Irrationality proofs, moduli spaces, and dinner parties

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Part I

History

Recall the Riemann zeta values

$$\zeta(n) = \sum_{k \geq 1} \frac{1}{k^n} \qquad \text{for } n \geq 2$$

Euler proved that $\zeta(2) = \frac{\pi^2}{6}$ and more generally

$$\zeta(2n) = -\frac{B_{2n}}{2} \frac{(2\pi i)^{2n}}{(2n)!} \qquad \text{for } n \geq 1$$

where B_m is the m^{th} Bernoulli number.

Folklore conjecture

The odd Riemann zeta values $\zeta(3), \zeta(5), \zeta(7), \ldots$ are algebraically independent over $\mathbb{Q}[\pi]$.

Very little is known.

A nearly complete list of qualitative known results:

1. (Lindemann 1882). The number π is transcendental. In particular the even values $\zeta(2n)$ are irrational.
2. (Apéry 1979). The number $\zeta(3)$ is irrational.
3. (Rivoal and Ball-Rivoal, 2000). The vector space spanned by odd zeta values is infinite-dimensional:

$$\dim_{\mathbb{Q}}\langle 1, \ldots, \zeta(2n+1), \ldots\rangle_{\mathbb{Q}} = \infty$$

4. (Zudilin, 2001). One out of the four numbers

$$\zeta(5), \zeta(7), \zeta(9), \zeta(11)$$

is irrational.

It is *not known* whether $\zeta(5) \notin \mathbb{Q}$, or $1, \zeta(2), \zeta(3)$ are linearly independent over $\mathbb{Q}$, nor is it known if $\zeta(3) \notin \pi^3\mathbb{Q}$.

Suppose that we can construct sequences of pairs of rational numbers a_n, b_n with the following properties:

1. There is a small number $0 < \varepsilon < 1$ such that

$$0 < \left|a_n\alpha - b_n\right| < \varepsilon^n$$

   for all sufficiently large n.

2. Let $d_n \in \mathbb{N}$ be the common denominator of a_n, b_n:

$$d_n a_n \in \mathbb{Z} \qquad d_n b_n \in \mathbb{Z}$$

   Assume that $d_n < D^n$ for some $D \in \mathbb{R}$.

3. D is not too big:

$$D\varepsilon < 1$$

Then α is irrational. It boils down to the following fact:

There is no integer n such that $0 < n < 1$

We only need to construct *small linear forms* in 1 and α whose denominators are not too big.

Proof (by contradiction). Suppose that α is rational, $\alpha = \frac{p}{q}$ where $p, q \in \mathbb{Z}$, $q > 0$. Assumption (1) then becomes

$$0 < \left|a_n \frac{p}{q} - b_n\right| < \varepsilon^n \qquad \text{for large } n$$

By multiplying through by q and d_n, we obtain

$$0 < \left|d_n a_n p - d_n b_n q\right| < q d_n \varepsilon^n < q D^n \varepsilon^n$$

Since by assumption (3) $D\varepsilon < 1$, the right-hand side tends to zero. Thus we can find a large n such that

$$0 < \left|\underbrace{(d_n a_n)}_{\in\mathbb{Z}} p - \underbrace{(d_n b_n)}_{\in\mathbb{Z}} q\right| < 1$$

But by (2), this is an integer between 0 and 1, contradiction.

First example: irrationality of $\log 2$

Let us define

$$f(x) = \frac{x(1-x)}{1+x} \qquad \text{and} \qquad \omega = \frac{dx}{1+x}$$

Consider the family of integrals

$$I_n = \int_0^1 f(x)^n \omega$$

By integrating by parts, one can show that

$$I_n = r_n \log 2 + s_n$$

where $r_n \in \mathbb{Z}$ is an integer, and $s_n \in \mathbb{Q}$ with denominator at most

$$d(n) := \mathrm{lcm}\,(1, 2, \ldots, n)$$

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Theorem (corollary of prime number theorem)

$$d(n) < e^{n(1+\epsilon)} \text{ where } e = 2.7181\cdots$$

Finally, $f(x)$ is positive on the interval $(0,1)$, and is bounded above by $|f(x)| \leq \max_{0<x<1} x(1-x) = \frac{1}{4}$. Therefore we have

$$0 < |I_n| < 4^{-n}$$

The irrationality criteria apply to the linear forms I_n, with

$$\varepsilon = \frac{1}{4}\ , \qquad D = e$$

and we check that $D\varepsilon \sim 0.679\cdots < 1$ and hence (3) holds.

Corollary : $\log 2$ is irrational

The whole difficulty in this game is to find approximations which satisfy the assumptions $(1), (2), (3)$.

Proof of irrationality of $\zeta(2)$ (Apéry, following Beukers)

Consider the family of integrals in two variables

$$I_n = \int_{0\le x,y\le 1} f^n \omega \ ,$$

$$\text{where } f = \frac{x(1-x)y(1-y)}{1-xy} \quad \text{and} \quad \omega = \frac{dxdy}{1-xy}$$

One can show that there is an $a_n \in \mathbb{Z}$, $b_n \in \mathbb{Q}$ such that

$$I_n = a_n \zeta(2) + b_n$$

where the denominator of b_n is bounded by $d(n)^2 \sim e^{2n}$, and

$$0 < I_n < \varepsilon^n$$

where $\varepsilon = \frac{5\sqrt{5}-11}{2}$. The irrationality of $\zeta(2)$ follows since

$$\frac{5\sqrt{5}-11}{2} e^2 = 0.66627 < 1$$

Proof of irrationality of $\zeta(3)$ (Apéry, following Beukers)

Consider the family of integrals in three variables:

$$I_n = \int_{0\le x,y,z\le 1} f^n \omega \ ,$$

where $f = \dfrac{x(1-x)y(1-y)z(1-z)}{1-(1-xy)z}$ and $\omega = \dfrac{dxdydz}{1-(1-xy)z}$

One can show that

$$I_n = a_n\zeta(3) + b_n$$

where the denominator of b_n is bounded by $d(n)^3 < e^{3n}$, and

$$0 < I_n < \varepsilon^n$$

where $\varepsilon = (\sqrt{2}-1)^4$. The irrationality of $\zeta(3)$ follows since

$$(\sqrt{2}-1)^4 e^3 = 0.59126\ldots < 1$$

Many people have tried to construct integrals that give linear combinations of 1 and $\zeta(5)$. The last inequality $D\varepsilon < 1$ fails.

Irrationality measures

Let $\alpha \notin \mathbb{Q}$ be irrational. The irrationality measure $\mu(\alpha)$ is the infimum of the set of real numbers ν such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^\nu}$$

has only finitely many solutions $p, q \in \mathbb{Z}$.

Necessarily $\mu(\alpha) \geq 2$.

Liouville numbers such as $\alpha = \sum_{k\geq 1} 10^{-k!}$ have $\mu(\alpha) = \infty$.

Roth's theorem: if α is algebraic irrational, then $\mu(\alpha) = 2$.

The best known bounds are

$$\mu(\zeta(2)) < 5.442 \quad \text{and} \quad \mu(\zeta(3)) < 5.514$$

are due Rhin and Viola by the group method.[1]

[1] The record for $\zeta(2)$ has recently been broken by Zudilin: $\mu(\zeta(2)) < 5.095$.

The group method

Let $h, i, j, k, l \geq 0$. Dixon in 1905 considered:

$$\int_{0 \leq x,y \leq 1} \frac{x^h(1-x)^i y^k (1-y)^j}{(1-xy)^{i+j-l}} \frac{dxdy}{1-xy}$$

Rhin and Viola (1996): these give linear forms in $1, \zeta(2)$. It has a large symmetry group of order 1440, which enables one to improve estimates of prime factors of denominators.

Rhin and Viola (2001):

$$\int_{0 \leq x,y,z \leq 1} \frac{x^h(1-x)^l y^k (1-y)^s z^j (1-z)^q}{(1-(1-xy)z)^{q+h-r}} \frac{dxdydz}{1-(1-xy)z},$$

where $h, j, k, l, q, r, s \geq 0$ subject to the constraints

$$j + q = l + s \qquad \text{and} \qquad k + r \geq h$$

It gives linear forms in $1, \zeta(3)$ and has group $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes \Sigma_5$.

Nesterenko's criterion for linear independence

Let $\alpha_1, \ldots, \alpha_r$ be real numbers. Suppose that we have linear forms

$$l_n = a_n^1 \alpha_1 + \ldots + a_n^r \alpha_r$$

such that a_n^i are *integers* and that

$$\begin{aligned} |a_n^i| &\leq \eta^n \qquad \text{for all } i, \text{ and large } n \\ \lim_{n\to\infty} |l_n|^{1/n} &= \varepsilon \end{aligned}$$

where $0 < \varepsilon < 1$. Then

$$\dim_{\mathbb{Q}} \langle \alpha_1, \ldots, \alpha_r \rangle > 1 - \frac{\log \varepsilon}{\log \eta}$$

Idea is now to construct linear forms in $1, \zeta(2), \zeta(3), \ldots, \zeta(n)$ and apply the above. Unfortunately, the linear forms are not good enough to prove independence; we already know the subspace

$$\langle 1, \zeta(2), \zeta(4), \ldots, \zeta(2k) \rangle_{\mathbb{Q}}$$

has dimension $k+1$ by Lindemann. Want to kill $\zeta(2n)$'s.

The linear forms of Ball and Rivoal

A breakthrough in 2000 was the introduction of very-well poised hypergeometric series. Fischler (after Zlobin) found the following integral representation for the linear forms of Ball-Rivoal:

$$\int_{[0,1]^{a-1}} \frac{\prod_{j=1}^{a-1} x_j^{rn}(1-x_j)^n dx_j}{(1-x_1x_2\dots x_{a-1})^{rn+1}\prod_{2\le 2j\le a-2}(1-x_1x_2\dots x_{2j})^{n+1}}$$

where $n \geq 0$, $a \geq 3$ and $1 \leq r < \frac{a}{2}$ are integers.

These integrals give small linear forms in

$$\begin{array}{ll} 1, \zeta(3), \zeta(5), \dots, \zeta(a-1) & \text{if } a \text{ even} \\ 1, \zeta(2), \zeta(4), \dots, \zeta(a-1) & \text{if } a \text{ odd} \end{array}$$

Applying Nesterenko's criterion to the first gives: the Ball-Rivoal theorem on odd zeta values. Applying it to the second gives another proof of the transcendence of π.

Picard-Fuchs recurrences

The linear forms occurring in Apéry's proof are of the form

$$a_n\zeta(3) + b_n$$

where a_n is the sequence of *integers*

$$a_1 = 1\ ,\ a_2 = 5\ ,\ a_3 = 73\ ,\ a_4 = 1445\ ,\ a_5 = 33001$$

The sequences a_n and b_n are solutions to the recurrence relation:

$$(n+1)^3 u_{n+1} - (34n^3 + 51n^2 + 27n + 5)u_n + n^3\, u_{n-1} = 0$$

It is remarkable that such a recurrence has a solution which are all integers! There are numerous interpretations of this recurrence relation as a Picard-Fuchs equation of a family of varieties. Interesting connections with modular forms. The coefficients satisfy many congruence and super-congruence relations . . .

Part II

Geometry

Moduli space of curves of genus 0

Let $n \geq 3$. The configuration space of n-points in $\mathbb{P}^1$ is

$$\mathcal{C}^n = \{(z_1, \ldots, z_n) \in \mathbb{P}^1 : z_i \text{ distinct}\}$$

The group PSL_2 acts on $\mathbb{P}^1$ by projective transformations

$$z \mapsto \frac{az+b}{cz+d} \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2 \ .$$

It acts diagonally on $\mathcal{C}^n$. The moduli space $\mathfrak{M}_{0,n}$ of genus 0 curves with n ordered marked points is the quotient

$$\mathfrak{M}_{0,n} = \mathcal{C}^n / \mathrm{PSL}_2 \ .$$

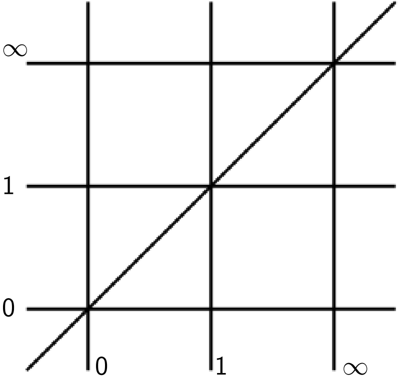
We can always put $z_1 = 0, z_{N-1} = 1, z_N = \infty$. Therefore $\mathfrak{M}_{0,n}$ is the complement of hyperplanes

$$\mathfrak{M}_{0,n} = \{(t_1, \ldots, t_{n-3}) \in \mathbb{A}^{n-3} \text{ such that } t_i \neq 0, 1 \text{ and distinct}\}$$

I claim that most (possibly all) known irrationality results for zeta values are related to $\mathfrak{M}_{0,n}(\mathbb{R})$.

Examples

For $n = 3$, $\mathfrak{M}_{0,3}$ is just a point.
For $n = 4$, $\mathfrak{M}_{0,4} \cong \mathbb{P}^1 \backslash \{0, 1, \infty\}$
Here is a picture of $\mathfrak{M}_{0,5}$:



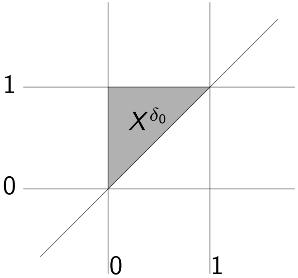
The group Σ_n acts on $\mathfrak{M}_{0,n}$ by permuting the marked points.

The points of $\mathfrak{M}_{0,n}(\mathbb{R})$ are in one-to-one correspondence with n distinct marked points on a circle $\mathbb{R} \cup \{\infty\}$ up to automorphisms. A *cell* is a connected component of $\mathfrak{M}_{0,n}(\mathbb{R})$.

$$\text{Cells } \mathfrak{M}_{0,n}(\mathbb{R}) \leftrightarrow \text{ dihedral orderings on } \{1, \ldots, n\}$$

The *standard cell* is the connected component corresponding to the standard dihedral ordering δ_0 on $\{1, \ldots, n\}$:

$$X^{\delta_0} = \{(t_1, \ldots, t_{n-3}) \in \mathbb{R}^{n-3} : 0 < t_1 < \ldots < t_{n-3} < 1 < \infty\}$$



The symmetric group Σ_n permutes the set of cells X^{δ}.

A class of integrals

A class of integrals (periods) of $\mathfrak{M}_{0,n}$ is given by

$$I = \int_{X^{\delta_0}} \omega$$

where $\omega \in \Omega^{n-3}(\mathfrak{M}_{0,n}; \mathbb{Q})$ is a regular algebraic $n-3$-form.
It is a linear combination of integrals

$$\int_{0<t_1<\ldots<t_{n-3}<1} \prod_{i=1}^{n-3} t_i^{a_i}(1-t_i)^{b_i} \prod_{1\leq i<j\leq n-3} (t_i - t_j)^{c_{ij}} dt_1 \ldots dt_{n-3}$$

where a_i, b_i, c_{ij} are integers. Assume that it converges.

Theorem (B. 2006)

I is a $\mathbb{Q}$-linear combination of multiple zeta values

$$\zeta(n_1, \ldots, n_r) = \sum_{0<k_1<\ldots<k_r} \frac{1}{k_1^{n_1} \ldots k_r^{n_r}}$$

where $n_r \geq 2$ and $n_1 + \ldots + n_r \leq n-3$.

A general construction

The proof of the theorem is effective (algorithms by B.-Bogner, E. Panzer). In principle it gives bounds, e.g., on denominators.

Now take

$$I_n = \int_{X^{\delta_0}} f^n \omega$$

where f vanishes along the boundary of X^{δ_0}. Then I_n tends to zero very fast as $n \to \infty$, and typically we obtain very small linear forms

$$I_n = a_n^1 \zeta_1 + a_n^2 \zeta_2 + \ldots + a_n^r \zeta_r$$

where ζ_i are multiple zeta values. This gives a huge supply of good linear forms for which Nesterenko's condition applies.

We could go a very long way if one could understand:

Vanishing problem

Find conditions on f, ω to force certain coefficients a_n^i to vanish.

Cohomological interpretation

Let $\overline{\mathfrak{M}}_{0,n}$ be the Deligne-Mumford-Knudsen compactification. The singularities of $f^n\omega$ define a boundary divisor A, the Zariski closure of the boundary of X^{δ_0} defines a boundary divisor B.

The integral I is a period of

$$m(A,B) = H^{n-3}(\overline{\mathfrak{M}}_{0,n-3}\backslash A, B\backslash(A\cap B))$$

If $\mathrm{gr}^W_{2k} m(A,B) = 0$ then no MZV's of weight k appear.

Vanishing problem (v2)

Find $A, B \subset \overline{\mathfrak{M}}_{0,n}\backslash\mathfrak{M}_{0,n}$ such that

$$\mathrm{gr}^W_{\bullet} m(A,B) = \mathbb{Q}(0) \oplus \mathbb{Q}(6-2n)$$

This would certainly give linear forms in $1, \zeta(n-3)$.

Theorem

This is possible for $n=5$ (trivial) and $n=6$ (tricky).

I do not know if it is possible for any $n \geq 7$.

Part III

Dinner Parties

Cellular integrals

Consider two dihedral orderings (δ, δ') on $\{1, \ldots, n\}$. They correspond to two connected components on $\mathfrak{M}_{0,n}(\mathbb{R})$.

Define an n-form on the configuration space $\mathcal{C}^n$ by:

$$\widetilde{\omega}_{\delta'} = \pm \frac{dz_1 \ldots dz_n}{\prod_{i \in \mathbb{Z}/n\mathbb{Z}} (z_{\delta'_i} - z_{\delta'_{i+1}})}$$

It is PSL_2-invariant and descends to a form $\omega_{\delta'} \in \Omega^{n-3}(\mathfrak{M}_{0,n})$. Now define a rational function on $\mathcal{C}^n$ by:

$$\widetilde{f}_{\delta/\delta'} = \pm \prod_{i \in \mathbb{Z}/n\mathbb{Z}} \frac{z_{\delta_i} - z_{\delta_{i+1}}}{z_{\delta'_i} - z_{\delta'_{i+1}}} \ .$$

It descends to a rational function $f_{\delta/\delta'} \in \Omega^0(\mathfrak{M}_{0,n})$.

Define the *basic cellular integrals* to be

$$I_{\delta/\delta'}(N) = \int_{X^\delta} f^N_{\delta/\delta'} \omega_{\delta'} \qquad \text{for } N \geq 0$$

Example:

Let $N = 5$, and $\delta = (1, 2, 3, 4, 5), \delta' = (1, 3, 5, 2, 4)$. Then

$$\widetilde{f}_{\delta/\delta'}(z) = \frac{(z_1 - z_2)(z_2 - z_3)(z_3 - z_4)(z_4 - z_5)(z_5 - z_1)}{(z_1 - z_3)(z_3 - z_5)(z_5 - z_2)(z_2 - z_4)(z_4 - z_1)}$$

Set $z_1 = 0, z_2 = t_1, z_3 = t_2, z_4 = 1$ and let z_5 go to ∞. We get

$$f_{\delta/\delta'}(t) = \frac{t_1(t_1 - t_2)(t_2 - 1)}{t_2(1 - t_1)} \quad \text{and} \quad \omega_{\delta'} = \frac{dt_1 dt_2}{t_2(1 - t_1)}$$

The family of basic cellular integrals are

$$I_{\delta/\delta'}(N) = \int_{0<t_1<t_2<1} \left(\frac{t_1(t_1 - t_2)(t_2 - 1)}{t_2(1 - t_1)}\right)^N \frac{dt_1 dt_2}{t_2(1 - t_1)}$$

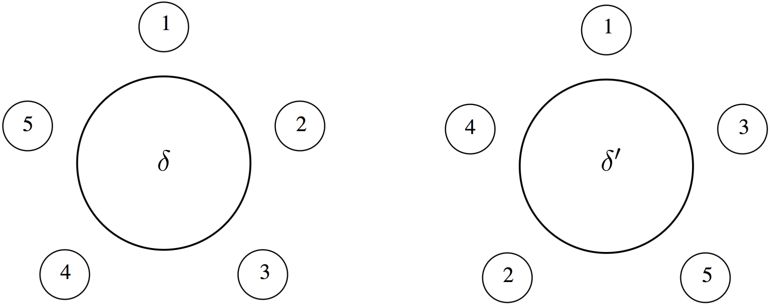
They give back exactly the Apéry linear forms in $1, \zeta(2)$.

Warning

The integral $I_{\delta/\delta'}(N)$ does not always converge! We want to understand for which δ, δ' it converges.

The dinner table problem

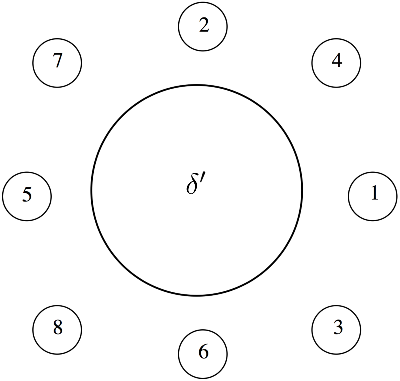
Suppose that we have N guests for dinner, sitting on a round table. It is boring to talk to the same person for the whole duration of the meal, so after the main course, we should permute the guests around in such a way that no-one is sitting next to someone they previously sat next to.



The first solution is for $N = 5$, and is unique.

The enumeration of dinner table seating plans was computed by Poulet in 1919. We actually need a variant where consecutive blocks of k guests don't sit next to each other.

The dinner table problem is $k = 2$. We need $k = \lfloor \frac{N}{2} \rfloor$.



This seating plan for 8 guests is bad for us: a block of four consecutive guests $1, 2, 3, 4$ (and $5, 6, 7, 8$) are sitting together.

Geometric meaning

The domain of integration is simply the cell X^δ. The form $\omega_{\delta'}$ has singularities contained in the boundary of $X^{\delta'}$. The rational function $f_{\delta/\delta'}$ vanishes along the boundary of X^δ and has poles along the boundary of $X^{\delta'}$.

Recall that the symmetric group Σ_n acts on $\mathfrak{M}_{0,n}$. Two pairs of dihedral orderings are equivalent if

$$(\delta, \delta') \sim (\sigma\delta, \sigma\delta') \qquad \text{for some } \sigma \in \Sigma$$

Call the equivalence class a *configuration*. Equivalent configurations give the same cellular integrals. A configuration (δ, δ') is *convergent* if $I^N_{\delta/\delta'}$ is finite for all N.

We can always assume that $\delta = \delta_0$ from now on.

As $N \to \infty$, the integrals $I_{\delta/\delta'}(N)$ tend to zero very fast. By a previous theorem, they give linear forms in multiple zeta values.

Observation

The multiple zeta values of sub-maximal weight always vanish.

Enumeration of convergent configurations:

N	4	5	6	7	8	9	10	11
#	0	1	1	5	17	105	771	7028

Theorem

For $N = 5, 6$ there is a unique class of convergent configurations. The basic cellular integrals give back exactly Apéry's proofs of the irrationality of $\zeta(2)$ and $\zeta(3)$, respectively.

Starting with $N = 8$ we find linear forms involving products such as $\zeta(2)\zeta(3)$ as well as $\zeta(5)$.

Ball-Rivoal's theorem and Lindemann's theorem

Theorem

Let $m \geq 3$. The family of convergent configurations (δ_0, π)

$$\pi^m_{odd} = (2m, 2, 2m-1, 3, 2m-2, 4, \dots, m, 1, m+1)$$

gives Ball-Rivoal's forms in $1, \zeta(3), \zeta(5), \dots, \zeta(2m-3)$. The family

$$\pi^m_{even} = (2m+1, 2, 2m, 3, 2m-1, 4, \dots, m+2, 1, m+1)$$

gives back their linear forms in $1, \zeta(2), \zeta(4), \dots, \zeta(2m-2)$.

There appears to be a whole zoo of configurations with interesting vanishing properties. For instance, the dual configuration $(\pi^m_{odd}, \delta_0) \sim (\delta_0, (\pi^m_{odd})^{-1})$ yields new linear forms in

$$1, \pi^2, \pi^4, \dots, \pi^{2m-6}, \zeta_{2m-3}$$

Can one do a p-adic or single-valued version to kill the π^{2n}'s?

Generalised cellular integrals

We can introduce parameters into the cellular integrals by

$$\widetilde{f}_{\delta/\delta'} = \pm \prod_{i\in\mathbb{Z}/n\mathbb{Z}} \frac{(z_{\delta_i} - z_{\delta_{i+1}})^{a_{i,i+1}}}{(z_{\delta'_i} - z_{\delta'_{i+1}})^{b_{i,i+1}}}$$

where $a_{i,i+1}, b_{i,i+1}$ are integers chosen such that the expression is homogeneous in each z_i. Each basic cellular integral on $\mathfrak{M}_{0,n}$ spawns a large family of integrals with n parameters.

Theorem

The generalised cellular integrals, for $N = 5$ and $N = 6$ are equivalent to Rhin and Viola's integrals for $\zeta(2)$ and $\zeta(3)$.

The dinner party game generates almost all irrationality results.

The generalised integrals for π^m_{odd} give a huge family of integrals that appears to give linear forms in odd zetas, with a rich symmetry group. Can one improve on Ball-Rivoal's theorem?

Picard-Fuchs recurrences

Every family of basic cellular integrals $I_\pi(N)$ satisfies a Picard-Fuchs recurrence equation. Some properties:

1. (Poincaré duality). The family $I_{\pi^\vee}$ of the dual configuration $\pi^\vee$ satisfies the dual (homogeneous) Picard-Fuchs equation.
2. (Products). Given certain π_1, π_2, one can find (several) convergent configurations π such that

   $$I_\pi(N) = I_{\pi_1}(N) I_{\pi_2}(N) \qquad \text{for all } N \geq 0$$

   This gives a partial multiplication law.
3. (Relations). Sometimes, for non-equivalent π, π' we have

   $$I_\pi(N) = I_{\pi'}(N) \quad \text{for all } N \geq 0$$

   When does this happen?