# Periods, Galois theory and particle physics

Francis Brown
All Souls College, Oxford

Gergen Lectures,
21st-24th March 2016

# Recap

Recall that we are interested in periods, which are integrals

$$I = \int_\sigma \omega$$

where $\omega$ is a regular differential $n$ form on some $n$-dimensional algebraic variety $X$, $\sigma$ is a smooth chain with boundary in $Z(\mathbb{C})$, where $Z \subset X$, and everything is defined with rational coefficients.

The set of such integrals generates a ring $P$, where

$$\mathbb{Q} \subset \overline{\mathbb{Q}} \subset P \subset \mathbb{C}$$

We studied numerous examples in $P$, including multiple zeta values, and amplitudes in quantum field theory.

Today: the goal is to replace $P$ with a ring of formally defined periods $P^{\mathfrak{m}}$, and define a group $G$ which acts upon it.

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# Classical Galois group

The Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is the group of symmetries of algebraic numbers. It is the group of maps $\sigma : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}$ such that:

$$\begin{aligned} \sigma(\lambda_1\alpha + \lambda_2\beta) &= \lambda_1\sigma(\alpha) + \lambda_2\sigma(\beta) \, , \\ \sigma(\alpha\beta) &= \sigma(\alpha)\sigma(\beta) \, , \end{aligned}$$

where $\alpha, \beta \in \overline{\mathbb{Q}}$, and $\lambda_1, \lambda_2 \in \mathbb{Q}$ (view $\overline{\mathbb{Q}} \subset \mathbb{C}$).

It permutes the roots of any polynomial: if $P \in \mathbb{Q}[x]$ and $\alpha \in \mathbb{C}$ such that $P(\alpha) = 0$ then $P(\sigma(\alpha)) = 0$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

The set of Galois conjugates $\{\sigma(\alpha) : \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\}$ is finite.

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# A first attempt....

Since algebraic numbers are examples of periods, one can ask if there exists a Galois theory of periods?

We would like a group $G_P$ which acts as automorphisms of $P$:

$$\sigma(\lambda_1 p_1 + \lambda_2 p_2) = \lambda_1 \sigma(p_1) + \lambda_2 \sigma(p_2)$$

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for all $\lambda_1, \lambda_2 \in \mathbb{Q}$, and $p_1, p_2 \in P$. Its restriction to $\overline{\mathbb{Q}}$ should give back the classical Galois group. We would also like

$$p \in P \text{ is in } \mathbb{Q} \quad \iff \quad \sigma(p) = p \text{ for all } \sigma \in G_P$$

If $p$ has finitely many conjugates, then $\prod_\sigma (X - \sigma(p))$ is $G_P$-invariant and will be in $\mathbb{Q}[X]$. So $p$ is in fact algebraic.

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This seems hopeless. However, there is a work-around:

1. Instead of considering a group acting on periods $P$, we can study the group which acts on their defining data (i.e., an integrand and a domain of integration). The latter can be replaced with finite linear algebra (cohomology).
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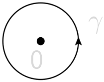
# Periods and cohomology

Let $X$ be a smooth affine variety over $\mathbb{Q}$. For example, the locus $\{(x_1, \ldots, x_n) : f_i = 0, g_j \neq 0\}$, where $f_i, g_j$ are finite sets of polynomials in $\mathbb{Q}[x_1, \ldots, x_n]$.

Its complex points $X(\mathbb{C})$ is a smooth manifold. The ordinary singular (Betti) homology $H_n(X(\mathbb{C}); \mathbb{Q})$ is defined by closed chains modulo boundaries. It is a finite-dimensional vector space over $\mathbb{Q}$.

*Example.* Let $X = \mathbb{P}^1 \backslash \{0, \infty\}$, the Riemann sphere punctured at two points. It satisfies $X(\mathbb{C}) = \mathbb{C}^\times$.

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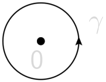


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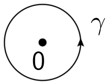


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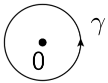


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# Algebraic de Rham cohomology for affine $X$

Let $\Omega^n_X$ denote the ring of global regular differential $n$-forms defined over $\mathbb{Q}$. The ordinary differential makes it into a complex

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \Omega^2_X \xrightarrow{d} \dots \xrightarrow{d} \Omega^n_X \longrightarrow 0$$

The cohomology of this complex (closed forms modulo exact forms) defines the algebraic de Rham cohomology

$$H^i_{dR}(X;\mathbb{Q})$$

It is a finite-dimensional vector space over $\mathbb{Q}$.

*Example.* For $X = \mathbb{P}^1 \backslash \{0, \infty\}$ we have $n = 1$, $\mathcal{O}_X = \mathbb{Q}[x, x^{-1}]$ and $\Omega^1_X = \mathbb{Q}[x, x^{-1}]dx$. The complex is

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# Comparison isomorphism

Integration pairs homology (cycles) with differential forms.

$$\begin{array}{rcl} H^n_{dR}(X(\mathbb{C})) \otimes_{\mathbb{Q}} H_n(X(\mathbb{C})) & \longrightarrow & \mathbb{C} \\ \omega \otimes \gamma & \mapsto & \int_\gamma \omega \end{array}$$

A better way to state this is via

**Theorem (Grothendieck - de Rham comparison isomorphism)**

There is a natural isomorphism

$$\mathrm{comp} : H^n_{dR}(X;\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H^n_B(X;\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$$

Betti cohomology $H^n_B(X;\mathbb{Q})$ is defined to be the dual of homology $H_n(X(\mathbb{C});\mathbb{Q})^\vee$. The map comp is $\omega \mapsto (\gamma \mapsto \int_\gamma \omega)$.

In particular, de Rham and Betti cohomology have the same dimension.

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$$\begin{array}{rcl} H^n_{dR}(X(\mathbb{C})) \otimes_{\mathbb{Q}} H_n(X(\mathbb{C})) & \longrightarrow & \mathbb{C} \\ \omega \otimes \gamma & \mapsto & \int_\gamma \omega \end{array}$$

A better way to state this is via

**Theorem (Grothendieck - de Rham comparison isomorphism)**

There is a natural isomorphism

$$\mathrm{comp} : H^n_{dR}(X;\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H^n_B(X;\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$$

Betti cohomology $H^n_B(X;\mathbb{Q})$ is defined to be the dual of homology $H_n(X(\mathbb{C});\mathbb{Q})^\vee$. The map comp is $\omega \mapsto (\gamma \mapsto \int_\gamma \omega)$.

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# Example: $2\pi i$

Return to our example $X = \mathbb{P}^1 \backslash \{0, \infty\}$. We calculated $H_1(X(\mathbb{C}); \mathbb{Q}) = \mathbb{Q}[\gamma]$, and $H^1_{dR}(X; \mathbb{Q}) = \mathbb{Q}[\frac{dx}{x}]$.

Cauchy's theorem is equivalent to the statement

$$2\pi i = \int_\gamma \frac{dx}{x} \tag{1}$$

The comparison isomorphism is therefore given by

$$\begin{aligned} \mathrm{comp}: \quad H^1_{dR}(X; \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} &\xrightarrow{\sim} H^1_B(X) \otimes_{\mathbb{Q}} \mathbb{C} \\ \left[\frac{dx}{x}\right] \otimes 1 &\mapsto [\gamma]^\vee \otimes 2\pi i \end{aligned}$$

The integral (1) is entirely encoded by the data:

$$\left[\frac{dx}{x}\right] \in H^1_{dR}(X; \mathbb{Q}) \ , \ [\gamma] \in (H^1_B(X))^\vee$$

and the map comp which tells us how to do the integration.

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# ‘Algebraic’ version of $2\pi i$

We can therefore replace the *number* $2\pi i$ with the data

$$((H^1_{dR}(X;\mathbb{Q}), H^1_B(X), \mathrm{comp}), \left[\frac{dx}{x}\right], [\gamma])$$

where $X = \mathbb{P}^1 \backslash \{0, \infty\}$. We can recover the period $2\pi i$ via

$$2\pi i = \langle \mathrm{comp}(\left[\frac{dx}{x}\right]), [\gamma] \rangle \ .$$

The geometry in the integral is represented by linear algebra data $(H^1_{dR}(X;\mathbb{Q}), H^1_B(X), \mathrm{comp})$. It consists of

$$(V_{dR}, V_B, c)$$

where $V_{dR}$, and $V_B$ are finite-dimensional $\mathbb{Q}$-vector spaces, and $c$ is an isomorphism $c : V_{dR} \otimes \mathbb{C} \xrightarrow{\sim} V_B \otimes \mathbb{C}$ . This can be formalised.

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# Matrix coefficients

Consider the category of triples

$$\mathcal{T}: \qquad \text{Objects} \qquad V = (V_{dR}, V_B, c)$$

with morphisms given by compatible linear maps. A *matrix coefficient* is an equivalence class

$$[V, \omega, \gamma] \quad \text{where} \quad \omega \in V_{dR}, \gamma \in V_B^\vee$$

modulo the relations

1. (Linearity) For $\lambda_1, \lambda_2 \in \mathbb{Q}$,

   $$[V, \lambda_1\omega_1 + \lambda_2\omega_2, \gamma] = \lambda_1[V, \omega_1, \gamma] + \lambda_2[V, \omega_2, \gamma]$$

   and similarly for $\gamma$.
2. (Functoriality) For all morphisms $\phi : V \to V'$ in $\mathcal{T}$,

   $$[V, \omega, (\phi_B)^\vee(\gamma) = [V', \phi_{dR}(\omega), \gamma]$$

These relations mirror (but are more flexible than) the relations between elementary periods discussed in the first lecture.

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# Motivic periods

Think of an equivalence class $[V, \omega, \gamma]$ as a 'disembodied integral' $\int_\gamma \omega$. Then the relations correspond to our usual intuition about manipulating integrals. There is a multiplication

$$[V_1, \omega_1, \gamma_1] \times [V_2, \omega_2, \gamma_2] = [V_1 \otimes V_2, \omega_1 \otimes \omega_2, \gamma_1 \otimes \gamma_2] .$$

The set of matrix coefficients forms a ring of '$\mathcal{T}$-periods' $\mathcal{P}_\mathcal{T}$. It is equipped with a period homomorphism

$$\begin{array}{rcccl} \mathrm{per} & : & \mathcal{P}_\mathcal{T} & \longrightarrow & \mathbb{C} \\ & & [V, \omega, \gamma] & \mapsto & \langle \mathrm{comp}\,(\omega), \gamma \rangle \end{array}$$

Warning: many $\mathcal{T}$-periods have nothing to do with periods in the sense of the first lecture. Only *some of them* actually come from period integrals. These are the ones where

$$(V_{dR}, V_B, c) \quad \text{are the cohomology of an algebraic variety}$$

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# Galois theory of motivic periods

Tannaka theory naturally provides a Galois group $\mathcal{G}$ acting on the ring $\mathcal{P}_{\mathcal{T}}$. It is an affine group scheme, for simplicity we shall only look at its rational points $\mathcal{G}(\mathbb{Q})$. This is just a group.

The group $\mathcal{G}$ is the (Tannaka) group of linear symmetries of $\mathcal{T}$. It is the largest group which acts linearly on $V_{dR}$ for all objects $(V_{dR}, V_B, c)$ in $\mathcal{T}$ in a compatible way.
In particular, for every such object, we get a homomorphism

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It acts on integrals by changing the 'differential' form $\omega \in V_{dR}$. It is clear that $\mathcal{G}$ will preserve the subspace of matrix coefficients which come from geometry, because $V$ is unchanged in both sides.

# Example: Galois action on $(2\pi i)^{\mathfrak{m}}$

Recall $X = \mathbb{P}^1 \backslash \{0, \infty\})$ and $2\pi i$ was replaced by the data:

$$[H^1(X), [\tfrac{dx}{x}], [\gamma]]$$

where $H^1(X) = (H^1(X)_{dR}, H^1_B(X), \mathrm{comp})$. The group $\mathcal{G}$ acts on the de Rham vector space

$$H^1(X)_{dR} = \mathbb{Q}[\tfrac{dx}{x}]$$

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# What's going on?

We started out with a ring $P$ of periods

$$\mathbb{Q} \subset P \subset \mathbb{C}$$

defined by elementary algebraic integrals

$$\int_\gamma \omega \ .$$

A naive attempt to study automorphisms of $P$ doesn't seem to work, and we run into difficult transcendence questions.

Instead, we replaced $P$ with a new ring $\mathcal{P}^{\mathfrak{m}} \subset \mathcal{P}_{\mathcal{T}}$ which *tautologically* carries an action by its group $\mathcal{G}$ of linear symmetries.

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# Some more examples: logarithm

What happened to Stokes' formula? This can be captured by including relative cohomology. This is necessary when we consider integrals over chains which have non-trivial boundary.

If $Z \subset X$ there is a notion of relative singular homology (chains with boundaries contained in $Z(\mathbb{C})$). There is a long exact sequence

$$\to H_i(Z(\mathbb{C})) \to H_i(X(\mathbb{C})) \to H_i(X(\mathbb{C}), Z(\mathbb{C})) \to$$

Likewise, there is a notion of relative algebraic de Rham cohomology, consisting of closed forms whose restriction to $Z$ vanishes. It sits in a long exact sequence

$$\longrightarrow H^i_{dR}(X, Z) \longrightarrow H^i_{dR}(X) \longrightarrow H^i_{dR}(Z) \longrightarrow H^{i+1}_{dR}(X, Z) \longrightarrow$$

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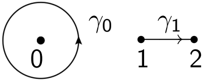
Let $X = \mathbb{P}^1 \backslash \{0, \infty\}$ and $Z = \{1, 2\}$. Then $H_1(X(\mathbb{C}), Z(\mathbb{C}))$ is a two-dimensional vector space, generated by the class of $\gamma_0$ and the path $\gamma_1$ whose endpoints are contained in $Z(\mathbb{C})$.

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A *period matrix* is obtained by pairing the forms with the cycles

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Define the motivic version of log(2) to be

$$\log^{\mathfrak{m}}(2) = [H^1(X, Z), [\frac{dx}{x}], [\gamma_1]]$$

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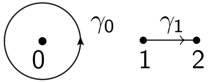


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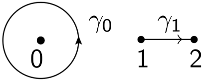


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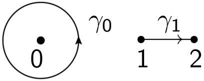


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The group $\mathcal{G}(\mathbb{Q})$ acts on $H^1_{dR}(\mathbb{P}^1 \backslash \{0,\infty\}, \{1,2\})$. We have the following short exact sequence in the category $\mathcal{T}$:

$$0 \to \widetilde{H}^0_\bullet(\{1,2\}) \to H^1_\bullet(\mathbb{P}^1 \backslash \{0,\infty\}, \{1,2\}) \to H^1_\bullet(\mathbb{P}^1 \backslash \{0,\infty\}) \to 0$$

The group action must respect it. We already established how $\mathcal{G}(\mathbb{Q})$ acts on the right-hand factor. It acts trivially on the left-hand factor. Therefore it fixes $[dx]$ and sends

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for some $\nu_g \in \mathbb{Q}$. So $\mathcal{G}(\mathbb{Q})$ acts on the 'motivic' version of the period matrix by multiplication on the right:

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The programme:

1. Take an interesting family of period integrals $I$.
2. Express them as a pairing between de Rham and Betti cohomology of algebraic varieties.
3. Define the corresponding motivic periods $I^{\mathfrak{m}}$ and compute the action of the Galois group $\mathcal{G}$ on them.

Note that the motivic period depends on the choice of integral representation. Conjecturally, it should not depend.

Proving that two motivic periods $I_1^{\mathfrak{m}}$ and $I_2^{\mathfrak{m}}$ are equal is a stronger statement than proving that the numbers $I_1$ and $I_2$ are equal. But it is much *easier* than proving equality in the Kontsevich-Zagier sense (using standard relations).

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2. Express them as a pairing between de Rham and Betti cohomology of algebraic varieties.
3. Define the corresponding motivic periods $I^{\mathfrak{m}}$ and compute the action of the Galois group $\mathcal{G}$ on them.

Note that the motivic period depends on the choice of integral representation. Conjecturally, it should not depend.

Proving that two motivic periods $I_1^{\mathfrak{m}}$ and $I_2^{\mathfrak{m}}$ are equal is a stronger statement than proving that the numbers $I_1$ and $I_2$ are equal. But it is much *easier* than proving equality in the Kontsevich-Zagier sense (using standard relations).

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# Multiple zeta values

The programme has been completely carried out for MZV's.

**Theorem (B. 2012)**

There exist motivic versions of multiple zeta values

$$\zeta^{\mathfrak{m}}(n_1, \ldots, n_r) \in \mathcal{P}^{\mathfrak{m}}$$

for all $n_1, \ldots, n_r \geq 1$, $n_r \geq 2$, whose periods are $\zeta(n_1, \ldots, n_r)$. They satisfy the regularised double shuffle equations (shuffle, stuffle products, and regularisation relation).

The *motivic* relations between MZV's are those satisfied by the $\zeta^{\mathfrak{m}}$. We don't know if there are more motivic relations than the regularised double shuffle equations.

The group $\mathcal{G}$ acts on the ring $\mathcal{Z}^{\mathfrak{m}}$ generated by the $\zeta^{\mathfrak{m}}$.

We get a group $\mathcal{G}^{MZV}$ acting on $\mathcal{Z}^{\mathfrak{m}}$. What does it look like?

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The even motivic zeta values satisfy a version of Euler's identity

$$\zeta^{\mathfrak{m}}(2n) = -\frac{B_{2n}}{2}\frac{((2\pi i)^{\mathfrak{m}})^{2n}}{(2n)!} \ .$$

Independently, we can prove that $\mathcal{G}$ acts via

$$\zeta^{\mathfrak{m}}(2n) \mapsto \lambda_g^{2n}\zeta^{\mathfrak{m}}(2n) \ .$$

The period matrix for an odd zeta value resembles log(2):

$$\begin{pmatrix} 1 & \zeta(2n+1) \\ 0 & (2\pi i)^{2n+1} \end{pmatrix}$$

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$$\zeta^{\mathfrak{m}}(2n+1) \mapsto \lambda_g^{2n+1}\zeta^{\mathfrak{m}}(2n+1) + s_g^{(2n+1)}$$

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# Infinitesimal Galois action

Choose an element of $\mathcal{G}$ which transforms the odd zeta value

$$\zeta^{\mathfrak{m}}(2n+1) \mapsto \zeta^{\mathfrak{m}}(2n+1) + 1$$

and fixes all the other ones. Denote its logarithm by $\sigma_{2n+1}$. We obtain elements corresponding to each odd zeta value:

$$\sigma_3, \sigma_5, \sigma_7, \ldots$$

These act on the space of motivic MZV's:

$$\begin{array}{cccc} \sigma_{2n+1} & : & \mathcal{Z}^{\mathfrak{m}} & \rightarrow & \mathcal{Z}^{\mathfrak{m}} \\ & & \zeta^{\mathfrak{m}}(2p+1) & \mapsto & \delta_{n,p}\ . \end{array}$$

They act as derivations: $\sigma(\xi_1\xi_2) = \sigma(x_1)\xi_2 + \xi_1\sigma(\xi_2)$ and act trivially on $(2\pi i)^{\mathfrak{m}}$.

Think of $\sigma_{2n+1}$ as 'differentiation with respect to $\zeta(2n+1)$'.

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# A model

Consider the graded $\mathbb{Q}$-vector space $M$ spanned by words in letters

$$f_3, f_5, f_7, \ldots$$

where $f_{2n+1}$ is in weight $2n+1$. Consider the polynomial ring $M[f_2]$ where $f_2$ has weight 2. It is an algebra for the shuffle product.

| Weight | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| | 1 | $\emptyset$ | $f_2$ | $f_3$ | $f_2^2$ | $f_5$<br>$f_3 f_2$ | $f_2^3$<br>$f_3 f_3$ | $f_7$<br>$f_5 f_2$<br>$f_3 f_2^2$ | $f_2^4$<br>$f_3 f_3 f_2$<br>$f_5 f_3$<br>$f_3 f_5$ |
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Look familiar from lecture 1? Define an action

$$\sigma_{2n+1} f_{i_1} \ldots f_{i_r} f_2^k = (\sigma_{2n+1} f_{i_1}) f_{i_2} \ldots f_{i_r} f_2^k$$

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# Structure theorem

**Theorem (B. 2012)**

There is an isomorphism of algebras

$$\phi : \mathcal{Z}^{\mathfrak{m}} \xrightarrow{\sim} M[f_2] \tag{2}$$

which commutes with the action of $\sigma_{2n+1}$ on both sides. It satisfies

$$\phi(\zeta^{\mathfrak{m}}(2n+1)) = f_{2n+1} \qquad \text{and} \qquad \phi(\zeta^{\mathfrak{m}}(2)) = f_2$$

In fact, there is a canonical such isomorphism!

This theorem completely elucidates the structure of motivic multiple zeta values. If $d_k$ is the dimension of $M[f_2]$ in weight $k$, it is a nice exercise to check that for all $k$,

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# Examples

The $f$-alphabet representation of an MZV reduces the problem of relations to linear algebra.

For example, in weight 5 we have a 2-dimensional vector space:

$$\begin{aligned} \phi(\zeta^{\mathfrak{m}}(2,3)) &= \frac{-11}{2} f_5 + 3 f_3 f_2 \\ \phi(\zeta^{\mathfrak{m}}(3,2)) &= \frac{9}{2} f_5 - 2 f_3 f_2 \end{aligned}$$

The map $\phi$ automatically respects all algebraic relations between motivic MZV's. For example, the stuffle relation

$$\zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(2) = \zeta^{\mathfrak{m}}(5) + \zeta^{\mathfrak{m}}(2,3) + \zeta^{\mathfrak{m}}(3,2)$$

is respected by $\phi$ and trivially checked in the model $M[f_2]$.

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The point is that the Galois point of view of MZV's enables us to cut through the complicated relations (which are not yet understood fully) and see the structure lying underneath. Indeed, we do not know what all the relations between motivic MZV's are!

**The full theorem**

The Galois group of motivic multiple zeta values is the semi-direct product of the multiplicative group $\mathbb{G}_m$ with a pro-algebraic group $U$ whose graded Lie algebra is free on generators

$$\sigma_3, \sigma_5, \sigma_7, \dots .$$

The elements $\zeta^{\mathfrak{m}}(n_1, \dots, n_r)$ for $n_i \in \{2, 3\}$ form a basis of $\mathcal{Z}^{\mathfrak{m}}$.

The ring of multiple zeta values is the first prototypical example of a Galois theory of periods. It 'grows out of' the single zeta values.

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# Conjectures

We expect that a relation holds between MZV's if and only if it holds between their motivic versions.

**Transcendence Conjecture**

The period homomorphism is injective, hence an isomorphism:

$$\text{per} : \mathcal{Z}^{\mathfrak{m}} \xrightarrow{\sim} \mathcal{Z}$$

This implies the main transcendence conjecture for MZV's, and would mean that the action of $\mathcal{G}^{MZV}$ is transported to numbers.

**Algebraic Conjecture**

The ring of motivic multiple zeta values is isomorphic to the ring generated by symbols $Z(n_1, \ldots, n_r)$ where $n_1, \ldots, n_r \geq 1$ and $n_r \geq 2$, modulo the regularised double shuffle equations.

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# Summary

1. Choose a natural integral representation for a period $I$, and interpret it as a pairing between cohomology theories.
2. Define its 'motivic version' $I^{\mathfrak{m}}$: it is an equivalence class of the defining data.
3. For free, we get a group which acts on such objects, and respects all the relations between them. Most of the time, we do not need to know these relations.
4. We can use this group action to define invariants. For MZV's this took the form of an assignment:

$$\text{Motivic MZV} \longrightarrow \text{ a word in } f_{2i+1}, f_2$$

where $f_n$ corresponds to $\zeta(n)$. It automatically respects the relations, and hey-presto, the simple structure underlying the ring of MZV's naturally emerges.

Next time: apply to general periods, and to amplitudes in physics.

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