Periods, Galois theory and particle physics

Francis Brown
All Souls College, Oxford

Gergen Lectures,
21st-24th March 2016

Reminders

We are interested in periods

$$I = \int_\gamma \omega$$

where ω is a regular algebraic n form on a smooth affine variety X of dimension n over $\mathbb{Q}$, and $\gamma \subset X(\mathbb{C})$ is a cycle which can have a boundary. We assume $\partial\gamma \subset Z(\mathbb{C})$ where $Z \subset X$ smooth and normal crossing over $\mathbb{Q}$.

Examples: MZV's and Feynman integrals in high-energy physics.

Grothendieck's conjectures on motives suggest there should be a Galois theory of periods (André, Kontsevich), so we can hope to study such integrals via group theory. One cheap way to set this up is via motivic periods.

Reminders

We are interested in periods

$$I = \int_\gamma \omega$$

where ω is a regular algebraic n form on a smooth affine variety X of dimension n over $\mathbb{Q}$, and $\gamma \subset X(\mathbb{C})$ is a cycle which can have a boundary. We assume $\partial\gamma \subset Z(\mathbb{C})$ where $Z \subset X$ smooth and normal crossing over $\mathbb{Q}$.

Examples: MZV's and Feynman integrals in high-energy physics.

Grothendieck's conjectures on motives suggest there should be a Galois theory of periods (André, Kontsevich), so we can hope to study such integrals via group theory. One cheap way to set this up is via motivic periods.

Reminders

We are interested in periods

$$I = \int_\gamma \omega$$

where ω is a regular algebraic n form on a smooth affine variety X of dimension n over $\mathbb{Q}$, and $\gamma \subset X(\mathbb{C})$ is a cycle which can have a boundary. We assume $\partial\gamma \subset Z(\mathbb{C})$ where $Z \subset X$ smooth and normal crossing over $\mathbb{Q}$.

Examples: MZV's and Feynman integrals in high-energy physics.

Grothendieck's conjectures on motives suggest there should be a Galois theory of periods (André, Kontsevich), so we can hope to study such integrals via group theory. One cheap way to set this up is via motivic periods.

Reminders

We are interested in periods

$$I = \int_\gamma \omega$$

where ω is a regular algebraic n form on a smooth affine variety X of dimension n over $\mathbb{Q}$, and $\gamma \subset X(\mathbb{C})$ is a cycle which can have a boundary. We assume $\partial\gamma \subset Z(\mathbb{C})$ where $Z \subset X$ smooth and normal crossing over $\mathbb{Q}$.

Examples: MZV's and Feynman integrals in high-energy physics.

Grothendieck's conjectures on motives suggest there should be a Galois theory of periods (André, Kontsevich), so we can hope to study such integrals via group theory. One cheap way to set this up is via motivic periods.

Reminders 2

We encode our integral by algebraic data:

1. A finite-dimensional $\mathbb{Q}$-vector space $H^n_{dR}(X,Z)$ (closed algebraic forms which vanish on Z, modulo exact forms).
2. A finite-dimensional $\mathbb{Q}$-vector space $H^n_B(X,Z) = H_n(X(\mathbb{C}), Z(\mathbb{C}))^\vee$ (closed cochains modulo coboundaries).
3. Integration is encoded by the comparison isomorphism

$$\mathrm{comp} : H^n_{dR}(X,Z) \otimes \mathbb{C} \xrightarrow{\sim} H^n_B(X,Z) \otimes \mathbb{C}$$

where $H^n_B(X,Z)$ is the dual vector space of $H_n(X(\mathbb{C}), Z(\mathbb{C}))$.

Denote this data by

$$H^n(X,Z) \quad := \quad (H^n_{dR}(X,Z), H^n_B(X,Z), \mathrm{comp})$$

The integral itself only depends on the classes of ω, γ

$$[\omega] \in H^n_{dR}(X,Z) \qquad\qquad [\gamma] \in H^n_B(X,Z)^\vee$$

The integral I is equal to $\langle \mathrm{comp}([\omega]), [\gamma] \rangle \in \mathbb{C}$.

Reminders 2

We encode our integral by algebraic data:

1. A finite-dimensional $\mathbb{Q}$-vector space $H^n_{dR}(X, Z)$ (closed algebraic forms which vanish on Z, modulo exact forms).
2. A finite-dimensional $\mathbb{Q}$-vector space $H^n_B(X, Z) = H_n(X(\mathbb{C}), Z(\mathbb{C}))^\vee$ (closed cochains modulo coboundaries).
3. Integration is encoded by the comparison isomorphism

$$\mathrm{comp} : H^n_{dR}(X, Z) \otimes \mathbb{C} \xrightarrow{\sim} H^n_B(X, Z) \otimes \mathbb{C}$$

where $H^n_B(X, Z)$ is the dual vector space of $H_n(X(\mathbb{C}), Z(\mathbb{C}))$.

Denote this data by

$$H^n(X, Z) \quad := \quad (H^n_{dR}(X, Z), H^n_B(X, Z), \mathrm{comp})$$

The integral itself only depends on the classes of ω, γ

$$[\omega] \in H^n_{dR}(X, Z) \qquad\qquad [\gamma] \in H^n_B(X, Z)^\vee$$

The integral I is equal to $\langle \mathrm{comp}([\omega]), [\gamma] \rangle \in \mathbb{C}$.

Reminders 2

We encode our integral by algebraic data:

1. A finite-dimensional $\mathbb{Q}$-vector space $H^n_{dR}(X, Z)$ (closed algebraic forms which vanish on Z, modulo exact forms).
2. A finite-dimensional $\mathbb{Q}$-vector space $H^n_B(X, Z) = H_n(X(\mathbb{C}), Z(\mathbb{C}))^\vee$ (closed cochains modulo coboundaries).
3. Integration is encoded by the comparison isomorphism

$$\mathrm{comp} : H^n_{dR}(X, Z) \otimes \mathbb{C} \xrightarrow{\sim} H^n_B(X, Z) \otimes \mathbb{C}$$

where $H^n_B(X, Z)$ is the dual vector space of $H_n(X(\mathbb{C}), Z(\mathbb{C}))$.

Denote this data by

$$H^n(X, Z) \quad := \quad (H^n_{dR}(X, Z), H^n_B(X, Z), \mathrm{comp})$$

The integral itself only depends on the classes of ω, γ

$$[\omega] \in H^n_{dR}(X, Z) \qquad\qquad [\gamma] \in H^n_B(X, Z)^\vee$$

The integral I is equal to $\langle \mathrm{comp}([\omega]), [\gamma] \rangle \in \mathbb{C}$.

Reminders 2

We encode our integral by algebraic data:

1. A finite-dimensional $\mathbb{Q}$-vector space $H^n_{dR}(X,Z)$ (closed algebraic forms which vanish on Z, modulo exact forms).
2. A finite-dimensional $\mathbb{Q}$-vector space $H^n_B(X,Z) = H_n(X(\mathbb{C}), Z(\mathbb{C}))^\vee$ (closed cochains modulo coboundaries).
3. Integration is encoded by the comparison isomorphism

$$\mathrm{comp} : H^n_{dR}(X,Z) \otimes \mathbb{C} \xrightarrow{\sim} H^n_B(X,Z) \otimes \mathbb{C}$$

where $H^n_B(X,Z)$ is the dual vector space of $H_n(X(\mathbb{C}), Z(\mathbb{C}))$.

Denote this data by

$$H^n(X,Z) \quad := \quad (H^n_{dR}(X,Z), H^n_B(X,Z), \mathrm{comp})$$

The integral itself only depends on the classes of ω, γ

$$[\omega] \in H^n_{dR}(X,Z) \qquad\qquad [\gamma] \in H^n_B(X,Z)^\vee$$

The integral I is equal to $\langle \mathrm{comp}([\omega]), [\gamma] \rangle \in \mathbb{C}$.

Reminders 2

We encode our integral by algebraic data:

1. A finite-dimensional $\mathbb{Q}$-vector space $H^n_{dR}(X, Z)$ (closed algebraic forms which vanish on Z, modulo exact forms).
2. A finite-dimensional $\mathbb{Q}$-vector space $H^n_B(X, Z) = H_n(X(\mathbb{C}), Z(\mathbb{C}))^\vee$ (closed cochains modulo coboundaries).
3. Integration is encoded by the comparison isomorphism

$$\mathrm{comp} : H^n_{dR}(X, Z) \otimes \mathbb{C} \xrightarrow{\sim} H^n_B(X, Z) \otimes \mathbb{C}$$

where $H^n_B(X, Z)$ is the dual vector space of $H_n(X(\mathbb{C}), Z(\mathbb{C}))$.

Denote this data by

$$H^n(X, Z) \quad := \quad (H^n_{dR}(X, Z), H^n_B(X, Z), \mathrm{comp})$$

The integral itself only depends on the classes of ω, γ

$$[\omega] \in H^n_{dR}(X, Z) \qquad\qquad [\gamma] \in H^n_B(X, Z)^\vee$$

The integral I is equal to $\langle \mathrm{comp}([\omega]), [\gamma] \rangle \in \mathbb{C}$.

We encode our integral by algebraic data:

1. A finite-dimensional $\mathbb{Q}$-vector space $H^n_{dR}(X, Z)$ (closed algebraic forms which vanish on Z, modulo exact forms).
2. A finite-dimensional $\mathbb{Q}$-vector space $H^n_B(X, Z) = H_n(X(\mathbb{C}), Z(\mathbb{C}))^\vee$ (closed cochains modulo coboundaries).
3. Integration is encoded by the comparison isomorphism
$$\mathrm{comp} : H^n_{dR}(X, Z) \otimes \mathbb{C} \xrightarrow{\sim} H^n_B(X, Z) \otimes \mathbb{C}$$
where $H^n_B(X, Z)$ is the dual vector space of $H_n(X(\mathbb{C}), Z(\mathbb{C}))$.

Denote this data by
$$H^n(X, Z) \quad := \quad (H^n_{dR}(X, Z), H^n_B(X, Z), \mathrm{comp})$$
The integral itself only depends on the classes of ω, γ
$$[\omega] \in H^n_{dR}(X, Z) \qquad\qquad [\gamma] \in H^n_B(X, Z)^\vee$$
The integral I is equal to $\langle \mathrm{comp}([\omega]), [\gamma] \rangle \in \mathbb{C}$.

Reminders 3

The 'motivic' version of I is the equivalence class

$$I^{\mathfrak{m}} := [H^n(X, Z), [\omega], [\gamma]]$$

where the equivalence relation is: bilinearity in $[\omega]$, $[\gamma]$, and naturality with respect to morphisms in the category of triples (V_{dR}, V_B, c), where $V_B, V_{dR} \in \mathrm{Vec}_{\mathbb{Q}}$, $c : V_{dR} \otimes \mathbb{C} \cong V_B \otimes \mathbb{C}$. Note that the morphisms are not required to be 'geometric'.

The set of $I^{\mathfrak{m}}$ forms a ring $\mathcal{P}^{\mathfrak{m}}$ equipped with a homomorphism

$$\begin{aligned} \mathrm{per} : \mathcal{P}^{\mathfrak{m}} &\longrightarrow P \\ I^{\mathfrak{m}} &\longmapsto I \end{aligned}$$

Version of Grothendieck's 'Period' conjecture

The period homomorphism is an isomorphism

Reminders 3

The 'motivic' version of I is the equivalence class

$$I^{\mathfrak{m}} := [H^n(X,Z), [\omega], [\gamma]]$$

where the equivalence relation is: bilinearity in $[\omega]$, $[\gamma]$, and naturality with respect to morphisms in the category of triples (V_{dR}, V_B, c), where $V_B, V_{dR} \in \mathrm{Vec}_{\mathbb{Q}}$, $c : V_{dR} \otimes \mathbb{C} \cong V_B \otimes \mathbb{C}$. Note that the morphisms are not required to be 'geometric'.

The set of $I^{\mathfrak{m}}$ forms a ring $\mathcal{P}^{\mathfrak{m}}$ equipped with a homomorphism

$$\begin{array}{rcl} \mathrm{per} : \mathcal{P}^{\mathfrak{m}} & \longrightarrow & P \\ I^{\mathfrak{m}} & \mapsto & I \end{array}$$

Version of Grothendieck's 'Period' conjecture

The period homomorphism is an isomorphism

The 'motivic' version of I is the equivalence class

$$I^{\mathfrak{m}} := [H^n(X, Z), [\omega], [\gamma]]$$

where the equivalence relation is: bilinearity in $[\omega]$, $[\gamma]$, and naturality with respect to morphisms in the category of triples (V_{dR}, V_B, c), where $V_B, V_{dR} \in \mathrm{Vec}_{\mathbb{Q}}$, $c : V_{dR} \otimes \mathbb{C} \cong V_B \otimes \mathbb{C}$. Note that the morphisms are not required to be 'geometric'.

The set of $I^{\mathfrak{m}}$ forms a ring $\mathcal{P}^{\mathfrak{m}}$ equipped with a homomorphism

$$\begin{array}{rcl} \mathrm{per} : \mathcal{P}^{\mathfrak{m}} & \longrightarrow & P \\ I^{\mathfrak{m}} & \mapsto & I \end{array}$$

Version of Grothendieck's 'Period' conjecture

The period homomorphism is an isomorphism

Reminders 4

The theory of Tannakian categories automatically endows $\mathcal{P}^{\mathfrak{m}}$ with an action of an affine group scheme $\mathcal{G}$:

$$\mathcal{G} \times \mathcal{P}^{\mathfrak{m}} \longrightarrow \mathcal{P}^{\mathfrak{m}}$$

It acts on $I^{\mathfrak{m}}$ as follows: by definition of $\mathcal{G}$ there is a homomorphism $\mathcal{G} \to \mathrm{GL}(H^n_{dR}(X, Z))$, and so

$$gI^{\mathfrak{m}} = g\left[H^n(X, Z), [\omega], [\gamma]\right] = \left[H^n(X, Z), g[\omega], [\gamma]\right]$$

Think of this as a Galois group of motivic periods.

Last time we computed

$$g(2\pi i)^{\mathfrak{m}} = \lambda_g (2\pi i)^{\mathfrak{m}}$$

$$g \log^{\mathfrak{m}}(2) = \lambda_g \log^{\mathfrak{m}}(2) + \nu_g$$

for some non-trivial $\lambda_g : \mathcal{G} \to \mathbb{G}_m$ and $\lambda_g, \nu_g : \mathcal{G} \to \mathbb{G}_a \rtimes \mathbb{G}_m$.

The theory of Tannakian categories automatically endows $\mathcal{P}^{\mathfrak{m}}$ with an action of an affine group scheme $\mathcal{G}$:

$$\mathcal{G} \times \mathcal{P}^{\mathfrak{m}} \longrightarrow \mathcal{P}^{\mathfrak{m}}$$

It acts on $I^{\mathfrak{m}}$ as follows: by definition of $\mathcal{G}$ there is a homomorphism $\mathcal{G} \to \mathrm{GL}(H^n_{dR}(X,Z))$, and so

$$gI^{\mathfrak{m}} = g\,[H^n(X,Z),[\omega],[\gamma]] = [H^n(X,Z), g[\omega],[\gamma]]$$

Think of this as a Galois group of motivic periods.

Last time we computed

$$g(2\pi i)^{\mathfrak{m}} = \lambda_g (2\pi i)^{\mathfrak{m}}$$

$$g\log^{\mathfrak{m}}(2) = \lambda_g \log^{\mathfrak{m}}(2) + \nu_g$$

for some non-trivial $\lambda_g : \mathcal{G} \to \mathbb{G}_m$ and $\lambda_g, \nu_g : \mathcal{G} \to \mathbb{G}_a \rtimes \mathbb{G}_m$.

Reminders 4

The theory of Tannakian categories automatically endows $\mathcal{P}^{\mathfrak{m}}$ with an action of an affine group scheme $\mathcal{G}$:

$$\mathcal{G} \times \mathcal{P}^{\mathfrak{m}} \longrightarrow \mathcal{P}^{\mathfrak{m}}$$

It acts on $I^{\mathfrak{m}}$ as follows: by definition of $\mathcal{G}$ there is a homomorphism $\mathcal{G} \to \mathrm{GL}(H^n_{dR}(X, Z))$, and so

$$gI^{\mathfrak{m}} = g\,[H^n(X, Z), [\omega], [\gamma]] = [H^n(X, Z), g[\omega], [\gamma]]$$

Think of this as a Galois group of motivic periods.

Last time we computed

$$g(2\pi i)^{\mathfrak{m}} = \lambda_g (2\pi i)^{\mathfrak{m}}$$

$$g \log^{\mathfrak{m}}(2) = \lambda_g \log^{\mathfrak{m}}(2) + \nu_g$$

for some non-trivial $\lambda_g : \mathcal{G} \to \mathbb{G}_m$ and $\lambda_g, \nu_g : \mathcal{G} \to \mathbb{G}_a \rtimes \mathbb{G}_m$.

Weights

In fact, the cohomology of algebraic varieties has a lot of extra structure (e.g., a mixed Hodge structure). It is useful to refine the defining data in a minimal way as follows:

1. $H^n_{dR}(X,Z)$ and $H^n_B(X,Z)$ both have a natural increasing filtration W called the weight filtration.
2. The comparison isomorphism

$$\mathrm{comp} : H^n_{dR}(X,Z) \otimes \mathbb{C} \longrightarrow H^n_B(X,Z) \otimes \mathbb{C}$$

respects the weight filtrations on both sides.

The upshot is that we can enhance the ring $\mathcal{P}^{\mathfrak{m}}$ so that it too has a weight filtration $W_n\mathcal{P}^{\mathfrak{m}}$, which is preserved by $\mathcal{G}$. Our motivic period $I^{\mathfrak{m}}$ is in W_m if and only if $[\omega] \in W_m H^n_{dR}(X,Z)$. We have

$$W_{-1}\mathcal{P}^{\mathfrak{m}} = 0 \qquad \text{and} \qquad W_0\mathcal{P}^{\mathfrak{m}} \overset{Thm}{=} \overline{\mathbb{Q}} \ .$$

Weights

In fact, the cohomology of algebraic varieties has a lot of extra structure (e.g., a mixed Hodge structure). It is useful to refine the defining data in a minimal way as follows:

1. $H^n_{dR}(X, Z)$ and $H^n_B(X, Z)$ both have a natural increasing filtration W called the weight filtration.
2. The comparison isomorphism

$$\mathrm{comp} : H^n_{dR}(X, Z) \otimes \mathbb{C} \longrightarrow H^n_B(X, Z) \otimes \mathbb{C}$$

respects the weight filtrations on both sides.

The upshot is that we can enhance the ring $\mathcal{P}^{\mathfrak{m}}$ so that it too has a weight filtration $W_n\mathcal{P}^{\mathfrak{m}}$, which is preserved by $\mathcal{G}$. Our motivic period $I^{\mathfrak{m}}$ is in W_m if and only if $[\omega] \in W_m H^n_{dR}(X, Z)$. We have

$$W_{-1}\mathcal{P}^{\mathfrak{m}} = 0 \qquad \text{and} \qquad W_0\mathcal{P}^{\mathfrak{m}} \overset{Thm}{=} \overline{\mathbb{Q}} \, .$$

Weights

In fact, the cohomology of algebraic varieties has a lot of extra structure (e.g., a mixed Hodge structure). It is useful to refine the defining data in a minimal way as follows:

1. $H^n_{dR}(X,Z)$ and $H^n_B(X,Z)$ both have a natural increasing filtration W called the weight filtration.
2. The comparison isomorphism
$$\mathrm{comp} : H^n_{dR}(X,Z) \otimes \mathbb{C} \longrightarrow H^n_B(X,Z) \otimes \mathbb{C}$$
respects the weight filtrations on both sides.

The upshot is that we can enhance the ring $\mathcal{P}^{\mathfrak{m}}$ so that it too has a weight filtration $W_n\mathcal{P}^{\mathfrak{m}}$, which is preserved by $\mathcal{G}$. Our motivic period $I^{\mathfrak{m}}$ is in W_m if and only if $[\omega] \in W_m H^n_{dR}(X,Z)$. We have

$$W_{-1}\mathcal{P}^{\mathfrak{m}} = 0 \qquad \text{and} \qquad W_0\mathcal{P}^{\mathfrak{m}} \overset{Thm}{=} \overline{\mathbb{Q}} \, .$$

Weights

In fact, the cohomology of algebraic varieties has a lot of extra structure (e.g., a mixed Hodge structure). It is useful to refine the defining data in a minimal way as follows:

1. $H^n_{dR}(X,Z)$ and $H^n_B(X,Z)$ both have a natural increasing filtration W called the weight filtration.
2. The comparison isomorphism

$$\mathrm{comp} : H^n_{dR}(X,Z)\otimes\mathbb{C} \longrightarrow H^n_B(X,Z)\otimes\mathbb{C}$$

   respects the weight filtrations on both sides.

The upshot is that we can enhance the ring $\mathcal{P}^{\mathfrak{m}}$ so that it too has a weight filtration $W_n\mathcal{P}^{\mathfrak{m}}$, which is preserved by $\mathcal{G}$. Our motivic period $I^{\mathfrak{m}}$ is in W_m if and only if $[\omega]\in W_mH^n_{dR}(X,Z)$. We have

$$W_{-1}\mathcal{P}^{\mathfrak{m}}=0 \qquad \text{and} \qquad W_0\mathcal{P}^{\mathfrak{m}} \overset{Thm}{=} \overline{\mathbb{Q}}\,.$$

What can we do with all this?

Every motivic period defines a representation of $\mathcal{G}$. For example, $(2i\pi)^{\mathfrak{m}}$ gives rise to a one-dimensional representation

$$g \mapsto \lambda_g : \mathcal{G} \longrightarrow \mathrm{GL}_1$$

and $\log^{\mathfrak{m}}(2)$ to a two-dimensional one

$$\begin{array}{ccc} \mathcal{G} & \longrightarrow & \mathrm{GL}_2 \\ g & \mapsto & \begin{pmatrix} 1 & \nu_g \\ 0 & \lambda_g \end{pmatrix} \end{array}$$

We obtain a new invariant: $\mathrm{rank}\,(I^{\mathfrak{m}})$ is the dimension of the representation of $\mathcal{G}$ that $I^{\mathfrak{m}}$ generates.

Define the *Galois conjugates* of $I^{\mathfrak{m}}$ to be elements in this representation (in the case of an algebraic number this would be a linear combination of its Galois conjugates).

From numbers to representations

What can we do with all this?

Every motivic period defines a representation of $\mathcal{G}$. For example, $(2i\pi)^{\mathfrak{m}}$ gives rise to a one-dimensional representation

$$g \mapsto \lambda_g : \mathcal{G} \longrightarrow \mathrm{GL}_1$$

and $\log^{\mathfrak{m}}(2)$ to a two-dimensional one

$$\begin{array}{ccc} \mathcal{G} & \longrightarrow & \mathrm{GL}_2 \\ g & \mapsto & \begin{pmatrix} 1 & \nu_g \\ 0 & \lambda_g \end{pmatrix} \end{array}$$

We obtain a new invariant: $\mathrm{rank}\,(I^{\mathfrak{m}})$ is the dimension of the representation of $\mathcal{G}$ that $I^{\mathfrak{m}}$ generates.

Define the *Galois conjugates* of $I^{\mathfrak{m}}$ to be elements in this representation (in the case of an algebraic number this would be a linear combination of its Galois conjugates).

From numbers to representations

What can we do with all this?

Every motivic period defines a representation of $\mathcal{G}$. For example, $(2i\pi)^{\mathfrak{m}}$ gives rise to a one-dimensional representation

$$g \mapsto \lambda_g : \mathcal{G} \longrightarrow \mathrm{GL}_1$$

and $\log^{\mathfrak{m}}(2)$ to a two-dimensional one

$$\begin{array}{ccc} \mathcal{G} & \longrightarrow & \mathrm{GL}_2 \\ g & \mapsto & \begin{pmatrix} 1 & \nu_g \\ 0 & \lambda_g \end{pmatrix} \end{array}$$

We obtain a new invariant: $\mathrm{rank}\,(I^{\mathfrak{m}})$ is the dimension of the representation of $\mathcal{G}$ that $I^{\mathfrak{m}}$ generates.

Define the *Galois conjugates* of $I^{\mathfrak{m}}$ to be elements in this representation (in the case of an algebraic number this would be a linear combination of its Galois conjugates).

Caveats

If we start with a period p and wish to define its motivic version $p^{\mathfrak{m}}$ we have to *choose* some natural integral representation for it. This works perfectly well in practice.

Given two different integral representations $p_1 = p_2$ for p, it may not at all be obvious to prove that $p_1^{\mathfrak{m}} = p_2^{\mathfrak{m}}$, and involves some geometry. This is exactly the point of the period conjecture.

A further subtlety remains: given a choice of integral representation for p, how do we write down the varieties X and Z? I will not discuss this for lack of space, but a motto is:

'arithmetically interesting periods require singular integrals'.

Caveats

If we start with a period p and wish to define its motivic version $p^{\mathfrak{m}}$ we have to *choose* some natural integral representation for it. This works perfectly well in practice.

Given two different integral representations $p_1 = p_2$ for p, it may not at all be obvious to prove that $p_1^{\mathfrak{m}} = p_2^{\mathfrak{m}}$, and involves some geometry. This is exactly the point of the period conjecture.

A further subtlety remains: given a choice of integral representation for p, how do we write down the varieties X and Z? I will not discuss this for lack of space, but a motto is:

'arithmetically interesting periods require singular integrals'.

Caveats

If we start with a period p and wish to define its motivic version $p^{\mathfrak{m}}$ we have to *choose* some natural integral representation for it. This works perfectly well in practice.

Given two different integral representations $p_1 = p_2$ for p, it may not at all be obvious to prove that $p_1^{\mathfrak{m}} = p_2^{\mathfrak{m}}$, and involves some geometry. This is exactly the point of the period conjecture.

A further subtlety remains: given a choice of integral representation for p, how do we write down the varieties X and Z? I will not discuss this for lack of space, but a motto is:

'arithmetically interesting periods require singular integrals'.

Caveats

If we start with a period p and wish to define its motivic version $p^{\mathfrak{m}}$ we have to *choose* some natural integral representation for it. This works perfectly well in practice.

Given two different integral representations $p_1 = p_2$ for p, it may not at all be obvious to prove that $p_1^{\mathfrak{m}} = p_2^{\mathfrak{m}}$, and involves some geometry. This is exactly the point of the period conjecture.

A further subtlety remains: given a choice of integral representation for p, how do we write down the varieties X and Z? I will not discuss this for lack of space, but a motto is:

> 'arithmetically interesting periods require singular integrals'.

Filtration by unipotency

There is a decomposition

$$1 \longrightarrow U \longrightarrow \mathcal{G} \longrightarrow R \longrightarrow 1$$

where R is (pro)-reductive. Its category of representations is semi-simple. The kernel U is a pro-unipotent affine group scheme.

Say that $p \in \mathcal{P}^{\mathfrak{m}}$ is a *pure* period if U acts trivially on it. Its representation is a direct sum of irreducible representations of R.

Say that $p \in \mathcal{P}^{\mathfrak{m}}$ is a period of a *simple extension* if $[U, U]$ acts trivially on it. Its representation is an extension of two semi-simple representations, i.e., in some basis, the image of $\mathcal{G}$ looks like

$$\left(\begin{array}{c|c} \star & \star \\ \hline 0 & \star \end{array}\right)$$

These are the first steps in a filtration by degree of unipotency, defined in terms of the lower central series of U.

Filtration by unipotency

There is a decomposition

$$1 \longrightarrow U \longrightarrow \mathcal{G} \longrightarrow R \longrightarrow 1$$

where R is (pro)-reductive. Its category of representations is semi-simple. The kernel U is a pro-unipotent affine group scheme.

Say that $p \in \mathcal{P}^{\mathfrak{m}}$ is a *pure* period if U acts trivially on it. Its representation is a direct sum of irreducible representations of R.

Say that $p \in \mathcal{P}^{\mathfrak{m}}$ is a period of a *simple extension* if $[U, U]$ acts trivially on it. Its representation is an extension of two semi-simple representations, i.e., in some basis, the image of $\mathcal{G}$ looks like

$$\left(\begin{array}{c|c} \star & \star \\ \hline 0 & \star \end{array}\right)$$

These are the first steps in a filtration by degree of unipotency, defined in terms of the lower central series of U.

Filtration by unipotency

There is a decomposition

$$1 \longrightarrow U \longrightarrow \mathcal{G} \longrightarrow R \longrightarrow 1$$

where R is (pro)-reductive. Its category of representations is semi-simple. The kernel U is a pro-unipotent affine group scheme.

Say that $p \in \mathcal{P}^{\mathfrak{m}}$ is a *pure* period if U acts trivially on it. Its representation is a direct sum of irreducible representations of R.

Say that $p \in \mathcal{P}^{\mathfrak{m}}$ is a period of a *simple extension* if $[U, U]$ acts trivially on it. Its representation is an extension of two semi-simple representations, i.e., in some basis, the image of $\mathcal{G}$ looks like

$$\left(\begin{array}{c|c} \star & \star \\ \hline 0 & \star \end{array}\right)$$

These are the first steps in a filtration by degree of unipotency, defined in terms of the lower central series of U.

Filtration by unipotency

There is a decomposition

$$1 \longrightarrow U \longrightarrow \mathcal{G} \longrightarrow R \longrightarrow 1$$

where R is (pro)-reductive. Its category of representations is semi-simple. The kernel U is a pro-unipotent affine group scheme.

Say that $p \in \mathcal{P}^{\mathfrak{m}}$ is a *pure* period if U acts trivially on it. Its representation is a direct sum of irreducible representations of R.

Say that $p \in \mathcal{P}^{\mathfrak{m}}$ is a period of a *simple extension* if $[U, U]$ acts trivially on it. Its representation is an extension of two semi-simple representations, i.e., in some basis, the image of $\mathcal{G}$ looks like

$$\left(\begin{array}{c|c} \star & \star \\ \hline 0 & \star \end{array}\right)$$

These are the first steps in a filtration by degree of unipotency, defined in terms of the lower central series of U.

Vista

Unip. degree	Examples
0	Pure periods: π, elliptic integrals, ... (classical)
1	Periods of simple extensions: $\log 2, \zeta(2n+1), \ldots$ (values of L-functions)
≥ 2	Multiple periods: MZV's, Feynman amplitudes, ... *(unknown)* — *Unexplored territory* —

Historically we focus only on the top two rows of 'abelian' periods where U acts via its abelianisation U^{ab}. But all the interesting 'multiple' and non-abelian phenomena are lower down in the picture in degrees ≥ 2. This is new territory.

Vista

Unip. degree	Examples
0	Pure periods: π, elliptic integrals, ... (classical)
1	Periods of simple extensions: $\log 2, \zeta(2n+1), \ldots$ (values of L-functions)
≥ 2	Multiple periods: MZV's, Feynman amplitudes, ... *(unknown)* — *Unexplored territory* —

Historically we focus only on the top two rows of 'abelian' periods where U acts via its abelianisation U^{ab}. But all the interesting 'multiple' and non-abelian phenomena are lower down in the picture in degrees ≥ 2. This is new territory.

More invariants

Using group theory we can assign many more invariants or 'measures of complexity' of motivic periods.

In fact, the representation generated by a motivic period actually carries a mixed Hodge structure, so we can assign *Hodge numbers* to periods. For example, $(2\pi i)^{\mathfrak{m}}$ is of type $(1,1)$.

Note that in general the weight is a filtration on periods, not a grading, so we can only say that a period is of weight 'at most' k.

However, for some sub-classes of periods (for example, those with Hodge numbers only of type (p,p)), the weight is a grading. This is why, in the case of multiple zeta values, we do actually have a weight grading, and it makes sense to speak of the weight of an MZV. The Hodge-theoretic weight is double the MZV-weight.

More invariants

Using group theory we can assign many more invariants or 'measures of complexity' of motivic periods.

In fact, the representation generated by a motivic period actually carries a mixed Hodge structure, so we can assign *Hodge numbers* to periods. For example, $(2\pi i)^{\mathfrak{m}}$ is of type $(1,1)$.

Note that in general the weight is a filtration on periods, not a grading, so we can only say that a period is of weight 'at most' k.

However, for some sub-classes of periods (for example, those with Hodge numbers only of type (p,p)), the weight is a grading. This is why, in the case of multiple zeta values, we do actually have a weight grading, and it makes sense to speak of the weight of an MZV. The Hodge-theoretic weight is double the MZV-weight.

More invariants

Using group theory we can assign many more invariants or 'measures of complexity' of motivic periods.

In fact, the representation generated by a motivic period actually carries a mixed Hodge structure, so we can assign *Hodge numbers* to periods. For example, $(2\pi i)^{\mathfrak{m}}$ is of type $(1,1)$.

Note that in general the weight is a filtration on periods, not a grading, so we can only say that a period is of weight 'at most' k.

However, for some sub-classes of periods (for example, those with Hodge numbers only of type (p,p)), the weight is a grading. This is why, in the case of multiple zeta values, we do actually have a weight grading, and it makes sense to speak of the weight of an MZV. The Hodge-theoretic weight is double the MZV-weight.

More invariants

Using group theory we can assign many more invariants or 'measures of complexity' of motivic periods.

In fact, the representation generated by a motivic period actually carries a mixed Hodge structure, so we can assign *Hodge numbers* to periods. For example, $(2\pi i)^{\mathfrak{m}}$ is of type $(1, 1)$.

Note that in general the weight is a filtration on periods, not a grading, so we can only say that a period is of weight 'at most' k.

However, for some sub-classes of periods (for example, those with Hodge numbers only of type (p, p)), the weight is a grading. This is why, in the case of multiple zeta values, we do actually have a weight grading, and it makes sense to speak of the weight of an MZV. The Hodge-theoretic weight is double the MZV-weight.

Example

The motivic multiple zeta value $\zeta^{\mathfrak{m}}(3,5)$ gives a representation of dimension 3, and unipotency degree 2. The multiple zeta value $\zeta^{\mathfrak{m}}(3,5)$ defines a 3-dimensional representation of $\mathcal{G}$:

$$g^{-1}\begin{pmatrix} 1 \\ \zeta^{\mathfrak{m}}(3) \\ \zeta^{\mathfrak{m}}(3,5) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \sigma_g^{(3)} & \lambda_g^3 & 0 \\ \sigma_g^{(3,5)} & -5\lambda_g^3\sigma_g^{(5)} & \lambda_g^8 \end{pmatrix}\begin{pmatrix} 1 \\ \zeta^{\mathfrak{m}}(3) \\ \zeta^{\mathfrak{m}}(3,5) \end{pmatrix}$$

Therefore $\zeta^{\mathfrak{m}}(3,5)$ has Galois conjugates any linear combination of

$$1, \zeta^{\mathfrak{m}}(3), \zeta^{\mathfrak{m}}(3,5)$$

Using this one can show that $\zeta^{\mathfrak{m}}(3,5)$ cannot be a polynomial in the single motivic zeta values.

By contrast, $\zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(5)$ defines a 4-dimensional representation, and has conjugates $1, \zeta^{\mathfrak{m}}(3), \zeta^{\mathfrak{m}}(5), \zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(5)$.

Example

The motivic multiple zeta value $\zeta^{\mathfrak{m}}(3,5)$ gives a representation of dimension 3, and unipotency degree 2. The multiple zeta value $\zeta^{\mathfrak{m}}(3,5)$ defines a 3-dimensional representation of $\mathcal{G}$:

$$g^{-1}\begin{pmatrix} 1 \\ \zeta^{\mathfrak{m}}(3) \\ \zeta^{\mathfrak{m}}(3,5) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \sigma_g^{(3)} & \lambda_g^3 & 0 \\ \sigma_g^{(3,5)} & -5\lambda_g^3\sigma_g^{(5)} & \lambda_g^8 \end{pmatrix}\begin{pmatrix} 1 \\ \zeta^{\mathfrak{m}}(3) \\ \zeta^{\mathfrak{m}}(3,5) \end{pmatrix}$$

Therefore $\zeta^{\mathfrak{m}}(3,5)$ has Galois conjugates any linear combination of

$$1, \zeta^{\mathfrak{m}}(3), \zeta^{\mathfrak{m}}(3,5)$$

Using this one can show that $\zeta^{\mathfrak{m}}(3,5)$ cannot be a polynomial in the single motivic zeta values.

By contrast, $\zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(5)$ defines a 4-dimensional representation, and has conjugates $1, \zeta^{\mathfrak{m}}(3), \zeta^{\mathfrak{m}}(5), \zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(5)$.

Example

The motivic multiple zeta value $\zeta^{\mathfrak{m}}(3,5)$ gives a representation of dimension 3, and unipotency degree 2. The multiple zeta value $\zeta^{\mathfrak{m}}(3,5)$ defines a 3-dimensional representation of $\mathcal{G}$:

$$g^{-1}\begin{pmatrix} 1 \\ \zeta^{\mathfrak{m}}(3) \\ \zeta^{\mathfrak{m}}(3,5) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \sigma_g^{(3)} & \lambda_g^3 & 0 \\ \sigma_g^{(3,5)} & -5\lambda_g^3\sigma_g^{(5)} & \lambda_g^8 \end{pmatrix}\begin{pmatrix} 1 \\ \zeta^{\mathfrak{m}}(3) \\ \zeta^{\mathfrak{m}}(3,5) \end{pmatrix}$$

Therefore $\zeta^{\mathfrak{m}}(3,5)$ has Galois conjugates any linear combination of

$$1, \zeta^{\mathfrak{m}}(3), \zeta^{\mathfrak{m}}(3,5)$$

Using this one can show that $\zeta^{\mathfrak{m}}(3,5)$ cannot be a polynomial in the single motivic zeta values.

By contrast, $\zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(5)$ defines a 4-dimensional representation, and has conjugates $1, \zeta^{\mathfrak{m}}(3), \zeta^{\mathfrak{m}}(5), \zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(5)$.

Example

The motivic multiple zeta value $\zeta^{\mathfrak{m}}(3,5)$ gives a representation of dimension 3, and unipotency degree 2. The multiple zeta value $\zeta^{\mathfrak{m}}(3,5)$ defines a 3-dimensional representation of $\mathcal{G}$:

$$g^{-1}\begin{pmatrix} 1 \\ \zeta^{\mathfrak{m}}(3) \\ \zeta^{\mathfrak{m}}(3,5) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \sigma_g^{(3)} & \lambda_g^3 & 0 \\ \sigma_g^{(3,5)} & -5\lambda_g^3\sigma_g^{(5)} & \lambda_g^8 \end{pmatrix}\begin{pmatrix} 1 \\ \zeta^{\mathfrak{m}}(3) \\ \zeta^{\mathfrak{m}}(3,5) \end{pmatrix}$$

Therefore $\zeta^{\mathfrak{m}}(3,5)$ has Galois conjugates any linear combination of

$$1, \zeta^{\mathfrak{m}}(3), \zeta^{\mathfrak{m}}(3,5)$$

Using this one can show that $\zeta^{\mathfrak{m}}(3,5)$ cannot be a polynomial in the single motivic zeta values.

By contrast, $\zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(5)$ defines a 4-dimensional representation, and has conjugates $1, \zeta^{\mathfrak{m}}(3), \zeta^{\mathfrak{m}}(5), \zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(5)$.

Recap on f-alphabet

Recall that the Lie algebra of U acts on $\mathcal{Z}^{\mathfrak{m}}$, the ring of motivic MZV's via derivations

$$\sigma_3, \sigma_5, \sigma_7, \ldots,$$

where σ_{2n+1} is 'differentiation with respect to $\zeta^{\mathfrak{m}}(2n+1)$'.

Last time we encoded this structure via an isomorphism

$$\phi : \mathcal{Z}^{\mathfrak{m}} \longrightarrow M[f_2]$$

where M is the graded $\mathbb{Q}$-vector space generated by words in letters $f_3, f_5, \ldots$, equipped with the shuffle product. This isomorphism is compatible with the action of the σ_{2n+1} on both sides.

In this representation we can read off the invariants very easily. The unipotency degree is just the number of letters f_{odd}. The Galois conjugates are obtained by slicing letters off from the left.

Recap on f-alphabet

Recall that the Lie algebra of U acts on $\mathcal{Z}^{\mathfrak{m}}$, the ring of motivic MZV's via derivations

$$\sigma_3, \sigma_5, \sigma_7, \ldots,$$

where σ_{2n+1} is 'differentiation with respect to $\zeta^{\mathfrak{m}}(2n+1)$'.

Last time we encoded this structure via an isomorphism

$$\phi : \mathcal{Z}^{\mathfrak{m}} \longrightarrow M[f_2]$$

where M is the graded $\mathbb{Q}$-vector space generated by words in letters $f_3, f_5, \ldots$, equipped with the shuffle product. This isomorphism is compatible with the action of the σ_{2n+1} on both sides.

In this representation we can read off the invariants very easily. The unipotency degree is just the number of letters f_{odd}. The Galois conjugates are obtained by slicing letters off from the left.

Recap on f-alphabet

Recall that the Lie algebra of U acts on $\mathcal{Z}^{\mathfrak{m}}$, the ring of motivic MZV's via derivations

$$\sigma_3, \sigma_5, \sigma_7, \ldots,$$

where σ_{2n+1} is 'differentiation with respect to $\zeta^{\mathfrak{m}}(2n+1)$'.

Last time we encoded this structure via an isomorphism

$$\phi : \mathcal{Z}^{\mathfrak{m}} \longrightarrow M[f_2]$$

where M is the graded $\mathbb{Q}$-vector space generated by words in letters $f_3, f_5, \ldots$, equipped with the shuffle product. This isomorphism is compatible with the action of the σ_{2n+1} on both sides.

In this representation we can read off the invariants very easily. The unipotency degree is just the number of letters f_{odd}. The Galois conjugates are obtained by slicing letters off from the left.

Decomposition of motivic periods

Example:

$$\begin{array}{rcl} \zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(5) & \mapsto & f_3 \,⧢\, f_5 = f_3 f_5 + f_5 f_3 \\ \zeta^{\mathfrak{m}}(3,5) & \mapsto & -5 f_5 f_3 \end{array}$$

which encodes the fact that $\sigma_5 \zeta^{\mathfrak{m}}(3,5) = -5\zeta^{\mathfrak{m}}(3)$, $\sigma_5 \zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(5) = \zeta^{\mathfrak{m}}(3)$, etc.

A version of this generalises to all motivic periods. One can define an injective 'decomposition' homomorphism

$$\Phi : \mathrm{gr}^{\mathcal{U}} \mathcal{P}^{\mathfrak{m}} \hookrightarrow T^c(H^1(U)) \otimes_{\mathbb{Q}} \mathcal{P}^{\mathfrak{m},ss}$$

where $\mathcal{P}^{\mathfrak{m},ss}$ is the ring of pure (semi-simple) periods, and $\mathcal{U}$ is the filtration by unipotency. The notation T^c is tensor coalgebra (shuffle algebra). This gives a way to attach words in a certain alphabet of symbols to arbitrary motivic periods.

Decomposition of motivic periods

Example:

$$\begin{aligned} \zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(5) &\mapsto f_3 \,ш\, f_5 = f_3 f_5 + f_5 f_3 \\ \zeta^{\mathfrak{m}}(3,5) &\mapsto -5 f_5 f_3 \end{aligned}$$

which encodes the fact that $\sigma_5 \zeta^{\mathfrak{m}}(3,5) = -5\zeta^{\mathfrak{m}}(3)$, $\sigma_5 \zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(5) = \zeta^{\mathfrak{m}}(3)$, etc.

A version of this generalises to all motivic periods. One can define an injective 'decomposition' homomorphism

$$\Phi : \mathrm{gr}^{\mathcal{U}} \mathcal{P}^{\mathfrak{m}} \hookrightarrow T^c(H^1(U)) \otimes_{\mathbb{Q}} \mathcal{P}^{\mathfrak{m},ss}$$

where $\mathcal{P}^{\mathfrak{m},ss}$ is the ring of pure (semi-simple) periods, and $\mathcal{U}$ is the filtration by unipotency. The notation T^c is tensor coalgebra (shuffle algebra). This gives a way to attach words in a certain alphabet of symbols to arbitrary motivic periods.

Towards a classification

These tools give a language with which to talk about periods, are the beginning of a classification of periods by 'types'.

This all generalises to families of motivic periods. These have many extra structures: a Gauss-Manin connection, differential Galois groups, monodromy groups, and so on.

I believe that when we see periods or families of periods arising in a mathematical formula, we should ask how the Galois group $\mathcal{G}$ acts. This group action should have some meaning.

Can we apply these ideas to amplitudes in quantum field theory?

Towards a classification

These tools give a language with which to talk about periods, are the beginning of a classification of periods by 'types'.

This all generalises to families of motivic periods. These have many extra structures: a Gauss-Manin connection, differential Galois groups, monodromy groups, and so on.

I believe that when we see periods or families of periods arising in a mathematical formula, we should ask how the Galois group $\mathcal{G}$ acts. This group action should have some meaning.

Can we apply these ideas to amplitudes in quantum field theory?

Towards a classification

These tools give a language with which to talk about periods, are the beginning of a classification of periods by 'types'.

This all generalises to families of motivic periods. These have many extra structures: a Gauss-Manin connection, differential Galois groups, monodromy groups, and so on.

I believe that when we see periods or families of periods arising in a mathematical formula, we should ask how the Galois group $\mathcal{G}$ acts. This group action should have some meaning.

Can we apply these ideas to amplitudes in quantum field theory?

Motivic amplitudes in physics

Recall that in the second lecture, we attached to every Feynman graph an *amplitude*

$$I_G(q,m) = \int_\sigma \frac{1}{(\Psi_G)^{2h_G}} \left(\frac{\Psi_G}{\Xi_G(q,m)} \right)^{N_G - 2h_G} \Omega_G$$

where $\Psi_G, \Xi_G(q,m)$ were explicitly-defined polynomials with coefficients in $\mathbb{Q}$. It is clearly a family of periods.

Theorem (B. 2015)

When the integral converges, and the momenta and masses are non-degenerate, we can construct a *canonical* motivic amplitude $I^{\mathfrak{m}}_G(q,m)$ (a family of motivic periods) such that

$$\mathrm{per}\, I^{\mathfrak{m}}_G(q,m) = I_G(q,m) \ .$$

The proof requires doing a canonical local resolution of singularities in order to write the integral as a period of variation of cohomology of a family of smooth varieties relative to a normal crossing divisor.

Motivic amplitudes in physics

Recall that in the second lecture, we attached to every Feynman graph an *amplitude*

$$I_G(q,m) = \int_\sigma \frac{1}{(\Psi_G)^{2h_G}} \left(\frac{\Psi_G}{\Xi_G(q,m)} \right)^{N_G - 2h_G} \Omega_G$$

where $\Psi_G, \Xi_G(q,m)$ were explicitly-defined polynomials with coefficients in $\mathbb{Q}$. It is clearly a family of periods.

Theorem (B. 2015)

When the integral converges, and the momenta and masses are non-degenerate, we can construct a *canonical* motivic amplitude $I_G^{\mathfrak{m}}(q,m)$ (a family of motivic periods) such that

$$\mathrm{per}\, I_G^{\mathfrak{m}}(q,m) = I_G(q,m) \ .$$

The proof requires doing a canonical local resolution of singularities in order to write the integral as a period of variation of cohomology of a family of smooth varieties relative to a normal crossing divisor.

Motivic amplitudes in physics

Recall that in the second lecture, we attached to every Feynman graph an *amplitude*

$$I_G(q,m) = \int_\sigma \frac{1}{(\Psi_G)^{2h_G}} \left(\frac{\Psi_G}{\Xi_G(q,m)} \right)^{N_G - 2h_G} \Omega_G$$

where $\Psi_G, \Xi_G(q,m)$ were explicitly-defined polynomials with coefficients in $\mathbb{Q}$. It is clearly a family of periods.

Theorem (B. 2015)

When the integral converges, and the momenta and masses are non-degenerate, we can construct a *canonical* motivic amplitude $I^{\mathfrak{m}}_G(q,m)$ (a family of motivic periods) such that

$$\mathrm{per}\, I^{\mathfrak{m}}_G(q,m) = I_G(q,m)\ .$$

The proof requires doing a canonical local resolution of singularities in order to write the integral as a period of variation of cohomology of a family of smooth varieties relative to a normal crossing divisor.

Cosmic Galois group

As a consequence, to every graph we can assign a representation of the group $\mathcal{G}$, and invariants such as a weight filtration etc.

Definition

Let $\mathcal{FP}^{\mathfrak{m}}_{Q,M}$ be the vector space generated by conjugates of motivic amplitudes of graphs with Q external momenta, and M distinct particle masses. The group $\mathcal{G}$ acts on $\mathcal{FP}^{\mathfrak{m}}_{Q,M}$ via a quotient $\mathcal{C}_{Q,M}$:

$$\mathcal{C}_{Q,M} \times \mathcal{FP}^{\mathfrak{m}}_{Q,M} \longrightarrow \mathcal{FP}^{\mathfrak{m}}_{Q,M} \ .$$

We could call $\mathcal{C}_{Q,M}$ the *cosmic Galois group*.

The phrase 'cosmic Galois group' is due to Cartier, who, based on the folklore conjecture for ϕ^4 theory (lecture 2), speculated that the Grothendieck-Teichmuller group (which is conjectured to be the Galois group of MZV's), would act on amplitudes. Although the folklore conjecture is expected to be false (lecture 2), the idea of a Galois symmetries acting on amplitudes in physics is salvageable.

Cosmic Galois group

As a consequence, to every graph we can assign a representation of the group $\mathcal{G}$, and invariants such as a weight filtration etc.

Definition

Let $\mathcal{FP}^{\mathfrak{m}}_{Q,M}$ be the vector space generated by conjugates of motivic amplitudes of graphs with Q external momenta, and M distinct particle masses. The group $\mathcal{G}$ acts on $\mathcal{FP}^{\mathfrak{m}}_{Q,M}$ via a quotient $\mathcal{C}_{Q,M}$:

$$\mathcal{C}_{Q,M} \times \mathcal{FP}^{\mathfrak{m}}_{Q,M} \longrightarrow \mathcal{FP}^{\mathfrak{m}}_{Q,M} \ .$$

We could call $\mathcal{C}_{Q,M}$ the *cosmic Galois group*.

The phrase 'cosmic Galois group' is due to Cartier, who, based on the folklore conjecture for ϕ^4 theory (lecture 2), speculated that the Grothendieck-Teichmuller group (which is conjectured to be the Galois group of MZV's), would act on amplitudes. Although the folklore conjecture is expected to be false (lecture 2), the idea of a Galois symmetries acting on amplitudes in physics is salvageable.

Cosmic Galois group

As a consequence, to every graph we can assign a representation of the group $\mathcal{G}$, and invariants such as a weight filtration etc.

Definition

Let $\mathcal{FP}^{\mathfrak{m}}_{Q,M}$ be the vector space generated by conjugates of motivic amplitudes of graphs with Q external momenta, and M distinct particle masses. The group $\mathcal{G}$ acts on $\mathcal{FP}^{\mathfrak{m}}_{Q,M}$ via a quotient $\mathcal{C}_{Q,M}$:

$$\mathcal{C}_{Q,M} \times \mathcal{FP}^{\mathfrak{m}}_{Q,M} \longrightarrow \mathcal{FP}^{\mathfrak{m}}_{Q,M} \ .$$

We could call $\mathcal{C}_{Q,M}$ the *cosmic Galois group*.

The phrase 'cosmic Galois group' is due to Cartier, who, based on the folklore conjecture for ϕ^4 theory (lecture 2), speculated that the Grothendieck-Teichmuller group (which is conjectured to be the Galois group of MZV's), would act on amplitudes. Although the folklore conjecture is expected to be false (lecture 2), the idea of a Galois symmetries acting on amplitudes in physics is salvageable.

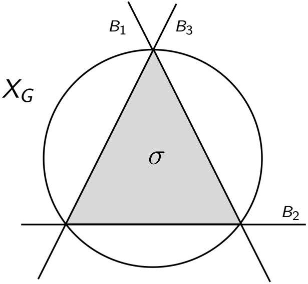
The graph 'motive' (Bloch-Esnault-Kreimer 2007)

Recall that

$$I_G = \int_\sigma \omega_G \qquad \text{where} \qquad \omega_G = \frac{\Omega_G}{\Psi_G^2}$$

How to interpret this as a period? Consider the graph hypersurface, and coordinate hyperplanes in projective space:

$$X_G = V(\Psi_G) \subset \mathbb{P}^{N_G - 1} \quad , \quad B_i = V(\alpha_i) \subset \mathbb{P}^{N_G-1}$$



$$\omega_G \in \Omega^{N_G-1}(\mathbb{P}^{N_G-1} \backslash X_G) \qquad \text{and} \qquad \partial\sigma \subset B = \cup_i B_i \ .$$

The graph motive (II)

The naive 'motive' (or rather, mixed Hodge structure) is

$$H^{N_G-1}(\mathbb{P}^{N_G-1}\backslash X_G, B\backslash(B\cap X_G)$$

However, in reality, the domain of integration σ meets the singular locus X_G so we must do some blow-ups. B-E-K construct an explicit local resolution of singularities $\pi : P \to \mathbb{P}^{N_G-1}$ and define

$$mot(G) = H^{N_G-1}(P\backslash\widetilde{X}_G, \widetilde{B}\backslash(\widetilde{B}\cap\widetilde{X}_G))$$

We can use this to define the motivic period

$$I^{\mathfrak{m}} = [mot(G), \pi^*\omega, \overline{\pi^{-1}(\sigma)}]^{\mathfrak{m}}$$

The general case, where there are masses and momenta is a lot more subtle and involves the family of hypersurfaces defined by the zero locus of the other graph polynomial $\Xi_G(q,m)$.

Stability theorem

Recall that the Feynman amplitude is given by a very specific integral. There is no reason why the Galois conjugates of an amplitude should still be an amplitude.

However, the cosmic Galois group preserves a slightly bigger space of periods which includes the space of amplitudes with numerators.

Theorem

The space of amplitudes of bounded weight is finite-dimensional

$$\dim_{\mathbb{Q}} W_k \mathcal{FP}^{\mathfrak{m}}_{Q,M} < \infty$$

This is surprising since there are infinitely many graphs, but only finitely many periods occur. The upshot is that if one computes $W_k \mathcal{FP}^{\mathfrak{m}}_{Q,M}$ for some k, then this imposes a constraint on the possible Galois conjugates of amplitudes for *infinitely many graphs*, to all loop orders.

Stability theorem

Recall that the Feynman amplitude is given by a very specific integral. There is no reason why the Galois conjugates of an amplitude should still be an amplitude.

However, the cosmic Galois group preserves a slightly bigger space of periods which includes the space of amplitudes with numerators.

Theorem

The space of amplitudes of bounded weight is finite-dimensional

$$\dim_{\mathbb{Q}} W_k \mathcal{FP}^{\mathfrak{m}}_{Q,M} < \infty$$

This is surprising since there are infinitely many graphs, but only finitely many periods occur. The upshot is that if one computes $W_k \mathcal{FP}^{\mathfrak{m}}_{Q,M}$ for some k, then this imposes a constraint on the possible Galois conjugates of amplitudes for *infinitely many graphs*, to all loop orders.

Stability theorem

Recall that the Feynman amplitude is given by a very specific integral. There is no reason why the Galois conjugates of an amplitude should still be an amplitude.

However, the cosmic Galois group preserves a slightly bigger space of periods which includes the space of amplitudes with numerators.

Theorem

The space of amplitudes of bounded weight is finite-dimensional

$$\dim_{\mathbb{Q}} W_k \mathcal{FP}^{\mathfrak{m}}_{Q,M} < \infty$$

This is surprising since there are infinitely many graphs, but only finitely many periods occur. The upshot is that if one computes $W_k \mathcal{FP}^{\mathfrak{m}}_{Q,M}$ for some k, then this imposes a constraint on the possible Galois conjugates of amplitudes for *infinitely many graphs*, to all loop orders.

The proof uses a magical feature of the geometry of graph hypersurfaces which boils down to the partial factorisation identities for graph polynomials. This implies that the graph motives assemble into some kind of operad.

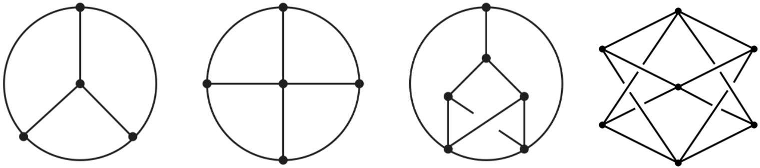
Let me illustrate all this theory with some remarkable computations due to Panzer and Schnetz for ϕ^4 amplitudes (2016).

Examples in massless ϕ^4

Primitive, log-divergent graphs in ϕ^4 theory:

$$I_G = \int_\sigma \frac{\Omega_G}{\Psi_G^2}$$

Examples at $3, 4, 5, 6$ loops:



I_G : $6\zeta(3)$ $20\zeta(5)$ $36\zeta(3)^2$ $N_{3,5}$

where $N_{3,5} = \frac{27}{5}\zeta(5,3) + \frac{45}{4}\zeta(5)\zeta(3) - \frac{261}{20}\zeta(8)$.

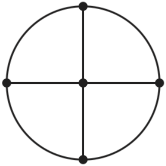
The idea is to replace MZV's with motivic MZV's, and compute the Galois action using what we know about motivic MZV's.

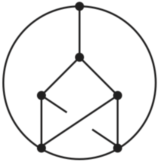
Examples in massless ϕ^4

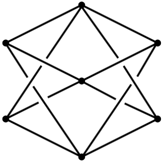
Primitive, log-divergent graphs in ϕ^4 theory:

$$I_G = \int_\sigma \frac{\Omega_G}{\Psi_G^2}$$

Examples at $3, 4, 5, 6$ loops:







I_G : $6\zeta(3)$ $20\zeta(5)$ $36\zeta(3)^2$ $N_{3,5}$

where $N_{3,5} = \frac{27}{5}\zeta(5,3) + \frac{45}{4}\zeta(5)\zeta(3) - \frac{261}{20}\zeta(8)$.

The idea is to replace MZV's with motivic MZV's, and compute the Galois action using what we know about motivic MZV's.

Galois action conjecture

Conjecture (Panzer and Schnetz 2016)

The space of motivic amplitudes of primitive, log-divergent graphs in ϕ^4 is closed under the action of the cosmic Galois group $\mathcal{C}_{0,0}$.

Panzer and Schnetz played the following game

- Replace the MZV's in an amplitude I_G with their motivic versions. This assumes the period conjecture.
- Calculate their Galois conjugates.
- Try to express every Galois conjugate as a linear combination of $I_{G'}$ for graphs G' in ϕ^4 which are primitive, log-divergent.

They verified that the conjecture is true in every single known example. There are about 250 such examples going up to 10 loops. The conjecture is equivalent to some extraordinary identities between the coefficients of MZV's occurring as amplitudes.

Galois action conjecture

Conjecture (Panzer and Schnetz 2016)

The space of motivic amplitudes of primitive, log-divergent graphs in ϕ^4 is closed under the action of the cosmic Galois group $\mathcal{C}_{0,0}$.

Panzer and Schnetz played the following game

- Replace the MZV's in an amplitude I_G with their motivic versions. This assumes the period conjecture.
- Calculate their Galois conjugates.
- Try to express every Galois conjugate as a linear combination of $I_{G'}$ for graphs G' in ϕ^4 which are primitive, log-divergent.

They verified that the conjecture is true in every single known example. There are about 250 such examples going up to 10 loops. The conjecture is equivalent to some extraordinary identities between the coefficients of MZV's occurring as amplitudes.

Galois action conjecture

Conjecture (Panzer and Schnetz 2016)

The space of motivic amplitudes of primitive, log-divergent graphs in ϕ^4 is closed under the action of the cosmic Galois group $\mathcal{C}_{0,0}$.

Panzer and Schnetz played the following game

- Replace the MZV's in an amplitude I_G with their motivic versions. This assumes the period conjecture.
- Calculate their Galois conjugates.
- Try to express every Galois conjugate as a linear combination of $I_{G'}$ for graphs G' in ϕ^4 which are primitive, log-divergent.

They verified that the conjecture is true in every single known example. There are about 250 such examples going up to 10 loops. The conjecture is equivalent to some extraordinary identities between the coefficients of MZV's occurring as amplitudes.

Galois action conjecture

Conjecture (Panzer and Schnetz 2016)

The space of motivic amplitudes of primitive, log-divergent graphs in ϕ^4 is closed under the action of the cosmic Galois group $\mathcal{C}_{0,0}$.

Panzer and Schnetz played the following game

- Replace the MZV's in an amplitude I_G with their motivic versions. This assumes the period conjecture.
- Calculate their Galois conjugates.
- Try to express every Galois conjugate as a linear combination of $I_{G'}$ for graphs G' in ϕ^4 which are primitive, log-divergent.

They verified that the conjecture is true in every single known example. There are about 250 such examples going up to 10 loops. The conjecture is equivalent to some extraordinary identities between the coefficients of MZV's occurring as amplitudes.

Galois action conjecture

Conjecture (Panzer and Schnetz 2016)

The space of motivic amplitudes of primitive, log-divergent graphs in ϕ^4 is closed under the action of the cosmic Galois group $\mathcal{C}_{0,0}$.

Panzer and Schnetz played the following game

- Replace the MZV's in an amplitude I_G with their motivic versions. This assumes the period conjecture.
- Calculate their Galois conjugates.
- Try to express every Galois conjugate as a linear combination of $I_{G'}$ for graphs G' in ϕ^4 which are primitive, log-divergent.

They verified that the conjecture is true in every single known example. There are about 250 such examples going up to 10 loops. The conjecture is equivalent to some extraordinary identities between the coefficients of MZV's occurring as amplitudes.

Galois action conjecture

Conjecture (Panzer and Schnetz 2016)

The space of motivic amplitudes of primitive, log-divergent graphs in ϕ^4 is closed under the action of the cosmic Galois group $\mathcal{C}_{0,0}$.

Panzer and Schnetz played the following game

- Replace the MZV's in an amplitude I_G with their motivic versions. This assumes the period conjecture.
- Calculate their Galois conjugates.
- Try to express every Galois conjugate as a linear combination of $I_{G'}$ for graphs G' in ϕ^4 which are primitive, log-divergent.

They verified that the conjecture is true in every single known example. There are about 250 such examples going up to 10 loops. The conjecture is equivalent to some extraordinary identities between the coefficients of MZV's occurring as amplitudes.

First examples

There is no graph which would correspond to an MZV of weight 2 (it would have $2\frac{1}{2}$ loops!). So $\zeta^{\mathfrak{m}}(2)$ trivially doesn't occur as an amplitude in ϕ^4.

If the conjecture is true, this means that we should never see

$$\zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(2) \quad , \quad \zeta^{\mathfrak{m}}(5)\zeta^{\mathfrak{m}}(2) \quad , \quad \ldots, \quad \zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(5)\zeta^{\mathfrak{m}}(2) \quad , \ldots$$

because they would have a Galois conjugate $\zeta^{\mathfrak{m}}(2)$, which is not an amplitude. This is a constraint on amplitudes *to all orders*.

Similarly, there is no graph which corresponds to $\zeta^{\mathfrak{m}}(2)^2$, because there simply aren't many graphs at low loop orders. So this rules out another infinite class of periods:

$$\zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(2)^2 \quad , \quad \zeta^{\mathfrak{m}}(5)\zeta^{\mathfrak{m}}(2)^2 \quad \ldots$$

First examples

There is no graph which would correspond to an MZV of weight 2 (it would have $2\frac{1}{2}$ loops!). So $\zeta^{\mathfrak{m}}(2)$ trivially doesn't occur as an amplitude in ϕ^4.

If the conjecture is true, this means that we should never see

$$\zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(2) \quad , \quad \zeta^{\mathfrak{m}}(5)\zeta^{\mathfrak{m}}(2) \quad , \quad \ldots, \quad \zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(5)\zeta^{\mathfrak{m}}(2) \quad , \ldots$$

because they would have a Galois conjugate $\zeta^{\mathfrak{m}}(2)$, which is not an amplitude. This is a constraint on amplitudes *to all orders*.

Similarly, there is no graph which corresponds to $\zeta^{\mathfrak{m}}(2)^2$, because there simply aren't many graphs at low loop orders. So this rules out another infinite class of periods:

$$\zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(2)^2 \quad , \quad \zeta^{\mathfrak{m}}(5)\zeta^{\mathfrak{m}}(2)^2 \quad \ldots$$

First examples

There is no graph which would correspond to an MZV of weight 2 (it would have $2\frac{1}{2}$ loops!). So $\zeta^{\mathfrak{m}}(2)$ trivially doesn't occur as an amplitude in ϕ^4.

If the conjecture is true, this means that we should never see

$$\zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(2) \quad , \quad \zeta^{\mathfrak{m}}(5)\zeta^{\mathfrak{m}}(2) \quad , \quad \ldots , \quad \zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(5)\zeta^{\mathfrak{m}}(2) \quad , \ldots$$

because they would have a Galois conjugate $\zeta^{\mathfrak{m}}(2)$, which is not an amplitude. This is a constraint on amplitudes *to all orders*.

Similarly, there is no graph which corresponds to $\zeta^{\mathfrak{m}}(2)^2$, because there simply aren't many graphs at low loop orders. So this rules out another infinite class of periods:

$$\zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(2)^2 \quad , \quad \zeta^{\mathfrak{m}}(5)\zeta^{\mathfrak{m}}(2)^2 \quad \ldots$$

The conjecture in action I

Look at all graphs with $1, 3, 4, 5, 6$ loops. We know their amplitudes are MZV's, and we also know how to predict their weight in these cases (which is tricky).

Loops	Weights	Possible MZV's
1	1	1
3	3	f_3
4	5	$f_5 \quad f_3 f_2$
5	7	$f_7 \quad f_5 f_2 \quad f_3 f_2^2$
	6	$f_3^2 \quad f_2^3$
6	9	$f_9 \quad f_7 f_2 \quad f_5 f_2^2 \quad f_3 f_2^3 \quad f_3^3$
	8	$f_3 f_5 \quad f_5 f_3 \quad f_3^2 f_2 \quad f_2^4$

Trivially: no amplitudes of weights 2 and 4 $\Rightarrow$ no f_2, f_2^2.
Easy calculation using graph identities $\Rightarrow$ no f_2^3.
The remaining numbers actually appear as amplitudes.

The conjecture in action I

Look at all graphs with $1, 3, 4, 5, 6$ loops. We know their amplitudes are MZV's, and we also know how to predict their weight in these cases (which is tricky).

Loops	Weights	Possible MZV's
1	1	1
3	3	f_3
4	5	f_5
5	7	f_7 $\quad$ $f_3 f_2^2$
	6	f_3^2 $\quad$ f_2^3
6	9	f_9 $\quad$ $f_5 f_2^2$ $\quad$ $f_3 f_2^3$ $\quad$ f_3^3
	8	$f_3 f_5$ $\quad$ $f_5 f_3$ $\quad$ f_2^4

Trivially: no amplitudes of weights 2 and 4 $\Rightarrow$ no f_2, f_2^2.
Easy calculation using graph identities $\Rightarrow$ no f_2^3.
The remaining numbers actually appear as amplitudes.

The conjecture in action I

Look at all graphs with $1, 3, 4, 5, 6$ loops. We know their amplitudes are MZV's, and we also know how to predict their weight in these cases (which is tricky).

Loops	Weights	Possible MZV's					
1	1				1		
3	3				f_3		
4	5			f_5			
5	7		f_7				
	6				f_3^2	f_2^3	
6	9	f_9				$f_3 f_2^3$	f_3^3
	8		$f_3 f_5$	$f_5 f_3$		f_2^4	

Trivially: no amplitudes of weights 2 and 4 $\Rightarrow$ no f_2, f_2^2.
Easy calculation using graph identities $\Rightarrow$ no f_2^3.
The remaining numbers actually appear as amplitudes.

The conjecture in action I

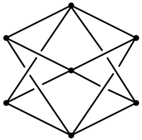
Look at all graphs with $1, 3, 4, 5, 6$ loops. We know their amplitudes are MZV's, and we also know how to predict their weight in these cases (which is tricky).

Loops	Weights	Possible MZV's
1	1	1
3	3	f_3
4	5	f_5
5	7	f_7
	6	f_3^2
6	9	f_9 $\quad f_3^3$
	8	$f_3 f_5 \quad f_5 f_3 \quad f_2^4$

Trivially: no amplitudes of weights 2 and 4 $\Rightarrow$ no f_2, f_2^2.
Easy calculation using graph identities $\Rightarrow$ no f_2^3.
The remaining numbers actually appear as amplitudes.

A more interesting example

Recall that $I_G = 32 \times (-\frac{216}{5}\zeta(5,3) - 81\zeta(5)\zeta(3) + \frac{552}{5}\zeta(8))$, where

$G =$ 

There are 2 primitive 6-loop amplitudes at weight 8 but 3 possible periods, so they can't all occur. In fact, the number $\zeta(5,3)$ always occurs in the combination $-\frac{216}{5}\zeta(5,3) + \frac{552}{5}\zeta(8)$.

Now take an amplitude at higher loop order, e.g. at 7 loops:

$$I_{P_{7,8}} = \frac{22383}{20}\zeta(11) + \ldots + \frac{3024}{5}\zeta(3)\zeta(5,3) - \frac{7308}{5}\zeta(3)\zeta(8)$$

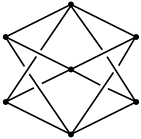
The Galois conjecture correctly predicts the ratio

$$\frac{3024/5}{7308/5} = \frac{216/5}{522/5} \ ,$$

plus 3 other constraints. The number of constraints *increases* with the number of loops!

A more interesting example

Recall that $I_G = 32 \times (-\frac{216}{5}\zeta(5,3) - 81\zeta(5)\zeta(3) + \frac{552}{5}\zeta(8))$, where

$G =$ 

There are 2 primitive 6-loop amplitudes at weight 8 but 3 possible periods, so they can't all occur. In fact, the number $\zeta(5,3)$ always occurs in the combination $-\frac{216}{5}\zeta(5,3) + \frac{552}{5}\zeta(8)$.

Now take an amplitude at higher loop order, e.g. at 7 loops:

$$I_{P_{7,8}} = \frac{22383}{20}\zeta(11) + \ldots + \frac{3024}{5}\zeta(3)\zeta(5,3) - \frac{7308}{5}\zeta(3)\zeta(8)$$

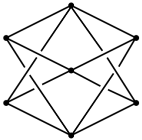
The Galois conjecture correctly predicts the ratio

$$\frac{3024/5}{7308/5} = \frac{216/5}{522/5} ,$$

plus 3 other constraints. The number of constraints *increases* with the number of loops!

A more interesting example

Recall that $I_G = 32 \times (-\frac{216}{5}\zeta(5,3) - 81\zeta(5)\zeta(3) + \frac{552}{5}\zeta(8))$, where

$G =$ 

There are 2 primitive 6-loop amplitudes at weight 8 but 3 possible periods, so they can't all occur. In fact, the number $\zeta(5,3)$ always occurs in the combination $-\frac{216}{5}\zeta(5,3) + \frac{552}{5}\zeta(8)$.

Now take an amplitude at higher loop order, e.g. at 7 loops:

$$I_{P_{7,8}} = \frac{22383}{20}\zeta(11) + \ldots + \frac{3024}{5}\zeta(3)\zeta(5,3) - \frac{7308}{5}\zeta(3)\zeta(8)$$

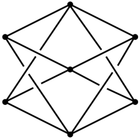
The Galois conjecture correctly predicts the ratio

$$\frac{3024/5}{7308/5} = \frac{216/5}{522/5},$$

plus 3 other constraints. The number of constraints *increases* with the number of loops!

A more interesting example

Recall that $I_G = 32 \times (-\frac{216}{5}\zeta(5,3) - 81\zeta(5)\zeta(3) + \frac{552}{5}\zeta(8))$, where

$G =$ 

There are 2 primitive 6-loop amplitudes at weight 8 but 3 possible periods, so they can't all occur. In fact, the number $\zeta(5,3)$ always occurs in the combination $-\frac{216}{5}\zeta(5,3) + \frac{552}{5}\zeta(8)$.

Now take an amplitude at higher loop order, e.g. at 7 loops:

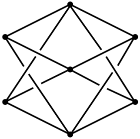
$$I_{P_{7,8}} = \frac{22383}{20}\zeta(11) + \ldots + \frac{3024}{5}\zeta(3)\zeta(5,3) - \frac{7308}{5}\zeta(3)\zeta(8)$$

The Galois conjecture correctly predicts the ratio

$$\frac{3024/5}{7308/5} = \frac{216/5}{522/5} ,$$

plus 3 other constraints. The number of constraints *increases* with the number of loops!

Recall that $I_G = 32 \times (-\frac{216}{5}\zeta(5,3) - 81\zeta(5)\zeta(3) + \frac{552}{5}\zeta(8))$, where

$G =$ 

There are 2 primitive 6-loop amplitudes at weight 8 but 3 possible periods, so they can't all occur. In fact, the number $\zeta(5,3)$ always occurs in the combination $-\frac{216}{5}\zeta(5,3) + \frac{552}{5}\zeta(8)$.

Now take an amplitude at higher loop order, e.g. at 7 loops:

$$I_{P_{7,8}} = \frac{22383}{20}\zeta(11) + \ldots + \frac{3024}{5}\zeta(3)\zeta(5,3) - \frac{7308}{5}\zeta(3)\zeta(8)$$

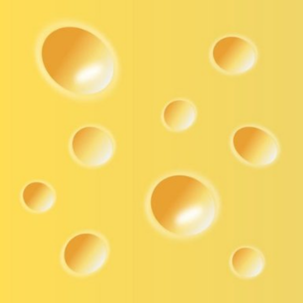
The Galois conjecture correctly predicts the ratio

$$\frac{3024/5}{7308/5} = \frac{216/5}{522/5} \ ,$$

plus 3 other constraints. The number of constraints *increases* with the number of loops!

Swiss cheese

The motivic amplitudes in ϕ^4 theory span a vector space P_{ϕ^4}. It is conjecturally stable under the group $\mathcal{C}_{0,0}$. But P_{ϕ^4} is 'full of holes' (there are few small graphs).



Each hole engenders infinitely many more holes.

Some examples of a different nature

This does not seem to be an isolated phenomenon:

1. Stieberger and Schlotterer expressed *tree-level superstring amplitudes* in terms of motivic multiple zeta values, and showed that this spectacularly simplifies the answer. Interpretation: superstring amplitudes are compatible with the Galois action.
2. Write the known amplitudes for the *anomalous magnetic moment of the electron* in terms of motivic periods. One should be able to define the motivic version of $\frac{g-2}{2}$. Experimentally it shows compatibility with action of the Galois group of periods. This is equivalent to some non-trivial constraints.

Question: why do these diverse theories exhibit compatibility with the action of the Galois group? Could there be a physical meaning to the action of the Cosmic Galois group?

Some examples of a different nature

This does not seem to be an isolated phenomenon:

1. Stieberger and Schlotterer expressed *tree-level superstring amplitudes* in terms of motivic multiple zeta values, and showed that this spectacularly simplifies the answer. Interpretation: superstring amplitudes are compatible with the Galois action.
2. Write the known amplitudes for the *anomalous magnetic moment of the electron* in terms of motivic periods. One should be able to define the motivic version of $\frac{g-2}{2}$. Experimentally it shows compatibility with action of the Galois group of periods. This is equivalent to some non-trivial constraints.

Question: why do these diverse theories exhibit compatibility with the action of the Galois group? Could there be a physical meaning to the action of the Cosmic Galois group?

Some examples of a different nature

This does not seem to be an isolated phenomenon:

1. Stieberger and Schlotterer expressed *tree-level superstring amplitudes* in terms of motivic multiple zeta values, and showed that this spectacularly simplifies the answer. Interpretation: superstring amplitudes are compatible with the Galois action.
2. Write the known amplitudes for the *anomalous magnetic moment of the electron* in terms of motivic periods. One should be able to define the motivic version of $\frac{g-2}{2}$. Experimentally it shows compatibility with action of the Galois group of periods. This is equivalent to some non-trivial constraints.

Question: why do these diverse theories exhibit compatibility with the action of the Galois group? Could there be a physical meaning to the action of the Cosmic Galois group?

Some examples of a different nature

This does not seem to be an isolated phenomenon:

1. Stieberger and Schlotterer expressed *tree-level superstring amplitudes* in terms of motivic multiple zeta values, and showed that this spectacularly simplifies the answer. Interpretation: superstring amplitudes are compatible with the Galois action.
2. Write the known amplitudes for the *anomalous magnetic moment of the electron* in terms of motivic periods. One should be able to define the motivic version of $\frac{g-2}{2}$. Experimentally it shows compatibility with action of the Galois group of periods. This is equivalent to some non-trivial constraints.

Question: why do these diverse theories exhibit compatibility with the action of the Galois group? Could there be a physical meaning to the action of the Cosmic Galois group?

Conclusion

- The theory of motivic periods provides a simple way to apply methods from group theory to the study of periods. It gives an elementary perspective on the theory of motives.
- When applied to high-energy physics, it gives an organising principle for much of the known structure of amplitudes. Emergence of a mysterious 'cosmic' group of symmetries acting on amplitudes in quantum field theory.
- Next stage: go beyond the MZV regime. Joint project with Richard Hain: study of multiple modular periods built out of iterated integrals of modular forms on $\mathcal{M}_{1,1}$. Connects with:
  1. the arithmetic of modular forms
  2. special values of L-functions (Beilinson's conjectures)
  3. structure of mapping class groups
  4. outer space and the outer automorphisms of free groups
  5. Super-string amplitudes in genus 1, . . .

Conclusion

- The theory of motivic periods provides a simple way to apply methods from group theory to the study of periods. It gives an elementary perspective on the theory of motives.
- When applied to high-energy physics, it gives an organising principle for much of the known structure of amplitudes. Emergence of a mysterious 'cosmic' group of symmetries acting on amplitudes in quantum field theory.
- Next stage: go beyond the MZV regime. Joint project with Richard Hain: study of multiple modular periods built out of iterated integrals of modular forms on $\mathcal{M}_{1,1}$. Connects with:
  1. the arithmetic of modular forms
  2. special values of L-functions (Beilinson's conjectures)
  3. structure of mapping class groups
  4. outer space and the outer automorphisms of free groups
  5. Super-string amplitudes in genus 1, ...

Conclusion

- The theory of motivic periods provides a simple way to apply methods from group theory to the study of periods. It gives an elementary perspective on the theory of motives.
- When applied to high-energy physics, it gives an organising principle for much of the known structure of amplitudes. Emergence of a mysterious 'cosmic' group of symmetries acting on amplitudes in quantum field theory.
- Next stage: go beyond the MZV regime. Joint project with Richard Hain: study of multiple modular periods built out of iterated integrals of modular forms on $\mathcal{M}_{1,1}$. Connects with:
  1. the arithmetic of modular forms
  2. special values of L-functions (Beilinson's conjectures)
  3. structure of mapping class groups
  4. outer space and the outer automorphisms of free groups
  5. Super-string amplitudes in genus 1, ...

Conclusion

- The theory of motivic periods provides a simple way to apply methods from group theory to the study of periods. It gives an elementary perspective on the theory of motives.
- When applied to high-energy physics, it gives an organising principle for much of the known structure of amplitudes. Emergence of a mysterious 'cosmic' group of symmetries acting on amplitudes in quantum field theory.
- Next stage: go beyond the MZV regime. Joint project with Richard Hain: study of multiple modular periods built out of iterated integrals of modular forms on $\mathcal{M}_{1,1}$. Connects with:
  1. the arithmetic of modular forms
  2. special values of L-functions (Beilinson's conjectures)
  3. structure of mapping class groups
  4. outer space and the outer automorphisms of free groups
  5. Super-string amplitudes in genus 1, ...