)$, we obtain $I\cap I'\subseteq J$ and $I\cap I'\subseteq I\cap J'$. Hence $I'\cap J\subseteq J\cap J'$ and since $J'\neq J$ and $|J'|\leq|J|$, we get $J\cap I'\neq J$. By repeating the previous argument we obtain that there exists $j\in J$ such that $\sigma_{J\backslash\{j\}}$ is a subquotient of $\mathrm{Ind}_I^K\chi^s$. Since $\mathrm{Ind}_I^K\chi$ and $\mathrm{Ind}_I^K\chi^s$ have the same irreducible summands, we are done. □

Proposition 14.7. *Let ρ be generic. Let d such that $|\mathcal{D}(\rho)|=2^d$ (see §11). If ρ is irreducible then $\dim_{\overline{\mathbb{F}}_p}D_0(\rho)^{I_1}=3^f-1$. If ρ is split then $\dim_{\overline{\mathbb{F}}_p}D_0(\rho)^{I_1}=3^f+1$. If ρ is reducible non-split then $\dim_{\overline{\mathbb{F}}_p}D_0(\rho)^{I_1}=2^{f-d}3^d$.*

Proof. If ρ is reducible non-split, note that $d=|\mathcal{S}|$ where $\mathcal{S}$ corresponds to the maximal weight of $\mathcal{D}(\rho)$ (see §11). Lemma 14.1 implies that it is enough to count the number of distinct characters in $(\oplus_{J\subseteq\mathcal{S}}\mathrm{inj}\,\sigma_J)^{I_1}$. Let $J\subseteq\mathcal{S}$ and suppose that $|J|<f$ then Lemmas 14.4 and 14.6 imply that there are $2^{f-|J|}$ characters which occur in $(\mathrm{inj}\,\sigma_J)^{I_1}$ and do not occur in $(\mathrm{inj}\,\sigma_{J'})^{I_1}$ for any $J'\subseteq\mathcal{S}$, $|J'|\leq|J|$, $J'\neq J$. If $|J|=f$ and ρ is reducible (resp. irreducible) then there are 2 (resp. 0) characters satisfying the above condition. Now there are $\binom{d}{k}$ subsets of $\mathcal{S}$ of cardinality k. Hence, if $d<f$ we obtain:

$$\dim_{\overline{\mathbb{F}}_p}D_0(\rho)^{I_1}=\sum_{k=0}^{d}\binom{d}{k}2^{f-k}=2^{f-d}(2+1)^d=2^{f-d}3^d$$

and if $d=f$ we obtain:

$$\dim_{\overline{\mathbb{F}}_p}D_0(\rho)^{I_1}=\pm1+\sum_{k=0}^{f}\binom{f}{k}2^{f-k}=3^f\pm1$$

where $+$ corresponds to the reducible case and $-$ to the irreducible case. □

We now assume ρ is generic tame and work out explicitly all Jordan-Hölder factors of $D_0(\rho)$ and those which "contribute" to $D_0(\rho)^{I_1}$. Fix $\sigma\in\mathcal{D}(\rho)$ and write $\sigma=(\lambda_0(r_0),\cdots,\lambda_{f-1}(r_{f-1}))\otimes\det^{e(\lambda)(r_0,\cdots,r_{f-1})}\eta$ with $\lambda=(\lambda_i(x_i))_i$ as in Lemma 11.2 or Lemma 11.4. If ρ is reducible (resp. irreducible) one defines $\mu_\lambda\in\mathcal{I}(y_0,\cdots,y_{f-1})$ as follows:

(i) $\mu_{\lambda,i}(y_i):=p-1-y_i$ if $\lambda_i(x_i)\in\{p-3-x_i,x_i\}$ (resp. if $i>0$ and $\lambda_0(x_0)\in\{p-2-x_0,x_0-1\}$)

(ii) $\mu_{\lambda,i}(y_i):=p-3-y_i$ if $\lambda_i(x_i)\in\{p-2-x_i,x_i+1\}$ (resp. if $i>0$ and $\lambda_0(x_0)\in\{p-1-x_0,x_0\}$).

For $\mu \in \mathcal{I}(y_0, \cdots, y_{f-1})$, define $\mu \circ \lambda := (\mu_i(\lambda_i(x_i))_i$ and $e(\mu \circ \lambda) \in \bigoplus_{i=0}^{f-1} \mathbb{Z}x_i$ as in Lemma 3.1 according to whether $\mu_{f-1}(\lambda_{f-1}(x_{f-1})) \in \mathbb{Z}+x_{f-1}$ or $\mathbb{Z} - x_{f-1}$.

Theorem 14.8. *Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^f}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous generic representation as in Definition 11.7 and assume ρ is tame, i.e. either irreducible or split. Fix $\sigma \in \mathcal{D}(\rho)$ and λ the corresponding f-tuple.*

(i) The irreducible subquotients of $D_{0,\sigma}(\rho)$ are exactly the (all distinct) weights:

$$(\mu_0(\lambda_0(r_0)), \cdots, \mu_{f-1}(\lambda_{f-1}(r_{f-1}))) \otimes \det{}^{e(\mu\circ\lambda)(r_0,\cdots,r_{f-1})}\eta \qquad (28)$$

for $\mu \in \mathcal{I}(y_0, \cdots, y_{f-1})$ such that μ and μ_λ are compatible (see Definition 4.10) forgetting the weights such that $\mu_i(\lambda_i(r_i)) < 0$ or $\mu_i(\lambda_i(r_i)) > p-1$ for some i.

(ii) The graded pieces of the socle filtration on $D_{0,\sigma}(\rho)$ are:

$$D_{0,\sigma}(\rho)_i = \bigoplus_{\ell(\mu)=i} \tau$$

for $0 \le i \le f-1$ and weights τ as in (28) with $\ell(\mu)$ as in §4.

Proof. We may embed $D_{0,\sigma}(\rho)$ inside $\mathrm{inj}\,\sigma$. By Lemma 3.2, all weights of $D_{0,\sigma}(\rho)$ are of the type (28) for certain $\mu \in \mathcal{I}(y_0, \cdots, y_{f-1})$. We provide a proof only for ρ split, leaving the completely analogous irreducible case to the reader. Let $\mu \in \mathcal{I}(y_0, \cdots, y_{f-1})$ which is not compatible with μ_λ, assume $0 \le \mu_i(\lambda_i(r_i)) \le p-1$ for all i and let τ be the corresponding weight (28). There is $j \in \{0, \cdots, f-1\}$ such that either $\lambda_j(x_j) \in \{p-3-x_j, x_j\}$ and $\mu_j(y_j) \in \{p-3-y_j, y_j+1\}$ or $\lambda_j(x_j) \in \{p-2-x_j, x_j+1\}$ and $\mu_j(y_j) \in \{p-1-y_j, y_j-1\}$. In the first case, define $\mu' = (\mu'_i(y_i))_i$ by $\mu'_i(y_i) := y_i$ if $i \notin \{j-1, j\}$, $\mu'_{j-1}(y_{j-1}) := p-2-y_{j-1}$ and $\mu'_j(y_j) := y_j+1$. In the second case, define $\mu' = (\mu'_i(y_i))_i$ by $\mu'_i(y_i) := y_i$ if $i \notin \{j-1, j\}$, $\mu'_{j-1}(y_{j-1}) := p-2-y_{j-1}$ and $\mu'_j(y_j) := y_j-1$. Let τ' be the corresponding weight (28). Then one checks that in both cases $\tau' \in \mathcal{D}(\rho)$, $\tau' \neq \sigma$ and τ' is a subquotient of $I(\sigma, \tau)$ (using Corollary 4.11 for the latter). Hence τ cannot appear in $D_{0,\sigma}(\rho)$ by multiplicity 1. Conversely, if μ is compatible with μ_λ and $\mu \neq (y_0, \cdots, y_{f-1})$, then the weight (28) is never in $\mathcal{D}(\rho)$ as is immediately checked. By maximality of $D_{0,\sigma}(\rho)$ together with Corollary 4.11, this implies (i). (ii) follows from Proposition 13.4 and Corollary 4.11. □

Remark 14.9. *(i)* If ρ is split and $\sigma = (r_0, \cdots, r_{f-1}) \otimes \eta$ or $\sigma = (p-3-r_0, \cdots, p-3-r_{f-1}) \otimes \det^{\sum_{i=0}^{f-1} p^i(r_i+1)}\eta$, then $D_{0,\sigma}(\rho) \cong \mathrm{Ind}_B^\Gamma \chi$ where χ^s is the action of I on σ^{I_1}.

(ii) One can prove that, if ρ is split (resp. irreducible), then $D_{0,\sigma}(\rho)$ is always the image under $\otimes_{\overline{\mathbb{Z}}_p}\overline{\mathbb{F}}_p$ of a $\overline{\mathbb{Z}}_p$-lattice in a principal series (resp. a cuspidal representation) of Γ over $\overline{\mathbb{Q}}_p$. In particular, one has $\dim_{\overline{\mathbb{F}}_p} D_{0,\sigma}(\rho) = p^f + 1$ (resp. $p^f - 1$) for all $\sigma \in \mathcal{D}(\rho)$.

If $S \in \mathrm{Rep}_\Gamma$ is multiplicity free and τ is an irreducible subquotient of S, we say that τ^U has a lift in S^U or contributes to S^U if and only if the surjection $U(\tau) \twoheadrightarrow \tau$ induces a surjection $U(\tau)^U \twoheadrightarrow \tau^U$ where $U(\tau) \subseteq S$ is the unique subrepresentation with cosocle τ.

Corollary 14.10. *Keep the notations of Theorem 14.8. The irreducible subquotients τ of $D_{0,\sigma}(\rho)$ such that τ^{I_1} has a lift in $D_{0,\sigma}(\rho)^{I_1}$ are exactly the weights (28) such that $\mu_i(y_i) \in \{p-2-y_i, p-1-y_i, y_i, y_i+1\}$.*

Proof. As usual, we only prove the case where ρ is reducible. Set $\mathbf{s} := (\lambda_0(r_0), \cdots, \lambda_{f-1}(r_{f-1}))$ and note that one has $0 \le \lambda_i(r_i) \le p-2$ for all i as $\sigma \in \mathcal{D}(\rho)$. By Proposition 3.6 and the fact $D_{0,\sigma}(\rho)$ is multiplicity free, we may embed $D_{0,\sigma}(\rho)$ into:

$$V_{\mathbf{2p-2-s}} \otimes \det^{\sum_{i=0}^{f-1} p^i \lambda_i(r_i)} \det^{e(\lambda)(r_0,\cdots,r_{f-1})}\eta$$

(see §3 for $V_{\mathbf{2p-2-s}}$). Using Proposition 4.13 and twisting, it is thus enough to prove that the set of weights as in the statement coincides with the set of weights:

$$\{V_{\mathbf{s}(\boldsymbol{\varepsilon})} \otimes \det^{e(\boldsymbol{\varepsilon})+e(\lambda)(r_0,\cdots,r_{f-1})+\sum_{i=0}^{f-1} p^i \lambda_i(r_i)}\eta, \boldsymbol{\varepsilon} \in \Sigma(D_{0,\sigma}(\rho)) \cap \Sigma'_{\mathbf{s}}\}.$$

Denote by $\mathcal{I}'(y_0, \cdots, y_{f-1})$ the subset of $\mathcal{I}(y_0, \cdots, y_{f-1})$ of f-tuples $\mu = (\mu_0(y_0), \cdots, \mu_{f-1}(y_{f-1}))$ such that $\mu_i(y_i) \in \{p-2-y_i, p-1-y_i, y_i, y_i+1\}$ for all i. The bijection $\mathcal{I}(y_0, \cdots, y_{f-1}) \xrightarrow{\sim} \Sigma$ in the proof of Corollary 4.11 obviously induces a bijection $\mathcal{I}'(y_0, \cdots, y_{f-1}) \xrightarrow{\sim} \Sigma' \simeq \Sigma'_{\mathbf{s}}$. Moreover, we have (see proof of Corollary 4.11):

$$(\mu_i(\lambda_i(r_i)) \otimes \det^{e(\mu)(\lambda_i(r_i))} \cong V_{\mathbf{s}(\boldsymbol{\varepsilon})} \otimes \det^{e(\boldsymbol{\varepsilon})+\sum_{i=0}^{f-1} p^i \lambda_i(r_i)}.$$

(i) of Theorem 14.8 implies the equality of the two sets of weights. □

15 Decomposition of generic Diamond diagrams

For ρ a continuous generic Galois representation, we study the decomposition of the family of basic 0-diagrams $(D_0(\rho), \{\ \})$ of Theorem 13.8.

Let $\mathcal{S}$ be a subset of $\{0, \cdots, f-1\}$ and define $\delta_r(\mathcal{S})$ (resp. $\delta_i(\mathcal{S})$) as follows (with the convention $f-1+1=0$): $i \in \delta_r(\mathcal{S})$ if and only if $i+1 \in \mathcal{S}$ (resp. if $0 < i$, $i \in \delta_i(\mathcal{S})$ if and only if $i+1 \in \mathcal{S}$ and $0 \in \delta_i(\mathcal{S})$ if and only if $1 \notin \mathcal{S}$). One defines in an obvious way $\delta_r^n(\mathcal{S})$ and $\delta_i^n(\mathcal{S})$ for $n \in \mathbb{Z}$. If $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^f}) \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is a continuous tamely ramified generic Galois representation and $\sigma \in \mathcal{D}(\rho)$ corresponds to $\mathcal{S}$ (see §11), we write $\delta^n(\sigma)$ for the unique weight in $\mathcal{D}(\rho)$ corresponding to $\delta_r^n(\mathcal{S})$ if ρ is reducible, $\delta_i^n(\mathcal{S})$ if ρ is irreducible.

Fix ρ generic as in Definition 11.7 and tamely ramified and let $\sigma \in \mathcal{D}(\rho)$. Let $\lambda \in \mathcal{RD}(x_0, \cdots, x_{f-1})$ or $\mathcal{ID}(x_0, \cdots, x_{f-1})$ give rise to σ via Lemma 11.2 or Lemma 11.4 and $\mathcal{S} \subseteq \{0, \cdots, f-1\}$ correspond to σ and λ (see §11). Let τ be an irreducible subquotient of $D_{0,\sigma}(\rho)$ such that τ^{I_1} has a lift in $D_{0,\sigma}(\rho)^{I_1}$ and write τ as in (28) for a $\mu \in \mathcal{I}(y_0, \cdots, y_{f-1})$. Note that by Corollary 14.10, one has $\mu_i(y_i) \in \{p-2-y_i, p-1-y_i, y_i, y_i+1\}$. If ρ is reducible, define:

$$\mathcal{S}^- := \{i \in \mathcal{S} \text{ and } \mu_{i-1}(\lambda_{i-1}(x_{i-1})) \in \{x_{i-1}, x_{i-1}+1, p-1-x_{i-1}\}\}$$
$$\mathcal{S}^+ := \{i \notin \mathcal{S} \text{ and } \mu_{i-1}(\lambda_{i-1}(x_{i-1})) \in \{p-3-x_{i-1}, p-2-x_{i-1}, x_{i-1}+2\}\}.$$

If ρ is irreducible, define $\mathcal{S}^+$ and $\mathcal{S}^-$ in the same way except that $1 \in \mathcal{S}^-$ (resp. $\mathcal{S}^+$) iff $1 \in \mathcal{S}$ and $\mu_0(\lambda_0(x_0)) \in \{x_0-1, x_0, p-x_0\}$ (resp. $1 \notin \mathcal{S}$ and $\mu_0(\lambda_0(x_0)) \in \{p-2-x_0, p-1-x_0, x_0+1\}$).

Lemma 15.1. *Assume ρ is reducible (hence split), then $|\mathcal{S}^-| = |\mathcal{S}^+|$.*

Proof. As $\lambda \in \mathcal{RD}(x_0, \cdots, x_{f-1})$ and as μ and μ_λ are compatible by (i) of Theorem 14.8, we have:

$$\mathcal{S}^- = \{i, \lambda_{i-1}(x_{i-1}) = p-3-x_{i-1} \text{ or } p-2-x_{i-1} \text{ and } \mu_{i-1}(y_{i-1}) = p-2-y_{i-1}\}$$
$$\amalg \{i, \lambda_{i-1}(x_{i-1}) = p-2-x_{i-1} \text{ and } \mu_{i-1}(y_{i-1}) = y_{i-1}+1\}$$

and likewise:

$$\mathcal{S}^+ = \{i, \lambda_{i-1}(x_{i-1}) = x_{i-1} \text{ or } x_{i-1}+1 \text{ and } \mu_{i-1}(y_{i-1}) = p-2-y_{i-1}\}$$
$$\amalg \{i, \lambda_{i-1}(x_{i-1}) = x_{i-1}+1 \text{ and } \mu_{i-1}(y_{i-1}) = y_{i-1}+1\}.$$

But if $\mu_i(y_i) = p-2-y_i$ and $\lambda_i(x_i) \in \{p-3-x_i, p-2-x_i\}$, we get from the compatibility of μ and μ_λ that the smallest $j \geq 1$ such that $\mu_{i+j}(y_{i+j}) = y_{i+j}+1$ must be such that $\lambda_{i+j}(x_{i+j}) = x_{i+j}+1$ (otherwise, some index between i and $i+j$ would contradict the compatibility). This implies:

$$|\{i, \lambda_i(x_{i-1}) = p-3-x_{i-1} \text{ or } p-2-x_{i-1} \text{ and } \mu_i(y_{i-1}) = p-2-y_{i-1}\}| =$$
$$|\{i, \lambda_i(x_{i-1}) = x_{i-1}+1 \text{ and } \mu_i(y_{i-1}) = y_{i-1}+1\}|.$$

Likewise, we have:

$$|\{i, \lambda_i(x_{i-1}) = x_{i-1} \text{ or } x_{i-1}+1 \text{ and } \mu_i(y_{i-1}) = p-2-y_{i-1}\}| = \\ |\{i, \lambda_i(x_{i-1}) = p-2-x_{i-1} \text{ and } \mu_i(y_{i-1}) = y_{i-1}+1\}|.$$

All this obviously implies $|\mathcal{S}^-| = |\mathcal{S}^+|$. □

Lemma 15.1 is wrong when ρ is irreducible.

Lemma 15.2. *With the previous notations, the unique weight $w \in \mathcal{D}(\rho)$ such that $\ell(w, \tau^{[s]}) = \ell(\rho, \tau^{[s]})$ corresponds to the subset $\delta_r((\mathcal{S} \setminus \mathcal{S}^-) \cup \mathcal{S}^+)$ (resp. $\delta_i((\mathcal{S} \setminus \mathcal{S}^-) \cup \mathcal{S}^+)$) if ρ is reducible (resp. irreducible).*

Proof. Note that there is indeed such a w thanks to Corollary 13.6 and that $(\mathcal{S} \setminus \mathcal{S}^-) \cap \mathcal{S}^+ = \emptyset$. Assume first ρ is reducible (hence split) and let $\widetilde{\lambda} \in \mathcal{RD}(x_0, \cdots, x_{f-1})$ correspond to the subset $\delta_r((\mathcal{S} \setminus \mathcal{S}^-) \cup \mathcal{S}^+)$. Let $(\mu(y_i))_i \in \mathcal{I}(y_0, \cdots, y_{f-1})$ as previously (corresponding to τ) and define $(\widetilde{\mu}_i(y_i))_i$ by the formula:

$$p - 1 - \mu_i(\lambda_i(x_i)) = \widetilde{\mu}_i(\widetilde{\lambda}_i(x_i)), \quad \forall\ i. \tag{29}$$

By (i) of Theorem 14.8, μ is compatible with μ_λ in the sense of Definition 4.10. By (i) of Theorem 4.10 again, it is enough to prove that $(\widetilde{\mu}_i(y_i))_i \in \mathcal{I}(y_0, \cdots, y_{f-1})$ and $(\widetilde{\mu}, \mu_{\widetilde{\lambda}})$ are compatible. This is horrible but easy combinatorics. Assume $i+1 \in (\mathcal{S} \setminus \mathcal{S}^-) \cup \mathcal{S}^+$, then $\widetilde{\lambda}_i(x_i) \in \{p-3-x_i, x_i+1\}$. If $i+1 \in \mathcal{S} \setminus \mathcal{S}^-$, one must have $\lambda_i(x_i) \in \{p-3-x_i, p-2-x_i\}$ and some conditions on μ which then imply the following cases:

(i) $\lambda_i(x_i) = p-3-x_i$, $\mu_(y_i) = y_i$ which gives using (29) $\widetilde{\lambda}_i(x_i) = x_{i-1}+1, \widetilde{\mu}_{i-1}(y_{i-1}) = y_i+1$ or $\widetilde{\lambda}_i(x_i) = p-3-x_i, \widetilde{\mu}_i(y_i) = p-1-y_i$

(ii) $\lambda_i(x_i) = p-3-x_i$, $\mu_i(y_i) = p-1-y_i$ which gives using (29) $\widetilde{\lambda}_i(x_i) = x_i+1, \widetilde{\mu}_i(y_i) = p-2-y_i$ or $\widetilde{\lambda}_i(x_i) = p-3-x_i, \widetilde{\mu}_i(y_i) = y_i$

(iii) $\lambda_i(x_i) = p-2-x_i$, $\mu_i(y_i) = y_i$ which gives using (29) $\widetilde{\lambda}_i(x_i) = x_i+1, \widetilde{\mu}_i(y_i) = y_i$ or $\widetilde{\lambda}_i(x_i) = p-3-x_i, \widetilde{\mu}_i(y_i) = p-2-y_i$.

If $i+1 \in \mathcal{S}^+$, one must have $\lambda_i(x_i) \in \{x_i, x_i+1\}$ and some conditions on μ which then imply the following cases:

(i) $\lambda_i(x_i) = x_i$, $\mu_i(y_i) = p-2-y_i$ which gives using (29) $\widetilde{\lambda}_i(x_i) = x_i+1, \widetilde{\mu}_i(y_i) = y_i$ or $\widetilde{\lambda}_i(x_i) = p-3-x_i, \widetilde{\mu}_i(y_i) = p-2-y_i$

(ii) $\lambda_i(x_i) = x_i+1$, $\mu_i(y_i) = p-2-y_i$ which gives using (29) $\widetilde{\lambda}_i(x_i) = x_i+1, \widetilde{\mu}_i(y_i) = y_i+1$ or $\widetilde{\lambda}_i(x_i) = p-3-x_i, \widetilde{\mu}_i(y_i) = p-1-y_i$

(iii) $\lambda_i(x_i) = x_i + 1$, $\mu_i(y_i) = y_i + 1$ which gives using (29) $\widetilde{\lambda}_i(x_i) = x_i + 1, \widetilde{\mu}_i(y_i) = p - 2 - y_i$ or $\widetilde{\lambda}_i(x_i) = p - 3 - x_i, \widetilde{\mu}_i(y_i) = y_i$.

Assume now $i+1 \notin (\mathcal{S}\setminus\mathcal{S}^-)\cup\mathcal{S}^+$, then $\widetilde{\lambda}_i(x_i) \in \{x_i, p-2-x_i\}$. If $i+1 \in \mathcal{S}^-$, one must have $\lambda_i(x_i) \in \{p-3-x_i, p-2-x_i\}$ and some conditions on μ which then imply the following cases:

(i) $\lambda_i(x_i) = p - 3 - x_i$, $\mu_i(y_i) = p - 2 - y_i$ which gives using (29) $\widetilde{\lambda}_i(x_i) = x_i, \widetilde{\mu}_i(y_i) = p - 2 - y_i$ or $\widetilde{\lambda}_i(x_i) = p - 2 - x_i, \widetilde{\mu}_i(y_i) = y_i$

(ii) $\lambda_i(x_i) = p - 2 - x_i$, $\mu_i(y_i) = p - 2 - y_i$ which gives using (29) $\widetilde{\lambda}_i(x_i) = x_i, \widetilde{\mu}_i(y_i) = p - 1 - y_i$ or $\widetilde{\lambda}_i(x_i) = p - 2 - x_i, \widetilde{\mu}_i(y_i) = y_i + 1$

(iii) $\lambda_i(x_i) = p - 2 - x_i$, $\mu_i(y_i) = y_i + 1$ which gives using (29) $\widetilde{\lambda}_i(x_i) = x_i, \widetilde{\mu}_i(y_i) = y_i$ or $\widetilde{\lambda}_i(x_i) = p - 2 - x_i, \widetilde{\mu}_i(y_i) = p - 2 - y_i$.

If $i + 1 \notin \mathcal{S}$ and $i + 1 \notin \mathcal{S}^+$, one must have $\lambda_i(x_i) \in \{x_i, x_i + 1\}$ and some conditions on μ which then imply the following cases:

(i) $\lambda_i(x_i) = x_i$, $\mu_i(y_i) = y_i$ which gives using (29) $\widetilde{\lambda}_i(x_i) = x_i, \widetilde{\mu}_i(y_i) = p - 1 - y_i$ or $\widetilde{\lambda}_i(x_i) = p - 2 - x_i, \widetilde{\mu}_i(y_i) = y_i + 1$

(ii) $\lambda_i(x_i) = x_i$, $\mu_i(y_i) = p - 1 - y_i$ which gives using (29) $\widetilde{\lambda}_i(x_i) = x_i, \widetilde{\mu}_i(y_i) = y_i$ or $\widetilde{\lambda}_i(x_i) = p - 2 - x_i, \widetilde{\mu}_i(y_i) = p - 2 - y_i$

(iii) $\lambda_i(x_i) = x_i + 1$, $\mu_i(y_i) = y_i$ which gives using (29) $\widetilde{\lambda}_i(x_i) = x_i, \widetilde{\mu}_i(y_i) = p - 2 - y_i$ or $\widetilde{\lambda}_i(x_i) = p - 2 - x_i, \widetilde{\mu}_i(y_i) = y_i$.

If $(\widetilde{\mu}_i(y_i))_i \in \mathcal{I}(y_0, \cdots, y_{f-1})$, we see that $\widetilde{\mu}$ and $\mu_{\widetilde{\lambda}}$ are compatible in all of the above cases. Let us now check $(\widetilde{\mu}_i(y_i))_i \in \mathcal{I}(y_0, \cdots, y_{f-1})$. Assume $\widetilde{\mu}_i(y_i) = y_i + 1$. From the above list, we have four possibilities for $(\lambda_i(x_i), \mu_i(y_i))$ and $(\widetilde{\lambda}_i(x_i), \widetilde{\mu}_i(y_i))$:

$$\begin{aligned}
(p-3-x_i, y_i) &\quad \text{and} \quad (x_i+1, y_i+1)\\
(x_i+1, p-2-y_i) &\quad \text{and} \quad (x_i+1, y_i+1)\\
(x_i, y_i) &\quad \text{and} \quad (p-2-x_i, y_i+1)\\
(p-2-x_i, p-2-y_i) &\quad \text{and} \quad (p-2-x_i, y_i+1).
\end{aligned}$$

In the first case, we have $(\lambda_{i+1}(x_{i+1}), \mu_{i+1}(y_{i+1})) \in \{(p-3-x_{i+1}, y_{i+1}), (p-3-x_{i+1}, p-2-y_{i+1}), (x_{i+1}+1, y_{i+1}), (x_{i+1}+1, p-2-y_{i+1})\}$ which, again from the above list and the fact $\widetilde{\lambda} \in \mathcal{RD}(x_0, \cdots, x_{f-1})$, yields:

$$(\widetilde{\lambda}_{i+1}(x_{i+1}), \widetilde{\mu}_{i+1}(y_{i+1})) \in \{(x_{i+1}, p-2-y_{i+1}), (p-2-x_{i+1}, y_{i+1})\}.$$

We see $\widetilde{\mu}_{i+1}(y_{i+1}) \in \{p-2-y_{i+1}, y_{i+1}\}$. The 3 other cases yield the same conclusion, hence we always have $\widetilde{\mu}_{i+1}(y_{i+1}) \in \{p-2-y_{i+1}, y_{i+1}\}$. Examining $\widetilde{\mu}_i(y_i) = y_i$ yields in the same way $\widetilde{\mu}_{i+1}(y_{i+1}) \in \{p-2-y_{i+1}, y_{i+1}\}$. For $\widetilde{\mu}_i(y_i) \in \{p-2-y_i, p-1-y_i\}$ a similar check yields $\widetilde{\mu}_{i+1}(y_{i+1}) \in \{p-1-y_{i+1}, y_{i+1}+1\}$. This implies $(\widetilde{\mu}_i(y_i))_i \in \mathcal{I}(y_0, \cdots, y_{f-1})$. We leave the somewhat analogous case ρ irreducible to the reader. □

In the case $\tau = \sigma$, Lemma 15.2 was noted independently by Buzzard.

Recall that, if ρ is a continuous reducible generic Galois representation (not necessarily split), there is a maximal element $\sigma^{\max}$ in $\mathcal{D}(\rho)$ for $\leq$ (see §11).

Lemma 15.3. *Let ρ be a continuous reducible generic Galois representation. Let $\rho^{\rm ss}$ be the semi-simplification of ρ and $\sigma^{\max} \in \mathcal{D}(\rho)$ the unique maximal weight. Let τ be any weight such that $\ell(\rho, \tau) < +\infty$ and $\sigma \in \mathcal{D}(\rho)$ such that $I(\rho, \tau) = I(\sigma, \tau)$. Then $\ell(\rho^{\rm ss}, \tau) < +\infty$ and, if $\sigma' \in \mathcal{D}(\rho^{\rm ss})$ is such that $I(\rho^{\rm ss}, \tau) = I(\sigma', \tau)$, we have $\sigma = \sigma' \cap \sigma^{\max}$ in $\mathcal{D}(\rho^{\rm ss})$.*

Proof. As $\mathcal{D}(\rho) \subseteq \mathcal{D}(\rho^{\rm ss})$, it is clear that $\ell(\rho, \tau) < +\infty$ implies $\ell(\rho^{\rm ss}, \tau) < +\infty$. We first prove $\sigma \leq \sigma'$. By (ii) of Lemma 12.8 applied to $\rho^{\rm ss}$, $I(\sigma, \tau)$ contains σ' hence we have $I(\sigma, \sigma') \subseteq I(\sigma, \tau)$. As in the proof of Lemma 12.6, we go from σ to σ' inside $I(\sigma, \tau)$ by applying to σ several sequences $p-2-\cdot, p-2-\cdot-\pm 1, \cdots, \cdot \pm 1$ such that the successive sets of indices that are affected are disjoint, or one full sequence $(\cdots, p-2-\cdot-\pm 1, \cdots)$. Let $\mu \in \mathcal{I}(y_0, \cdots, y_{f-1})$ be the unique element corresponding to these sequences and recall that $\mathcal{S}(\mu) = \{i, \mu_i(y_i) = p-2-y_i-\pm 1 \text{ or } y_i \pm 1\}$ and that σ' is the negative of σ within $\mathcal{S}(\mu)$ (see Lemma 12.4). Let $\mathcal{S}, \mathcal{S}' \subseteq \{0, \cdots, f-1\}$ correspond to σ, σ'. Assume that we don't have $\sigma \leq \sigma'$, or equivalently that we have $\mathcal{S}(\mu) \cap \mathcal{S} \neq \emptyset$. Let μ' be the unique element of $\mathcal{I}(y_0, \cdots, y_{f-1})$ such that:

(i) $\mathcal{S}(\mu') = \mathcal{S}(\mu) \cap \mathcal{S}$

(ii) μ and μ' are compatible (Definition 4.10).

Then, by Corollary 4.11 applied to $I(\sigma, \sigma')$, μ' corresponds to a unique irreducible component σ'' of $I(\sigma, \sigma') \subseteq I(\sigma, \tau)$ which is distinct from σ. Moreover, from Lemma 12.4, one easily derives that this weight is still in $\mathcal{D}(\rho^{\rm ss})$ (it is the negative of σ within $\mathcal{S}(\mu')$). But we have $\sigma'' < \sigma$ as $\mathcal{S}(\mu') \subseteq \mathcal{S}$ by assumption, so σ'' is also still in $\mathcal{D}(\rho)$. The definition of σ then implies $\sigma = \sigma''$, which is a contradiction. Hence we have $\sigma \leq \sigma'$. As $\sigma \leq \sigma^{\max}$, we have $\sigma \leq \sigma' \cap \sigma^{\max} \leq \sigma'$. In particular, $\sigma' \cap \sigma^{\max}$ is obtained from σ by applying sequences $p-2-\cdot, p-2-\cdot-\pm 1, \cdots, \cdot\pm 1$ with support (in the sense of §12) contained in $\mathcal{S}(\mu)$. As $\sigma'\cap\sigma^{\max}$, $\sigma'\in\mathcal{D}(\rho^{\rm ss})$, we get from Lemma 12.6 that these sequences are compatible with μ and then from Corollary 4.11 that $\sigma'\cap\sigma^{\max}$ is a component of $I(\sigma,\sigma')\subseteq I(\sigma,\tau)$. But $\sigma'\cap\sigma^{\max}\in\mathcal{D}(\rho)$ since $\sigma'\cap\sigma^{\max}\leq\sigma^{\max}$ and $\sigma^{\max}\in\mathcal{D}(\rho)$. Thus, we must have $\sigma=\sigma'\cap\sigma^{\max}$. □

We are now ready to prove:

Theorem 15.4. *Let $\rho:\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^f})\to\mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous generic Galois representation and $(D_0(\rho),\{\ \})$ as in Theorem 13.8.*

(i) Assume ρ is indecomposable, then $(D_0(\rho),\{\ \})$ cannot be written as the direct sum of two non-zero families of basic 0-diagrams (Definition 13.7).

(ii) Assume ρ is reducible split, then we have:

$$(D_0(\rho),\{\ \})=\bigoplus_{\ell=0}^{f}(D_{0,\ell}(\rho),\{\ \}) \tag{30}$$

where $D_{0,\ell}(\rho):=\oplus_{\ell(\sigma)=\ell}D_{0,\sigma}(\rho)$ (see (27) for $D_{0,\sigma}(\rho)$). Moreover, for each ℓ, $(D_{0,\ell}(\rho),\{\ \})$ cannot be written as the direct sum of two non-zero families of basic 0-diagrams.

Proof. The whole proof is again easy but tedious combinatorics and we only give details in the reducible case. Let us start with (ii). Note that there is a unique $\sigma\in\mathcal{D}(\rho)$ such that $\ell(\sigma)=0$ (resp. $\ell(\sigma)=f$), namely $\sigma=\sigma_0:=(r_0,\cdots,r_{f-1})\otimes\eta$ (resp. $\sigma=\sigma_f:=(p-3-r_0,\cdots,p-3-r_{f-1})\otimes\det^{r_0+1+p(r_1+1)+\cdots+p^{f-1}(r_{f-1}+1)}\eta$). We have to prove that the unique pairing $\{\ \}$ on the I-eigencharacters of $D_0(\rho)^{I_1}$ preserves the I-eigencharacters of $D_{0,\ell}(\rho)^{I_1}$ for each ℓ, and doesn't preserve those corresponding to any strict non-zero K-direct factor of $D_{0,\ell}(\rho)$. This is straightforward if $\ell=0$ (resp. $\ell=f$) as $D_{0,0}(\rho)=\mathrm{Ind}_B^{\Gamma}\chi_0$ (resp. $D_{0,f}(\rho)=\mathrm{Ind}_B^{\Gamma}\chi_f$) where χ_0^s (resp. χ_f^s) is the character giving the action of I on $\sigma_0^{I_1}$ (resp. $\sigma_f^{I_1}$), see (i) of Remark 14.9. Fix $\sigma\in\mathcal{D}(\rho)$, $\sigma\notin\{\sigma_0,\sigma_f\}$ and let $\mathcal{S}\subset\{0,\cdots,f-1\}$ correspond to σ. Let τ be an irreducible subquotient of $D_{0,\sigma}(\rho)$ as in Corollary 14.10 contributing to I_1-invariants. Let $\mathcal{S}^-$ and $\mathcal{S}^+$ as before with τ and let $\widetilde{\sigma}\in\mathcal{D}(\rho)$ correspond to $\delta_r((\mathcal{S}\setminus\mathcal{S}^-)\cup\mathcal{S}^+)$. By Lemma 15.2, $\tau^{[s]}$ sits in $D_{0,\widetilde{\sigma}}(\rho)$ and by Lemma 15.1, $\ell(\sigma)=\ell(\widetilde{\sigma})$, hence $\{\ \}$ preserves the I-eigencharacters of $D_{0,\ell(\sigma)}(\rho)^{I_1}$. For $\ell\in\{1,\cdots,f-1\}$, let $\sigma_\ell\in\mathcal{D}(\rho)$ correspond to the subset $\{1,2,\cdots,\ell\}$. We are going to prove that one can always "go" from $D_{0,\sigma}(\rho)^{I_1}$ to $D_{0,\sigma_{\ell(\sigma)}}(\rho)^{I_1}$ using $\chi\mapsto\chi^s$. By Lemma 15.2 applied successively to $\tau=\sigma$,

$\tau = \delta(\sigma)$ etc., we can assume $0 \notin \mathcal{S}$ and $1 \in \mathcal{S}$. Write $\mathcal{S} = \amalg_{\alpha=0}^{r} \mathcal{S}_\alpha$ with $\mathcal{S}_\alpha := \{i_\alpha + 1, \cdots, i_\alpha + j_\alpha\}$, $i_\alpha \in \{0, \cdots, f-1\}$, $i_\alpha \notin \mathcal{S}$ and $i_\alpha + j_\alpha < i_{\alpha+1}$ (so $i_0 = 0$ and $\sum_{\alpha=0}^{r} j_\alpha = \ell(\sigma)$). If $r = 0$, we are done as $\sigma = \sigma_{\ell(\sigma)}$ in that case. Assume $r > 0$ and define $\mu := (\mu_i(y_i))_i \in \mathcal{I}(y_0, \cdots, y_{f-1})$ as follows:

$$\begin{aligned} \mu_0(y_0) &:= p - 2 - y_0 \\ \mu_i(y_i) &:= p - 1 - y_i,\ 1 \leq i \leq j_0 - 1 \\ \mu_{j_0}(y_{j_0}) &:= y_{j_0} + 1 \\ \mu_i(y_i) &:= y_i,\ i > j_0. \end{aligned}$$

Let τ be the irreducible subquotient of $D_{0,\sigma}(\rho)$ corresponding to μ as in (i) of Theorem 14.8. We have $\mathcal{S}^- = \{1\}$, $\mathcal{S}^+ = \{j_0 + 1\}$ and, by Lemma 15.2, $\tau^{[s]}$ sits in $D_{0,\sigma^{(1)}}(\rho)$ where $\sigma^{(1)}$ corresponds to $\mathcal{S}^{(1)} := \mathcal{S}_0 \amalg \delta_r(\mathcal{S} \setminus \mathcal{S}_0)$. If $j_0 + 1 \notin \mathcal{S}^{(1)}$, we start again with $\tau^{(1)}$ inside $D_{0,\sigma^{(1)}}(\rho)$ corresponding to the same μ and get that ${\tau^{(1)}}^{[s]}$ sits in $D_{0,\sigma^{(2)}}(\rho)$ where $\sigma^{(2)}$ corresponds to $\mathcal{S}^{(2)} := \mathcal{S}_0 \amalg \delta_r^2(\mathcal{S} \setminus \mathcal{S}_0)$. Repeating this again, one reaches $\mathcal{S}^{(i_1 - j_0)} := \mathcal{S}_0 \amalg \delta_r^{i_1 - j_0}(\mathcal{S} \setminus \mathcal{S}_0) = \amalg_{\alpha=0}^{r-1} \mathcal{S}_\alpha^{(i_1 - j_0)}$ with $\mathcal{S}_\alpha^{(i_1 - j_0)}$ as before. In particular, r has strictly decreased. By an obvious induction, we can "reach" like this $r = 0$, that is $\sigma_{\ell(\sigma)}$. All this implies that $\{\ \}$ doesn't preserve any strict non-zero K-direct factor of $D_{0,\ell(\sigma)}(\rho)$. Let us now prove (i). The case ρ irreducible is analogous to (ii) and we leave the details to the reader. Let us assume ρ is reducible non-split and let $\sigma \in \mathcal{D}(\rho)$. We are going to prove that one can always "go" from $D_{0,\sigma}(\rho)^{I_1}$ to $D_{0,\sigma_0}(\rho)^{I_1}$ using $\chi \mapsto \chi^s$. By using Lemma 15.3 and Lemma 15.2 "backwards" and since $D_{0,\sigma}(\rho^{\rm ss}) \subseteq D_{0,\sigma}(\rho)$, we can (and do) replace σ by $\delta^{-n}(\sigma)$ for the biggest integer n such that $\delta^{-n}(\sigma) \leq \sigma^{\max}$. Consider now the weight $\delta^{-1}(\sigma)^{[s]}$. By Lemma 15.2 applied to $\rho^{\rm ss}$ and $\tau = \delta^{-1}(\sigma) \in \mathcal{D}(\rho^{\rm ss})$, $\delta^{-1}(\sigma)^{[s]}$ is a Jordan-Hölder component in $D_{0,\sigma}(\rho^{\rm ss}) \subseteq D_{0,\sigma}(\rho)$ such that $(\delta^{-1}(\sigma)^{[s]})^{I_1}$ contributes to $D_{0,\sigma}(\rho)^{I_1}$. By Lemma 15.3 applied to $\tau = \delta^{-1}(\sigma)$, $\delta^{-1}(\sigma)$ is a Jordan-Hölder component in $D_{0,\delta^{-1}(\sigma)\cap\sigma^{\max}}(\rho)$. Moreover, $\ell(\delta^{-1}(\sigma) \cap \sigma^{\max}) < \ell(\delta^{-1}(\sigma)) = \ell(\sigma)$ as $\delta^{-1}(\sigma) \notin \mathcal{D}(\rho)$. Thus, replacing σ by $\delta^{-1}(\sigma) \cap \sigma^{\max}$, we see that $\ell(\sigma)$ has strictly decreased. By an obvious induction, we can "reach" like this $\ell(\sigma) = 0$. This finishes the proof. $\square$

16 Generic Diamond diagrams for $f \in \{1, 2\}$

We completely describe the family of basic 0-diagrams $(D_0(\rho), \{\ \})$ attached to a continuous generic $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^f}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ for $f = 1$ and $f = 2$.

We write a finite dimensional indecomposable representation S of Γ over $\overline{\mathbb{F}}_p$ as follows:

$$S_0 — S_1 — S_2 — \cdots — S_n$$

where $(S_i)_i$ are the graded pieces of the socle filtration (see introduction).

Let us start with $f = 1$. Twisting if necessary, we can assume that p acts trivially on $\det(\rho)$ and that the restriction of ρ to inertia has one of the following forms:

(i) $\begin{pmatrix} \omega^{r_0+1} & * \\ 0 & 1 \end{pmatrix}$ with $* \neq 0$

(ii) $\begin{pmatrix} \omega^{r_0+1} & 0 \\ 0 & 1 \end{pmatrix}$

(iii) $\begin{pmatrix} \omega_2^{r_0+1} & 0 \\ 0 & \omega_2^{p(r_0+1)} \end{pmatrix}$

where ω stands for ω_1 (the reduction modulo p of the cyclotomic character) and where $1 \leq r_0 \leq p-4$ in the first two cases and $1 \leq r_0 \leq p-2$ in the last (remember ρ is generic!). The corresponding $D_0(\rho)$ is:

(i) $$\mathrm{Sym}^{r_0}\overline{\mathbb{F}}_p^2 \quad — \quad \begin{matrix} \mathrm{Sym}^{p-1-r_0}\overline{\mathbb{F}}_p^2 \otimes \det^{r_0} \\ \oplus \\ \mathrm{Sym}^{p-3-r_0}\overline{\mathbb{F}}_p^2 \otimes \det^{r_0+1} \end{matrix}$$

(ii) $$\begin{matrix} \mathrm{Sym}^{r_0}\overline{\mathbb{F}}_p^2 & — & \mathrm{Sym}^{p-1-r_0}\overline{\mathbb{F}}_p^2 \otimes \det^{r_0} \\ & \oplus & \\ \mathrm{Sym}^{p-3-r_0}\overline{\mathbb{F}}_p^2 \otimes \det^{r_0+1} & — & \mathrm{Sym}^{r_0+2}\overline{\mathbb{F}}_p^2 \otimes \det^{-1} \end{matrix}$$

(iii) $$\begin{matrix} \mathrm{Sym}^{r_0}\overline{\mathbb{F}}_p^2 & — & \mathrm{Sym}^{p-3-r_0}\overline{\mathbb{F}}_p^2 \otimes \det^{r_0+1} \\ & \oplus & \\ \mathrm{Sym}^{p-1-r_0}\overline{\mathbb{F}}_p^2 \otimes \det^{r_0} & — & \mathrm{Sym}^{r_0-2}\overline{\mathbb{F}}_p^2 \otimes \det . \end{matrix}$$

(If a weight has a negative entry, we just forget it.) The reader can easily find the unique pairing $\{\chi, \chi^s\}$ and check directly that $D_0(\rho)^{I_1}$ has dimension 2 in case (i), 4 in case (ii) and again 2 in case (iii). We let Π act on $D_0(\rho)^{I_1}$ in the unique possible way (up to isomorphism of $\mathfrak{K}_1$-representations) and let $D_1(\rho)$ be the resulting $\mathfrak{K}_1$-representation. Up to isomorphism of basic 0-diagrams, the reader can check that an IZ-equivariant injection $r : D_1(\rho) \hookrightarrow D_0(\rho)$ depends on one scalar in $\overline{\mathbb{F}}_p^\times$ in case (i), on two scalars in $\overline{\mathbb{F}}_p^\times$ in case (ii) and is unique in case (iii).

We go on with the slightly more involved case $f = 2$. We will distinguish the following cases on the restriction of ρ to inertia (after some possible twist):

(ia) $\begin{pmatrix} \omega_2^{r_0+1+p(r_1+1)} & * \\ 0 & 1 \end{pmatrix}$ with $* \neq 0$ and $\mathcal{D}(\rho) = \{(r_0, r_1)\}$

(ib) $\begin{pmatrix} \omega_2^{r_0+1+p(r_1+1)} & * \\ 0 & 1 \end{pmatrix}$ with $* \neq 0$ and $\mathcal{D}(\rho) = \{(r_0, r_1), (p-2-r_0, r_1+1) \otimes \det^{r_0+p(p-1)}\}$

(ic) $\begin{pmatrix} \omega_2^{r_0+1+p(r_1+1)} & * \\ 0 & 1 \end{pmatrix}$ with $* \neq 0$ and $\mathcal{D}(\rho) = \{(r_0, r_1), (r_0+1, p-2-r_1) \otimes \det^{p-1+pr_1}\}$

(ii) $\begin{pmatrix} \omega_2^{r_0+1+p(r_1+1)} & 0 \\ 0 & 1 \end{pmatrix}$

(iii) $\begin{pmatrix} \omega_4^{r_0+1+p(r_1+1)} & 0 \\ 0 & \omega_4^{p^2(r_0+1)+p^3(r_1+1)} \end{pmatrix}$

where $0 \leq r_0, r_1 \leq p-3$ with $(r_0, r_1) \notin \{(0,0), (p-3, p-3)\}$ in the first four cases and $1 \leq r_0 \leq p-2$, $0 \leq r_1 \leq p-3$ in the last (note that if ρ is reducible non-split then $\mathcal{D}(\rho)$ always comes from a subset of Galois type as in Definition 11.5). The corresponding $D_0(\rho)$ is (we don't write the twists by powers of det for each weight, one can recover them from the usual formulas of §11; moreover if a weight has a negative entry, we just forget it):

(ia)

$$(r_0, r_1) \text{ — } S_1 \text{ — } S_2$$

where S_1 is given by :

$$(p-2-r_0, r_1+1) \oplus (r_0-1, p-2-r_1) \oplus (p-2-r_0, r_1-1) \oplus (r_0+1, p-2-r_1)$$

and S_2 by:

$$(p-1-r_0, p-3-r_1) \oplus (p-1-r_0, p-1-r_1) \oplus (p-3-r_0, p-1-r_1) \oplus (p-3-r_0, p-3-r_1)$$

(ib)

$$\big((r_0,r_1) — S_1 — S_2\big) \;\oplus\; \big((p-2-r_0,r_1+1) — S_1' — S_2'\big)$$

where:

$$\begin{aligned}
S_1 &:= (r_0-1,p-2-r_1)\oplus(p-2-r_0,r_1-1)\oplus(r_0+1,p-2-r_1)\\
S_2 &:= (p-1-r_0,p-1-r_1)\oplus(p-3-r_0,p-1-r_1)\\
S_1' &:= (p-3-r_0,p-3-r_1)\oplus(r_0,r_1+2)\oplus(p-1-r_0,p-3-r_1)\\
S_2' &:= (r_0+1,p-4-r_1)\oplus(r_0-1,p-4-r_1)
\end{aligned}$$

(ic)

$$\big((r_0,r_1) — S_1 — S_2\big) \;\oplus\; \big((r_0+1,p-2-r_1) — S_1' — S_2'\big)$$

where:

$$\begin{aligned}
S_1 &:= (p-2-r_0,r_1+1)\oplus(r_0-1,p-2-r_1)\oplus(p-2-r_0,r_1-1)\\
S_2 &:= (p-1-r_0,p-3-r_1)\oplus(p-1-r_0,p-1-r_1)\\
S_1' &:= (p-3-r_0,p-3-r_1)\oplus(r_0+2,r_1)\oplus(p-3-r_0,p-1-r_1)\\
S_2' &:= (p-4-r_0,r_1+1)\oplus(p-4-r_0,r_1-1)
\end{aligned}$$

(ii)

$$\begin{array}{rcccl}
(r_0,r_1) & — & S_1 & — & (p-1-r_0,p-1-r_1)\\
 & & \oplus & & \\
(p-2-r_0,r_1+1) & — & S_1' & — & (r_0-1,p-4-r_1)\\
 & & \oplus & & \\
(r_0+1,p-2-r_1) & — & S_1'' & — & (p-4-r_0,r_1-1)\\
 & & \oplus & & \\
(p-3-r_0,p-3-r_1) & — & S_1''' & — & (r_0+2,r_1+2)
\end{array}$$

where:

$$\begin{aligned}
S_1 &:= (p-2-r_0,r_1-1)\oplus(r_0-1,p-2-r_1)\\
S_1' &:= (r_0,r_1+2)\oplus(p-1-r_0,p-3-r_1)\\
S_1'' &:= (p-3-r_0,p-1-r_1)\oplus(r_0+2,r_1)\\
S_1''' &:= (r_0+1,p-4-r_1)\oplus(p-4-r_0,r_1+1)
\end{aligned}$$

(iii)

$$\begin{array}{ccccc}
(r_0,r_1) & — & S_1 & — & (p-3-r_0,p-1-r_1) \\
 & & \oplus & & \\
(r_0-1,p-2-r_1) & — & S_1' & — & (p-r_0,r_1-1) \\
 & & \oplus & & \\
(p-1-r_0,p-3-r_1) & — & S_1'' & — & (r_0-2,r_1+2) \\
 & & \oplus & & \\
(p-2-r_0,r_1+1) & — & S_1''' & — & (r_0+1,p-4-r_1)
\end{array}$$

where:

$$\begin{array}{rcl}
S_1 & := & (p-2-r_0,r_1-1)\oplus(r_0+1,p-2-r_1) \\
S_1' & := & (r_0-2,r_1)\oplus(p-1-r_0,p-1-r_1) \\
S_1'' & := & (r_0-1,p-4-r_1)\oplus(p-r_0,r_1+1) \\
S_1''' & := & (p-3-r_0,p-3-r_1)\oplus(r_0,r_1+2).
\end{array}$$

The reader can easily find the unique pairing $\{\chi,\chi^s\}$ and check directly that $D_0(\rho)^{I_1}$ has dimension 4 in case (ia), 6 in cases (ib), (ic), 10 in case (ii) and 8 in case (iii). Defining the $\mathfrak{K}_1$-representation $D_1(\rho)$ as previously, the reader can check that, up to isomorphism of basic 0-diagrams, an IZ-equivariant injection $r : D_1(\rho) \hookrightarrow D_0(\rho)$ depends on two scalars in $\overline{\mathbb{F}}_p^\times$ in case (ia), (ib) and (ic), on four scalars in $\overline{\mathbb{F}}_p^\times$ in case (ii) and on one scalar in $\overline{\mathbb{F}}_p^\times$ in case (iii).

17 The representation $R(\sigma)$

For σ a weight, we define and start studying a K-representation $R(\sigma)$ which is a subrepresentation of $\text{c-Ind}_{\mathfrak{K}_0}^G\,\sigma$ and which will contain all the Γ-representations $D_{0,\delta(\sigma)}(\rho)$ for generic tame ρ. Although it might not be strictly necessary, we assume $p>2$ and $\chi\neq\chi^s$ where χ is the character giving the action of I on σ^{I_1}.

We fix $\sigma=(r_0,\cdots,r_{f-1})\otimes\eta$ a weight as above, i.e. such that not all r_i are equal to zero and not all r_i are equal to $p-1$. We extend the K-action on σ to a $\mathfrak{K}_0$-action by making p act trivially and we let χ be the character giving the action of I on σ^{I_1}. Following the notations of [4], for $g\in G$ and $v\in\sigma$ we denote by $[g,v]\in\text{c-Ind}_{\mathfrak{K}_0}^G\,\sigma$ the unique fonction with support in $\mathfrak{K}_0g^{-1}$ which sends g^{-1} to v. Let $r:=r_0+pr_1+\cdots+p^{f-1}r_{f-1}$ and recall that any element of σ can be seen as a polynomial over $\overline{\mathbb{F}}_p$ in the variables

$x^{r-i}y^i$ for $i = i_0 + pi_1 + \cdots + p^{f-1}i_{f-1}$ with $0 \leq i_j \leq r_j$. We first define $\widetilde{R}(\sigma)$ to be the K-subrepresentation of c-$\mathrm{Ind}_{\mathfrak{K}_0}^G \sigma$ generated by the elements:

$$\left(\left[\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}, x^{r-i}y^i\right], i \in \left\{\sum_{j\in J} p^j, J \subseteq J_\sigma\right\}\right)$$

where $J_\sigma := \{i \in \{0, \cdots, f-1\}, r_i \geq 1\}$ and with the convention $\sum_{j\in J} p^j = 0$ when $J = \emptyset$. An easy calculation shows that this is the same as the K-subrepresentation of c-$\mathrm{Ind}_{\mathfrak{K}_0}^G \sigma$ generated by the elements:

$$\Big(\left[\left(\begin{smallmatrix} p & [\lambda_0] \\ 0 & 1 \end{smallmatrix}\right), x^{r-i}y^i\right], \left[\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right), x^{r-i}y^i\right], i \in \{\textstyle\sum_{j\in J} p^j, J \subseteq J_\sigma\}, \lambda_0 \in \mathbb{F}_q\Big).$$

For $J \subseteq J_\sigma$, we define $\mathrm{Fil}^J \widetilde{R}(\sigma)$ to be the K-subrepresentation of $\widetilde{R}(\sigma)$ generated by the element:

$$\left[\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}, x^{r-i}y^i\right] \text{ for } i = \sum_{j\in J} p^j.$$

An easy calculation gives $\mathrm{Fil}^{J'} \widetilde{R}(\sigma) \subsetneq \mathrm{Fil}^J \widetilde{R}(\sigma)$ if $J' \subsetneq J$ hence we have $\mathrm{Fil}^{J_\sigma} \widetilde{R}(\sigma) = \widetilde{R}(\sigma)$.

Lemma 17.1. *(i) For $J \subseteq J_\sigma$, we have:*

$$\frac{\mathrm{Fil}^J \widetilde{R}(\sigma)}{\sum_{J' \subsetneq J} \mathrm{Fil}^{J'} \widetilde{R}(\sigma)} = \mathrm{Ind}_B^\Gamma \chi^s \alpha^{\sum_{i\in J} p^i}.$$

(ii) For $J \subsetneq J_\sigma$ and $j \in J_\sigma \setminus J$, the K-representation:

$$\mathrm{Gr}^{J,j} := \frac{\mathrm{Fil}^{J\cup\{j\}} \widetilde{R}(\sigma)}{\sum_{\substack{J' \subsetneq J\cup\{j\} \\ J' \neq J}} \mathrm{Fil}^{J'} \widetilde{R}(\sigma)}$$

is an extension:

$$0 \longrightarrow \mathrm{Ind}_B^\Gamma \chi^s \alpha^{\sum_{i\in J} p^i} \longrightarrow \mathrm{Gr}^{J,j} \longrightarrow \mathrm{Ind}_B^\Gamma \chi^s \alpha^{p^j + \sum_{i\in J} p^i} \longrightarrow 0 \quad (31)$$

which is isomorphic to the induction from I to K of the extension of I-representations $0 \to \chi^s \alpha^{\sum_{i\in J} p^i} \to * \to \chi^s \alpha^{p^j + \sum_{i\in J} p^i} \to 0$ where the action of I is given in a basis (v, w) of $*$ such that $v \in \chi^s \alpha^{\sum_{i\in J} p^i}$ by:

$$\begin{pmatrix} a & b \\ pc & d \end{pmatrix} w = (\chi^s \alpha^{p^j + \sum_{i\in J} p^i})\left(\begin{pmatrix} a & b \\ pc & d \end{pmatrix}\right)\Big((c/a)^{p^j} v + w\Big). \quad (32)$$

Proof. Straightforward and left to the reader. □

Definition 17.2. *Let σ, χ and $J \subseteq J_\sigma$ as previously and set $r_i^J := r_i$ if $i \notin J$, $r_i^J := r_i - 2$ if $i \in J$ (hence $-1 \leq r_i^J \leq p-3$). We say that an irreducible subquotient of $\mathrm{Ind}_B^\Gamma \chi^s \alpha^{\sum_{i\in J} p^i}$ is special if it is of the form:*

$$\big((\theta_0(r_0^J), \cdots, \theta_{f-1}(r_{f-1}^J)) \otimes \det{}^{e(\theta)(r_0^J,\cdots,r_{f-1}^J)}\big)\det{}^{\sum_{i\in J} p^i}\eta \tag{33}$$

where $(\theta_i(x_i))_i \in \mathcal{P}(x_0, \cdots, x_{f-1})$ (see §2) with $\theta_i(x_i) \in \{p-2-x_i, x_i\}$ if $i \in J$.

Example 17.3. If $f = 1$, we have $J_\sigma = \{0\}$. The special irreducible subquotients of $\mathrm{Ind}_B^\Gamma \chi^s$ are σ and $\sigma^{[s]}$. If $r_0 \geq 2$, the special irreducible subquotient of $\mathrm{Ind}_B^\Gamma \chi^s\alpha$ is $\mathrm{Sym}^{r_0-2}\overline{\mathbb{F}}_p^2 \otimes \det\eta$ and if $r_0 = 1$, $\mathrm{Ind}_B^\Gamma \chi^s\alpha$ has no special irreducible subquotient.

Lemma 17.4. *The Γ-representation $\mathrm{Ind}_B^\Gamma \chi^s \alpha^{\sum_{i\in J} p^i}$ has all its irreducible subquotients special if and only if $J = \emptyset$.*

Proof. If $J = \emptyset$, it follows from Lemma 2.2 that all irreducible subquotients are special. Assume $J \neq \emptyset$. If $r_i \geq 2$ for all $i \in J$, it again follows from the same lemma that all subquotients can't be special (e.g. the cosocle if the representation is indecomposable). Assume $r_i = 1$ for some $i \in J$. As $r_i - 2 = -1$, one checks the socle of $\mathrm{Ind}_B^\Gamma \chi^s \alpha^{\sum_{i\in J} p^i}$ is a weight $(s_0, \cdots, s_{f-1}) \otimes \psi$ with $s_i \in \{p-1, p-2\}$. Hence the cosocle is a weight $(t_0, \cdots, t_{f-1}) \otimes \psi$ with $t_i \in \{0, 1\}$. But as $r_i^J = -1$, any special subquotient (33) is such that $\theta_i(x_i) = p-2-x_i$ i.e. such that $\theta_i(r_i^J) = p-1$. As $p > 2$, $t_i \neq \theta_i(r_i^J)$ and the cosocle is again never special. □

Lemma 17.5. *Assume that $J \neq \emptyset$ and that $\mathrm{Ind}_B^\Gamma \chi^s \alpha^{\sum_{i\in J} p^i}$ has at least one irreducible special subquotient. Let i and j be consecutive elements in J (with possibly $i = j$ if $|J| = 1$). If $r_i = 1$, there is $s \in \{0, \cdots, f-1\}$ such that $i+1 \leq s \leq j-1$ (modulo f) and $r_s > 0$ (and $s \notin J$).*

Proof. Indeed, if this was not the case, then any special subquotient of $\mathrm{Ind}_B^\Gamma \chi^s \alpha^{\sum_{i\in J} p^i}$ as in (33) would necessarily be such that $\theta_i(x_i) = p-2-x_i$ (as $r_i^J = -1$) and $\theta_s(x_s) \neq x_s - 1$ for $i+1 \leq s \leq j-1$ (as $r_s^J = r_s = 0$). As $(\theta_i(x_i))_i \in \mathcal{P}(x_0, \cdots, x_{f-1})$, this implies $\theta_j(x_j) \in \{p-1-x_j, x_j-1\}$. But this is impossible because $j \in J$ implies $\theta_j(x_j) \in \{p-2-x_j, x_j\}$. □

Lemma 17.6. *Let $J' \subsetneq J \subseteq J_\sigma$ and set $\zeta^s := \chi^s \alpha^{\sum_{i\in J'} p^i}$. Let $(s_0, \cdots, s_{f-1}) \otimes \psi$ be the socle of $\mathrm{Ind}_B^\Gamma \zeta^s$ and set $s_i^{J\setminus J'} := s_i$ if $i \notin J \setminus J'$ and $s_i^{J\setminus J'} := s_i - 2$ if $i \in J \setminus J'$. Any special irreducible subquotient of:*

$$\mathrm{Ind}_B^\Gamma \chi^s \alpha^{\sum_{i\in J} p^i} = \mathrm{Ind}_B^\Gamma \zeta^s \alpha^{\sum_{i\in J\setminus J'} p^i}$$

can be written as:

$$(\theta_0(s_0^{J\setminus J'}),\cdots,\theta_{f-1}(s_{f-1}^{J\setminus J'}))\otimes \det{}^{e(\theta)(s_0^{J\setminus J'},\cdots,s_{f-1}^{J\setminus J'})}\det{}^{\sum_{i\in J\setminus J'}p^i}\psi \tag{34}$$

for $\theta\in\mathcal{P}(x_0,\cdots,x_{f-1})$ with $\theta_i(x_i)\in\{p-2-x_i,x_i\}$ if $i\in J\setminus J'$.

Proof. Note first that, as $J'\subsetneq J$ and $\chi\neq\chi^s$, we can't have $\zeta=\zeta^s$ and the socle of $\mathrm{Ind}_B^\Gamma\zeta^s$ is irreducible. If $f=1$, there is nothing to prove as $J'=\emptyset$ in that case. Assume $f>1$. If $r_i\geq 2$ for all $i\in J'$, this follows immediately from Definition 33. If $r_i=1$ for some $i\in J'$, this easily follows from Definition 33 together with Lemma 17.5 (note that, using Lemma 17.5 if there is $i\in J'$ such that $r_i=1$, one has $s_i=r_i\geq 1$ if $i\notin J'$). □

Beware that, conversely, all subquotients as in (34) are not necessarily special in the sense of Definition 33.

Lemma 17.7. *Assume that $J\neq\emptyset$ and let τ be a special irreducible subquotient of $\mathrm{Ind}_B^\Gamma\chi^s\alpha^{\sum_{i\in J}p^i}$. Then τ doesn't occur in $\mathrm{Ind}_B^\Gamma\chi^s\alpha^{\sum_{i\in J'}p^i}$ for $J'\subsetneq J$.*

Proof. Set $\zeta^s:=\chi^s\alpha^{\sum_{i\in J'}p^i}$. By Lemma 17.6, any special irreducible subquotient of $\mathrm{Ind}_B^\Gamma\zeta^s\alpha^{\sum_{i\in J\setminus J'}p^i}$ can be written as:

$$(\theta_0(s_0^{J\setminus J'}),\cdots,\theta_{f-1}(s_{f-1}^{J\setminus J'}))\otimes \det{}^{e(\theta)(s_0^{J\setminus J'},\cdots,s_{f-1}^{J\setminus J'})}\det{}^{\sum_{i\in J\setminus J'}p^i}\psi$$

for $\theta\in\mathcal{P}(x_0,\cdots,x_{f-1})$ with $\theta_i(x_i)\in\{p-2-x_i,x_i\}$ if $i\in J\setminus J'$. By Lemma 2.2, the irreducible subquotients of $\mathrm{Ind}_B^\Gamma\zeta^s$ are:

$$(\lambda_0(s_0),\cdots,\lambda_{f-1}(s_{f-1}))\otimes\det{}^{e(\lambda)(s_0,\cdots,s_{f-1})}\psi$$

for $\lambda\in\mathcal{P}(x_0,\cdots,x_{f-1})$. If a special subquotient of $\mathrm{Ind}_B^\Gamma\zeta^s\alpha^{\sum_{i\in J\setminus J'}p^i}$ also occurs in $\mathrm{Ind}_B^\Gamma\zeta^s$, then by considerations of determinants as in the proof of Lemma 12.8, one can check this implies $\lambda_i(x_i)=\theta_i(x_i-2)$ as formal expressions of x_i for any $i\in J\setminus J'$. This is impossible as $\lambda\in\mathcal{P}(x_0,\cdots,x_{f-1})$, and thus one can't have $\lambda_i(x_i)\in\{x_i-2,p-x_i\}$. □

Lemma 17.8. *Let τ be a special irreducible subquotient of $\mathrm{Ind}_B^\Gamma\chi^s\alpha^{\sum_{i\in J}p^i}$ and $U(\tau)\subseteq\mathrm{Fil}^J\widetilde{R}(\sigma)$ the unique subrepresentation with cosocle τ (which is well defined by Lemma 17.7). Then all irreducible subquotients of $U(\tau)$ are special.*

Proof. If $f=1$, this follows from Example 17.3 so we can assume $f>1$. First, for any $J\subseteq J_\sigma$ and any special subquotient τ of $\mathrm{Ind}_B^\Gamma\chi^s\alpha^{\sum_{i\in J}p^i}$ corresponding to some $\lambda\in\mathcal{P}(x_0,\cdots,x_{f-1})$ by Lemma 2.2, one checks (using Lemma 17.5 if $r_i=1$ for some $i\in J$) that the unique subrepresentation of $\mathrm{Ind}_B^\Gamma \chi^s\alpha^{\sum_{i\in J}p^i}$ with cosocle τ has only special irreducible subquotients, namely all the weights corresponding by Lemma 2.2 to the f-tuples $\lambda' \in \mathcal{P}(x_0,\cdots,x_{f-1})$ with $\lambda' \leq \lambda$ in the sense of §2. Using this inductively, it is enough to prove the following statement: all weights τ' of $\mathrm{Ind}_B^\Gamma \chi^s\alpha^{\sum_{i\in J'}p^i}$ with $J' \subsetneq J$ which could possibly be involved in a non-trivial K-extension with τ are already automatically special. Define ζ^s and $(s_0,\cdots,s_{f-1})\otimes\psi$ as in Lemma 17.6. Using the notations of the proof of Lemma 17.7, write:

$$\begin{aligned}\tau &= (\theta_0(s_0^{J\setminus J'}),\cdots,\theta_{f-1}(s_{f-1}^{J\setminus J'}))\otimes \det{}^{e(\theta)(s_0^{J\setminus J'},\cdots,s_{f-1}^{J\setminus J'})}\det{}^{\sum_{i\in J\setminus J'}p^i}\psi\\ \tau' &= (\lambda_0(s_0),\cdots,\lambda_{f-1}(s_{f-1}))\otimes\det{}^{e(\lambda)(s_0,\cdots,s_{f-1})}\psi\end{aligned}$$

with $\theta,\lambda\in\mathcal{P}(x_0,\cdots,x_{f-1})$ and $\theta_i(x_i)\in\{p-2-x_i,x_i\}$ if $i\in J\setminus J'$. The weights distinct from τ and possibly involved in a K-extension with τ are described in Corollary 5.6. Consider first the extensions ∓ 2, i.e. cases (a) and (c) of (ii) of Corollary 5.6. In order for a weight τ' to be involved in such an extension with τ, the only possibilities are $J = J'\amalg\{j\}$ (for some j), $\theta_i(x_i)=\lambda_i(x_i)$ for $i\neq j$ and $\theta_j(x_j-2)=\lambda_j(x_j)-2$ with $\theta_j(x_j-2)=x_j-2$ or $\theta_j(x_j-2)=\lambda_j(x_j)+2$ with $\theta_j(x_j-2)=p-2-(x_j-2)$ (note that we are dealing with weights, not formal weights, but considerations of determinant as in the proof of Lemma 12.8 show this is actually equivalent). But in both cases, we have $\lambda_j(x_j)=\theta_j(x_j)$ and τ' is thus necessarily the weight $(\theta_0(s_0),\cdots,\theta_{f-1}(s_{f-1}))\otimes\det^{e(\theta)(s_0,\cdots,s_{f-1})}\psi$. It is certainly special in $\mathrm{Ind}_B^\Gamma\zeta^s$ if τ is special in $\mathrm{Ind}_B^\Gamma\zeta^s\alpha^{\sum_{i\in J\setminus J'}p^i}$. Consider now the extensions ∓ 1, i.e. cases (a) and (b) of (i) of Corollary 5.6. In order for a weight τ' to be involved in such an extension with τ, the only possibilities are $J = J'\amalg\{j\}$ (for some j), $\theta_i(x_i)=\lambda_i(x_i)$ for $i\notin\{j-1,j\}$, $\theta_{j-1}(x_{j-1})=p-2-\lambda_{j-1}(x_{j-1})$ and either $\theta_j(x_j-2)=\lambda_j(x_j)-1$ with $\theta_j(x_j-2)=x_j-2$ or $\theta_j(x_j-2)=\lambda_j(x_j)+1$ with $\theta_j(x_j-2)=p-2-(x_j-2)$. Let us suppose $\theta_j(x_j-2)=x_j-2$, then the weight τ is up to twist:

$$(\theta_0(s_0),\cdots,\theta_{j-1}(s_{j-1}),s_j-2,\cdots,\theta_{f-1}(s_{f-1}))$$

whereas the weight τ' is up to twist:

$$(\theta_0(s_0),\cdots,p-2-\theta_{j-1}(s_{j-1}),s_j-1,\cdots,\theta_{f-1}(s_{f-1})).$$

As τ is special, the weight τ can also be written up to twist:

$$(\theta'_0(r_0^J),\cdots,\theta'_{f-1}(r_{f-1}^J))$$

with θ' as in (33) such that $\theta'_j(x_j)=x_j$ (as $r_j^J=r_j-2=s_j-2$). As $r_i^{J'}=r_i^J$ if $i\neq j$ and $r_j^{J'}=r_j$, the weight τ' can thus be rewritten up to twist:

$$(\theta'_0(r_0^{J'}),\cdots,p-2-\theta'_{j-1}(r_{j-1}^{J'}),r_j^{J'}-1,\cdots,\theta'_{f-1}(r_{f-1}^{J'}))$$

which is special in $\mathrm{Ind}_B^\Gamma \zeta^s$ by Definition 17.2 and an easy calculation in $\mathcal{P}(x_0, \cdots, x_{f-1})$. The other case is analogous and left to the reader. No other weight distinct from τ can possibly be involved in a K-extension with τ which has a central character. $\square$

Definition 17.9. *We define $R(\sigma)$ to be the following subrepresentation of $\widetilde{R}(\sigma)$:*

$$R(\sigma) := \sum_\tau U(\tau)$$

for all J and all subrepresentations $U(\tau)$ as in Lemma 17.8.

Example 17.10. Assume $f = 1$. If $r_0 = 1$, we have $R(\sigma) = \mathrm{Ind}_B^\Gamma \chi^s$. If $r_0 \geq 2$, $R(\sigma)$ is an extension:

$$0 \to \mathrm{Ind}_B^\Gamma \chi^s \to R(\sigma) \to \mathrm{Sym}^{r_0-2}\overline{\mathbb{F}}_p^2 \otimes \det\eta \to 0.$$

We will see in §18 that this extension is non-split.

Recall that the set $\mathcal{I}(x_0, \cdots, x_{f-1})$ was defined in §3.

Lemma 17.11. *The irreducible subquotients of $R(\sigma)$ are exactly the (all distinct) weights:*

$$(\mu_0(r_0), \cdots, \mu_{f-1}(r_{f-1})) \otimes \det^{e(\mu)(r_0,\cdots,r_{f-1})}\eta$$

for $\mu_i(x_i) := \lambda_i(p-1-x_i)$ with $\lambda \in \mathcal{I}(x_0, \cdots, x_{f-1})$ and $e(\mu)$ defined in the usual way (forgetting the weights such that $\mu_i(r_i) < 0$ or $\mu_i(r_i) > p-1$ for some i). In particular, they occur in $R(\sigma)$ with multiplicity 1.

Proof. Let μ be as in the statement and set:

$$J := \{i \in \{0, \cdots, f-1\}, \mu_i(x_i) \in \{x_i - 2, p - x_i\}\}.$$

Let $\theta_i(x_i) := \mu_i(x_i)$ if $i \notin J$ and $\theta_i(x_i) := \mu_i(x_i + 2)$ if $i \in J$. Then it is straightforward to check that $\theta \in \mathcal{P}(x_0, \cdots, x_{f-1})$ and that:

$$(\mu_0(r_0), \cdots, \mu_{f-1}(r_{f-1})) \otimes \det^{e(\mu)(r_0,\cdots,r_{f-1})}\eta = (\theta_0(r_0^J), \cdots, \theta_{f-1}(r_{f-1}^J)) \otimes \det^{e(\theta)(r_0^J,\cdots,r_{f-1}^J)}\eta \quad (35)$$

with r_i^J as in Definition 17.2. Hence any weight as in the statement is special and thus occurs in $R(\sigma)$. Conversely, going backwards on (35), any weight as in (33) corresponds to a unique μ as above such that $J = \{i, \mu_i(x_i) \in \{p - x_i, x_i - 2\}\}$. $\square$

Lemma 17.12. *For a weight* $\tau = (\mu_0(r_0), \cdots, \mu_{f-1}(r_{f-1})) \otimes \det^{e(\mu)(r_0,\cdots,r_{f-1})}\eta$ *in* $R(\sigma)$*, define:*

$$\begin{aligned} J(\tau) &:= \{i \in \{0, \cdots, f-1\}, \mu_i(x_i) \in \{x_i - 2, p - x_i\}\} \\ K(\tau) &:= \{i \in \{0, \cdots, f-1\}, \mu_i(x_i) \in \{x_i - 1, x_i - 2, p - x_i, p - 1 - x_i\}\}. \end{aligned}$$

(i) The set $J := J(\tau)$ *is the unique* $J \subseteq J_\sigma$ *such that* τ *is a special subquotient of* $\mathrm{Ind}_B^\Gamma \chi^s \alpha^{\sum_{i\in J} p^i}$*.*

(ii) If a non-split K*-extension* $0 \to \tau' \to \epsilon \to \tau \to 0$ *occurs as a subquotient in* $\mathrm{Fil}^{J(\tau)}\widetilde{R}(\sigma)$*, then it occurs as a subquotient of* $R(\sigma)$ *and we have* $J(\tau') \subseteq J(\tau)$ *and* $K(\tau') \subseteq K(\tau)$*. Moreover, if* ϵ *is a* Γ*-extension, then we have either* $J(\tau') = J(\tau)$ *and* $K(\tau') \subsetneq K(\tau)$ *or* $J(\tau') \subsetneq J(\tau)$ *and* $K(\tau') = K(\tau)$*, and* $|J(\tau) \cup K(\tau)| = |J(\tau') \cup K(\tau')| + 1$ *in both cases.*

Proof. If $f = 1$, this follows from Example 17.10 so we can assume $f > 1$. (i) follows from Lemma 17.11 and its proof (see (35)). Let us prove (ii). By the definition of $U(\tau)$ in Lemma 17.8, the non-split extension ϵ must be a quotient of $U(\tau)$ and hence is also a subquotient of $R(\sigma)$. Write $\tau' = (\mu'_0(r_0), \cdots, \mu'_{f-1}(r_{f-1})) \otimes \det^{e(\mu')(r_0,\cdots,r_{f-1})}\eta$ with μ' as in Lemma 17.11. Assume ϵ sits in $\mathrm{Ind}_B^\Gamma \chi^s \alpha^{\sum_{i\in J(\tau)} p^i}$ (i.e. $J(\tau') = J(\tau)$). If $r_j \geq 2$ for all $j \in J(\tau)$, then it directly follows from Theorem 2.4 (together with Lemma 2.2) that $K(\tau') \subsetneq K(\tau)$ and $|K(\tau)| = |K(\tau')| + 1$. If $r_j = 1$ for some $j \in J(\tau)$, this is still true but one has to use Lemma 17.5. Assume now that τ' comes from a distinct parabolic induction inside $\widetilde{R}(\sigma)$, we are then exactly in the situation of the proof of Lemma 17.8. If the extension ϵ is of type ± 2 (i.e. it is not a Γ-extension), going back to this proof, we see that we necessarily have $j \in J(\tau)$ such that $(\mu'_j(x_j) = x_j,\ \mu_j(x_j) = x_j - 2)$ or $(\mu'_j(x_j) = p - 2 - x_j,\ \mu_j(x_j) = p - x_j)$ and $\mu'(x_i) = \mu(x_i)$ for $i \neq j$. In both cases, we have $J(\tau') \subsetneq J(\tau)$ and $K(\tau') \subsetneq K(\tau)$. Assume now that the extension ϵ is of type ± 1 (i.e. is a Γ-extension). Again, by the proof of Lemma 17.8, we necessarily have $j \in J(\tau)$ such that $\mu'_{j-1}(x_{j-1}) = p - 2 - \mu_{j-1}(x_{j-1})$ and either $(\mu'_j(x_j) = x_j - 1,\ \mu_j(x_j) = x_j - 2)$ or $(\mu'_j(x_j) = p - 1 - x_j,\ \mu_j(x_j) = p - x_j)$ (and $\mu'_i(x_i) = \mu_i(x_i)$ for $i \notin \{j-1, j\}$). The operation $p - 2 - \cdot$ preserving $J(\tau')$ and $K(\tau')$, we see that we have $J(\tau') \subsetneq J(\tau)$ (with $|J(\tau)| = |J(\tau')| + 1$) and $K(\tau') = K(\tau)$. □

18 The extension Lemma

In this section, we crucially use that we are working with Witt vectors. We keep the assumptions of §17 ($p > 2$, $\chi \neq \chi^s$) and prove that $R(\sigma)$ contains many non-split extensions. The existence of these non-split extensions will imply the irreducibility of some G-representations (§19).

We start with three easy lemmas.

Lemma 18.1. *Let τ, τ' be two distinct weights, $Q_{\tau'}$ (resp. S_τ) a representation of Γ on a finite dimensional $\overline{\mathbb{F}}_p$-vector space with socle τ' (resp. cosocle τ), and R a Γ-extension $0 \to Q_{\tau'} \to R \to S_\tau \to 0$. Assume that, if w' (resp. w) is an irreducible subquotient of $Q_{\tau'}$ (resp. S_τ) with $(w', w) \neq (\tau', \tau)$, we have $\mathrm{Ext}^1_\Gamma(w, w') = 0$. Then R is obtained by push-forward along $\tau' \hookrightarrow Q_{\tau'}$ and pull-back along $S_\tau \twoheadrightarrow \tau$ from a Γ-extension $0 \to \tau' \to \epsilon \to \tau \to 0$.*

Proof. For any irreducible constituent w' of $Q_{\tau'}$ distinct from the socle τ', we have $\mathrm{Ext}^1_\Gamma(S_\tau, w') = 0$. By the usual long exact sequence for Hom and Ext, we derive a surjection $\mathrm{Ext}^1_\Gamma(S_\tau, \tau') \twoheadrightarrow \mathrm{Ext}^1_\Gamma(S_\tau, Q_{\tau'})$. For any irreducible constituent w of S_τ distinct from the cosocle τ, we have $\mathrm{Ext}^1_\Gamma(w, \tau') = 0$. We derive again a surjection $\mathrm{Ext}^1_\Gamma(\tau, \tau') \twoheadrightarrow \mathrm{Ext}^1_\Gamma(S_\tau, \tau')$. This implies the statement. □

Lemma 18.2. *Let τ', τ be two weights and ϵ a Γ-extension $0 \to \tau' \to \epsilon \to \tau \to 0$. Let $F \in \epsilon$ be a non-zero H-eigenvector with eigencharacter χ where χ is the action of I on τ^{I_1}. Assume that χ doesn't occur as an H-eigencharacter on τ' and that $\langle \Gamma \cdot F \rangle$ contains τ'. Then ϵ is non-split.*

Proof. Note that τ and τ' are necessarily distinct because of the assumption on χ. If ϵ was split, as χ doesn't occur in τ' we would have that F necessarily belongs to τ via a splitting $\tau \hookrightarrow \epsilon$. This would imply $\langle \Gamma \cdot F \rangle = \tau$ which contradicts $\tau' \subset \langle \Gamma \cdot F \rangle$. □

Lemma 18.3. *Let c be an integer between 0 and $p-2$. Then the following equality holds in $\mathbb{F}_p$:*

$$\sum_{n=0}^{c} \binom{c}{n} \frac{\binom{p}{c+1-n}}{p} = \frac{1}{c+1}.$$

Proof. Exercise. □

The following lemma is the main result of this section and its proof is a long computation.

Lemma 18.4. *Let $\tau' := (t'_0, \cdots, t'_{f-1}) \otimes \eta_{\tau'}$ and $\tau := (t_0, \cdots, t_{f-1}) \otimes \eta_\tau$ be two irreducible subquotients of $R(\sigma)$. If $f > 1$, assume there is $i \in \{0, \cdots, f-1\}$ such that $t_i = p - 2 - t'_i$, $t_{i+1} = t'_{i+1} \pm 1$ and $\eta_\tau = \eta_{\tau'} \det^{p^i(t'_i+1) - 1/2(1\pm1)p^{i+1}}$ (with $i+1 = 0$ if $i = f-1$). If $f = 1$, assume $t_0 = p - 2 - t'_0 \pm 1$ and*

$\eta_\tau = \eta_{\tau'}\det^{t_0'+1-1/2(1\pm 1)p}$. *Then either the unique non-split* Γ*-extension* $0 \to \tau' \to \epsilon \to \tau \to 0$ *or the unique non-split* Γ*-extension* $0 \to \tau \to \epsilon \to \tau' \to 0$ *occurs as a subquotient of* $R(\sigma)$.

Proof. We divide the proof into 6 parts (i) to (vi).
(i) If τ' and τ occur as subquotients of the same $\mathrm{Ind}_B^\Gamma \chi^s\alpha^{\sum_{i\in J}p^i}$, then the result follows from the structure of such Γ-representations (see Theorem 2.4). So assume that τ' is in $\mathrm{Ind}_B^\Gamma \chi^s\alpha^{\sum_{i\in J'}p^i}$ and τ in $\mathrm{Ind}_B^\Gamma \chi^s\alpha^{\sum_{i\in J}p^i}$ with J' and J distinct. Switching τ' and τ if necessary, the same proof as for (ii) of Lemma 17.12 implies that we can assume $J' \subsetneq J$. We first assume $f > 1$. Using notations as in Lemmas 17.6 and 17.8 and twisting everything by ψ^{-1}, the same proof as the second half of the proof of Lemma 17.8 shows we can assume $J = J' \amalg \{j\}$, $\tau' \in \mathrm{Ind}_B^\Gamma \zeta^s$, $\tau \in \mathrm{Ind}_B^\Gamma \zeta^s\alpha^{p^j}$ and:

$$\begin{aligned}\tau' &= (\lambda_0(s_0),\cdots,\lambda_{f-1}(s_{f-1}))\otimes \det^{e(\lambda)(s_0,\cdots,s_{f-1})}\\ \tau &= (\theta_0(s_0),\cdots,\theta_j(s_j-2),\cdots,\theta_{f-1}(s_{f-1}))\otimes \det^{e(\theta)(s_0,\cdots,s_j-2,\cdots,s_{f-1})}\det^{p^j}\end{aligned}$$

where $(s_0,\cdots,s_{f-1})$ is the socle of $\mathrm{Ind}_B^\Gamma \zeta^s$ with $s_j \geq 1$ (we have $\zeta \neq \zeta^s$, see the proof of Lemma 17.6), where $\theta, \lambda \in \mathcal{P}(x_0,\cdots,x_{f-1})$ with $\theta_i(x_i) = \lambda_i(x_i)$ if $i \notin \{j-1,j\}$, $\theta_j(x_j-2) \in \{x_j-2, p-x_j\}$, $\theta_{j-1}(x_{j-1}) = p-2-\lambda_{j-1}(x_{j-1})$ and one of the following two possibilities occurs:

$$\begin{array}{lllllll}\text{case } -1 & : & \lambda_j(x_j) & = & x_j-1 & \theta_j(x_j-2) & = & x_j-2\\ \text{case } +1 & : & \lambda_j(x_j) & = & p-1-x_j & \theta_j(x_j-2) & = & p-x_j.\end{array}$$

(ii) As in §2, define:

$$\begin{aligned}J(\lambda) &:= \{i\in\{0,\cdots,f-1\}, \lambda_i(x_i)\in\{p-2-x_i, p-1-x_i\}\}\\ J(\theta) &:= \{i\in\{0,\cdots,f-1\}, \theta_i(x_i)\in\{p-2-x_i, p-1-x_i\}\}\end{aligned}$$

and note that $j-1 \in J(\lambda)$, $J(\theta) = J(\lambda)\setminus\{j-1\}$ and $j \in J(\theta)$ (or $J(\lambda)$) if and only if $\theta_j(x_j-2) = p-x_j$. With the notations of Lemma 17.1, let us work inside the representation:

$$0 \to \mathrm{Ind}_B^\Gamma \zeta^s \to \mathrm{Gr}^{J',j} \to \mathrm{Ind}_B^\Gamma \zeta^s\alpha^{p^j} \to 0$$

which is isomorphic by (ii) of Lemma 17.1 to $\mathrm{Ind}_I^K(\overline{\mathbb{F}}_p v \oplus \overline{\mathbb{F}}_p w)$, the action of I being given as in (32) with $\zeta^s\alpha^{p^j}$ instead of $\chi^s\alpha^{p^j+\sum_{i\in J}p^i}$. Let us denote by ϕ_v (resp. ϕ_w) the unique function in $\mathrm{Ind}_I^K(\overline{\mathbb{F}}_p v \oplus \overline{\mathbb{F}}_p w)$ with support in I sending 1 to v (resp. w). Let χ_τ be the character giving the action of H on τ^{I_1}. We define $F \in \mathrm{Ind}_I^K(\overline{\mathbb{F}}_p v \oplus \overline{\mathbb{F}}_p w)$ as follows:

$$\begin{aligned}\text{case } -1: F &:= \sum_{\lambda\in\mathbb{F}_q}\lambda^{\sum_{i\in J(\theta)}p^i(p-1-\theta_i(s_i))}\begin{pmatrix}[\lambda] & 1\\ 1 & 0\end{pmatrix}\phi_w + \varepsilon(\tau)(-1)^{p^j}\phi_w\\ \text{case } +1: F &:= \sum_{\lambda\in\mathbb{F}_q}\lambda^{p^j(p-1-\theta_j(s_j-2))+\sum_{i\in J(\theta)\setminus\{j\}}p^i(p-1-\theta_i(s_i))}\begin{pmatrix}[\lambda] & 1\\ 1 & 0\end{pmatrix}\phi_w\end{aligned}$$

where $\varepsilon(\tau) := 1$ if $\zeta^s \alpha^{p^j} = (\zeta^s \alpha^{p^j})^s$ (which implies $s_j = 2$ and τ 1-dimensional) and $\varepsilon(\tau) := 0$ otherwise. There is a conflict of notations between $\lambda \in \mathcal{P}(x_0, \cdots, x_{f-1})$ and $\lambda \in \mathbb{F}_q$ but there is no possible confusion between the two. The element F is an H-eigenvector of eigenvalue χ_τ and its image in $\mathrm{Ind}_B^\Gamma \zeta^s \alpha^{p^j}$ maps to a basis of τ^{I_1} in any quotient of $\mathrm{Ind}_B^\Gamma \zeta^s \alpha^{p^j}$ where τ is a subrepresentation. If $s_j \geq 2$, this follows directly from Lemma 2.7 or Lemma 2.6 applied to τ and $\mathrm{Ind}_B^\Gamma \zeta^s \alpha^{p^j}$. If $s_j = 1$ (which implies $\theta_j(x_j - 2) = p - x_j$ and we are in the case $+1$), this is still true but requires a small computation together with Lemma 17.5 (the set $J(\theta)$ is then strictly larger than the set corresponding to τ in (ii) of Lemma 2.7, for instance it contains j, but the extra indices i of $J(\theta)$ are harmless since they are all such that $\theta_i(s_i) = p-1$ if $i \neq j$ or $i = j$ and $\theta_j(s_j - 2) = p-1$ and thus $p-1-\theta_i(s_i) = p-1-\theta_j(s_j-2) = 0$). We are going to prove that $\langle K \cdot F \rangle$ contains τ' (as a subquotient).
(iii) Consider first the case -1, which implies $s_j \geq 2$ and $j \notin J(\theta)$. First, F is fixed by K_1 in any quotient of $\mathrm{Ind}_I^K(\overline{\mathbb{F}}_p v \oplus \overline{\mathbb{F}}_p w)$ coming by push-forward from a quotient of $\mathrm{Ind}_I^K(\overline{\mathbb{F}}_p v) = \mathrm{Ind}_B^\Gamma \zeta^s$ containing τ' as subrepresentation. Indeed, any matrix of K_1 acts on F by adding to F a linear combination of the following vectors:

$$\sum_{\lambda \in \mathbb{F}_q} \lambda^{\sum_{i \in J(\theta)} p^i(p-1-\theta_i(s_i))} \begin{pmatrix} [\lambda] & 1 \\ 1 & 0 \end{pmatrix} \phi_v$$
$$\sum_{\lambda \in \mathbb{F}_q} \lambda^{p^j + \sum_{i \in J(\theta)} p^i(p-1-\theta_i(s_i))} \begin{pmatrix} [\lambda] & 1 \\ 1 & 0 \end{pmatrix} \phi_v$$
$$\sum_{\lambda \in \mathbb{F}_q} \lambda^{2p^j + \sum_{i \in J(\theta)} p^i(p-1-\theta_i(s_i))} \begin{pmatrix} [\lambda] & 1 \\ 1 & 0 \end{pmatrix} \phi_v + \varepsilon(\tau)\phi_v.$$

Using Lemma 2.7 together with Theorem 2.4, one checks that these vectors are zero in any quotient of $\mathrm{Ind}_I^K \zeta^s$ containing τ' as subobject (use $s_j \geq 2$, $J(\theta) \subsetneq J(\lambda)$, $j \notin J(\lambda)$ and $\theta_i(s_i) = \lambda_i(s_i)$ for $i \in J(\theta)$). A computation yields now for $\delta \in \mathbb{F}_q$:

$$\begin{pmatrix} 1 & [\delta] \\ 0 & 1 \end{pmatrix} F = \sum_{\lambda \in \mathbb{F}_q} (\lambda - \delta)^{\sum_{i \in J(\theta)} p^i(p-1-\theta_i(s_i))} \begin{pmatrix} [\lambda] & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ p[X] & 1 \end{pmatrix} \phi_w + \varepsilon(\tau)(-1)^{p^j} \phi_w$$

where:

$$X := \sum_{s=1}^{p-1} \frac{\binom{p}{s}}{p} \lambda^{p^{-1}(p-s)} (-\delta)^{p^{-1}s}.$$

Note that X comes from the addition law $[\lambda] + [-\delta] \equiv [\lambda - \delta] - p[X]$ (p^2) in $W(\mathbb{F}_q)$. Using (32), we obtain for $\left(\begin{smallmatrix} 1 & [\delta] \\ 0 & 1 \end{smallmatrix}\right) F$, up to multiplication by a non-zero scalar:

$$\sum_{\lambda\in\mathbb{F}_q}(\lambda-\delta)^{\sum_{i\in J(\theta)}p^i(p-1-\theta_i(s_i))}\left(\sum_{s=1}^{p-1}\frac{\binom{p}{s}}{p}\lambda^{p^{j-1}(p-s)}(-\delta)^{p^{j-1}s}\right)\begin{pmatrix}[\lambda] & 1\\ 1 & 0\end{pmatrix}\phi_v+$$
$$\sum_{\lambda\in\mathbb{F}_q}(\lambda-\delta)^{\sum_{i\in J(\theta)}p^i(p-1-\theta_i(s_i))}\begin{pmatrix}[\lambda] & 1\\ 1 & 0\end{pmatrix}\phi_w+\varepsilon(\tau)(-1)^{p^j}\phi_w.$$

Rewriting this $\sum_{t=0}^{p^f-1}\delta^t A_t$ and varying δ in $\mathbb{F}_q$, we get that all the elements A_t are in $\langle K\cdot F\rangle$, in particular the element $A_{p^{j-1}}$ which is, up to multiplication by a non-zero scalar and since $J(\lambda)=J(\theta)\amalg\{j-1\}$:

$$\sum_{\lambda\in\mathbb{F}_q}\lambda^{\sum_{i\in J(\lambda)\setminus\{j-1\}}p^i(p-1-\lambda_i(s_i))}\lambda^{p^{j-1}(p-1)}\begin{pmatrix}[\lambda] & 1\\ 1 & 0\end{pmatrix}\phi_v\in\operatorname{Ind}_B^\Gamma\zeta^s.$$

By Lemma 2.7, this element generates τ' inside $\operatorname{Ind}_B^\Gamma\zeta^s$.
(iv) Consider now the case $+1$. Here, one can check using Lemma 2.7 and calculations analogous to those of the case -1 that F is now I_1-invariant (and not just K_1-invariant) in any quotient of $\operatorname{Ind}_I^K(\overline{\mathbb{F}}_pv\oplus\overline{\mathbb{F}}_pw)$ coming by push-forward from a quotient of $\operatorname{Ind}_I^K(\overline{\mathbb{F}}_pv)$ containing τ' as subobject. We will thus need the action of $\left(\begin{smallmatrix}[\delta] & 1\\ 1 & 0\end{smallmatrix}\right)\in K$. Using the equality (for $\lambda\neq 0$):

$$\begin{pmatrix}[\delta] & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}[\lambda] & 1\\ 1 & 0\end{pmatrix}=\begin{pmatrix}[\lambda^{-1}]+[\delta] & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}[\lambda] & 0\\ 0 & -[\lambda^{-1}]\end{pmatrix}\begin{pmatrix}1 & [\lambda^{-1}]\\ 0 & 1\end{pmatrix}$$

and the fact that F is I_1-invariant and that ϕ_w is an H-eigenvector, we get for $\delta\in\mathbb{F}_q$:

$$\begin{pmatrix}[\delta] & 1\\ 1 & 0\end{pmatrix}F=\varepsilon(\tau)\phi_w\pm\sum_{\lambda\in\mathbb{F}_q^\times}\lambda^{b(\tau)+p^j(2-s_j)-\sum_{i\neq j}p^is_i}\begin{pmatrix}[\lambda^{-1}+\delta] & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ pY & 1\end{pmatrix}\phi_w$$

where we didn't bother to check the sign, where:

$$\begin{aligned}Y &:= -\sum_{s=1}^{p-1}\frac{\binom{p}{s}}{p}\lambda^{-p^{-1}(p-s)}\delta^{p^{-1}s}\\ b(\tau) &:= p^j(p-1-\theta_j(s_j-2))+\sum_{i\in J(\theta)\setminus\{j\}}p^i(p-1-\theta_i(s_i))\end{aligned}$$

and where $\varepsilon(\tau):=1$ if τ is the socle of $\operatorname{Ind}_B^\Gamma\zeta^s\alpha^{p^j}$ and 0 otherwise (compare with (4) and (5)). Note that, in the case $+1$, we have $\zeta^s\alpha^{p^j}\neq(\zeta^s\alpha^{p^j})^s$,

$\varepsilon(\tau) = 1$ implies $s_j = 1$ and the term with $\lambda = 0$ in F is non-zero if and only if $\varepsilon(\tau) = 1$. This can be rewritten as follows (using (32) for the action of $\left(\begin{smallmatrix} 1 & 0 \\ pY & 1 \end{smallmatrix}\right)$):

$$\pm \sum_{\lambda \in \mathbb{F}_q^\times} \lambda^{c(\tau)} \left(\sum_{s=1}^{p-1} \frac{\binom{p}{s}}{p} \lambda^{p^{j-1}(p-s)} \delta^{p^{j-1}s} \right) \begin{pmatrix} [\lambda + \delta] & 1 \\ 1 & 0 \end{pmatrix} \phi_v$$
$$\pm \sum_{\lambda \in \mathbb{F}_q^\times} \lambda^{c(\tau)} \begin{pmatrix} [\lambda + \delta] & 1 \\ 1 & 0 \end{pmatrix} \phi_w + \varepsilon(\tau) \phi_w$$

where:

$$c(\tau) := -p^j + \sum_{i \neq j} p^i s_i - \sum_{i \in J(\theta) \backslash \{j\}} p^i (p - 1 - \theta_i(s_i)).$$

Using $\theta \in \mathcal{P}(x_0, \cdots, x_{f-1})$ and $\theta_j(x_j) = p - 2 - x_j$, a small computation gives $c(\tau) \equiv d(\tau)$ modulo $p^f - 1$ where:

$$d(\tau) := \sum_{i \in J(\theta)} p^i (p-1) + \sum_{i \notin J(\theta)} p^i \theta_i(s_i),$$

hence one finally gets (changing λ into $\lambda - \delta$):

$$\begin{pmatrix} [\delta] & 1 \\ 1 & 0 \end{pmatrix} F = \pm \sum_{\lambda \in \mathbb{F}_q} (\lambda - \delta)^{d(\tau)} \left(\sum_{s=1}^{p-1} \frac{\binom{p}{s}}{p} \lambda^{p^{j-1}(p-s)} (-\delta)^{p^{j-1}s} \right) \begin{pmatrix} [\lambda] & 1 \\ 1 & 0 \end{pmatrix} \phi_v$$
$$\pm \sum_{\lambda \in \mathbb{F}_q} (\lambda - \delta)^{d(\tau)} \begin{pmatrix} [\lambda] & 1 \\ 1 & 0 \end{pmatrix} \phi_w + \varepsilon(\tau) \phi_w. \quad (36)$$

Consider the expression:

$$(\lambda - \delta)^{p^{j-1}\theta_{j-1}(s_{j-1})} \Big(\sum_{s=1}^{p-1} \frac{\binom{p}{s}}{p} \lambda^{p^{j-1}(p-s)} (-\delta)^{p^{j-1}s} \Big), \quad (37)$$

the coefficient of $\delta^{p^{j-1}(\theta_{j-1}(s_{j-1})+1)}$ in (37) is (up to sign):

$$\left(\sum_{n=0}^{\theta_{j-1}(s_{j-1})} \binom{\theta_{j-1}(s_{j-1})}{n} \frac{\binom{p}{\theta_{j-1}(s_{j-1})+1-n}}{p} \right) \lambda^{p^{j-1}(p-1)} = \frac{1}{\theta_{j-1}(s_{j-1}) + 1} \lambda^{p^{j-1}(p-1)}$$

where the equality comes from Lemma 18.3 (note that one always has $0 \leq \theta_{j-1}(s_{j-1}) \leq p - 2$). In particular, it is never zero. Now writing:

$$d(\tau) = \sum_{i \in J(\theta)} p^i (p-1) + \sum_{\substack{i \notin J(\theta) \\ i \neq j-1}} p^i \theta_i(s_i) + p^{j-1} \theta_{j-1}(s_{j-1}),$$

we deduce that the coefficient of $\delta^{p^{j-1}(\theta_{j-1}(s_{j-1})+1)}$ in (36) is, up to multiplication by a non-zero scalar, the element:

$$\sum_{\lambda\in\mathbb{F}_q}\lambda^{\sum_{i\in J(\theta)}p^i(p-1)+\sum_{\substack{i\notin J(\theta)\\ i\neq j-1}}p^i\theta_i(s_i)+p^{j-1}(p-1)}\begin{pmatrix}[\lambda] & 1\\ 1 & 0\end{pmatrix}\phi_v.$$

Varying δ in $\mathbb{F}_q$ as for the case -1, we get that this element belongs to $\langle K\cdot F\rangle$. But since $J(\lambda)=J(\theta)\amalg\{j-1\}$, this element is precisely:

$$\sum_{\lambda\in\mathbb{F}_q}\lambda^{\sum_{i\in J(\lambda)}p^i(p-1)+\sum_{i\notin J(\lambda)}p^i\lambda_i(s_i)}\begin{pmatrix}[\lambda] & 1\\ 1 & 0\end{pmatrix}\phi_v$$

which generates τ' inside $\mathrm{Ind}_B^\Gamma\zeta^s$ by Lemma 2.7. In all cases, we have that τ' occurs as a subquotient of $\langle K\cdot F\rangle$.
(v) Let now S_τ be the unique subrepresentation of $\mathrm{Ind}_B^\Gamma\zeta^s\alpha^{p^j}$ with cosocle τ, by the definition of F, we have a surjection of Γ-representations $\langle K\cdot F\rangle\twoheadrightarrow S_\tau$ (see (ii)). Denote by $Q'_{\tau'}\subseteq\mathrm{Ind}_B^\Gamma\zeta^s$ its kernel (which contains τ' as a subquotient), $Q_{\tau'}$ the unique quotient of $Q'_{\tau'}$ with socle τ' and R the corresponding quotient of $\langle K\cdot F\rangle$ obtained by push-forward. We thus have an exact sequence of Γ-representations $0\to Q_{\tau'}\to R\to S_\tau\to 0$ which is a subquotient of $\mathrm{Ind}_I^K(\overline{\mathbb{F}}_p v\oplus\overline{\mathbb{F}}_p w)$. Let (w',w) be irreducible constituents of respectively $Q_{\tau'}$ and S_τ. By Lemma 17.8, w is special as τ is. Assume $\mathrm{Ext}^1_\Gamma(w,w')\neq 0$, then by Lemma 17.8, w' is also special and by (ii) of Lemma 17.12, we have:

$$K(\tau')\subseteq K(w')=K(w)\subseteq K(\tau).$$

But the same proof as for the last part of (ii) of Lemma 17.12 shows $K(\tau')=K(\tau)$, hence $K(\tau')=K(w')$ and $K(w)=K(\tau)$. By (ii) of Lemma 17.12 again, this implies $(w',w)=(\tau',\tau)$. We can thus apply Lemma 18.1 saying that R contains as a subquotient a Γ-extension of τ by τ'. One can easily check that χ_τ doesn't occur as an H-eigenvalue in τ'. We can thus apply Lemma 18.2 saying that this extension is non-split. This finishes the proof for $f>1$.
(vi) Assume finally $f=1$. Going back to the beginning of (i), from Example 17.10 we can assume $r_0\geq 2$, $\tau'=\mathrm{Sym}^{p-1-r_0}\overline{\mathbb{F}}_p^2\otimes\det^{r_0}\eta$ and $\tau=\mathrm{Sym}^{r_0-2}\overline{\mathbb{F}}_p^2\otimes\det\eta$. A completely analogous computation as the one in (iii) with $F:=\sum_{\lambda\in\mathbb{F}_p}\left(\begin{smallmatrix}[\lambda] & 1\\ 1 & 0\end{smallmatrix}\right)\phi_w-\varepsilon(\tau)\phi_w$ ($\varepsilon(\tau)=1$ if $r_0=2$ and $\varepsilon(\tau)=0$ otherwise) shows that $\langle K\cdot F\rangle$ contains τ', and hence by (v) that the corresponding extension is non-split. $\square$

19 Generic Diamond diagrams and representations of GL_2

We prove the main results of the paper (Theorems 1.4 and 1.5). Since there are no generic ρ if $p = 2$ (see the end of §11), we can assume $p > 2$ all along.

We start with several lemmas. Recall $\delta(\sigma)$ was defined in §15 and $R(\sigma)$ in Definition 17.9.

Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^f}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous generic tame representation. Let $\sigma \in \mathcal{D}(\rho)$ and denote by χ the action of I on σ^{I_1}. By Lemma 15.2 applied to $\tau = \sigma$, the weight $\delta(\sigma)$ is a component of $\mathrm{Ind}_B^\Gamma \chi^s$. Writing $\sigma := (s_0, \cdots, s_{f-1}) \otimes \theta$, it is thus of the form:

$$\delta(\sigma) = (\xi_0(s_0), \cdots, \xi_{f-1}(s_{f-1})) \otimes \det{}^{e(\xi)(s_0,\cdots,s_{f-1})}\theta$$

for a unique $\xi \in \mathcal{P}(x_0, \cdots, x_{f-1})$. Set $\mathcal{S}(\xi) := \{i \in \{0, \cdots, f-1\}, \xi_i(x_i) \in \{x_i - 1, p-1-x_i\}\}$ as in §4. Note that $\mathcal{S}(\xi)$ determines uniquely ξ in $\mathcal{P}(x_0, \cdots, x_{f-1})$.

Lemma 19.1. *Keep the previous notations and let $\lambda \in \mathcal{RD}(x_0, \cdots, x_{f-1})$ or $\mathcal{ID}(x_0, \cdots, x_{f-1})$ correspond to σ via Lemma 11.2 or 11.4.*

(i) Assume ρ is reducible, we have:

$$\mathcal{S}(\xi) = \{i \in \{0, \cdots, f-1\}, \lambda_i(x_i) \in \{p-2-x_i, x_i+1\}\}.$$

(ii) Assume ρ is irreducible. If $\lambda_0(x_0) \in \{p-2-x_0, x_0-1\}$, we have:

$$\mathcal{S}(\xi) = \{i \in \{1, \cdots, f-1\}, \lambda_i(x_i) \in \{p-2-x_i, x_i+1\}\}$$

and if $\lambda_0(x_0) \in \{p-1-x_0, x_0\}$, we have:

$$\mathcal{S}(\xi) = \{i \in \{1, \cdots, f-1\}, \lambda_i(x_i) \in \{p-2-x_i, x_i+1\}\} \amalg \{0\}.$$

Proof. With the usual notations for ρ as in Definition 11.7, recall we have $\sigma = (\lambda_0(r_0), \cdots, \lambda_{f-1}(r_{f-1})) \otimes \det^{e(\lambda)(r_0,\cdots,r_{f-1})}\eta$. (i) Let $\zeta \in \mathcal{P}(x_0, \cdots, x_{f-1})$ be the unique element such that $\mathcal{S}(\zeta) := \{i \in \{0, \cdots, f-1\}, \lambda_i(x_i) \in \{p-2-x_i, x_i+1\}\}$ with $\mathcal{S}(\zeta)$ defined as in §4. A straightforward computation shows that applying ζ to λ, that is computing $(\zeta_i(\lambda_i(x_i))_i$, pushes all sequences $p-2-\cdot, p-3-\cdot, p-3-\cdot, \cdot+1$ on λ one step to the left. By definition of $\delta(\sigma)$, we have thus:

$$\delta(\sigma) = (\zeta_0(\lambda_0(r_0)), \cdots, \zeta_{f-1}(\lambda_{f-1}(r_{f-1}))) \otimes \det{}^{e(\zeta)(\lambda_i(r_i))}\det{}^{e(\lambda)(r_i)}\eta$$

which implies $\zeta = \xi$ and hence yields (i). (ii) is analogous. $\square$

Keep the previous notations and define $\mu_\xi \in \mathcal{I}(y_0, \cdots, y_{f-1})$ as follows:

(i) $\mu_{\xi,i}(y_i) := p - 1 - y_i$ if $\xi_i(x_i) \in \{x_i - 1, x_i\}$

(i) $\mu_{\xi,i}(y_i) := p - 3 - y_i$ if $\xi_i(x_i) \in \{p - 2 - x_i, p - 1 - x_i\}$.

Lemma 19.2. *We keep the previous notations.*

(i) The irreducible subquotients of $D_{0,\delta(\sigma)}(\rho)$ are exactly the (all distinct) weights:

$$(\mu_0(\xi_0(s_0)), \cdots, \mu_{f-1}(\xi_{f-1}(s_{f-1}))) \otimes \det^{e(\mu\circ\xi)(s_0,\cdots,s_{f-1})}\theta \tag{38}$$

for $\mu \in \mathcal{I}(y_0, \cdots, y_{f-1})$ such that μ and μ_ξ are compatible (see Definition 4.10) forgetting the weights such that $\mu_i(\xi_i(s_i)) < 0$ or $\mu_i(\xi_i(s_i)) > p - 1$ for some i.

(ii) The graded pieces of the socle filtration on $D_{0,\delta(\sigma)}(\rho)$ are:

$$D_{0,\delta(\sigma)}(\rho)_i = \bigoplus_{\ell(\mu)=i} \tau$$

for $0 \leq i \leq f - 1$ and weights τ as in (38) with $\ell(\mu)$ as in §4.

Proof. Let $\delta_r(\lambda)$ (resp. $\delta_i(\lambda)$) be the f-tuple of $\mathcal{RD}(x_0, \cdots, x_{f-1})$ (resp. $\mathcal{ID}(x_0, \cdots, x_{f-1})$) associated to $\delta(\sigma)$. From Theorem 14.8 applied to $\delta(\sigma)$, it is enough to prove the following:

(i) $\xi_i(x_i) \in \{x_i - 1, x_i\}$ if and only if $\delta_r(\lambda)_i(x_i) \in \{p - 3 - x_i, x_i\}$ (resp. for $i > 0$ and $\delta_i(\lambda)_0(x_0) \in \{p - 2 - x_0, x_0 - 1\}$)

(ii) $\xi_i(x_i) \in \{p-2-x_i, p-1-x_i\}$ if and only if $\delta_r(\lambda)_i(x_i) \in \{p-2-x_i, x_i+1\}$ (resp. for $i > 0$ and $\delta_i(\lambda)_0(x_0) \in \{p - 1 - x_0, x_0\}$).

But this very easily follows from the equality $\xi_i(\lambda_i(x_i)) = \delta_r(\lambda)_i(x_i)$ (resp. $\xi_i(\lambda_i(x_i)) = \delta_i(\lambda)_i(x_i)$), from Lemma 19.1 and from $\delta_r(\lambda) \in \mathcal{RD}(x_0, \cdots, x_{f-1})$ (resp. $\delta_i(\lambda) \in \mathcal{ID}(x_0, \cdots, x_{f-1})$). □

Lemma 19.3. *We keep the previous notations. Let τ be an irreducible subquotient of $D_{0,\delta(\sigma)}(\rho)$. Then τ is a subquotient of $R(\sigma)$.*

Proof. We write $\sigma^{[s]} = (s'_0, \cdots, s'_{f-1}) \otimes \theta'$. Equivalently by Lemma 17.11, it is enough to prove there is $\lambda \in \mathcal{I}(y_0, \cdots, y_{f-1})$ such that $\tau = (\lambda_i(s'_i)) \otimes \det^{e(\lambda)(s'_i)}\theta'$. By (i) of Theorem 14.8 or of Lemma 19.2, we have:

$$\begin{aligned}
\tau &= (\nu_0(\xi_0(s_0)), \cdots, \nu_{f-1}(\xi_{f-1}(s_{f-1}))) \otimes \det^{e(\nu\circ\xi)(s_0,\cdots,s_{f-1})}\theta \\
\sigma^{[s]} &= (\nu'_0(\xi_0(s_0)), \cdots, \nu'_{f-1}(\xi_{f-1}(s_{f-1}))) \otimes \det^{e(\nu'\circ\xi)(s_0,\cdots,s_{f-1})}\theta
\end{aligned}$$

with $\nu, \nu' \in \mathcal{I}(y_0, \cdots, y_{f-1})$ and compatible in the sense of Definition 4.10. Let $\nu'^{-1} \in \mathcal{I}(y_0, \cdots, y_{f-1})$ be the unique f-tuple such that $\nu'(\nu'^{-1}(y_i)) = y_i$. From the compatibility of ν and ν', one checks that the unique f-tuple $(\lambda_i(y_i))_i$ such that $\lambda_i(y_i) := \nu_i(\nu_i'^{-1}(y_i))$ is in $\mathcal{I}(y_0, \cdots, y_{f-1})$. This λ gives the result. Note that one has $\nu_i'^{-1}(y_i) = p - 1 - \xi_i(y_i)$. □

If τ is an irreducible subquotient of $D_{0,\delta}(\sigma)(\rho)$, by Lemma 19.3 it is in $R(\sigma)$ and one can attach to it a well-defined f-tuple μ as in Lemma 17.11.

Lemma 19.4. *We keep the previous notations. Let τ be an irreducible subquotient of $D_{0,\delta(\sigma)}(\rho)$, μ its corresponding f-tuple as in Lemma 17.11, $i(\tau)$ the unique integer such that $\tau \in D_{0,\delta(\sigma)}(\rho)_{i(\tau)}$ and $j(\tau) := |K(\tau) \setminus J(\tau)|$ with $J(\tau)$ and $K(\tau)$ as in Lemma 17.12. Then we have:*

$$i(\tau) = j(\tau) + 2|J(\tau)| + i(\sigma^{[s]}) - (f + 1). \tag{39}$$

Proof. From (i) of Lemma 19.2, τ is of the form:

$$\tau = (\nu_0(\xi_0(s_0)), \cdots, \nu_{f-1}(\xi_{f-1}(s_{f-1}))) \otimes \det{}^{e(\nu\circ\xi)(s_0,\cdots,s_{f-1})}\theta$$

for a unique $\nu \in \mathcal{I}(y_0, \cdots, y_{f-1})$. Define $\mathcal{S}(\nu)$ and $\ell(\nu)$ as in §4, we have $i(\tau) = |\mathcal{S}(\nu)|$ by (ii) of Lemma 19.2. From Theorem 2.4 and the fact that the Γ-representation $I(\delta(\sigma), \sigma^{[s]})$ (inside $D_{0,\delta(\sigma)}(\rho)$) is the unique quotient of $\mathrm{Ind}_B^\Gamma \chi^s$ of socle $\delta(\sigma)$, we have $f + 1 - i(\sigma^{[s]}) = |\mathcal{S}(\xi)|$. Recall from (i) of Lemma 19.2 that ν and ξ satisfy the conditions:

$$\begin{aligned}
\xi_i(x_i) = x_i - 1 \quad &\Rightarrow \quad \nu_i(y_i) \in \{y_i - 1, p-1-y_i, p-2-y_i, y_i\}\\
\xi_i(x_i) = p-1-x_i \quad &\Rightarrow \quad \nu_i(y_i) \in \{y_i + 1, p-3-y_i, p-2-y_i, y_i\}\\
\nu_i(y_i) \in \{p-1-y_i, y_i-1\} \quad &\Rightarrow \quad \xi_i(x_i) \in \{x_i - 1, x_i\}\\
\nu_i(y_i) \in \{p-3-y_i, y_i+1\} \quad &\Rightarrow \quad \xi_i(x_i) \in \{p-2-x_i, p-1-x_i\}.
\end{aligned}$$

Moreover, if $\lambda \in \mathcal{I}(y_0, \cdots, y_{f-1})$ is such that $\mu_i(y_i) = \lambda(p - 1 - y_i)$ (see Lemma 17.11), then λ is as in the proof of Lemma 19.3 and from this proof we get $\mu = \nu \circ \xi$. The above conditions on ν and ξ then immediately imply by a short computation:

$$J(\tau) = \mathcal{S}(\xi) \cap \mathcal{S}(\nu) \quad \text{and} \quad K(\tau) = \mathcal{S}(\xi) \cup \mathcal{S}(\nu)$$

where we recall $J(\tau) := \{i, \mu_i(x_i) \in \{x_i - 2, p - x_i\}\}$ and $K(\tau) := \{i, \mu_i(x_i) \in \{x_i - 1, x_i - 2, p - x_i, p - 1 - x_i\}\}$. We thus have:

$$\begin{aligned}
j(\tau) &= |(\mathcal{S}(\xi) \cup \mathcal{S}(\nu)) \setminus \mathcal{S}(\xi) \cap \mathcal{S}(\nu)|\\
&= |\mathcal{S}(\xi) \cup \mathcal{S}(\nu)| - |\mathcal{S}(\xi) \cap \mathcal{S}(\nu)|\\
&= |\mathcal{S}(\xi)| + |\mathcal{S}(\nu)| - 2|\mathcal{S}(\xi) \cap \mathcal{S}(\nu)|\\
&= f + 1 - i(\sigma^{[s]}) + i(\tau) - 2|J(\tau)|.
\end{aligned}$$

□

Lemma 19.5. *Let* $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^f}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ *be a continuous tamely ramified generic Galois representation and* $\sigma \in \mathcal{D}(\rho)$*. There exists a unique quotient* $Q(\rho, \sigma^{[s]})$ *of* $R(\sigma)$ *such that:*

(i) $\mathrm{soc}_K Q(\rho, \sigma^{[s]}) \subseteq \bigoplus_{\sigma \in \mathcal{D}(\rho)} \sigma$

(ii) $Q(\rho, \sigma^{[s]})$ *contains the* Γ*-representation* $I(\rho, \sigma^{[s]})$*.*

Moreover, we have $\mathrm{soc}_K Q(\rho, \sigma^{[s]}) = \delta(\sigma)$.

Proof. Recall that $I(\rho, \sigma^{[s]}) = I(\delta(\sigma), \sigma^{[s]})$ (Lemma 15.2). We first prove there is a unique quotient of $\mathrm{Ind}_B^\Gamma \chi^s$ containing $I(\rho, \sigma^{[s]})$ and with K-socle contained in $\mathcal{D}(\rho)$: namely $I(\rho, \sigma^{[s]})$ itself. Indeed, consider such a quotient. If its K-socle has just one weight, then it is obviously $I(\rho, \sigma^{[s]})$. If not, let w be another weight of $\mathcal{D}(\rho)$ distinct from $\delta(\sigma)$ in this socle. From (ii) of Lemma 12.8 applied to $\tau = \sigma^{[s]}$ and $\sigma' = w$, we get that $\delta(\sigma)$ must be a constituent of $I(w, \sigma^{[s]})$ inside $\mathrm{Ind}_B^\Gamma \chi^s$. Hence w and $\delta(\sigma)$ cannot be in the same K-socle and the unique relevant quotient is thus $I(\rho, \sigma^{[s]})$. Let Q be a quotient of $R(\sigma)$ satisfying (i) and (ii) above. One easily checks using Lemma 17.11 that none of the irreducible Jordan-Hölder factors of $R(\sigma)/\mathrm{Ind}_B^\Gamma \chi^s$ are in $\mathcal{D}(\rho)$. From (i), this implies that Q induces a non-zero quotient of $\mathrm{Ind}_B^\Gamma \chi^s$. From (ii), we get that this non-zero quotient must contain $\delta(\sigma)$ in its socle as $\delta(\sigma)$ doesn't appear elsewhere in $R(\sigma)$ (use multiplicity 1 in Lemma 17.11). Thus, this induced quotient must be $I(\rho, \sigma^{[s]})$. Now let $\mathcal{K}$ be the kernel of $\mathrm{Ind}_B^\Gamma \chi^s \twoheadrightarrow I(\rho, \sigma^{[s]})$, we have a surjection $R(\sigma)/\mathcal{K} \twoheadrightarrow Q$. If $w' \in \mathrm{soc}_K(R(\sigma)/\mathcal{K})$, $w' \neq \delta(\sigma)$, then $w' \notin \mathcal{D}(\rho)$ as either w' is a subquotient of $R(\sigma)/\mathrm{Ind}_B^\Gamma \chi^s$ or a subquotient of $I(\rho, \sigma^{[s]})$. Hence w' maps to 0 in Q. One can thus replace $\mathcal{K}$ by $\mathcal{K} + \sum w'$ for all such w' and start again. We see in the end that Q is uniquely determined and that its K-socle is just $\delta(\sigma)$. □

Example 19.6. Assume $f = 1$ and write $\sigma := \mathrm{Sym}^{s_0}\overline{\mathbb{F}}_p^2 \otimes \theta$ with $s_0 \geq 1$. We have either $\delta(\sigma) = \sigma$ or $\delta(\sigma) = \sigma^{[s]}$. If $\delta(\sigma) = \sigma$, then $Q(\rho, \sigma^{[s]}) = R(\sigma)$ (see Example 19.6). If $\delta(\sigma) = \sigma^{[s]}$ and $s_0 > 1$, then $Q(\rho, \sigma^{[s]})$ is the unique quotient of $R(\sigma)$ which is a Γ-extension (non-split by Lemma 18.4):

$$0 \to \sigma^{[s]} \to Q(\rho, \sigma^{[s]}) \to \mathrm{Sym}^{s_0-2}\overline{\mathbb{F}}_p^2 \otimes \det\theta \to 0.$$

If $\delta(\sigma) = \sigma^{[s]}$ and $s_0 = 1$, then $Q(\rho, \sigma^{[s]}) \simeq \sigma^{[s]}$.

The following lemma is essential:

Lemma 19.7. *Let ρ, σ and $Q(\rho, \sigma^{[s]})$ be as in Lemma 19.5. The quotient $Q(\rho, \sigma^{[s]})$ contains the Γ-representation $D_{0,\delta(\sigma)}(\rho)$.*

Proof. If $f = 1$, the statement follows directly from Example 19.6 above and §16. We assume $f > 1$, write $\sigma := (s_0, \cdots, s_{f-1}) \otimes \theta$ and let $\tau := (\mu_0(s_0), \cdots, \mu_{f-1}(s_{f-1})) \otimes \det^{e(\mu)(s_0,\cdots,s_{f-1})}\theta$ be an irreducible subquotient of $D_{0,\delta(\sigma)}(\rho)$ with μ as in Lemma 19.4. It is enough to prove the following two facts: (i) any such τ is also a subquotient of $Q(\rho, \sigma^{[s]})$ and (ii) the unique K-subrepresentation $Q(\rho, \sigma^{[s]}, \tau)$ of $Q(\rho, \sigma^{[s]})$ with cosocle τ is a Γ-representation (recall $Q(\rho, \sigma^{[s]})$ is multiplicity free as it sits in $R(\sigma)$ and thus $Q(\rho, \sigma^{[s]}, \tau)$ is well defined). Indeed, from (ii), the last assertion of Lemma 19.5 and Corollary 3.12, we get $Q(\rho, \sigma^{[s]}, \tau) \simeq I(\delta(\sigma), \tau)$. From (i), we get that $Q(\rho, \sigma^{[s]})$ contains $I(\delta(\sigma), \tau)$ for all constituents τ of $D_{0,\delta(\sigma)}(\rho)$, and hence contains $D_{0,\delta(\sigma)}(\rho)$ by Proposition 13.4. Let us prove (i). Let $0 \to \tau' \to \epsilon \to \tau \to 0$ be a non-split (Γ-)extension that occurs as a subquotient of $D_{0,\delta(\sigma)}(\rho)$ (or equivalently as a quotient of $I(\delta(\sigma), \tau)$). By (ii) of Lemma 19.2 and the fact that the socle and cosocle filtrations on $I(\delta(\sigma), \tau)$ are the same (which follows from Corollary 4.9), we exactly have $i(\tau) = i(\tau') + 1$ (see Lemma 19.4 for notations). By Lemma 19.3, (i) of Corollary 5.6 and Lemma 18.4, either ϵ occurs in $R(\sigma)$ or the unique non-split $0 \to \tau \to * \to \tau' \to 0$ occurs. Moreover, by the beginning of the proof of Lemma 18.4, we have either $J(\tau) = J(\tau')$ or $J(\tau) = J(\tau') \amalg \{j\}$ or $J(\tau') = J(\tau) \amalg \{j\}$. If $J(\tau) = J(\tau')$, then (39) tells us $j(\tau) = j(\tau') + 1$ which implies only τ' can be a subobject by (ii) of Lemma 17.12 and thus ϵ occurs in $R(\sigma)$. If $J(\tau) = J(\tau') \amalg \{j\}$, the proof of Lemma 18.4 tells us that ϵ occurs in $R(\sigma)$. If $J(\tau') = J(\tau) \amalg \{j\}$, we must have $j(\tau) = j(\tau') + 3$ by (39) which is impossible by (ii) of Lemma 17.12. Thus ϵ always occurs in $R(\sigma)$, or equivalently in $U(\tau)$ (see Lemma 17.8). Starting again with τ' instead of τ, we see that $U(\tau)$ contains all the weights of $I(\delta(\sigma), \tau)$ and in particular $\delta(\sigma)$. Now if Q is a quotient of $R(\sigma)$ such that τ doesn't occur in Q, then $U(\tau)$ necessarily vanishes via the surjection $R(\sigma) \twoheadrightarrow Q$. In particular $\delta(\sigma)$ doesn't occur in Q. As $\delta(\sigma)$ is the socle of $Q(\rho, \sigma^{[s]})$, this can't happen for $Q = Q(\rho, \sigma^{[s]})$, which must thus contain τ as a subquotient. Let us now prove (ii). We claim that $Q(\rho, \sigma^{[s]}, \tau)$ contains no pair of distinct weights (w, w') corresponding to f-tuples (ν, ν') as in Lemma 17.11 with $\nu'_j(x_j) = \nu_j(x_j) \pm 2$ for one j and $\nu'_i(x_i) = \nu_i(x_i)$ for $i \neq j$. Assume there exists such a pair (w, w'). Swapping w and w' if necessary, we can assume $\nu_j(x_j) \in \{x_j - 2, p - x_j\}$, $\nu'_j(x_j) \in \{x_j, p - 2 - x_j\}$ and thus $J(w) = J(w') \amalg \{j\}$. From (ii) of Lemma 17.12, we get also $j \in J(\tau)$ hence $\mu_j(x_j) \in \{x_j - 2, p - x_j\}$. We write $\delta(\sigma) = (\xi_0(s_0), \cdots, \xi_{f-1}(s_{f-1})) \otimes \det^{e(\xi)(s_0,\cdots,s_{f-1})}\theta$ with $\xi \in \mathcal{P}(x_0, \cdots, x_{f-1})$ as previously and note that ξ is also the f-tuple associated by Lemma 17.11

to $\delta(\sigma)$ viewed as a constituent of $R(\sigma)$. We have seen in the proof of Lemma 19.4 that we have the equality $J(\tau) = \mathcal{S}(\xi) \cap \mathcal{S}(\mu)$ which implies $j \in \mathcal{S}(\xi)$ i.e. $\xi_j(x_j) \in \{x_j - 1, p - 1 - x_j\}$. Since $\mathrm{soc}_K\, Q(\rho, \sigma^{[s]}, \tau) = \delta(\sigma)$ by Lemma 19.5, there is a chain of non-split K-extensions leading from $\delta(\sigma)$ to w' inside $Q(\rho, \sigma^{[s]}, \tau)$ which implies $K(\delta(\sigma)) \subseteq K(w')$ by (ii) of Lemma 17.12. But this is impossible since $j \in K(\delta(\sigma))$ but $j \notin K(w')$ as $\nu'_j(x_j) \in \{x_j, p - 2 - x_j\}$. As $f > 1$, Corollary 5.7 applied to $W = Q(\rho, \sigma^{[s]}, \tau)$ tells us that $Q(\rho, \sigma^{[s]}, \tau)$ is a Γ-representation and we are done. □

Theorem 19.8. *Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^f}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous generic representation as in §11 such that p acts trivially on its determinant. Let $(D_0(\rho), D_1(\rho), r)$ be one of the basic 0-diagrams associated to ρ in §13 with $D_0(\rho)$ as in Theorem 13.8.*

- *(i) There exists a smooth admissible representation π of G such that:*
  (a) $\mathrm{soc}_K\, \pi = \bigoplus_{\sigma \in \mathcal{D}(\rho)} \sigma$
  (b) $(\pi^{K_1}, \pi^{I_1}, \mathrm{can}) \hookleftarrow (D_0(\rho), D_1(\rho), r)$
  (c) π is generated by $D_0(\rho)$.
- *(ii) If $(D_0(\rho), D_1(\rho), r)$ and $(D_0(\rho), D_1(\rho), r')$ are two non-isomorphic basic 0-diagrams associated to ρ, and if π, π' satisfy (a), (b), (c) of (i) respectively for $(D_0(\rho), D_1(\rho), r)$ and $(D_0(\rho), D_1(\rho), r')$, then π and π' are non-isomorphic.*

Proof. Let $D := (D_0(\rho), D_1(\rho), r)$. By Theorem 9.8, we have a smooth admissible G-representation Ω with K-socle $\mathcal{D}(\rho)$ and an injection $D \hookrightarrow \mathcal{K}(\Omega)$. We define $\pi \subseteq \Omega$ to be the subrepresentation generated by $D_0(\rho)$. By construction, it satisfies (a), (b) and (c) of (i). Assume $\pi \xrightarrow{\sim} \pi'$ where π and π' are as in (ii). If $D_0(\rho) \subset \pi$ is not sent to $D_0(\rho) \subset \pi'$, there is $\sigma \in \mathcal{D}(\rho)$ such that $D_{0,\sigma}(\rho) \subset \pi$ is not sent to $D_{0,\sigma}(\rho) \subset \pi'$. Consider the obvious induced map $D_{0,\sigma}(\rho) \oplus_\sigma D_{0,\sigma}(\rho) \to \pi'$. The representation $D_{0,\sigma}(\rho) \oplus_\sigma D_{0,\sigma}(\rho)$ contains $D_{0,\sigma}(\rho)/\sigma$ and the induced map $D_{0,\sigma}(\rho)/\sigma \to \pi'$ can't be zero because $D_{0,\sigma}(\rho) \subset \pi$ is not sent to $D_{0,\sigma}(\rho) \subset \pi'$ by assumption. This contradicts $\mathrm{soc}_K\, \pi' = \mathcal{D}(\rho)$ as the K-socle of $D_{0,\sigma}(\rho)/\sigma$ can't be in $\mathcal{D}(\rho)$ by construction of $D_0(\rho)$. Hence $D_0(\rho) \subset \pi$ is sent to $D_0(\rho) \subset \pi'$, and likewise with $D_0(\rho)^{I_1}$. Since $\pi \simeq \pi'$, this implies $(D_0(\rho), D_1(\rho), r) \simeq (D_0(\rho), D_1(\rho), r')$ which is impossible by assumption. Thus, we can't have $\pi \simeq \pi'$. □

By an exactly similar proof, we get:

Theorem 19.9. *Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^f}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous generic representation as in §11 such that ρ is split and p acts trivially on its determinant. Let $\ell \in \{0, \cdots, f\}$ and $(D_{0,\ell}(\rho), D_{1,\ell}(\rho), r_\ell)$ be one of the basic 0-diagrams associated to the family (30).*

(i) There exists a smooth admissible representation π_ℓ of G such that:
(a) $\mathrm{soc}_K \pi_\ell = \bigoplus_{\substack{\sigma\in\mathcal{D}(\rho)\\ \ell(\sigma)=\ell}} \sigma$
(b) $(\pi_\ell^{K_1}, \pi_\ell^{I_1}, \mathrm{can}) \hookleftarrow (D_{0,\ell}(\rho), D_{1,\ell}(\rho), r_\ell)$
(c) π_ℓ is generated by $D_{0,\ell}(\rho)$.

(ii) If $(D_{0,\ell}(\rho), D_{1,\ell}(\rho), r_\ell)$ and $(D_{0,\ell}(\rho), D_{1,\ell}(\rho), r'_\ell)$ are non-isomorphic (as basic 0-diagrams), and if π_ℓ, π'_ℓ are two representations satisfying (a), (b), (c) of (i) for $(D_{0,\ell}(\rho), D_{1,\ell}(\rho), r_\ell)$ and $(D_{0,\ell}(\rho), D_{1,\ell}(\rho), r'_\ell)$ respectively, then π_ℓ and π'_ℓ are non-isomorphic.

We now state an irreducibility result which is based on the results of §18.

Theorem 19.10. *(i) Let ρ be as in Theorem 19.8 and assume ρ is irreducible. Then any π satisfying (a), (b), (c) of (i) of Theorem 19.8 is irreducible and is a supersingular representation.*

(ii) Let ρ be as in Theorem 19.8 and assume ρ is split. Then any π_ℓ satisfying (a), (b), (c) of (i) of Theorem 19.9 is irreducible. Moreover, π_ℓ is a principal series if $\ell \in \{0, f\}$ and is a supersingular representation otherwise.

Proof. We start with (i). Let $\pi' \subseteq \pi$ be a non-zero subrepresentation and $\sigma \in \mathrm{soc}_K \pi'$. We prove that $D_{0,\delta(\sigma)}(\rho) \subseteq \pi'$. We have a non-zero map $\text{c-Ind}_{\mathfrak{K}_0}^G \sigma \to \pi'$ which induces a map $R(\sigma) \to \pi'$ upon restriction to $R(\sigma)$. Let $v \in \sigma^{I_1} \subset \text{c-Ind}_{\mathfrak{K}_0}^G \sigma$ and $v^s := \Pi v \in (\sigma^{[s]})^{I_1} \subset \text{c-Ind}_{\mathfrak{K}_0}^G \sigma$. Going back to the definition of $\widetilde{R}(\sigma)$ in §17 and using Lemma 17.4, note that $\langle K \cdot v^s \rangle = \mathrm{Ind}_B^\Gamma \chi^s \subseteq R(\sigma) \subseteq \widetilde{R}(\sigma) \subseteq \text{c-Ind}_{\mathfrak{K}_0}^G \sigma$. Let w^s be the image of v^s in π', the map $R(\sigma) \to \pi'$ induces $\mathrm{Ind}_B^\Gamma \chi^s \twoheadrightarrow \langle K \cdot w^s \rangle \subset \pi'$. But $\langle K \cdot w^s \rangle$ actually sits in $D_{0,\delta(\sigma)}(\rho) \subset \pi$ by Lemma 15.2, hence equals $I(\delta(\sigma), \sigma^{[s]}) = I(\rho, \sigma^{[s]})$ by construction of $D_{0,\delta(\sigma)}(\rho)$. Thus $R(\sigma) \to \pi'$ factors through a quotient containing $I(\rho, \sigma^{[s]})$ and with a K-socle contained in $\mathcal{D}(\rho)$ (as $\mathrm{soc}_K \pi' \subseteq \mathcal{D}(\rho)$). By Lemma 19.5, this quotient must be $Q(\rho, \sigma^{[s]})$, hence contains $D_{0,\delta(\sigma)}(\rho)$ by Lemma 19.7. We thus get $D_{0,\delta(\sigma)}(\rho) \subset \pi'$. Starting again with $\delta(\sigma)$ instead of σ, we obtain that π' contains $D_{0,\delta^2(\sigma)}(\rho)$ etc. As $\delta^n(\sigma) = \sigma$ for some $n > 0$, we get $D_{0,\sigma}(\rho) \subset \pi'$. As this is true for all $\sigma \in \mathrm{soc}_K \pi'$, we finally deduce, using that all weights of $\mathcal{D}(\rho)$ are distinct:

$$\bigoplus_{\sigma\in\mathrm{soc}_K \pi'} D_{0,\sigma}(\rho) = \pi' \cap \bigoplus_{\sigma\in\mathcal{D}(\rho)} D_{0,\sigma}(\rho),$$

the intersection being taken in π. This implies that $\oplus_{\sigma\in\mathrm{soc}_K \pi'} D_{0,\sigma}(\rho)^{I_1} \subseteq D_0(\rho)^{I_1}$ is preserved by the unique possible pairing $\{\ \}$ on $D_0(\rho)^{I_1}$. By (i) of Theorem 15.4, we thus get $\mathrm{soc}_K \pi' = \mathcal{D}(\rho) = \mathrm{soc}_K \pi$, hence $\pi' = \pi$: π is irreducible. If $f = 1$, we know from §16 and §10 that π is a supersingular representation. If $f > 1$, we have $\dim_{\overline{\mathbb{F}}_p} \pi^{I_1} > 2$ as $\mathrm{soc}_K \pi$ already contains 2^f weights, hence π is a supersingular representation. We prove (ii). The irreducibility of π_ℓ is proven by a completely analogous argument using (ii) of Theorem 15.4. If $\ell \in \{0, f\}$, the reader can easily check, using that $D_{0,0}(\rho)$ (resp. $D_{0,f}(\rho)$) has an irreducible socle and that $D_{0,0}(\rho)^{I_1}$ (resp. $D_{0,f}(\rho)^{I_1}$) is preserved by $\{\ \}$ inside $D_0(\rho)^{I_1}$, that the surjective m ap $\text{c-Ind}_{\mathfrak{K}_0}^G \sigma_0 \twoheadrightarrow \pi_0$ (resp. $\text{c-Ind}_{\mathfrak{K}_0}^G \sigma_f \twoheadrightarrow \pi_f$) cannot factor through $T(\text{c-Ind}_{\mathfrak{K}_0}^G \sigma_0)$ (resp. $T(\text{c-Ind}_{\mathfrak{K}_0}^G \sigma_f)$) (see (i) of Remark 14.9 and §6 for T). It implies that π_0 and π_f are (irreducible) principal series. If $\ell \neq 0$ and $\ell \neq f$, then one has $\dim_{\overline{\mathbb{F}}_p} \pi_\ell^{I_1} > 2$ (if $f = 2$ this easily follows from §16 and if $f > 2$, $\mathrm{soc}_K \pi_\ell$ has strictly more than 2 components), hence π_ℓ is a supersingular representation. □

For $F \neq \mathbb{Q}_p$, there exist non-isomorphic π satisfying (i) of Theorem 19.8 for the same basic 0-diagram ([22]), thus the conditions in (i) are in general not enough to isolate a single π. By enlarging $D_1(\rho)$, Y. Hu shows that for each π there exists a diagram that determines π ([21]), but the "enlarged" $D_1(\rho)$ is not (yet) explicitely known. Also, when $F \neq \mathbb{Q}_p$, it is in general not true that any π (resp. π_ℓ) as in (i) of Theorem 19.8 (resp. as in (i) of Theorem 19.9) satisfies $(\pi^{K_1}, \pi^{I_1}, \mathrm{can}) \cong (D_0(\rho), D_1(\rho), r)$ (resp. $(\pi_\ell^{K_1}, \pi_\ell^{I_1}, \mathrm{can}) \cong (D_{0,\ell}(\rho), D_{1,\ell}(\rho), r_\ell)$), but we believe that some of these π (resp. π_ℓ) do. When ρ is reducible split, any representation $\oplus_{\ell=0}^f \pi_\ell$ with π_ℓ as in (i) of Theorem 19.9 satisfies the conditions in (i) of Theorem 19.8 but any π as in (i) of Theorem 19.8 can't in general be decomposed as $\oplus_{\ell=0}^f \pi_\ell$ (for all this see [22]), although we expect the "good" π in that case to be of the form $\oplus_{\ell=0}^f \pi_\ell$. Likewise, when ρ is reducible non-split, we expect that, among the π constructed in (i) of Theorem 19.8, there are some (the "good" ones) which are indecomposable with G-socle π_0 and such that their other Jordan-Hölder factors are the π_ℓ, $1 \leq \ell \leq f$, with π_ℓ as in (i) of Theorem 19.9 for ρ^{ss}. All of this is true if $F = \mathbb{Q}_p$ (§20).

As a concluding remark, we hope that the local representations of G appearing as subobject in the cohomology modulo p of towers of Shimura varieties of p^n-level are at least among those constructed in Theorem 19.8 (ρ being the restriction to some decomposition group at p of some global irreducible Galois representation over $\overline{\mathbb{F}}_p$). For evidence in that direction, see [9].

20 The case $F = \mathbb{Q}_p$

We prove Theorem 1.6 for $F = \mathbb{Q}_p$.

Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous generic representation such that p acts trivially on its determinant.

Proposition 20.1. *Assume ρ is irreducible and write its restriction to inertia as:*

$$\begin{pmatrix} \omega_2^{r_0+1} & 0 \\ 0 & \omega_2^{p(r_0+1)} \end{pmatrix} \otimes \eta$$

for some character η and some r_0 with $1 \leq r_0 \leq p-2$. See η as a smooth character of $\mathbb{Q}_p^\times$ (via the local reciprocity map) by making p act trivially. Let $(D_0(\rho), D_1(\rho), r)$ be the unique basic 0-diagram associated to ρ in §13 (see §16 for unicity).

(i) There is a unique smooth admissible representation π of G such that:
(a) $\mathrm{soc}_K\,\pi = \bigoplus_{\sigma\in\mathcal{D}(\rho)} \sigma$
(b) $(\pi^{K_1}, \pi^{I_1}, \mathrm{can}) \hookleftarrow (D_0(\rho), D_1(\rho), r)$
(c) π is generated by $D_0(\rho)$.

(ii) This representation π is irreducible, isomorphic to $\pi(r_0, 0, \eta)$ (see Definition 6.2) and such that:

$$(D_0(\rho), D_1(\rho), r) \xrightarrow{\sim} (\pi^{K_1}, \pi^{I_1}, \mathrm{can}).$$

Proof. We have $\mathcal{D}(\rho) = \{\sigma, \sigma^{[s]}\}$ with $\sigma := (\mathrm{Sym}^{r_0}\overline{\mathbb{F}}_p^2) \otimes \eta \circ \det$ (see §16). We have already proven the existence and irreducibility of π as in (i) (see §19). The unicity of π in (i) follows from $\pi = H_0(D) = \pi(r_0, 0, \eta)$ (see Theorem 10.1) where $D := (\sigma \oplus \sigma^{[s]}, \sigma^{I_1} \oplus {\sigma^{[s]}}^{I_1}, \mathrm{can})$ is the unique irreducible basic subdiagram of $(D_0(\rho), D_1(\rho), r)$. For the rest of (ii), it follows from Lemmas 3.4 and 3.5 that $D_0(\rho)$ is the maximal K-invariant subspace of $\mathrm{inj}(\sigma \oplus \sigma^{[s]})$ such that the K-socle is isomorphic to $\sigma \oplus \sigma^{[s]}$ and the space of I_1-invariants is 2-dimensional. As π^{I_1} has dimension 2 by the second part of Theorem 10.1, this implies the injection $(D_0(\rho), D_1(\rho), r) \hookrightarrow (\pi^{K_1}, \pi^{I_1}, \mathrm{can})$ is an isomorphism. □

Proposition 20.2. *Assume ρ is reducible split and write its restriction to inertia as:*

$$\rho := \begin{pmatrix} \omega^{r_0+1} & 0 \\ 0 & 1 \end{pmatrix} \otimes \eta$$

for some character η and some r_0 with $1 \leq r_0 \leq p-4$ (recall $\omega = \omega_1$). See η as a smooth character of $\mathbb{Q}_p^\times$ (via the local reciprocity map) by making p act trivially. Let $(D_0(\rho), D_1(\rho), r)$ be one of the basic 0-diagrams associated to ρ in §13.

(i) There is a unique smooth admissible representation π of G such that:
(a) $\mathrm{soc}_K\, \pi = \bigoplus_{\sigma \in \mathcal{D}(\rho)} \sigma$
(b) $(\pi^{K_1}, \pi^{I_1}, \mathrm{can}) \hookleftarrow (D_0(\rho), D_1(\rho), r)$
(c) π is generated by $D_0(\rho)$.

(ii) This representation π is the direct sum of two irreducible principal series isomorphic to $\pi(r_0, \lambda_0, \eta)$ and $\pi(p-3-r_0, \lambda_1, \eta)$ (see Definition 6.2) for some scalars $\lambda_0, \lambda_1 \in \overline{\mathbb{F}}_p^\times$ depending on $(D_0(\rho), D_1(\rho), r)$ and is such that:

$$(D_0(\rho), D_1(\rho), r) \xrightarrow{\sim} (\pi^{K_1}, \pi^{I_1}, \mathrm{can}).$$

Proof. We have $\mathcal{D}(\rho) = \{\sigma_0, \sigma_1\}$ with $\sigma_0 := (\mathrm{Sym}^{r_0}\overline{\mathbb{F}}_p^2) \otimes \eta \circ \det$ and $\sigma_1 := (\mathrm{Sym}^{p-3-r_0}\overline{\mathbb{F}}_p^2) \otimes \eta \circ \det^{r_0+1}$ (see §16). Let χ_0 (resp. χ_1) be the character giving the action of I on $\sigma_0^{[s]}$ (resp. $\sigma_1^{[s]}$). We have (see (30) and §19 for the notations):

$$(D_0(\rho), D_1(\rho), r) = (D_{0,0}(\rho), D_{1,0}(\rho), r_0) \oplus (D_{0,1}(\rho), D_{1,1}(\rho), r_1)$$

where $(D_{0,0}(\rho), D_{1,0}(\rho), r_0)$ (resp. $(D_{0,1}(\rho), D_{1,1}(\rho,), r_1)$) is as in Example (iv) of §10 with $\chi = \chi_0$ (resp. $\chi = \chi_1$, see §16). For π as in (i), let π_0 (resp. π_1) be the G-subrepresentation generated by σ_0 (resp. σ_1), then $\pi_0 \simeq \pi(r_0, \lambda_0, \eta)$ (resp. $\pi_1 \simeq \pi(p-3-r_0, \lambda_1, \eta)$) for some scalars λ_i uniquely determined by $(D_0(\rho), D_1(\rho), r)$ (this follows for instance from §10 or from Proposition 6.8). As π is generated by $D_0(\rho)$, we thus have $\pi = \pi_0 \oplus \pi_1$. The rest of (ii) follows for instance from Proposition 6.8. □

To state the reducible non-split case, we need some further work.

Let r_0 be an integer, $1 \leq r_0 \leq p-3$, and $\lambda \in \overline{\mathbb{F}}_p^\times$. We first define a basic 0-diagram $D(r_0, \lambda) := (D_0(r_0, \lambda), D_1(r_0, \lambda), \mathrm{can})$. We define $D_0(r_0, \lambda)$ as the following $\mathfrak{K}_0$-representation where K_1 and p act trivially (see §16 for notations):

$$\mathrm{Sym}^{r_0}\overline{\mathbb{F}}_p^2 \quad — \quad \begin{array}{c} \mathrm{Sym}^{p-1-r_0}\overline{\mathbb{F}}_p^2 \otimes \det^{r_0} \\ \oplus \\ \mathrm{Sym}^{p-3-r_0}\overline{\mathbb{F}}_p^2 \otimes \det^{r_0+1} \end{array} .$$

Note that Proposition 3.6 and Corollary 3.11 imply that $D_0(r_0,\lambda)$ is isomorphic to $V_{2p-2-r_0}\otimes \det^{r_0}$. It follows from Proposition 4.13 that $D_0(r_0,\lambda)^{I_1}\cong \chi\oplus\chi^s$ where $\chi: H\to \overline{\mathbb{F}}_p^{\times}$ is the character given by $\chi((\begin{smallmatrix}\lambda & 0\\ 0 & \mu\end{smallmatrix})) := \mu^{r_0}$. Since $1\le r_0\le p-3$, we have $\chi\neq\chi^s$ and we let v_χ be a basis vector in $D_0(r_0,\lambda)^{I_1}$ for the eigencharacter χ. Set $v_{\chi^s} := \sum_{\mu\in\mathbb{F}_p}(\begin{smallmatrix}1 & [\mu]\\ 0 & 1\end{smallmatrix})n_s^{-1}v_\chi = \sum_{\mu\in\mathbb{F}_p}(\begin{smallmatrix}[\mu] & 1\\ 1 & 0\end{smallmatrix})v_\chi$. The set $\{v_\chi, v_{\chi^s}\}$ is a basis of $D_0(r_0,\lambda)^{I_1}$. We define a representation $D_1(r_0,\lambda)$ of $\mathfrak{K}_1$ on $D_0(r_0,\lambda)^{I_1}$ by setting $\Pi v_\chi := \lambda^{-1}v_{\chi^s}$ and $\Pi v_{\chi^s} := \lambda v_\chi$.

Theorem 20.3. *Let r_0,λ and $D(r_0,\lambda)$ be as above.*

(i) Let π be the unique non-split extension (see Corollary 8.3):

$$0\longrightarrow \pi(r_0,\lambda)\longrightarrow \pi\longrightarrow \pi(p-3-r_0,\lambda^{-1},\omega^{r_0+1})\longrightarrow 0,$$

then there exists an isomorphism of diagrams $(\pi^{K_1},\pi^{I_1},\mathrm{can})\cong D(r_0,\lambda)$.

(ii) Let τ be a smooth representation of G with a central character such that $\mathrm{soc}_K(\tau)\cong \mathrm{Sym}^{r_0}\overline{\mathbb{F}}_p^2$ and such that there exists an injection of diagrams $D(r_0,\lambda)\hookrightarrow(\tau^{K_1},\tau^{I_1},\mathrm{can})$, then the subspace $\langle G\cdot D_0(r_0,\lambda)\rangle$ of τ is isomorphic to π.

Proof. It follows from Theorem 9.8 that there exists an injection of diagrams $D(r_0,\lambda)\hookrightarrow \mathcal{K}(\Omega)$ where Ω is a smooth representation of G such that $\Omega|_K$ is an injective envelope of $\mathrm{Sym}^{r_0}\overline{\mathbb{F}}_p^2$ in $\mathrm{Rep}_{K,\chi}$. We first claim that the subspace $\pi' := \langle G\cdot D_0(r_0,\lambda)\rangle$ of Ω is isomorphic to the extension π of (i). Corollary 6.4 implies that the subspace $\overline{\mathbb{F}}_p v_\chi\oplus\overline{\mathbb{F}}_p v_{\chi^s}$ of Ω^{I_1} is stable under the action of $\mathcal{H}$ and isomorphic to $M(r_0,\lambda)$ as an $\mathcal{H}$-module. Proposition 6.8 implies that $\langle G\cdot v_\chi\rangle = \langle G\cdot v_{\chi^s}\rangle\cong\pi(r_0,\lambda)$. Now $\pi(r_0,\lambda)\cong \mathrm{Ind}_P^G\chi$ and hence $\pi(r_0,\lambda)^{K_1}\cong \mathrm{Ind}_I^K\chi$. Since $\Omega|_K$ is an injective envelope of $\mathrm{Sym}^{r_0}\overline{\mathbb{F}}_p^2$ in $\mathrm{Rep}_{K,\chi}$, Ω^{K_1} is an injective envelope of $\mathrm{Sym}^{r_0}\overline{\mathbb{F}}_p^2$ in Rep_Γ. Lemmas 3.4, 3.5, 3.8 imply that the image of $D_0(r_0,\lambda)$ via the composition $D_0(r_0,\lambda)\to \Omega^{K_1}/\pi(r_0,\lambda)^{K_1}\hookrightarrow \Omega/\pi(r_0,\lambda)$ is isomorphic to $(\mathrm{Sym}^{p-3-r_0}\overline{\mathbb{F}}_p^2)\otimes\det^{r_0+1}$. Let v be a basis for the I_1-invariants of this image. Since $F=\mathbb{Q}_p$ we have $\Omega^{I_1}=\pi(r_0,\lambda)^{I_1}$ and since $\Omega|_K$ is an injective object we obtain $(\Omega/\pi(r_0,\lambda))^{I_1}\cong \mathbb{R}^1\mathcal{I}(\pi(r_0,\lambda))$. Now H acts on v by a character $\chi\alpha^{-1}$. The assumption on r_0 implies that $\chi\alpha^{-1}\notin\{\chi,\chi^s\}$ thus it follows from Theorem 7.16 that the submodule $\langle v\cdot\mathcal{H}\rangle$ of $(\Omega/\pi(r_0,\lambda))^{I_1}$ is isomorphic to $M(p-3-r_0,\lambda^{-1},\omega^{r_0+1})$. Proposition 6.8 implies that $\langle G\cdot v\rangle$ is isomorphic to $\pi(p-3-r_0,\lambda^{-1},\omega^{r_0+1})$. Hence there exists an exact sequence:

$$0\longrightarrow \pi(r_0,\lambda)\longrightarrow \pi'\longrightarrow \pi(p-3-r_0,\lambda^{-1},\omega^{r_0+1})\longrightarrow 0. \qquad (40)$$

This sequence cannot be split as π' is a subspace of Ω and hence $\mathrm{soc}_K\,\pi' \cong \mathrm{Sym}^{r_0}\overline{\mathbb{F}}_p^2$. Corollary 8.3 implies then that $\pi' \cong \pi$. We thus obtain an injection $D(r_0,\lambda) \hookrightarrow (\pi^{K_1},\pi^{I_1},\mathrm{can})$. Suppose that this injection is not an isomorphism, then Lemma 3.4 implies $\pi^{K_1} \cong \Omega^{K_1}$ and hence $\mathrm{Sym}^{r_0}\overline{\mathbb{F}}_p^2$ occurs in π^{K_1} with multiplicity 2. This is impossible, since taking K_1-invariants of (40) yields an exact sequence $0 \longrightarrow \mathrm{Ind}_I^K\,\chi \longrightarrow \pi^{K_1} \longrightarrow \mathrm{Ind}_I^K\,\chi^s\alpha$. Hence we get (i). Since by Corollary 9.11 any τ as in (ii) can be embedded into Ω as above, we also get (ii). □

Proposition 20.4. *Assume ρ is reducible non-split and write its restriction to inertia as:*

$$\begin{pmatrix}\omega^{r_0+1} & * \\ 0 & 1\end{pmatrix} \otimes \eta$$

for some character η and some r_0 with $1 \leq r_0 \leq p-4$. See η as a smooth character of $\mathbb{Q}_p^\times$ (via the local reciprocity map) by making p act trivially. Let $(D_0(\rho), D_1(\rho), r)$ be one of the basic 0-diagrams associated to ρ in §13.

(i) There is a unique smooth admissible representation π of G such that:
(a) $\mathrm{soc}_K\,\pi = \bigoplus_{\sigma\in\mathcal{D}(\rho)}\sigma$
(b) $(\pi^{K_1},\pi^{I_1},\mathrm{can}) \hookleftarrow (D_0(\rho),D_1(\rho),r)$
(c) π is generated by $D_0(\rho)$.

(ii) This representation π is the unique non-split extension of $\pi(p-3-r_0,\lambda^{-1},\omega^{r_0+1}\eta)$ by $\pi(r_0,\lambda,\eta)$ for some scalar $\lambda \in \overline{\mathbb{F}}_p^\times$ depending on $(D_0(\rho),D_1(\rho),r)$ and is such that:

$$(D_0(\rho),D_1(\rho),r) \xrightarrow{\sim} (\pi^{K_1},\pi^{I_1},\mathrm{can}).$$

Proof. This follows from Theorem 20.3 and the fact that $(D_0(\rho),D_1(\rho),r)$ is isomorphic to $D(r_0,\lambda)$ up to twist for some $\lambda\in\overline{\mathbb{F}}_p^\times$ (see §16). □

Theorem 1.6 follows from all the previous propositions.

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