Modular forms in high-energy physics

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I. Particle physics
Collision of beam particles

Test the laws of physics by analysing particle tracks.
Perturbative Quantum Field theory

General framework describing fundamental forces and particles.

Every Feynman graph $G$ represents a possible particle interaction. Feynman \textit{amplitude} is a complex probability assigned to $G$. 
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Every Feynman graph $G$ represents a possible particle interaction. Feynman \textit{amplitude} is a complex probability assigned to $G$. 
The blue line (background) requires calculating a huge number of Feynman amplitudes.
II. Graphs and Numbers
Let $G = (V_G, E_G)$ be a connected graph. The graph polynomial

$$\Psi_G \in \mathbb{Z}[\alpha_e, e \in E(G)]$$

is a sum over spanning trees $T$ of $G$

$$\Psi_G = \sum_{T \subset G} \prod_{e \notin T} \alpha_e$$
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A tree $T \subset G$ is spanning if $V_T = V_G$. 
Example

\[ \psi_G = ? \]
Example

$\psi_G = \alpha_3 \alpha_4$
Example

\[ \Psi_G = \alpha_3\alpha_4 + \alpha_2\alpha_4 \]
In general, $G$ is homogeneous of degree $h_G$ ('loop number').

Physically relevant graphs have vertices of degree $\leq 4$. ($G$ in $\{1, 2, 3, 4\}$).

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\psi_G = \alpha_3 \alpha_4 + \alpha_2 \alpha_4 + \alpha_1 \alpha_4
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In general, $G$ is homogeneous of degree $h_G$ ('loop number').

Physically relevant graphs have vertices of degree $\nabla$. ('$G$ in $\nabla$').

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In general, \( \Psi_G \) is homogeneous of degree \( h_G \) ('loop number').

\[ \text{deg} \Psi_G = h_G \quad N_G = \#E(G) \]
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Physically relevant graphs have vertices of degree \( \leq 4 \). (‘\( G \) in \( \phi^4 \)’).
Feynman integrals

For convergence, assume

- $N_G = 2h_G$
- $N_\gamma > 2h_\gamma \text{ for all } \gamma \subset G.$
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The *residue* is the convergent integral

$$I_G = \int_\sigma \frac{\Omega_G}{\Psi^2_G} \in \mathbb{R}$$
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\Omega_G = \sum_{i=1}^{N_G} (-1)^i \alpha_i d\alpha_1 \wedge \ldots \wedge \hat{d\alpha_i} \wedge \ldots d\alpha_{N_G}
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Feynman integrals

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\[
\sigma = \{(\alpha_1 : \ldots : \alpha_{N_G}) \in \mathbb{P}^{N_G-1}(\mathbb{R}) \text{ such that } \alpha_i \geq 0\}
\]
We obtain a map

\[ I : \{\text{convergent graphs in } \phi^4\} \rightarrow \mathbb{R} \]
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Graphs and numbers

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\]

\[
I_G = \int_{\sigma} \frac{\alpha_2 d\alpha_1 - \alpha_1 d\alpha_2}{(\alpha_1 + \alpha_2)^2} = \int_{\alpha_1 \geq 0} \frac{d\alpha_1}{(\alpha_1 + 1)^2} = 1
\]
The Zoo

\[ I_G : \quad 6\zeta(3) \quad 20\zeta(5) \quad 36\zeta(3)^2 \quad N_{3,5} \]
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\[ N_{3,5} = \frac{27}{5} \zeta(5, 3) + \frac{45}{4} \zeta(5)\zeta(3) - \frac{261}{20} \zeta(8) \]
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Multiple Zeta Values, defined for \( n_1, \ldots, n_{r-1} \geq 1 \), and \( n_r \geq 2 \):

\[ \zeta(n_1, \ldots, n_r) = \sum_{1 \leq k_1 < k_2 < \ldots < k_r} \frac{1}{k_1^{n_1} \ldots k_r^{n_r}} \in \mathbb{R} \]
Folklore conjecture 90’s

The numbers $I_G$ are $\mathbb{Q}$-linear combinations of multiple zeta values.
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Closed formula known for only one infinite family:

\[ Z_5 \propto \zeta(2n - 3) \]
Folklore conjecture 90’s

The numbers $I_{G}$ are $\mathbb{Q}$-linear combinations of multiple zeta values.

Known to be true for some infinite classes of graphs.

Closed formula known for only one infinite family:

In general, very hard to compute the integrals even numerically because they are highly singular.
Properties I

1. Contraction-Deletion:

\[ \Psi_G = \alpha_e \Psi_{G\setminus e} + \Psi_{G/\setminus e} \]
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\[ \psi_\gamma \quad \psi_{G/\gamma} \quad R_{\gamma,G} \]

Determines \( \Psi_G \) essentially uniquely.
The graph polynomial is a determinant

\[ \Psi_G = \det(L_G) \]

where \( L_G \) is the reduced graph Laplacian matrix.
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Many identities between \( I_G \). For example:

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and planar duals, completion (Fourier transform), ...
III. Point-counting
Points over finite fields

Let \( f_1, \ldots, f_n \in \mathbb{Z}[x_1, \ldots, x_N] \). Let \( X \) denote the algebraic variety (affine scheme over \( \mathbb{Z} \)) defined by

\[
f_1 = \ldots = f_n = 0.
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For every prime power $q = p^e$, let

$$[X]_q = \#X(\mathbb{F}_q).$$
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For example,

$$[X]_p = \#\{(x_1, \ldots, x_N) : x_i \in \mathbb{F}_p, f_i(x_1, \ldots, x_N) \equiv 0 \mod p \text{ for all } i\}$$
Some general results

Serre: if \([X]_p = [Y]_p\) for a set of primes \(p\) of density 1, then

\[ [X]_{p^e} = [Y]_{p^e} \]

for all \(e \geq 1\), provided \(p \geq p_0\) sufficiently large.
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Grothendieck-Lefschetz trace formula:

\([X]_q = \sum_i (-1)^i Tr(F : H^i_c(X_{\overline{F}_q}, \mathbb{Q}_\ell))\]

Dwork, Deligne.
Graph hypersurfaces

Graph hypersurface:

\[ X_G \subset \mathbb{A}^{N_G} \]

zero locus of the graph polynomial \( \Psi_G \). Highly singular.
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Example:

$$\Psi_G = \alpha_1 + \alpha_2 \quad , \quad [G]_q = q$$
Examples

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Notice that

$$[G]_q \equiv 0 \mod q^2$$
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Notice that
\[
[G]_q \equiv 0 \mod q^2
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Question: is \([X_G]_q\) always a polynomial in \(q\)?
The periods detect extensions, but the trace of Frobenius only depends on the semi-simplification $\text{Ms}$ of a motive $M$. There exists an abelian category of mixed Tate motives over number fields (Levine, using Beilinson-Soulé vanishing via Borel). Their point-counting functions are polynomials in $q$. 

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**Diagram:**

- **I-adic Galois reps**
- **Mixed Motives over $\mathbb{Q}$**
- **Period integrals**

Connections:
- $I$ from **I-adic Galois reps** to **Mixed Motives over $\mathbb{Q}$**
- $B$ from **Mixed Motives over $\mathbb{Q}$** to **Period integrals**
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Philosophy

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Results

Stembridge (1998): True for all graphs $G$ with $N_G \leq 12$.

Belkale, Brosnan (2003): The function $G_q$ is of general type. Given any $X$, there exist graphs $G_1, \ldots, G_k$ such that $r_0[X] = \sum_{i=1}^k r_i[G_i]$ where $r_i \in \mathbb{Z}[q]$ are polynomials in $q$. Uses Mnev universality.

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IV. Modularity
Consider the quantities

$$[X]_p \mod p$$

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for all primes \( p \). They define an element

\(([X]_p \mod p)_p \in \mathbb{F}_2 \times \mathbb{F}_3 \times \mathbb{F}_5 \times \ldots\)
Consider the quantities

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Suppose \( X \) defined by one polynomial \( f(x_1, \ldots, x_N) \).

1. If \( \deg f = N \),

\[ [X]_p \equiv (\text{coeff. of } (x_1 \ldots x_N)^{p-1} \text{ in } f^{p-1}) \mod p \]
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2. (Chevalley-Warning theorem). If degree $f < N$ then

$$[X]_p \equiv 0 \mod p$$
For one of our graphs, $G$, define $c_G(p)$ as:

$$c_G(p) := \left[ G \right] \pmod{p^2}$$

If $G$ is a polynomial, then $c_G(p)$ is the coefficient $k$ taken modulo all primes. Therefore, $c_G = (k \pmod{2}, k \pmod{3}, k \pmod{5}, ...)$

Call such a sequence constant.
For \( G \) one of our graphs, \( [G]_p \equiv 0 \mod p^2 \). Define
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c_G(p) := \frac{[G]_p}{p^2} \mod p
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If $[G]_q \in \mathbb{Z}[q]$ polynomial then $c_G(p)$ is its coefficient $k$ of $q^2$ taken modulo all primes. Therefore

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Call such a sequence constant.
The $c$-invariant contains the relevant information about $I_G$. Conjecture: If $I_G = I_G^0$ then $c_G = c_G^0$. 

6$\zeta(3)$  

20$\zeta(5)$  

36$\zeta(3)^2$


**c-invariant examples**

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The $c_G$ invariant contains the relevant information about $I_G$.

Conjecture: If $I_G = I_{G'}$, then $c_G = c_{G'}$. 
For each of the two (convergent, $\phi^4$) graphs:

\[
G(p) \mod p \quad \alpha \quad \beta
\]

are Fourier coefficients of modular form of weight 3, level 7:

\[
X_n z^n = z^3 + 5z^7 + 7z^{11} + 15z^{15} + \ldots
\]
For each of the two (convergent, $\phi^4$) graphs:

\[ c_G(p) \equiv a_p \mod p \]
Modular graphs (w/ O. Schnetz, 2012)

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$a_p$ are Fourier coeffs. of modular form of weight 3, level 7:

\[
\sum a_n z^n = z \prod_{n \geq 1} ((1 - z^n)(1 - z^{7n}))^3 \\
= z - 3z^2 + 5z^4 - 7z^7 - 3z^8 + \ldots
\]
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1. Find a polynomial \( f \) with \( \deg f = \# \{\text{variables of } f\} \) s.t.

\[
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1. Find a polynomial $f$ with $\text{deg } f = \# \{\text{variables of } f\}$ s.t.

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2. Eliminate variables in the right order from $f$ to reduce the dimension. Uses Chevalley-Warning, combinatorics of $G$, ...
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$$F = b(a + c)(ac + bd) - ad(b + c)(c + d)$$

The zero locus of $F$ defines a singular $K_3$ surface.
Singular K3 surfaces (maximal Picard rank 20) over $\mathbb{Q}$ are modular. Modular forms of weight 3 with CM by $\mathbb{Q}(\sqrt{-d})$, and rational coefficients. Follows from Livné (1995), modularity of two-dimensional CM Galois representations. Elkies and Schütt: they all arise in this way (2013).
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**Rigid Calabi-Yau three-folds over $\mathbb{Q}$ are modular (\ldots, Gouvêa-Yui (2010)).** Uses proof of Serre’s modularity conjecture by Khare and Wintenberger.
V. Questions
More modular counter-examples in $\phi^4$ (O. Schnetz)

<table>
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<tr>
<th>weight</th>
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Adam Logan (2016) has proved three more entries.
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No modular forms of weight 2?
What does this mean for Feynman amplitudes?

The point-counting function depends on $M_{pt} = N_1 \times \prod_{i=0}^{\infty} H_i c(X_G; Q)$.

The period integral depends on a piece of $M_{int} = H N_1 \left( \bigcap_{P} N_1 \cap X_G \cap D \cap (D \setminus X_G) \right)$.

$X_G$ is the graph hypersurface (Bloch-Esnault-Kreimer).

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One can show that the 'modular' piece of $\text{M}_{\text{pt}}$ actually arises in precisely the piece of $\text{M}_{\text{int}}$ detected by the integral (Doryn).

Grothendieck's period conjecture $=\quad$ for modular $G$, $\text{I}_G$ is transcendental over the ring of MZV's.

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How do we construct realisations of motives of mixed modular type? What are their period integrals?
Which numbers and functions for quantum field theory?

- Grothendieck, Deligne, Ihara, Drinfeld, . . .
- $\mathbb{P}^1 \cap \{0, 1, 1\}$
- Iterated integrals
- Multiple zeta values
- Polylogarithms

Generate all amplitudes up to a certain number of loops, and infinite families of amplitudes in $\mathbb{N} = 4$ SYM, $\mathbb{Q}$ C D, $\mathbb{Q}$ E D, . . .

Modular examples beyond this regime (e.g. also with masses)

What are the geometric objects which describe QFT in general?
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The numbers and functions generated by a single space:

\[ \mathbb{P}^1 \setminus \{0, 1, \infty\} \]

\[ \begin{array}{c}
\infty \\
0 \\
1 \\
\end{array} \]

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