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MZV's and Differential Galois theory

(towards a Galois theory for some transcendental numbers)

IHES, 21st October 2011

I/. <u>Goal</u>: A Galois theory of certain transcendental numbers, or <u>periods</u>, such as $\zeta(n)$, $n \geq 2$

<u>Prototype</u>: Multiple Zeta Values: $n_i \geq 1$, $n_r \geq 2$

$$\zeta(n_1, \ldots, n_r) = \sum_{1 \leq k_1 < \ldots < k_r} \frac{1}{k_1^{n_1} \ldots k_r^{n_r}} \in \mathbb{R}$$

<u>Weight</u> $:= n_1 + \ldots + n_r$, let $\mathcal{Z} = \langle \text{MZVs} \rangle_{\mathbb{Q}}$ vector space.
Easy to see $\mathcal{Z}$ is an algebra: every product of MZV's is a $\mathbb{Z}$-linear combination of MZVs, e.g.

$$\zeta(m)\zeta(n) = \zeta(m,n) + \zeta(n,m) + \zeta(m+n) \qquad \text{(Euler)}$$

There are <u>very many</u> other relations between MZVs.

<u>Desiderata</u>:

$$\begin{array}{c} \mathcal{Z}[\frac{1}{2\pi i}] \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \\ G^{tr} \Big| \\ \overline{\mathbb{Q}} \end{array}$$

"Galois extension" of transcendental numbers, with Galois group G^{tr}.

G^{tr} should be a proalgebraic group over $\mathbb{Q}$, should be isomorphic to G^{MT}, the motivic Galois group of $MT(\mathbb{Z})$, category of mixed Tate motives over $\mathbb{Z}$, whose structure is known:

$$G^{MT} \cong G_u \rtimes \mathbb{G}_m$$

G_u prounipotent, with graded Lie algebra Lie G_u <u>free</u> and generated by $\sigma_3, \sigma_5, \sigma_7, \ldots$ with σ_{2n+1} in degree $-2n-1$.

<u>Folklore conjecture</u>: $\pi, \zeta(3), \zeta(5), \ldots$ are algebraically independent over $\mathbb{Q}$.

This is <u>hopeless</u>: the goal is out of reach.

<u>Idea</u>: Replace actual numbers with symbols

$$\zeta^{\mathfrak{m}}(n_1,\dots,n_r) \qquad \text{"motivic MZVs"}$$

elements of a certain weight-graded algebra $\mathcal{H}$.

$$\text{Action of } G_u \longleftrightarrow \text{Coaction by } \mathcal{O}(G_u)$$

$\mathcal{H}$ comes with
- <u>coaction</u> $\quad \mathcal{H} \xrightarrow{\Delta} \mathcal{O}(G_u) \otimes_{\mathbb{Q}} \mathcal{H}$
- <u>period</u> $\quad \mathcal{H} \xrightarrow{\text{per}} \mathcal{Z} \subset \mathbb{R}$, $\quad \zeta^{\mathfrak{m}}(n_1,\dots,n_r) \longmapsto \zeta(n_1,\dots,n_r)$

<u>Main point</u>: using $(\mathcal{H},\Delta,\text{per})$ we can do explicit calculations as in ordinary Galois theory, and can deduce new results about $\mathcal{Z}$.

<u>How to compute Δ</u>? I know three ways:

(1) Via the motivic fundamental group of $\mathbb{P}^1 \setminus \{0,1,\infty\}$
(2) Via mixed Hodge theory
(3) Via differential Galois theory

Today: approach (3). Involves:

(i) Deforming the numbers $\zeta(n_1,\dots,n_r)$ to functions $I(t_0;t_1,\dots,t_n;t_{n+1})$ (iterated integrals)

(ii) Computing the monodromy of I (encoded by Hopf algebra)

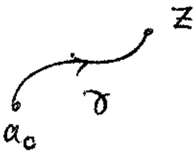
(iii) Specializing to obtain Δ.

II/. <u>Iterated Integrals</u> (K.T. Chen). M smooth manifold, $\omega_1,\dots,\omega_n$ 1-forms and $\gamma : (0,1) \longrightarrow M$ piecewise smooth path. Define $f_i(t)$ by $\gamma^*(\omega_i) = f_i(t)\,dt$ and define iterated integral by

$$\int_\gamma \omega_1 \dots \omega_n \;:=\; \int_{0<t_1\le t_2\le \dots \le t_n<1} f_1(t_1)\dots f_n(t_n)\,dt_1\dots dt_n$$

<u>Example</u>: $M = \mathbb{C} \setminus \Sigma$, Σ finite. Let $a_1,\dots,a_n \in \Sigma$, $a_0, z \in M$

$$I_\gamma(a_0; a_1,\dots,a_n; z) := \int_\gamma \frac{dt}{t-a_1} \dots \frac{dt}{t-a_n}$$



only depends on homotopy class of γ relative to a_0, z. So $I_\cdot(a_0; a_1,\dots,a_n; -)$ defines multivalued function of z. The space of all such I defines a Picard-Vessiot extension of the ring of regular functions on M, and is smallest such extension closed under primitives.

<u>Kontsevich</u>: $\Sigma = \{0,1\}$

$$\zeta(n_1,\dots,n_r) = (-1)^r I_{\gamma_1}(0; 1 0^{n_1-1} \dots 1 0^{n_r-1}; 1)$$

<u>Notation</u> $1 0^{n_1-1} 1 0^{n_2-1} \dots$ denotes $1, \underbrace{0,0,\dots,0}_{n_1-1}, 1, \underbrace{0,\dots,0}_{n_2-1}$, etc.

<u>Properties</u>

- <u>Shuffle</u> <u>product</u>:

$$\int_\gamma \omega_1 \dots \omega_r \int_\gamma \omega_{r+1} \dots \omega_{r+s} = \sum_{\sigma \in Ш(r,s)} \int_\gamma \omega_{\sigma(1)} \dots \omega_{\sigma(r+s)}$$

(r,s) shuffles

• Monodromy $\quad \alpha, \beta : (0,1) \longrightarrow M \quad$ composable

$$\int_{\beta\circ\alpha} \omega_1 \ldots \omega_n = \sum_{r=0}^{n} \int_{\alpha} \omega_1 \ldots \omega_r \int_{\beta} \omega_{r+1} \ldots \omega_n$$

where $\quad$ empty iterated integral $:= 1$

Monodromy encoded by deconcatenation coproduct on tensor algebra generated by 1-forms on M:

$$\omega_1 \otimes \ldots \otimes \omega_n \xmapsto{\Delta_{dec}} \sum_{r=0}^{n} (\omega_1 \otimes \ldots \otimes \omega_r) \otimes (\omega_{r+1} \otimes \ldots \otimes \omega_n)$$

III/.

Universal deformation. The integrals $I_\gamma(a_0; a_1, \ldots, a_n; -)$ were viewed as functions on $M = \mathbb{C} \setminus \Sigma$, punctured Riemann sphere. Now want to view as functions on universal curve of genus 0 with marked points (i.e. allow $a_1, \ldots, a_n$ to move)

$\mathcal{M}_{0,n+3}$ = moduli space of curves of genus 0 with $n+3$ marked points

$$\mathcal{M}_{0,n+3}(\mathbb{C}) = \{ (a_1, \ldots, a_n) \in (\mathbb{C} \setminus \{0,1\})^n \text{ st } a_i \neq a_j \ \ i<j \}$$

Forgetting last marked point:

$$\pi : \mathcal{M}_{0,n+4}(\mathbb{C}) \longrightarrow \mathcal{M}_{0,n+3}(\mathbb{C})$$

Fibers $\cong \mathbb{C} \setminus \{0, 1, a_1, \ldots, a_n\}$

The iterated integral on $\mathbb{C} \setminus \Sigma$ was encoded by

$$\frac{dt}{t-a_1} \otimes \dots \otimes \frac{dt}{t-a_n} \in \Omega_1^{\otimes n}\left(\mathcal{M}_{0,n+4}/\mathcal{M}_{0,n+3}\right)$$

$$\Big\downarrow \text{ Iterated integration in the fiber}$$

Multivalued function on $\mathcal{M}_{0,n+3}(\mathbb{C})$. It has global unipotent monodromy & moderate growth, therefore an iterated integral.

This is a very special case of Gauss-Manin connection for iterated integrals (or, reduced bar construction). (in progress)

<u>Example</u>:

$$\int_0^1 \frac{dt}{t-a_1}\frac{dt}{t-a_2} \underset{\substack{\text{Equality}\\ \text{of multivalued}\\ \text{functions}}}{\overset{\text{"bar Gauss-Manin"}}{\rightsquigarrow}} \int_\gamma \frac{ds_1}{s_1-1}\frac{ds_2}{s_2-1} + \frac{ds_2-ds_1}{s_2-s_1}\cdot\frac{ds_1}{s_1-1} - \frac{ds_2-ds_1}{s_2-s_1}\frac{ds_2}{s_2-1}$$

(γ path from some base point to $(a_1, a_2) \in \mathcal{M}_{0,5}(\mathbb{C})$)

Now apply deconcatenation coproduct to the right-hand side to compute monodromy of iterated integral
$I(0; a_1, \dots, a_n; 1)$, viewed as function of $(a_1, \dots, a_n) \in \mathcal{M}_{0,n+3}(\mathbb{C})$.

$\rightsquigarrow$ Defines a coproduct on $I(a_0; a_1, \dots, a_n; a_{n+1})$

IV/

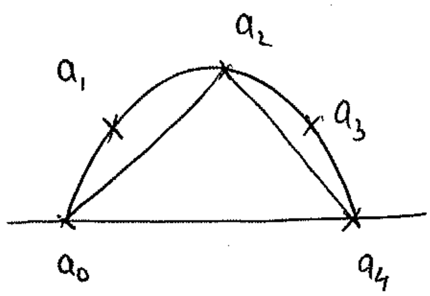
The coproduct (Goncharov's version)

$$\Delta I(a_0; a_1, \dots, a_n; a_{n+1}) = \sum_{0=i_0<i_1<\dots<i_k<i_{k+1}=n+1} \left(\prod_p I(\underbrace{a_{i_p}; a_{i_p+1}, \dots ; a_{i_{p+1}}}_{\text{consecutive}}) \right)$$

$$\otimes \; I(a_0; a_{i_1}, \dots, a_{i_k}; a_{n+1}) \qquad (*)$$

⇝ View the sum as a sum over all dissections of a polygon

Ex :



Typical term :

$$I(a_0; a_1; a_2)\, I(a_2; a_3; a_4)$$
$$\otimes \; I(a_0; a_2; a_4)$$

General : $\left(\prod I(\text{segments})\right) \otimes I(\text{vertices})$

Proof of coproduct : differentiate both sides.

Motivic MZVs will be defined by formal symbols $I^{m}(a_0; a_1, \dots, a_n; a_{n+1})$ with $a_i \in \{0,1\}$, modulo some relations.

Rmk : $I(a_0; a_1, \dots, a_n; a_{n+1})$ diverges for some choices of $a_i \in \{0,1\}$ but there is well-known procedure to regularize it.

V/

Definition of motivic MZVs

- Let $\mathcal{O} := \bigoplus_{a_i \in \{0,1\}} I^{m}(a_0; a_1, .., a_n; a_{n+1})$ + shuffle product

weight-graded algebra

- Let $\zeta^{m}(n_1, .., n_r) := (-1)^r I^{m}(0; 10^{n_1-1} \ldots 10^{n_r-1}; 1)$

- $$\begin{aligned} \mathrm{per}: \mathcal{O} &\longrightarrow \mathbb{R} \\ I^{m}(a_0; ..; a_{n+1}) &\longrightarrow I(a_0; \ldots; a_{n+1}) \quad \text{(regularized value)} \\ \zeta^{m}(n_1, .., n_r) &\longrightarrow \zeta(n_1, .., n_r) \end{aligned}$$

- Let $\bar{\mathcal{O}} = \mathcal{O} / \zeta^{m}(2)\mathcal{O}$

$$\Delta : \mathcal{O} \longrightarrow \bar{\mathcal{O}} \otimes_{\mathbb{Q}} \mathcal{O} \qquad \text{given by } (*)$$

- Define $R \subseteq \mathcal{O}$ ideal of "motivic" relations. It is the largest graded ideal $R = \oplus R_n$ s.t

$$R_n \subseteq \ker(\mathrm{per})$$

and $$\Delta R \subseteq \bar{\mathcal{O}} \otimes R + \bar{R} \otimes \mathcal{O}$$

<u>Define</u>: $\mathcal{H} = \mathcal{O}/R \qquad \mathcal{A} = \mathcal{H}/\zeta^{m}(2)\mathcal{H} \qquad \mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$

We have

$$\Delta : \mathcal{H} \longrightarrow \mathcal{A} \otimes_{\mathbb{Q}} \mathcal{H}$$

$$\mathrm{per} : \mathcal{H} \longrightarrow \mathbb{R}$$

$(\mathcal{H}, \Delta, \mathrm{per})$ is algebra of motivic MZVs.

<u>Remark</u>: Definition is completely <u>elementary</u>, the relations R can be computed algorithmically (conjecturally, R = standard relations)

Strictly speaking, the above defines "Hodge" MZVs. A posteriori the same as "motivic" MZVs

<u>Motivic input</u>: Borel's deep theorem computing $\dim_{\mathbb{Q}} K_{2n-1}(\mathbb{Q}) \otimes \mathbb{Q}$, implies, via the theory of mixed Tate motives, the following

<u>Theorem 1</u>: $n \geq 2$. $\xi \in \mathcal{H}_n$ is primitive $(\Delta \xi = \xi \otimes 1 + 1 \otimes \xi)$

$$\Updownarrow$$

$$\xi \in \mathbb{Q}\, \zeta^{m}(n)$$

VI/ <u>Applications</u>

<u>Theorem</u> 2 ('Mixed Tate motives over $\mathbb{Z}$', 2012)

$\{\zeta^{m}(n_1, \ldots, n_r) \text{ for } n_i = 2, 3\}$ is a basis for $\mathcal{H}$.

$\Rightarrow$ Hoffman basis conjecture for MZVs
"Every MZV is $\mathbb{Q}$-linear combination of $\zeta(n_1, \ldots, n_r)$, $n_i = 2, 3$"

$\Rightarrow$ Deligne - Ihara conjecture on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$

Roughly,

$$\begin{array}{c} \mathcal{H} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \\ | \\ \overline{\mathbb{Q}} \end{array}$$

is the desired Galois extension & has the required Galois group G^{MT} of the introduction.

The <u>desiderata</u> are fulfilled if we replace MZVs with motivic MZVs.

Pf of thm 2 is by induction on number of 3's. Initial step uses an identity for MZVs found by Zagier.

Application 2. An 'exact-numerical' algorithm for decomposing MZVs into a basis.

Ex: If Δ' is the reduced coproduct,

$$\Delta' \zeta^m(2,3) = 3\, \overline{\zeta^m(3)}\, \zeta^m(2)$$

$$\Delta' \zeta^m(3)\zeta^m(2) = \overline{\zeta^m(3)}\, \zeta^m(2)$$

$$\Rightarrow \quad \xi = \zeta^m(2,3) - 3\,\zeta^m(3)\zeta^m(2) \quad \text{is primitive.}$$

$$\text{Thm 1} \Rightarrow \quad \xi = c\,\zeta^m(5) \quad \text{for some } c \in \mathbb{Q}$$

$$\text{Period map} \Rightarrow \qquad c = \frac{\zeta(2,3) - \zeta(3)\zeta(2)}{\zeta(5)} = \frac{-11}{2}$$

$$\text{So} \qquad \zeta^m(2,3) = \frac{-11}{2}\,\zeta^m(5) + 3\,\zeta^m(3)\zeta^m(2)$$

- If we know c, we can compute R this way
- Thm 1 tells us $c \in \mathbb{Q}$, but the period map (regulator) is transcendental. This gives a numerical algorithm for calculating relations between motivic MZVs which is
  - Completely elementary (no use of thm 1, or motives)
  - Fast. Decomposes an MZV of weight 16 into a basis in a few seconds.
- Would like algebraic way to compute quotient of two regulators.

Application 3 Galois descent

Example: $\zeta^m(1,3,1,3,\ldots,1,3)$ is primitive

$$\Rightarrow \quad \zeta(\underbrace{1,3,\ldots,1,3}_{N}) \in \mathbb{Q}\,\pi^{2N}$$