A K3 IN ϕ^4

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Abstract. Inspired by Feynman integral computations in quantum field theory, Kontsevich conjectured in 1997 that the number of points of graph hypersurfaces over a finite field $\mathbb{F}_q$ is a (quasi-) polynomial in q. Stembridge verified this for all graphs with ≤ 12 edges, but in 2003 Belkale and Brosnan showed that the counting functions are of general type for large graphs. In this paper we give a sufficient combinatorial criterion for a graph to have polynomial point-counts, and construct some explicit counter-examples to Kontsevich's conjecture which are in ϕ^4 theory. Their counting functions are given modulo pq^2 ($q = p^n$) by a modular form arising from a certain singular K3 surface.

1. Introduction

We first recall the definition of graph hypersurfaces and the history of the point-counting problem, before explaining its relevance to Feynman integral calculations in perturbative Quantum Field Theory.

1.1. **Points on graph hypersurfaces.**

Let G be a connected graph, possibly with multiple edges and self-loops (an edge whose endpoints coincide). The graph polynomial of G is defined by associating a variable α_e, known as a Schwinger parameter, to every edge e of G and setting

$$\Psi_G(\alpha) = \sum_{T \subset G} \prod_{e \notin T} \alpha_e \in \mathbb{Z}[\alpha_e] \ , \tag{1}$$

where the sum is over all spanning trees T of G (connected subgraphs meeting every vertex of G which have no loops). These polynomials go back to the work of Kirchhoff in relation to the study of currents in electrical circuits [14].

The projective graph hypersurface X_G is defined to be the zero locus of Ψ_G in projective space $\mathbb{P}^{N_G-1}$, where N_G is the number of edges of G (although from §1.3 onwards, X_G will denote the zero locus in affine space $\mathbb{A}^{N_G}$). It is highly singular in general. For any prime power q, let $\mathbb{F}_q$ denote the field with q elements, and consider the point-counting function:

$$[X_G]_q : q \mapsto \# X_G(\mathbb{F}_q) \in \mathbb{N} \cup \{0\}.$$

In 1997, Kontsevich informally conjectured that this function might be polynomial in q for all graphs. This question was studied by Stanley, Stembridge and others, and in particular was proved for all graphs with at most twelve edges [24]. A dual statement was proved for various families of graphs obtained by deleting trees in complete graphs [23], [10]. But in [2], contrary to expectations, Belkale and Brosnan used Mnëv's universality theorem to prove that the $[X_G]_q$ are of general type in the following precise sense.

Theorem 1. *(Belkale-Brosnan). For every scheme Y of finite type over* Spec $\mathbb{Z}$*, there exist finitely many polynomials $p_i \in \mathbb{Z}[q]$ and graphs G_i such that*

$$s[Y]_q = \sum_i p_i [X_{G_i}]_q \ ,$$

where $[Y]_q$ denotes the point-counting function on Y, and s is a product of terms of the form $q^n - q$, where $n > 1$. In particular, $[X_G]_q$ is not always polynomial.

This does not imply that the point-counting functions $[X_{G_i}]_q$ themselves are arbitrary. The methods of [4] §4, for example, imply strong constraints on $[X_G]_q$. Moreover, Belkale and Brosnan's method constructs graphs G_i with very large numbers of edges ([2], Remark 9.4), and no explicit counter-example was known until recently, when Doryn [12] and Schnetz [18] independently constructed graphs which are quasi-polynomial (i.e., which become polynomial only after a finite extension of the base field or the exclusion of exceptional primes). It has since been hoped that various 'physicality' constraints on G might be sufficient to ensure the validity of Kontsevich's conjecture in this slightly weaker sense. However, the modular counter-examples we construct below show that this hope is completely false.

1.2. **Feynman integrals and motives.** The point-counting problem has its origin in the question of determining the arithmetic content of perturbative quantum field theories. For this, some convergency conditions are required on the graphs. A connected graph G is said to be primitive-divergent if:

$$\begin{array}{rcll} N_G & = & 2h_G & \\ N_\gamma & > & 2h_\gamma & \text{for all strict subgraphs } \gamma \subsetneq G \ , \end{array} \tag{2}$$

where h_γ denotes the number of loops (first Betti number) and N_γ the number of edges in a graph γ. In this case, the residue of G is defined by the absolutely convergent projective integral [25], [5]

$$I_G = \int_\sigma \frac{\Omega_{N_G}}{\Psi_G^2} \ , \tag{3}$$

where $\sigma = \{(\alpha_1 : \ldots : \alpha_{N_G}) \subset \mathbb{P}^{N_G-1}(\mathbb{R}) : \alpha_i \geq 0\}$ is the real coordinate simplex in projective space, and $\Omega_{N_G} = \sum_{i=1}^{N_G} (-1)^i d\alpha_1 \ldots \widehat{d\alpha_i} \ldots d\alpha_{N_G}$. This defines a map from the set of primitive-divergent graphs to positive real numbers. It is important to note that the quantities I_G are renormalization-scheme independent. We say that G is in ϕ^4 theory if every vertex of G has degree at most four. Even in this case, the numbers I_G are very hard to evaluate, and known analytically for only a handful of graphs. Despite the difficulties in computation, the remarkable fact was observed by Broadhurst, Kreimer [3], and later Schnetz [17], that every graph whose period is computable (either analytically or numerically to high precision) is consistent with being a multiple zeta value. This was the original motivation for Kontsevich's question.

The algebraic approach to this problem comes from the observation that the numbers I_G are periods in the sense of algebraic geometry. To make this precise, the integrand of (3) defines a cohomology class in $H^{N_G-1}(\mathbb{P}^{N_G-1}\backslash X_G)$, and the domain of integration a relative homology class in $H_{N_G-1}(\mathbb{P}^{N_G-1}, B)$ where $B = V(\prod_{i=1}^{N_G} \alpha_i)$, which contains the boundary of the simplex σ. Thus as a first approximation, one could consider the relative mixed Hodge structure

$$H^{N_G-1}(\mathbb{P}^{N_G-1}\backslash X_G, B\backslash(B\cap X_G)) \ . \tag{4}$$

For technical reasons related to the fact that σ meets X_G non-trivially, the integral I_G is not in fact a period of (4). One of the main constructions of [5] is to blow up boundary components of B to obtain a slightly different relative mixed Hodge structure called the graph motive M_G. The integral I_G is now a period of M_G. If M_G is of mixed Tate type (its weight graded pieces are of type (p,p)) and satisfies some ramification conditions, then by standard conjectures on mixed Tate motives (now proved [6]), it should follow that the period I_G is a multiple zeta value.

Although not explicitly stated in [5], it follows from the geometry underlying their construction and the relative cohomology spectral sequence that M_G is controlled by the absolute mixed Hodge structures $H^i(\mathbb{P}^i\backslash X_\gamma)$, where γ ranges over all minors (subquotients) of G. Thus the simplest way in which the period I_G could be a multiple zeta value is if the mixed Hodge structure M_γ were entirely of Tate type, or, even stronger, if $H^\bullet(\mathbb{P}^i\backslash X_\gamma)$ were of Tate type in all cohomological dimensions, for all minors γ of G. To simplify matters further, one can ask the somewhat easier question of whether the Euler characteristics of the X_γ's are of Tate type. In this way, one is led to consider the class of X_G in the Grothendieck ring of varieties $K_0(\mathrm{Var}_k)$ and ask if it is a polynomial in the Lefschetz motive $\mathbb{L} = [\mathbb{A}^1_k]$. This is surely the reasoning behind Kontsevich's original question, although it was formulated almost ten years before M_G was defined. Note, however, that there is *a priori* no way to construe information about I_G from the Grothendieck class $[X_G]$.

1.3. **Results and contents of the paper.** We begin in §2 by reviewing some algebraic and combinatorial properties of graph polynomials. In §3, we discuss implications for the class of the affine graph hypersurface $[X_G]$ in the Grothendieck ring of varieties $K_0(\mathrm{Var}_k)$, where k is a field. The first observation is the following:

Proposition 2. *Let G be any graph satisfying $h_G \leq N_G - 2$. Then there is an invariant $c_2(G) \in K_0(\mathit{Var}_k)/\mathbb{L}$ such that*

$$[X_G] \equiv c_2(G)\mathbb{L}^2 \mod \mathbb{L}^3 \ . \tag{5}$$

If G has a three-valent vertex, then $c_2(G)$ has a simple representative in $K_0(\mathit{Var}_k)$ as the class of the intersection of two explicit affine hypersurfaces.

For primitive-divergent graphs (2) this intersection satisfies a Calabi-Yau property in the sense that, after projectifying, the total degree is exactly one greater than the dimension of the ambient projective space.

The class $c_2(G)$ is much more tractable than the full class $[X_G]$. In order to exploit its combinatorial properties, we require the Chevalley-Warning theorem 25 on the point-counts of polynomials of small degree modulo q. We hope that this theorem lifts to the Grothendieck ring under some conditions on k (§3.3), but since this is unavailable, we are forced to pass to point-counts modulo q. Thus, denoting the corresponding counting functions by $[.]_q$, equation (5) gives

$$[X_G]_q \equiv c_2(G)_q q^2 \mod q^3 \ ,$$

where $c_2(G)_q$ is a map from prime powers q to $\mathbb{Z}/q\mathbb{Z}$.

In §3 we explain how to compute the invariant $c_2(G)$ by a simple algorithm ('denominator reduction') which reduces the problem to counting points on hypersurfaces of smaller and smaller dimension. This is the key to constructing non-Tate counter-examples, and stems from the following observation:

Theorem 3. *Let G be primitive divergent with at least five edges $e_1, \ldots, e_5$. Then*

$$c_2(G)_q \equiv -[{}^5\Psi_G(e_1, \ldots, e_5)]_q \mod q$$

where ${}^5\Psi_G(e_1, \ldots, e_5)$ is the 5-invariant [8] *of those edges.*

The 5-invariant is a certain resultant of polynomials derived from the graph polynomials of minors of G, and it follows from this theorem that the invariant $c_2(G)_q$ can be computed inductively by taking iterated resultants. This uses the Chevalley-Warning theorem in an essential way to kill parasite terms.

In §4 we use this result to deduce the following properties of $c_2(G)_q$ in the case when G is primitive-divergent graph in ϕ^4 theory:

1. If G is two-vertex reducible then $c_2(G)_q \equiv 0 \mod q$.
2. If G has weight-drop (in the sense of [9]), then $c_2(G)_q \equiv 0 \mod q$.
3. If G has vertex-width ≤ 3, then $c_2(G)_q \equiv c \mod q$ for some $c \in \mathbb{Z}$.
4. $c_2(G)_q$ is invariant under double triangle reduction.

For the definitions of these terms, see §4. All of these properties have some bearing on the residue I_G ([8], [9]). A further property is conjectural:

Conjecture 4. *$c_2(G)$ is invariant under the completion relation (*[17], [18]*).*

This is implied by the following stronger conjecture (see [18], remark 2.10 (2)):

Conjecture 5. *If $I_{G_1} = I_{G_2}$ for two graphs G_1, G_2 then $c_2(G_1) = c_2(G_2)$.*

In short, the invariant $c_2(G)$ detects all the known qualitative features of the residue I_G, but is much easier to compute. Intuitively, $c_2(G)_q$ should be closely related to the action of Frobenius on the framing of M_G, i.e., the smallest subquotient motive of M_G which is spanned by the Feynman differential form $\Omega_{N_G}\Psi_G^{-2}$.

In §5 we review the notion of vertex-width, which is a measure of the local connectivity of a graph, and prove Kontsevich's conjecture for an infinite family of graphs. Note that this result is valid in the Grothendieck ring $K_0(\mathrm{Var}_k)$.

Theorem 6. *Let G have vertex-width at most 3. Then $[\Psi_G]$ is a polynomial in $\mathbb{L}$.*

This family of graphs contains almost all the physically interesting cases at low loop orders. It was proved in [8] that a variant of the motive M_G is mixed Tate in this case, but the proof we give here is elementary and gives an effective way to compute the polynomial $[\Psi_G]$ by induction over the minors of G. It also enables one to compute the Grothendieck classes of any infinite family of graphs obtained by inserting triangles into a known graph (a problem raised in §13.2 of [5]). In §5.5 and §5.6, we carry this out for the wheels (also computed independently in [12]) and zig-zag graphs. These are the only two families of graphs for which a formula for the residue I_G is known, or conjectured.

The motivation for such computations is the hope that they will give combinatorial insight into the full structure of the motive M_G, and ultimately the action of the motivic Galois group, which would yield a lot of information about the periods. Currently there is not a single example where this has been successfully carried out.

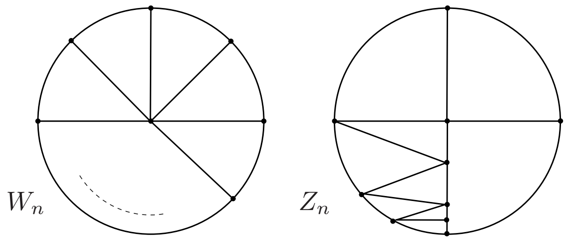

FIGURE 1. The wheels with spokes (left), and zig-zags (right).

In §6 we construct explicit counter-examples to Kontsevich's conjecture by computing the c_2-invariants of some graphs G of vertex width 4. The denominator reduction algorithm reduces $c_2(G)_q$ down to a determinant of graph polynomials of small degree derived from G. By a series of manipulations one can extract a polynomial which defines a surface of degree 4 in $\mathbb{P}^3$, whose minimal desingularization X is a K3 surface. In §7, we show that this surface has Néron-Severi group of maximal rank, and that its Picard lattice has discriminant -7. This proves that X is a singular $K3$ surface, which have been classified by Shioda and Inose [20]. The modularity of such surfaces is known by [15], and in this case $H^2_{tr}(X)$ is a submotive of the symmetric square of the first cohomology group of the elliptic curve:

$$E_{49A1}: \qquad y^2 + xy = x^3 - x^2 - 2x - 1 \ ,$$

which has complex-multiplication by $\mathbb{Q}(\sqrt{-7})$. We write down the modular form of weight 2 and level 49 whose coefficients give the point-counts on E_{49A1}. Its symmetric square is given by the following product of Dedekind η-functions:

$$\big(\eta(z)\eta(z^7)\big)^3 \ , \tag{6}$$

which defines a cusp form of weight 3 and level 7.

Theorem 7. *Let $q = p^n$. There exists a non-planar primitive-divergent graph in ϕ^4 theory with 8 loops, 16 edges, and vertex-width 4 such that*

$$c_2(G)_q \equiv -a_q^2 \equiv -b_q \mod p$$

where $q + 1 - a_q = [E_{49A1}]_q$ is the number of points on $E_{49A1}(\mathbb{F}_q)$, and b_q is the coefficient of z^q in (6)*. In particular,*

$$[X_G]_q \equiv -a_q^2 q^2 \equiv -b_q q^2 \mod pq^2 \tag{7}$$

cannot be a polynomial in q. Furthermore, there exists a ***planar*** *primitive-divergent graph in ϕ^4 theory with 9 loops, 18 edges, and vertex-width 4 with the same property.*

The fact that our counter-examples have vertex-width 4 shows that theorem 6 cannot be improved. In fact, within ϕ^4 theory the 8-loop graph is the smallest modular counter-example to Kontsevich's conjecture [18].

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2. Graph polynomials

Throughout this paper, G will denote any graph, possibly with multiple edges and self-loops. A subgraph of G will be a subset of edges of G. In this section, we make no assumptions about the primitive-divergence or otherwise of G.

2.1. **Matrix representation.** We recall some basic results from [8]. We will use the following matrix representation for the graph polynomial.

Definition 8. Choose an orientation on the edges of G, and for every edge e and vertex v of G, define the incidence matrix:

$$(\mathcal{E}_G)_{e,v} = \begin{cases} 1, & \text{if the edge } e \text{ begins at } v \text{ and does not end at } v, \\ -1, & \text{if the edge } e \text{ ends at } v \text{ and does not begin at } v, \\ 0, & \text{otherwise.} \end{cases}$$

Let A be the diagonal matrix with entries α_e, for $e \in E(G)$, and set

$$\widetilde{M}_G = \left(\begin{array}{c|c} A & \mathcal{E}_G \\ \hline -\mathcal{E}_G^T & 0 \end{array} \right)$$

where the first N_G rows and columns are indexed by the set of edges of G, and the remaining v_G rows and columns are indexed by the set of vertices of G, in some order. The matrix $\widetilde{M}_G$ has corank ≥ 1. Choose any vertex of G and let M_G denote the square $(N_G + v_G - 1) \times (N_G + v_G - 1)$ matrix obtained from it by deleting the row and column indexed by this vertex.

It follows from the matrix-tree theorem that the graph polynomial satisfies (§2.2 in [8])

$$\Psi_G = \det(M_G) \ .$$

If G has at least two components G_1 and G_2, then by permuting rows and columns $\widetilde{M}_G$ can be written as a direct sum of $\widetilde{M}_{G_1}$ and $\widetilde{M}_{G_2}$. In this case the graph polynomial Ψ_G vanishes, since $\operatorname{corank} \widetilde{M}_G = \operatorname{corank} \widetilde{M}_{G_1} + \operatorname{corank} \widetilde{M}_{G_2} \geq 2$.

Definition 9. Let I, J, K be subsets of the set of edges of G which satisfy $|I| = |J|$. Let $M_G(I,J)_K$ denote the matrix obtained from M_G by removing the rows (resp. columns) indexed by the set I (resp. J) and setting $\alpha_e = 0$ for all $e \in K$. Let

$$\Psi^{I,J}_{G,K} = \det M_G(I,J)_K \ . \tag{8}$$

We shall perpetuate the anachronism of [8] by referring to these polynomials as Dodgson polynomials.

It is clear that $\Psi^{\emptyset,\emptyset}_{G,\emptyset} = \Psi_G$, and $\Psi^{I,J}_{G,K} = \Psi^{J,I}_{G,K}$ because $I, J \subset E(G)$. If $K = \emptyset$, we will often drop it from the notation. We also write $\Psi^I_{G,K}$ as a shorthand for $\Psi^{I,I}_{G,K}$.

Since the matrix M_G depends on various choices, the polynomials $\Psi^{I,J}_{G,K}$ are only well-defined up to sign. In what follows, for any graph G, we shall fix a particular matrix M_G and this will fix all the signs in the polynomials $\Psi^{I,J}_{G,K}$ too.

Proposition 10. *The monomials which occur in $\Psi^{I,J}_{G,K}$ have coefficient ± 1, and are precisely the monomials which occur in both $\Psi^{I,I}_{G,J\cup K}$ and $\Psi^{J,J}_{G,I\cup K}$.*

Proof. See [8], proposition 23, §2.3. □

Definition 11. If $f = f_1 + f^1\alpha_1$ and $g = g_1 + g^1\alpha_1$ are polynomials of degree one in α_1, recall that their resultant is defined by:

$$[f,g]_{\alpha_1} = f^1 g_1 - f_1 g^1 \ . \tag{9}$$

We now state some identites between Dodgson polynomials which will be used in the sequel. The proofs can be found in ([8], §2.4-2.6).

2.2. **General identities.** The first set of identities only use symmetries of the matrix M_G, and therefore hold for any graph G.

(1) *The contraction-deletion formula.* It is clear from its definition that $\Psi^{I,J}_{G,K}$ is linear in every Schwinger variable α_e. When the index of e is larger than all elements of I and J (in general there are signs), we can write:

$$\Psi^{I,J}_{G,K} = \Psi^{Ie,Je}_{G,K}\alpha_e + \Psi^{I,J}_{G,Ke} \ .$$

The contraction-deletion relations state that

$$\Psi^{Ie,Je}_{G,K} = \Psi^{I,J}_{G\backslash e,K} \text{ and } \Psi^{I,J}_{G,Ke} = \Psi^{I,J}_{G/\!\!/ e,K} \ ,$$

where $G\backslash e$ is the graph obtained by deleting the edge e (but not its endpoints), and $G/\!\!/e$ denotes the graph obtained by contracting the edge e (and identifying its two endpoints). Note that we define the contraction of a self-loop to be the zero graph 0, for which we set $\Psi_0 = 0$.

(2) *Dodgson-type identities.* Let I, J be two subsets of edges of G such that $|I| = |J|$ and let $a, b, x \notin I \cup J \cup K$. Then the first (Dodgson) identity is:

$$\big[\Psi^{I,J}_{G,K}, \Psi^{Ia,Jb}_{G,K}\big]_x = \Psi^{Ix,Jb}_{G,K}\Psi^{Ia,Jx}_{G,K} \ .$$

Let I, J be two subsets of edges of G such that $|J| = |I| + 1$ and let $a, b, x \notin I \cup J \cup K$. Then the second identity is:

$$\big[\Psi^{Ia,J}_{G,K}, \Psi^{Ib,J}_{G,K}\big]_x = \pm\Psi^{Ix,J}_{G,K}\Psi^{Iab,Jx}_{G,K} \ .$$

Note that $\Psi^{I,I}_{G,K} = \Psi_{G\backslash I/\!\!/K}$. A graph obtained by contracting and deleting edges of G will be called a minor of G.

2.3. **Graph-specific identities.** The second set of identities depend on the particular combinatorics of a graph G, and follow from proposition 10 together with the fact that $\Psi_{G\backslash I} = 0$ if I contains the set of all edges which meet a given vertex (because $G\backslash I$ is not connected), and $\Psi_{G/\!\!/K} = 0$ if $h_1(K) > 0$ (because contracting edges of K one by one leads to the contraction of a self-loop).

(1) *Vanishing property for vertices.* Suppose that $E = \{e_1, \ldots, e_k\}$ is the set of edges which are adjacent to a given vertex of G. Then

$$\Psi^{I,J}_{G,K} = 0 \qquad \text{if} \qquad E \subset I \quad \text{or} \quad E \subset J \ .$$

(2) *Vanishing property for loops.* Suppose that $E = \{e_1, \ldots, e_k\}$ is a set of edges in G which contain a loop. Then

$$\Psi^{I,J}_{G,K} = 0 \quad \text{if} \quad (E \subset I \cup K \text{ or } E \subset J \cup K) \quad \text{and} \quad E \cap I \cap J = \emptyset \ .$$

2.4. **Local structure.** We use these to deduce the local structure of Ψ_G in some simple circumstances. Many more identities are derived in [8].

(1) *Local 2-valent vertex.* Suppose that G contains a 2-valent vertex, whose neighbouring edges are labelled $1,2$. Then

$$\Psi_G^{12} = 0 \quad \text{and} \quad \Psi_G^{1,2} = \Psi_{G,2}^1 = \Psi_{G,1}^2$$

which imply that $\Psi_G = \Psi_{G\backslash 1/\!/ 2}(\alpha_1+\alpha_2)+\Psi_{G/\!/\{1,2\}}$. In general, if $|I|=|J|$ are sets of edges such that $\{1,2\}\notin I\cup J\cup K$, then

$$\Psi_{G,K}^{1I,2J} = \Psi_{G\backslash 1/\!/ 2,K}^{I,J} = \Psi_{G\backslash 2/\!/ 1,K}^{I,J}\ .$$

(2) *Doubled edge.* Suppose that G contains doubled edges $1,2$ (i.e., two edges which have the same set of endpoints). Then

$$\Psi_{G,12} = 0 \quad \text{and} \quad \Psi_G^{1,2} = \Psi_{G,2}^1 = \Psi_{G,1}^2$$

which imply that $\Psi_G = \Psi_{G\backslash\{1,2\}}\alpha_1\alpha_2 + \Psi_{G\backslash 1/\!/ 2}(\alpha_1+\alpha_2)$. In general, if $|I|=|J|$ are sets of edges such that $\{1,2\}\notin I\cup J\cup K$, then

$$\Psi_{G,K}^{1I,2J} = \Psi_{G\backslash 1/\!/ 2,K}^{I,J} = \Psi_{G\backslash 2/\!/ 1,K}^{I,J}\ .$$

(3) *Local star.* Suppose that G contains a three-valent vertex, whose neighbouring edges are labelled $1,2,3$. Then we have ([8], Example 32)

$$\Psi_G^{123} = 0 \quad \text{and} \quad \Psi_{G,3}^{12} = \Psi_{G,2}^{13} = \Psi_{G,1}^{23}$$

which follow from contraction-deletion. Furthermore, for $\{a,b,c\}=\{1,2,3\}$ we have the identities

$$\Psi_G^{ab,bc} = \Psi_{G,c}^{ab} = \ldots = \Psi_{G,a}^{bc} \quad \text{and} \quad \Psi_{G,bc}^{a} = \Psi_{G,b}^{a,c} + \Psi_{G,c}^{a,b}\ .$$

These identities propagate to higher order Dodgson polynomials. Let $i,j\notin\{1,2,3\}$. Then for all $\{a,b,c\}=\{a',b',c'\}=\{1,2,3\}$, we have ([8], §7.4):

$$\Psi_G^{abc,aij} = 0 \quad \text{and} \quad \Psi_G^{aci,bcj} = \pm\Psi_{G\backslash\{a',b'\}/\!/ c'}^{i,j}\ .$$

(4) *Local triangle.* Suppose that G contains a triangle, with edges $1,2,3$. Then

$$\Psi_{G,123} = 0 \quad \text{and} \quad \Psi_{G,23}^1 = \Psi_{G,13}^2 = \Psi_{G,12}^3$$

which follow from contraction-deletion. Furthermore, for $\{a,b,c\}=\{1,2,3\}$ we have the identities ([8], Example 33):

$$\Psi_{G,c}^{a,b} = \Psi_{G,bc}^{a} = \ldots = \Psi_{G,ac}^{b} \quad \text{and} \quad \Psi_{G,c}^{ab} = \Psi_G^{ab,ac} + \Psi_G^{ab,bc}\ .$$

Now let $i,j\notin\{1,2,3\}$. For all $\{a,b,c\}=\{a',b',c'\}=\{1,2,3\}$, we have

$$\Psi_{G,c}^{ab,ij} = 0 \quad \text{and} \quad \Psi_{G,c}^{ai,bj} = \pm\Psi_{G\backslash a'/\!/\{b',c'\}}^{i,j}\ .$$

2.5. **The five-invariant.**

Definition 12. Let i,j,k,l,m denote any five distinct edges in a graph G. The five-invariant of these edges, denoted ${}^5\Psi_G(i,j,k,l,m)$ is defined to be the determinant

$${}^5\Psi_G(i,j,k,l,m) = \pm\det\begin{pmatrix} \Psi_{G,m}^{ij,kl} & \Psi_{G,m}^{ik,jl} \\ \Psi_G^{ijm,klm} & \Psi_G^{ikm,jlm}\end{pmatrix}$$

It can be shown that the five-invariant is well-defined, i.e., permuting the five indices i, j, k, l, m only modifies the right-hand determinant by a sign. In general, the 5-invariant is irreducible of degree 2 in each Schwinger variable α_e. However, in the case when three of the five edges i, j, k, l, m form a star or a triangle, it splits, i.e., factorizes into a product of Dodgson polynomials.

Example 13. Suppose that G contains a triangle a, b, c. Then

$$^5\Psi_G(a,b,c,i,j) = \pm \det \begin{pmatrix} \Psi^{ab,ij}_{G,c} & \Psi^{ai,bj}_{G,c} \\ \Psi^{abc,cij}_{G} & \Psi^{aci,bcj}_{G} \end{pmatrix} = \pm \Psi^{i,j}_{G\backslash a/\!\!/\{b,c\}} \Psi^{abc,cij}_{G} \ .$$

It factorizes because $\Psi^{ab,ij}_{G,c} = 0$ by the vanishing property for loops. By contraction-deletion, $\Psi^{ai,bj}_{G,c} = \Psi^{ai,bj}_{G/\!\!/c}$, and this is $\Psi^{i,j}_{G\backslash a/\!\!/\{b,c\}}$, by the last equation of §2.4, (2), since a, b form a doubled edge in the quotient graph $G/\!\!/c$.

2.6. **Denominator reduction.** Given a graph G and an ordering $e_1, \ldots, e_{N_G}$ on its edges, we can extract a sequence of higher invariants, as follows.

Definition 14. Define $D^5_G(e_1, \ldots, e_5) = {}^5\Psi_G(e_1, \ldots, e_5)$. Let $n \geq 5$ and suppose that we have defined $D^n_G(e_1, \ldots, e_n)$. Suppose furthermore that $D^n_G(e_1, \ldots, e_n)$ factorizes into a product of linear factors in α_{n+1}, i.e., it is of the form $(a\alpha_{n+1} + b)(c\alpha_{n+1} + d)$. Then we define

$$D^{n+1}_G(e_1, \ldots, e_{n+1}) = \pm(ad - bc) \ ,$$

to be the resultant of the two factors of $D^n_G(e_1, \ldots, e_n)$. A graph G for which the polynomials $D^n_G(e_1, \ldots, e_n)$ can be defined for all n is called *denominator-reducible*. It can happen that $D^n_G(e_1, \ldots, e_n)$ vanishes. Then G is said to have *weight-drop*.

For general graphs above a certain loop order and any ordering on their edges, there will come a point where $D^n_G(e_1, \ldots, e_n)$ is irreducible (typically for $n = 5$). Thus the generic graph is not denominator reducible. One can prove, as for the 5-invariant, that $D^n_G(e_1, \ldots, e_n)$ does not depend on the order of reduction of the variables, although it may happen that the intermediate terms $D^k_G(e_{i_1}, \ldots, e_{i_k})$ may factorize for some choices of orderings and not others.

3. The class of X_G in the Grothendieck Ring of Varieties

Let k be a field. The Grothendieck ring of varieties $K_0(\mathrm{Var}_k)$ is the free abelian group generated by isomorphism classes $[X]$, where X is a separated scheme of finite type over k, modulo the inclusion-exclusion relation $[X] = [X\backslash Z] + [Z]$, where $Z \subset X$ is a closed subscheme. It has the structure of a commutative ring induced by the product relation $[X \times_k Y] = [X] \times [Y]$, with unit $1 = [\mathrm{Spec}\, k]$. One defines the Lefschetz motive $\mathbb{L}$ to be the class of the affine line $[\mathbb{A}^1_k]$.

Remark 15. *We only consider affine varieties here. If $f_1, \ldots, f_\ell \in k[\alpha_1, \ldots, \alpha_n]$ are polynomials, we denote by $[f_1, \ldots, f_\ell]$ the class in $K_0(\mathrm{Var}_k)$ of the intersection of the hypersurfaces $V(f_1) \cap \ldots \cap V(f_\ell)$ in affine space $\mathbb{A}^n_k$. The dimension of the ambient affine space will usually be clear from the context.*

Let G be a graph. Since the graph polynomial Ψ_G (and, more generally, all Dodgson polynomials $\Psi^{I,J}_{G,K}$) is defined over $\mathbb{Z}$, we can view the element $[\Psi_G]$ in $K_0(\mathrm{Var}_k)$ for any field k. Most of the results below are valid in this generality. But at a certain point, we are obliged to switch to point-counting functions since we require the use of the Chevalley-Warning theorem (theorem 25). Recall that if k is a finite field, the point-counting map:

$$\begin{array}{rcl} \# : K_0(\mathrm{Var}_k) & \rightarrow & \mathbb{Z} \\ [X] & \mapsto & \#X(k) \end{array}$$

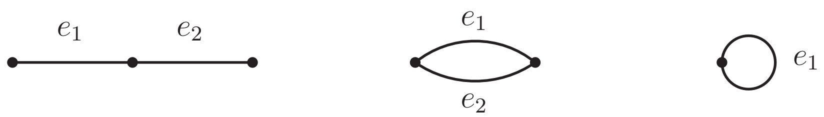
is well-defined, so results about the point-counts can be deduced from results in the Grothendieck ring, but not conversely (for example, it is not known if $\mathbb{L}$ is a zero-divisor). In this case, we shall denote by $[X]_q$ the point-counting function which associates to all prime powers q the integers $\#X(\mathbb{F}_q)$.

3.1. **Linear reductions.** The main observation of [24] is that the class in the Grothendieck ring of polynomials which are linear in many of their variables can be computed inductively by some simple reductions.

Lemma 16. *Let $f^1, f_1, g^1, g_1 \in k[\alpha_2, \ldots, \alpha_n]$ be polynomials. Then*
i). $[f^1\alpha_1 + f_1] = [f^1, f_1]\,\mathbb{L} + \mathbb{L}^{n-1} - [f^1]$
ii). $[f^1\alpha_1 + f_1, g^1\alpha_1 + g_1] = [f^1, f_1, g^1, g_1]\,\mathbb{L} + [f^1g_1 - f_1g^1] - [f^1, g^1]$

Various proofs of this lemma can be found in ([18], [5] §8, [24] lemma 2.3, or §3.4 of [1]). Note that the quantity $f^1g_1 - g^1f_1$ is nothing other than the resultant (9) with respect to α_1 of the polynomials $f^1\alpha_1 + f_1$ and $g^1\alpha_1 + g_1$.

Henceforth let G be connected. We call a graph simple if it has no vertices of valency ≤ 2 (below left), multiple edges (below middle), or self-loops (below right).



Lemma 17. *Let G be a graph with a subdivided edge e_1, e_2 (left). Then*

$$[\Psi_G] = \mathbb{L}[\Psi_{G/\!/e_1}]\ . \tag{10}$$

Let G be a graph with a doubled edge e_1, e_2 (middle). Then

$$[\Psi_G] = (\mathbb{L}-2)[\Psi_{G\backslash e_1}] + (\mathbb{L}-1)[\Psi_{G\backslash\{e_1,e_2\}}] + \mathbb{L}[\Psi_{G\backslash e_1/\!/e_2}] + \mathbb{L}^{N_G-2}\ . \tag{11}$$

Let G be a graph with a self-loop e_1 (right). Then

$$[\Psi_G] = (\mathbb{L}-1)[\Psi_{G\backslash e_1}] + \mathbb{L}^{N_G-1}\ . \tag{12}$$

Proof. These identities follow from the determination of the corresponding graph polynomials §2.4 (1), (2) and two applications of lemma 16 (see also [1], §4). □

An immediate consequence of the above lemma is that a smallest counter-example to Kontsevich's conjecture will be a simple graph. Stronger restrictions are derived in [24]. Iterating the operations (10) and (11) leads to explicit formulae for $[X_G]$ as a polynomial in $\mathbb{L}$ when G is a series-parallel graph [1].

Proposition-Definition 18. *Let G be a graph such that $h_G \leq N_G - 2$. Then there exists an element $c_2(G) \in K_0(\mathrm{Var}_k)/\mathbb{L}$ such that*

$$[\Psi_G] \equiv c_2(G)\mathbb{L}^2 \mod \mathbb{L}^3\ .$$

Proof. By induction we prove that for any $F \in k[\alpha_1, \ldots, \alpha_n]$ of degree $< n$ which is linear in every variable α_i, or any G satisfying $h_G \leq N_G - 2$ and any edge e of G, there exist $a, b, c \in K_0(\mathrm{Var}_k)$ such that

(1) $[F] \equiv a(F)\mathbb{L} \mod \mathbb{L}^2$
(2) $[\Psi_{G\backslash e}, \Psi_{G/\!\!/e}] \equiv b(G,e)\mathbb{L} \mod \mathbb{L}^2$
(3) $[\Psi_G] \equiv c(G)\mathbb{L}^2 \mod \mathbb{L}^3$.

Proof of (1). The case where $n \leq 2$ is obvious. By linearity, let $F = f^1\alpha_1 + f_1$. Lemma 16 *(i)* implies that $[F] \equiv [f^1, f_1]\mathbb{L} + \mathbb{L}^{n-1} - [f^1]$. Since f^1 satisfies the required condition on the degrees, we can define a inductively for $n > 2$ by:

$$a(F) = [f^1, f_1] - a(f^1) \ .$$

Proof of (2). From the contraction-deletion relations, we have $\Psi^1_G = \Psi^{12}_G\alpha_2 + \Psi^1_{G,2}$, and $\Psi_{G,1} = \Psi^2_{G,1}\alpha_2 + \Psi_{G,12}$. The first Dodgson identity gives

$$\Psi^1_{G,2}\Psi^2_{G,1} - \Psi^{12}_G\Psi_{G,12} = (\Psi^{1,2}_G)^2.$$

Inserting this into lemma 16 *(ii)* gives

$$[\Psi^1_G, \Psi_{G,1}] = [\Psi^{12}_G, \Psi^1_{G,2}, \Psi^2_{G,1}, \Psi_{G,12}]\,\mathbb{L} + [\Psi^{1,2}_G] - [\Psi^{12}_G, \Psi^2_{G,1}] \ .$$

We have $\deg \Psi^{1,2}_G = h_G - 1 < N_G - 2$ so (1) applies to $\Psi^{1,2}_G$. If G is not connected, define $b(G,1)$ to be 0. Otherwise, define b inductively by:

$$b(G,1) = a(\Psi^{1,2}_G) - b(G\backslash 2, 1) + [\Psi^{12}_G, \Psi^1_{G,2}, \Psi^2_{G,1}, \Psi_{G,12}] \ .$$

If $G\backslash 2$ is connected, Euler's formula shows that $G\backslash 2$ satisfies the required condition on the degree. The initial case when $h_G = 0$, i.e. G is a tree, is obvious.

Proof of (3). By contraction-deletion we write $\Psi_G = \Psi^1_G\alpha_1 + \Psi_{G,1}$. By lemma 16 *(i)*, $[\Psi_G] = [\Psi^1_G, \Psi_{G,1}]\mathbb{L} - [\Psi^1_G] + \mathbb{L}^{N_G-1}$, so for $N_G > 2$ we inductively define

$$c(G) = b(G,1) - c(G\backslash 1)$$

if G is connected, and set $c(G) = 0$ otherwise. The case $N_G = 2$ is obvious. □

Note that in the opposite case, if G is connected and satisfies $h_G > N_G - 2$, then G has at most two vertices and is essentially uninteresting.

Corollary 19. *Suppose that G has a 2-valent vertex and $h_G \leq N_G - 3$. Then $c_2(G) \equiv 0 \mod \mathbb{L}$.*

Proof. Using lemma 17, we can write $[\Psi_G] \equiv \mathbb{L}[\Psi_{G/\!\!/e}] \equiv 0 \mod \mathbb{L}^3$ since we have $h_{G/\!\!/e} = h_G \leq N_{G/\!\!/e} - 2$. □

Remark 20. *Below we give a formula for $c_2(G)$ when $2h_G \leq N_G$. The quantity $2h_G - N_G$ is connected to the physical 'superficial degree of divergence' in space-time dimension 4. Graphs with $2h_G < N_G$ are superficially convergent. The physically interesting case is superficial log-divergence $2h_G = N_G$. Primitive-divergent graphs belong to this class.*

3.2. **Three-valent vertices.** Our approach to studying $[\Psi_G]$ uses the existence of a vertex with low degree. Note that whenever

$$2h_G - 2 < N_G \tag{13}$$

the minimum vertex-degree is ≤ 3. To see this, note that Euler's formula for a connected graph implies that $N_G - V_G = h_G - 1$. If α denotes the average degree of the vertices of G, then $N_G = \frac{\alpha}{2}V_G$, and (13) implies that $\alpha < 4$.

The case of a two-valent vertex was dealt with in §2.4. The case of a three-valent vertex is more complicated but still implies that Ψ_G has a simple structure.

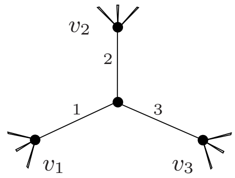


FIGURE 2. A three-valent vertex

Definition 21. Let v_1, v_2, v_3 be any three vertices in G which form a three-valent vertex as shown above. Following [8], we will use the notation:

$$f_0 = \Psi_{G\backslash\{1,2\}/\!\!/3} \ , \ f_1 = \Psi^{2,3}_{G,1} \ , \ f_2 = \Psi^{1,3}_{G,2} \ , f_3 = \Psi^{1,2}_{G,3} \ , \ f_{123} = \Psi_{G/\!\!/\{1,2,3\}} \ .$$

Lemma 22. *In this case, the graph polynomial of G has the following structure:*

$$\Psi_G = f_0(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) + (f_1 + f_2)\alpha_3 + (f_1 + f_3)\alpha_2 + (f_2 + f_3)\alpha_1 + f_{123}$$

where the polynomials f_i satisfy the equation

$$f_0 f_{123} = f_1 f_2 + f_1 f_3 + f_2 f_3 \ . \tag{14}$$

Proof. The general shape of the polynomial comes from the contraction-deletion relations, and §2.4 (3) (or ex. 32 in [8]). Equation (14) is merely a restatement of the first Dodgson identity for $G/\!\!/3$ which gives $(\Psi^{1,2}_{G,3})^2 = \Psi^1_{G,23}\Psi^2_{G,13} - \Psi^{12}_{G,3}\Psi_{G,123}$. Using the definitions of f_i this translates as

$$f_3^2 = (f_2 + f_3)(f_1 + f_3) - f_0 f_{123} \ . \tag{15}$$

□

Proposition 23. *Suppose that G contains a three-valent vertex, and let f_i be given by definition 21. Then*

$$[\Psi_G] = \mathbb{L}^{N_G - 1} + \mathbb{L}^3[f_0, f_1, f_2, f_3, f_{123}] - \mathbb{L}^2[f_0, f_1, f_2, f_3]$$

Proof. Let $\beta_i = f_0\alpha_i + f_i$, for $i = 1, 2, 3$. It follows from (14) that

$$f_0\Psi_G = \beta_1\beta_2 + \beta_1\beta_3 + \beta_2\beta_3 \ .$$

The right-hand side is the graph polynomial of the sunset graph (a triple edge). It defines a quadric in $\mathbb{A}^3$ whose class is $\mathbb{L}^2$. It follows that if U, and U' denote the open set $f_0 \neq 0$ in $\mathbb{A}^{N_G}$, and in $\mathbb{A}^{N_G-3}$ resp., we have $[X_G \cap U] = \mathbb{L}^2[U']$. On the complement $V(f_0)$, the graph polynomial Ψ_G reduces to the equation

$$(f_1 + f_2)\alpha_3 + (f_1 + f_3)\alpha_2 + (f_2 + f_3)\alpha_1 + f_{123}$$

which defines a family of hyperplanes in $\mathbb{A}^3$. Thus, consider the fiber of the projection $X_G \cap V(f_0) \to \mathbb{A}^{N_G-3} \cap V(f_0)$. In the generic case this is a hyperplane whose class is $\mathbb{L}^2$. Otherwise, f_1, f_2, f_3 vanish and there are two possibilities: if $f_{123} = 0$ the fiber is isomorphic to $\mathbb{A}^3$, otherwise it is empty. We have

$$[X_G \cap V(f_0)] = \mathbb{L}^3[f_0, f_1, f_2, f_3, f_{123}] + \mathbb{L}^2([f_0] - [f_0, f_1, f_2, f_3])$$

Writing $[X_G] = [X_G \cap U] + [X_G \cap V(f_0)]$ and $[U'] = \mathbb{L}^{N_G-3} - [f_0]$ gives the result. □

In particular, if G has a three-valent vertex and $N_G \geq 4$ then

$$c_2(G) \equiv -[f_0, f_1, f_2, f_3] \mod \mathbb{L} \ . \tag{16}$$

Lemma 24. *Let G satisfy $h_G + 3 \leq N_G$, where $N_G \geq 4$, and contain a three-valent vertex whose neighbouring edges are numbered* $1,2,3$. *Then*

$$c_2(G) \equiv [\Psi^{1,2}_{G,3}, \Psi^{13,23}_{G}] \mod \mathbb{L} \ . \tag{17}$$

Proof. We use the explicit expression for Ψ_G in lemma 22 and the relations in §2.4 (3). It follows from (14) and inclusion-exclusion that:

$$[f_0, f_3] = [f_0, f_1 f_2, f_3] = [f_0, f_1, f_3] + [f_0, f_2, f_3] - [f_0, f_1, f_2, f_3]$$

On the other hand, $[f_0, f_1 + f_3] = [f_0, f_1 + f_3, f_3^2]$ by equation (15), and so we have $[f_0, f_1 + f_3] = [f_0, f_1, f_3]$. By contraction-deletion, we can write

$$[f_0, f_1 + f_3] = [\Psi^{12}_{G,3}, \Psi^{2}_{G,13}] = [\Psi^{1}_{G'}, \Psi_{G',1}] \ ,$$

where $G' = G \backslash 2 /\!\!/ 3$. Either G' is not connected, or else $h_{G'} \leq N_{G'} - 2$ by the assumption on the loop number of G, and so the previous expression vanishes modulo $\mathbb{L}$ by statement (2) in the proof of proposition-definition 18. The same is true for $[f_0, f_1 + f_2]$ by symmetry. We have therefore shown that

$$-[f_0, f_1, f_2, f_3] \equiv [f_0, f_3] \mod \mathbb{L} \ .$$

The lemma follows from (16) and $\Psi^{1,2}_{G,3} = f_3$ and $\Psi^{13,23}_{G} = \Psi^{12}_{G,3} = f_0$. □

3.3. **Counting points over finite fields.** For any prime power q, let $\mathbb{F}_q$ denote the field with q elements. Given polynomials $P_1, \ldots, P_\ell \in \mathbb{Z}[\alpha_1, \ldots, \alpha_n]$, let

$$[P_1, \ldots, P_\ell]_q \in \mathbb{N} \cup \{0\}$$

denote the number of points on the affine variety $V(\overline{P_1}, \ldots, \overline{P_\ell}) \subset \mathbb{F}_q^n$, where $\overline{P}_i$ denotes the reduction of P_i modulo p (the characteristic of $\mathbb{F}_q$). Recall the Chevalley-Warning theorem (e.g., [22]) on the point-counts of polynomials of small degrees.

Theorem 25. *Let $P_1, \ldots, P_\ell \in \mathbb{Z}[\alpha_1, \ldots, \alpha_n]$ such that $\sum_{i=1}^{\ell} \deg P_i < n$. Then*

$$[P_1, \ldots, P_\ell]_q \equiv 0 \mod q \ .$$

It is natural to ask if the Chevalley-Warning theorem lifts to the Grothendieck ring of varieties. We were unable to find such a result in the literature.

Question 26. *For which fields k is the following statement true: Let $P_1, \ldots, P_\ell$ be polynomials satisfying the above condition on their degrees. Then $[V(P_1, \ldots, P_\ell)] \equiv 0 \mod \mathbb{L}$ in $K_0(Var_k)$?*

In an earlier version of this paper we cautiously conjectured this to be true for all C_1 fields (see the examples in [13]), which (as pointed out to us by a referee) would imply the result for all fields of finite characteristic. Lacking strong evidence for this, it is perhaps more prudent to assume k to be algebraically closed. In any case, since a geometric Chevalley-Warning theorem is unavailable, we henceforth work with point-counting functions rather than with elements in the Grothendieck ring of varieties. It turns out that for many of the results below, one can in fact circumvent this question by elementary arguments. Nevertheless, we now set

$$c_2(G)_q = [\Psi^{13,23}_{G}, \Psi^{1,2}_{G,3}]_q \mod q$$

viewed as a map from all prime powers q to $\mathbb{Z}/q\mathbb{Z}$, and where $1,2,3$ forms a three-valent vertex as above. Below we show that the formula remains valid for any set of three edges $1,2,3$. We have $[\Psi_G]_q \equiv c_2(G)_q q^2 \mod q^3$.

Lemma 27. *Suppose that $f = f^1\alpha_1 + f_1$ and $g = g^1\alpha_1 + g_1$ are polynomials in $\mathbb{Z}[\alpha_1, \dots, \alpha_n]$ such that $\deg f + \deg g \leq n$, which are linear in a variable α_1. Then*

$$[f, g]_q \equiv [f^1 g_1 - f_1 g^1]_q \mod q \ .$$

If the resultant has a non-trivial factorization $f^1 g_1 - f_1 g^1 = ab$, then

$$[f, g]_q \equiv -[a, b]_q \mod q \ .$$

Proof. By lemma 16 *(ii)*, $[f, g]_q = q[f^1, f_1, g^1, g_1]_q + [f^1 g_1 - f_1 g^1]_q - [f^1, g^1]_q$. Since $f^1, g^1 \in \mathbb{Z}[\alpha_2, \dots, \alpha_n]$ have total degree $\deg(f^1) + \deg(g^1) \leq n-2$, this is congruent to $[f^1 g_1 - f_1 g^1]_q$ mod q by theorem 25.

By inclusion-exclusion we have $[ab]_q = [a]_q + [b]_q - [a, b]_q$. The factorization is non-trivial if and only if $\deg a$ and $\deg b$ are strictly smaller than $\deg ab$. By theorem 25, this implies that $[a]_q$ and $[b]_q$ vanish mod q, giving the second statement. □

Corollary 28. *Let G be a connected graph such that $2h_G \leq N_G$, $N_G \geq 5$, and let $1, 2, 3$ be any distinct edges of G. Then*

$$[\Psi^{1,2}_{G,3}, \Psi^{13,23}_G]_q \equiv -[{}^5\Psi_G]_q \mod q \ ,$$

where the 5-invariant is taken with respect to any set of five edges of G. In particular, the point-counts of all 5-invariants are equivalent mod q.

Proof. First assume that the edges 1,2,3 are a subset of the edges in the 5-invariant ${}^5\Psi(1, 2, 3, 4, 5)$. We have $\deg(\Psi^{1,2}_{G,3}) = h_G - 1$ and $\deg \Psi^{13,23}_G = h_G - 2$ giving total degree $2h_G - 3$, whereas the ambient affine space has dimension $N_G - 3$. Applying the previous lemma to equation (17) gives

$$[\Psi^{1,2}_{G,3}, \Psi^{13,23}_G]_q \equiv -[\Psi^{13,24}_G, \Psi^{14,23}_G]_q \mod q$$

by the Dodgson identities. Applying the previous lemma one more time gives

$$-[\Psi^{13,24}_G, \Psi^{14,23}_G]_q \equiv -[{}^5\Psi_G(1, 2, 3, 4, 5)]_q \mod q$$

by definition of the five-invariant as a resultant. Since the edges 4 and 5 are arbitrary we see that the point-counts of 5-invariants are equivalent mod q whenever they have three edges in common. By considering chains of overlapping edge-sets the same is true for any 5-invariants. □

In particular, if G has a three-valent vertex, then $c_2(G)_q \equiv [\Psi^{13,23}_G, \Psi^{1,2}_{G,3}]_q \mod q$ for *any* three edges 1,2,3 of G which do not necessarily meet the three-valent vertex.

Theorem 29. *Let G be a connected graph with $2h_G \leq N_G$, $N_G \geq 5$. Suppose that $D^n_G(e_1, \dots, e_n)$ is the result of the denominator reduction after $n < N_G$ steps. Then*

$$c_2(G)_q \equiv (-1)^n [D^n_G(e_1, \dots, e_n)]_q \mod q \ . \tag{18}$$

If G has weight drop or $2h_G < N_G \geq 4$, then $c_2(G)_q \equiv 0 \mod q$.

Proof. Suppose first that G has a three-valent vertex. Equation (18) follows by induction from corollary 28 by applying the denominator reduction §2.6. There are two cases to consider: if the factorization in the denominator reduction is non-trivial and $n < N_G$ then the induction step follows from lemma 27. If the factorization in the denominator reduction is trivial (the denominator is of degree one in the reduction variable), then it follows from lemma 16 *i*).

If G does not have a three-valent vertex, then since $2h_G \leq N_G$, one of the following situations must hold (see the argument in §3.2): (i) G has a two-valent vertex, (ii) G has a one-valent vertex, or (iii) G has a self-loop connected to a single edge (forming a degenerate three-valent vertex).

In the case (i), we conclude from $2h_G \leq N_G$ and $N_G \geq 5$ that $h_G \leq N_G - 3$ and so $c_2(G)_q \equiv 0 \mod q$ by corollary 19. Likewise, the 5-invariant vanishes if it contains a two-valent vertex (see [8] lemma 92). The same argument holds trivially in the cases (ii) and (iii).

In case of a weight drop the right hand side of (18) vanishes, hence $c_2(G)_q \equiv 0 \mod q$. The 5-invariant is of degree $2h_G - 5$ in $\mathbb{A}^{N_G-5}$. If $2h_G < N_G \geq 5$ we have $c_2(G)_q \equiv 0 \mod q$ by theorem 25. If $2h_G < N_G = 4$ then $h_G \leq 1$. The hypersurface X_G is either empty or a hyperplane in $\mathbb{A}^4$, hence $c_2(G)_q \equiv 0 \mod q$. □

Notice that the proofs rely on the fact that the terms $D^n_G(e_1, \ldots, e_n)$ in the denominator reduction are of degree exactly equal to the dimension of the ambient space, and therefore lie on the limit of the Chevalley-Warning theorem (the Calabi-Yau condition for the associated projective varieties).

4. Properties of the c_2-invariant

We state some known and conjectural properties of the c_2-invariant of a graph. Throughout this section G is a graph with $2h_G \leq N_G$ and at least five edges.

4.1. **Triviality of $c_2(G)$.**

The following results follow from theorem 29.

Lemma 30. *If G as above has a doubled edge then $c_2(G)_q \equiv 0 \mod q$.*

Proof. If G has a doubled edge e_1, e_2, then any five-invariant ${}^5\Psi_G(i_1, \ldots, i_5)$ where $e_1, e_2 \in \{i_1, \ldots, i_5\}$ necessarily vanishes ([8] lemma 90). □

Recall that G is called 2-vertex reducible if there is a pair of distinct vertices such that removing them (and their incident edges) causes the graph to disconnect.

Proposition 31. *Let G as above be 2-vertex reducible. Then $c_2(G)_q \equiv 0 \mod q$.*

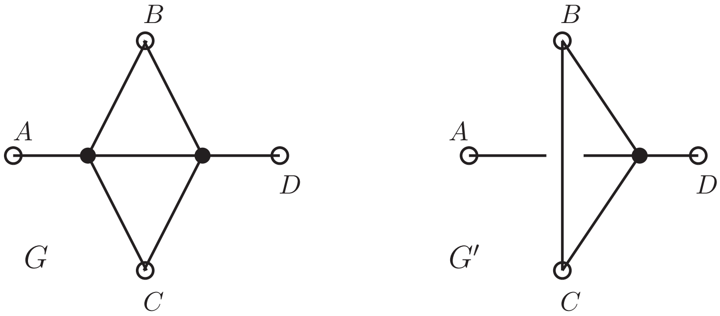
Proof. It is proved in [9], proposition 36, that such a graph has weight drop. □

Proposition 32. *If G is denominator reducible, and non-weight drop, then $c_2(G)_q \equiv (-1)^{N_G-1} \mod q$ except for finitely many primes p for which $c_2(G)_{p^n} \equiv 0 \mod p^n$.*

Proof. If G is non-weight drop denominator reducible then there exists a degree one homogeneous polynomial $D_G^{N_G-1}(e_1, \ldots, e_{N_G-1}) = c\alpha_{N_G}$ with $0 \neq c \in \mathbb{Z}$. For primes $p|c$ we have $[c\alpha_{N_G}]_{p^n} = p^n$, otherwise $[c\alpha_{N_G}]_q = 1$. The result follows from theorem 29. □

4.2. **Double triangle reduction.**

Consider a graph G which contains seven edges $e_1, \ldots, e_7$ arranged in the configuration shown below on the left (where anything may be attached to vertices A-D). The double triangle reduction of G is the graph G' obtained by replacing these seven edges with the configuration of five edges $e'_1, \ldots, e'_5$ as shown below on the right. The following theorem was proved in [9].



Theorem 33. *Let G' be a double triangle reduction of G. Then*

$$D^7_G(e_1,\ldots,e_7)=\pm D^5_{G'}(e'_1,\ldots,e'_5)\ .$$

Corollary 34. *Let G, G' be as above, with $2h_G \leq N_G$. Then*

$$c_2(G)_q \equiv c_2(G')_q \mod q\ .$$

Since the double-triangle reduction violates planarity, this is the first hint that the genus of a graph is *not* the right invariant for understanding its periods.

4.3. **The completion relation.** It follows from a simple application of Euler's formula that a primitive-divergent graph G in ϕ^4 with more than one loop has exactly four three-valent vertices $v_1,\ldots,v_4$, and all remaining vertices have valency 4. The completion of G is defined to be the graph $\widehat{G}$ obtained by adding a new vertex v to G and connecting it to $v_1,\ldots,v_4$ [17]. The resulting graph is 4-regular.

Conjecture 35. *Let G_1, G_2 be two primitive-divergent graphs in ϕ^4 and suppose that $\widehat{G_1}\cong\widehat{G_2}$. Then $c_2(G_1)\equiv c_2(G_2) \mod \mathbb{L}$.*

The motivation for this conjecture comes from the result [17] that the corresponding residues are the same: $I_{G_1}=I_{G_2}$. Once again, the completion relation does not respect the genus of a graph.

5. Mixed Tate families: Graphs of Vertex-width 3

When G contains sufficiently many triangles and three-valent vertices, we show that $[\Psi_G]\in K_0(\mathrm{Var}_k)$ is a polynomial in $\mathbb{L}$ which can be computed inductively.

5.1. **The vertex-width of a graph.** Throughout, G is a connected graph.

Definition 36. Let $\mathcal{O}$ be an ordering on the edges of G. It gives rise to a filtration

$$\emptyset = G_0\subset G_1\subset \ldots \subset G_{N-1}\subset G_N = G$$

of subgraphs of G, where G_i has exactly i edges. To any such filtration we obtain a sequence of integers $v_i^{\mathcal{O}}=$ number of vertices of $G_i\cap(G\backslash G_i)$. We say that G has *vertex-width* at most n if there exists an ordering $\mathcal{O}$ such that $v_i^{\mathcal{O}}\leq n$ for all i [8].

For example, a row of boxes with vertices $a_1,\ldots,a_n,b_1,\ldots,b_n$ and edges $\{a_i,b_i\}$, $\{a_i,a_{i+1}\}$, $\{b_i,b_{i+1}\}$, has vertex-width two. The wheels and zig-zag graphs (below) have vertex-width ≤ 3. Bounding the vertex-width is a strong constraint on a graph, and one can show that the set of planar graphs have arbitrarily high vertex-width. In [8] it was shown that the relative cohomology of the graph hypersurface for graphs of vertex width ≤ 3 is mixed Tate, and that the periods are multiple polylogarithms.

Here we explain how to compute the class of $[X_G]$ as a polynomial in $\mathbb{L}$ for such graphs. For this, it is not enough to consider only the classes $[\Psi_H] \in K_0(\mathrm{Var}_k)$, where H are minors of G, and we are forced to introduce a new invariant:

Definition 37. Let e_1, e_2, e_3 be any three edges in G which form a three-valent vertex. If $f_0, f_1, f_2, f_3, f_{123}$ are given by definition 21, we set

$$\langle G \rangle_{e_1,e_2,e_3} = [f_0, f_1, f_2, f_3, f_{123}] \tag{19}$$

in $\mathbb{A}^{N_G-3}$. Sometimes we shall write $\langle G \rangle_v$ if v is the 3-valent vertex meeting edges e_1, e_2, e_3.

We first consider recurrence relations for $[\Psi_G]$ (which also involve invariants $\langle H \rangle$ for minors H of G), and then recurrence relations for $\langle G \rangle$ (which also involve invariants $[\Psi_H]$ for minors H of G).

5.2. **Reduction of $[\Psi_G]$.** The two main cases are split triangles and split vertices.

5.2.1. *Split Vertices.* Let G be any graph containing a three-valent vertex (left), and let G' be the graph obtained by splitting that vertex in two (right). An empty (white) vertex indicates that there can be other edges connected to it which are not drawn on the diagram (anything can be attached to v_1, v_2, v_3).

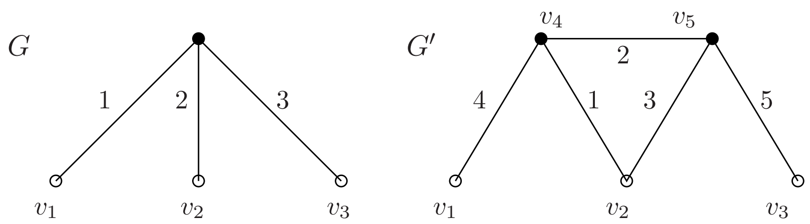


FIGURE 3

Theorem 38. *The class of the graph polynomial of G' can be written explicitly in terms of the invariant $\langle G \rangle_{1,2,3}$, and the classes of minors of G:*

$$[\Psi_{G'}]+(\mathbb{L}-\mathbb{L}^2)\big([\Psi_{G,2}]-[\Psi^{13}_{G,2}]\big)+(\mathbb{L}-1)[\Psi_G] = (\mathbb{L}^5-\mathbb{L}^4)\langle G \rangle_{1,2,3}+\mathbb{L}^{N_G-2}(\mathbb{L}^3+\mathbb{L}-1)$$

Proof. The structure of the graph polynomial of G' can be obtained as follows. Since v_4 is a three-valent vertex in G', it follows that $\Psi_{G'}$ must be of the shape given in lemma 22, for some polynomials $f'_0, f'_1, f'_2, f'_4, f'_{124}$ relative to the edges $1, 2, 4$. By contraction-deletion relations, one easily sees that

$$f'_0 = f_0(\alpha_3 + \alpha_5) + (f_2 + f_3) \quad , \quad f'_{124} = f_{123}\alpha_3 + (f_1 + f_2)\alpha_3\alpha_5$$

$$f'_1 = f_2\alpha_3 \quad , \quad f'_2 = f_3\alpha_3 + f_0\alpha_3\alpha_5 \quad , \quad f'_4 = f_{123} + f_1\alpha_3 + (f_1 + f_2)\alpha_5 \ ,$$

where $f_0, f_1, f_2, f_3, f_{123}$ satisfy (14) and are the invariants of the three-valent vertex formed by edges $1, 2, 3$ of G. By proposition 23 we know that $[\Psi_{G'}]$ is given by $\mathbb{L}^{N_{G'}-1} + \mathbb{L}^3[f'_0, f'_1, f'_2, f'_4, f'_{124}] - \mathbb{L}^2[f'_0, f'_1, f'_2, f'_4]$. The conclusion of the theorem follows by a brute force calculation by exploiting the inclusion-exclusion relations, identity (14), and reducing out the linear variables α_3, α_5 using lemma 16 *(ii)*. □

Any inductive procedure to compute the class of a split-vertex graph G' is blocked by the presence of an invariant $\langle G \rangle$. However, modulo $\mathbb{L}^4$ it drops out.

Corollary 39. *Suppose that $N_G \geq 6$. Then $c_2(G') \equiv c_2(G) \mod \mathbb{L}$. If $[\Psi_G]$ is of the form $[\Psi_G] \equiv c_3(G)\mathbb{L}^3 + c_2(G)\mathbb{L}^2 \mod \mathbb{L}^4$, then so is $[\Psi'_G]$ and we have:*

$$c_3(G') - c_3(G) \equiv c_2(G\backslash\{1,3\}/\!\!/2) - c_2(G/\!\!/2) - c_2(G) \mod \mathbb{L}$$

Proof. This follows from theorem 38 and proposition-definition 18. □

Iterating this corollary leads, for example, to an inductive way to compute the coefficient c_3 of $\mathbb{L}^3$ for certain classes of graphs which are polynomials in $\mathbb{L}$.

5.2.2. *Split triangles.* Let G' be a graph of the shape depicted below (right), and let G denote the subgraph obtained by deleting edges 4 and 5.

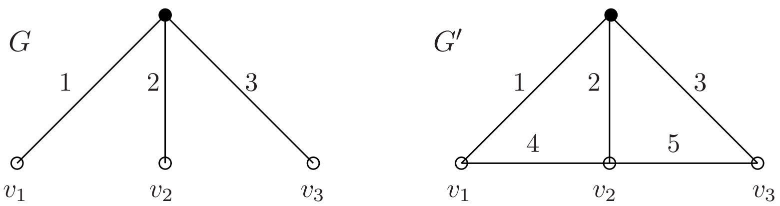


FIGURE 4

Theorem 40. *Let G', G be as above, and let $H = G'\backslash\{1,3\}/\!\!/2$, and $\triangle = G'\backslash 2/\!\!/3$. The following equation relates $[\Psi_{G'}]$ to the classes of minors of G', and $\langle G\rangle_{1,2,3}$:*

$$[\Psi_{G'}] + [\Psi^4_{G'}] + [\Psi^5_{G'}] + [\Psi^{45}_{G'}] + \mathbb{L}\big([\Psi_H] + [\Psi^4_H] + [\Psi^5_H] + [\Psi^{45}_H] + [\Psi^4_{\triangle,1}] + [\Psi^{45}_{\triangle,1}]\big)$$
$$-(\mathbb{L}^3 - \mathbb{L}^2)\big([\Psi_{H,4}] + [\Psi_{H,5}] + [\Psi^4_{H,5}] + [\Psi^5_{H,4}] + [\Psi_{H,45}]\big) - \sum_{T\subseteq\{1,4,5\}} [\Psi^T_\triangle]$$
$$= (\mathbb{L}^5 - \mathbb{L}^4)\langle G\rangle_{1,2,3} + (\mathbb{L}^4 + 3\,\mathbb{L}^2 - \mathbb{L} - 1)\,\mathbb{L}^{N_H-2}$$

where the sum is over all 8 subgraphs T of $\triangle$ obtained by deleting the edges $1,4,5$.

Proof. We omit the proof, which is similar to the proof of theorem 38. □

5.3. **Recurrence relations for $\langle G\rangle$.** It turns out that $\langle G\rangle$ satisfies recurrence relations with respect to a set of four edges. Consider the three graphs below:

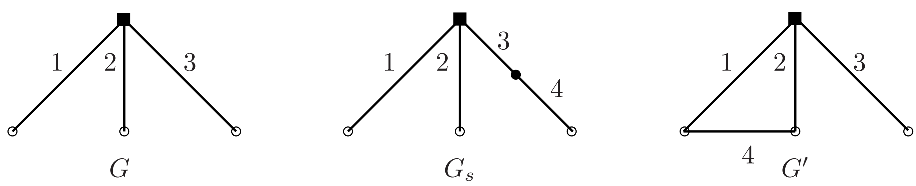


FIGURE 5

White vertices may have extra edges which are not shown, and the invariant $\langle G\rangle$ is taken with respect to a 3-valent vertex marked by a black square.

Lemma 41. *Let G_s be obtained from G by splitting the edge 3 as shown above (middle). Then $\langle G_s\rangle_{1,2,3} = \mathbb{L}\langle G\rangle_{1,2,3}$.*

Proof. Let $f'_0, f'_1, f'_2, f'_3, f'_{123}$ denote the polynomials in G_s with respect to the marked 3-valent vertex. By §2.4 (1) we have $f'_0 = f_0$, $f'_1 = f_1$, $f'_2 = f_2$, $f'_3 = f_3 + f_0\alpha_4$, $f'_{123} = f_{123} + (f_1 + f_2)\alpha_4$, where $f_0, f_1, f_2, f_3, f_{123}$ are the corresponding polynomials for G. It follows immediately from the definitions that

$$\langle G_s\rangle_{1,2,3} = [f_0, f_1, f_2, f_3+f_0\alpha_4, f_{123}+(f_1+f_2)\alpha_4] = \mathbb{L}[f_0, f_1, f_2, f_3, f_{123}] = \mathbb{L}\langle G\rangle_{1,2,3}.$$

□

Remark 42. *It can happen that two or more of the edges* $1, 2, 3$ *of* G *have two common endpoints (i.e., two or more of the white vertices can coincide). In this degenerate case the invariant* $\langle G\rangle$ *is easily expressed in terms of graph polynomials* $[\Psi_H]$*, where* H *is a strict minor of* G*, by §2.4 (2).*

Lemma 43. *Let* G' *be as indicated above, with edges* $1, 2, 3$ *forming a three-valent vertex and edges* $1, 2, 4$ *forming a triangle, and let* $G = G'\backslash\{4\}$*. Then*

$$\mathbb{L}\langle G'\rangle_{1,2,3} = (\mathbb{L}^2 - \mathbb{L})\langle G\rangle_{1,2,3} + [\Psi_{G,12}] + [\Psi^3_{G,12}] - \mathbb{L}^{N_G-2} \ .$$

Proof. Since G' has a three-valent vertex, $\Psi_{G'}$ has the general shape given by lemma 22 with coefficients $f'_0, f'_1, f'_2, f'_3, f'_{123}$ where, by contraction-deletion:

$$f'_0 = f_1+f_2+f_0\alpha_4\ , \quad f'_1 = f_1\alpha_4\ , \quad f'_2 = f_2\alpha_4\ , \quad f'_3 = f_{123}+f_3\alpha_4\ , \quad f'_{123} = f_{123}\alpha_4\ ,$$

and $f_0, f_1, f_2, f_3, f_{123}$ are the corresponding structure constants for G. On considering the two cases $\alpha_4 = 0$ and $\alpha_4 \neq 0$ we find

$$[f'_0, f'_1, f'_2, f'_3, f'_{123}] = (\mathbb{L} - 1)[f_0, f_1, f_2, f_3, f_{123}] + [f_1 + f_2, f_{123}]\ .$$

By definition 21, $[f_1 + f_2, f_{123}] = [\Psi^3_H, \Psi_{H,3}]$ where $H = G/\!\!/\{1, 2\}$. One concludes by applying lemma 16 *(i)*. □

Corollary 44. *Let* G' *contain a split vertex as depicted in figure 3. Then*

$$\mathbb{L}\langle G'\rangle_{1,2,4} = (\mathbb{L}^3 - \mathbb{L}^2)\langle G\rangle_{1,2,3} + [\Psi_{G,2}] + [\Psi^1_{G,2}] - \mathbb{L}^{N_G-2}\ .$$

Proof. This follows from applying lemma 43 and then lemma 41 to figure 3. □

5.4. **Graphs of vertex width ≤ 3.** The notion of vertex width is minor monotone.

Lemma 45. *Let* G *be a connected graph of vertex width* $\leq n$*, and let* H *be any connected minor of* G*. Then the vertex width of* H *is* $\leq n$*.*

Proof. Any ordering on the edges of G defines a (strict) filtration G_i of subgraphs of G. This induces a filtration H_i of subgraphs of H (which is not necessarily strict any more). Clearly $|\text{vertices}(H_i \cap (H\backslash H_i))| \leq |\text{vertices}(G_i \cap (G\backslash G_i))|$. □

We give a constructive proof of the following theorem (compare [8], §7.5)

Theorem 46. *If* G *has vertex-width at most 3, then* $[\Psi_G]$ *is a polynomial in* $\mathbb{L}$*.*

Proof. A graph of vertex width ≤ 3 comes with a filtration $G_i \subset G$ such that $G_i \cap (G\backslash G_i)$ has at most 3 vertices for all i. For every minor H of G there is an induced ordering on its edges. We say that H has an initial 3-valent vertex v, if v is 3-valent and formed by the first three edges in H.

We show that:

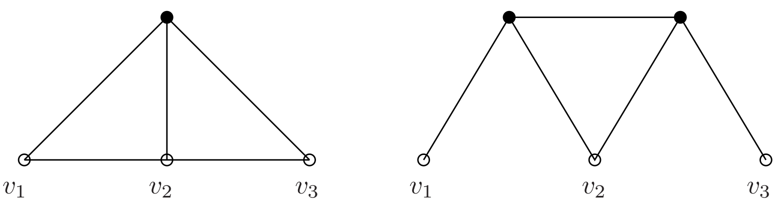
1. If G has an initial 3-valent vertex v, then $\langle G\rangle_v$ is a linear combination of $[\Psi_H]$ and $\langle H\rangle_{v'}$ with coefficients in $\mathbb{Z}[\mathbb{L}]$, where H are strict minors of G, and v' is an initial 3-valent vertex in H.

(2) $[\Psi_G]$ is a linear combination of $[\Psi_H]$ and $\langle H\rangle_{v'}$ with coefficients in $\mathbb{Z}[\mathbb{L}]$, where H are strict minors of G, and v' is an initial 3-valent vertex of H.

These two facts, together with the fact that the vertex-width is minor monotone, are enough to prove the theorem. Note that if H is not connected, then both $[\Psi_H]$ and $\langle H\rangle_{v'}$ vanish.

First we show (1). Consider the subgraph $G_4 \subset G$ defined by its first four edges, and let v be an initial 3-valent vertex in G_4. Suppose that $G_4 \cap (G\backslash G_4)$ consists of exactly 3 distinct vertices (the degenerate cases where there are ≤ 2 vertices are trivial by lemma 17 and left to the reader). Drawing the three vertices in white, we find ourselves in the two cases denoted G_s and G' of figure 5 (up to renumbering of the edges). Statement (1) follows from lemmas 41 and 43.

Now we prove (2). Consider the subgraph $G_5 \subset G$ defined by the first five edges of G. Assume that $G_5 \cap (G\backslash G_5)$ consists of three distinct vertices v_1, v_2, v_3, since the degenerate cases where there are fewer than three vertices are again trivial by lemma 17. If G_5 is not simple, then $[\Psi_G]$ trivially reduces to a linear combination of classes $[\Psi_H]$ where H are strict minors of G, with coefficients in $\mathbb{Z}[\mathbb{L}]$. This follows from lemma 17. If G_5 is a simple graph, then up to renumbering of the edges, there are only two cases, shown below:



The left-hand figure is a split triangle and is covered by theorem 40; the right-hand figure is a split vertex and is covered by theorem 38. Statement (2) holds in both cases, which completes the proof. □

5.5. **Example 1: wheels with n spokes.** We use the previous results to compute the classes $[W_n]$ for all n, where W_n denotes the wheel with n spokes graph pictured below (left). Let B_n denote the family of graphs obtained by contracting a spoke of W_n, which have exactly n vertices on the outer circle (right).

The graphs B_n are series-parallel reducible, so the classes $[B_n]$ can be computed using lemma 17. This also follows from the results of [1], theorem 5.10.

Lemma 47. *Let us set $b_0 = 0$, $b_1 = 1$, and $b_n = [B_n]$ for $n \geq 2$. If $B(t) = \sum_{n\geq 0} b_n t^n$ is the generating series for the family of graphs B_n, then we have*

$$B(t) = \frac{(1 + \frac{\mathbb{L}t}{1-\mathbb{L}^2 t})t}{1 - (\mathbb{L}-1)(1+\mathbb{L}t)\mathbb{L}t}\ . \tag{20}$$

Proof. We refer to the two edges e_1, e_2 indicated on the diagram above. Since e_1, e_2 form a doubled edge, we have by (11):

$$[B_n] = (\mathbb{L}-2)[B_n\backslash e_1] + (\mathbb{L}-1)[B_n\backslash\{e_1, e_2\}] + \mathbb{L}[B_n\backslash e_1 /\!\!/ e_2] + \mathbb{L}^{2n-3}$$

since B_n has $2n-1$ edges. Now $B_n\backslash e_1$ is isomorphic to the graph obtained from B_{n-1} by subdividing an outer edge, so $[B_n\backslash e_1] = \mathbb{L}[B_{n-1}]$ by (10). The graph $B_n\backslash\{e_1, e_2\}$ has an external leg, which provides a factor of $\mathbb{L}$, leaving, as before,

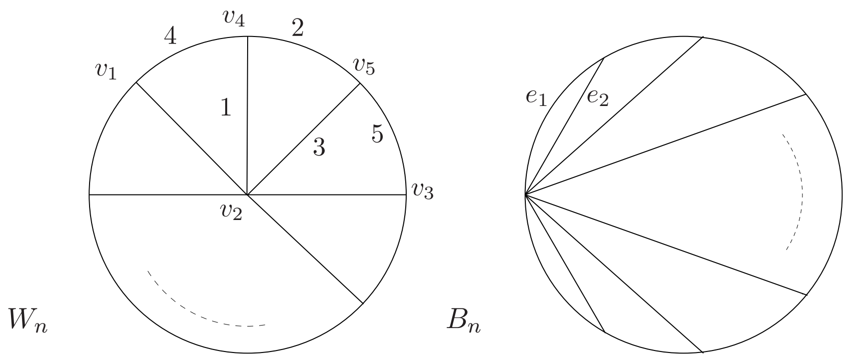


FIGURE 6. The wheels with spokes graphs W_n, and a related family B_n of series-parallel graphs.

a copy of B_{n-2} with a subdivided outer edge. Thus $[B_n \backslash \{e_1, e_2\}] = \mathbb{L}^2[B_{n-2}]$. Finally, we have $B_n \backslash e_1 /\!\!/ e_2 \cong B_{n-1}$, so we obtain

$$[B_n] = \mathbb{L}(\mathbb{L}-2)[B_{n-1}] + \mathbb{L}^2(\mathbb{L}-1)[B_{n-2}] + \mathbb{L}[B_{n-1}] + \mathbb{L}^{2n-3} \ .$$

We deduce that for all $n \geq 4$ we have:

$$b_n = \mathbb{L}(\mathbb{L}-1)b_{n-1} + \mathbb{L}^2(\mathbb{L}-1)b_{n-2} + \mathbb{L}^{2n-3} \ . \tag{21}$$

The constants b_0, b_1 are chosen such that the equation is valid for $n = 2, 3$, where $b_2 = \mathbb{L}^2$ and $b_3 = \mathbb{L}^2(\mathbb{L}^2 + \mathbb{L} - 1)$ by direct computation. The formula for the generating series then follows immediately from the recurrence relation (21). □

One has $b_2 = \mathbb{L}^2$, and

$$b_3 = \mathbb{L}^2(\mathbb{L}^2 + \mathbb{L} - 1) \quad , \quad b_4 = \mathbb{L}^3(\mathbb{L}^3 + 2\mathbb{L}^2 - 3\mathbb{L} + 1) \ ,$$
$$b_5 = \mathbb{L}^5(\mathbb{L}^3 + 3\mathbb{L}^2 - 5\mathbb{L} + 2) \quad , \quad b_6 = \mathbb{L}^5(\mathbb{L}^5 + 4\mathbb{L}^4 - 7\mathbb{L}^3 + 2\mathbb{L}^2 + 2\mathbb{L} - 1) \ .$$

Let v_1, v_2, v_5 denote any three vertices on W_n, joined by a three-valent vertex (v_4) as shown in the diagram above. Let us write $\langle W_n \rangle = \langle W_n \rangle_{v_4}$.

Lemma 48. *Let $\widehat{w}_n = 0$ for $n \leq 2$ and set $\widehat{w}_n = \langle W_n \rangle$ for $n \geq 3$. Denote the corresponding ordinary generating series by $\widehat{W}(t) = \sum_{n \geq 0} \widehat{w}_n t^n$. Then*

$$\widehat{W}(t) = \frac{(1 + \mathbb{L}t)t\, B(t) - \frac{t^2}{1-\mathbb{L}^2 t}}{\mathbb{L} - (\mathbb{L}-1)\mathbb{L}^2 t} \ . \tag{22}$$

Proof. We use corollary 44, applied to the graphs $G' = W_n$ with the edge and vertex labels on G' as shown above. Then $G \cong G' \backslash 1 /\!\!/ 2 \cong W_{n-1}$. We have

$$\mathbb{L}\, \widehat{w}_n = (\mathbb{L}^3 - \mathbb{L}^2)\, \widehat{w}_{n-1} + [\Psi^2_{G',45}] + [\Psi^{12}_{G',45}] - \mathbb{L}^{2n-4}$$

since W_n has $2n$ edges. Now $G' \backslash 2 /\!\!/ \{4, 5\}$ is isomorphic to B_{n-1} and $G' \backslash \{1, 2\} /\!\!/ \{4, 5\}$ gives the graph obtained from B_{n-2} by subdividing one outer edge. Therefore $[\Psi^{12}_{G',45}] = \mathbb{L}[B_{n-2}]$ by lemma 17. We deduce that for all $n \geq 4$,

$$\mathbb{L}\, \widehat{w}_n = (\mathbb{L}^3 - \mathbb{L}^2)\, \widehat{w}_{n-1} + b_{n-1} + \mathbb{L}b_{n-2} - \mathbb{L}^{2n-4} \ . \tag{23}$$

Using the fact that $\widehat{w}_3 = 1$ determines $\widehat{w}_n$ for $n = 0, 1, 2$. The formula for the generating series $\widehat{W}(t)$ then follows immediately from (23). □

Proposition 49. *Let $w_1 = \mathbb{L}$, $w_2 = \mathbb{L}^3$, and $w_n = [W_n]$ for $n \geq 3$. Let $W(t) = \sum_{n\geq 0} w_n t^n$ be the generating function for the wheels with spokes graphs. Then*

$$W(t) = \frac{(\mathbb{L}^4-\mathbb{L}^3)\widehat{W}(t) + (\mathbb{L}-1)(1-\mathbb{L}^2t^2)B(t) + \frac{(1-\mathbb{L}t^2+\mathbb{L}^2t^2)}{(1-\mathbb{L}^2t)}\mathbb{L}t}{1+(\mathbb{L}-1)t} \tag{24}$$

where $B(t)$, $\widehat{W}(t)$ are defined above.

Proof. The graphs W_1 and W_2 are series-parallel and therefore w_1 and w_2 are given by lemma 17. We apply theorem 38 to the graph $G' = W_n$ with the labelling on its edges depicted above. Since $G \cong W_{n-1}$, we deduce for all $n \geq 3$ that

$$[W_n]+(\mathbb{L}-\mathbb{L}^2)\big([\Psi_{G,2}]-[\Psi^{13}_{G,2}]\big)+(\mathbb{L}-1)[W_{n-1}] = (\mathbb{L}^5-\mathbb{L}^4)\langle W_{n-1}\rangle+\mathbb{L}^{2n-4}(\mathbb{L}^3+\mathbb{L}-1)$$

As before, $G/\!\!/\{2\} \cong B_{n-1}$, and $G\backslash\{1,3\}/\!\!/\{2\}$ is isomorphic to the graph obtained from B_{n-3} by subdividing two outer edges. It follows that $[G\backslash\{1,3\}/\!\!/\{2\}] = \mathbb{L}^2[B_{n-3}]$, giving

$$w_n + (\mathbb{L}-1)w_{n-1} + (\mathbb{L}-\mathbb{L}^2)\big(b_{n-1}-\mathbb{L}^2 b_{n-3}\big) = (\mathbb{L}^5-\mathbb{L}^4)\widehat{w}_{n-1} + \mathbb{L}^{2n-4}(\mathbb{L}^3+\mathbb{L}-1)$$

The formula for the generating function follows from this. □

Corollary 50. *Let $c_i(W_n)$ denote the coefficient of $\mathbb{L}^i$ in $[W_n]$. Then $c_2(W_n) = -1$, $c_{2n-1}(W_n) = 1$ and $c_{2n-2}(W_n) = 0$ for all n. The outermost non-trivial coefficients are $c_3(W_3) = 1$, and $c_3(W_n) = n$ for all $n \geq 4$, and $c_{2n-3}(W_n) = \binom{n}{2}$ for all $n \geq 3$.*

The following curious identity follows from the explicit description of $W(t)$:

$$[W_n] - [W_n\backslash O] - [W_n/\!\!/O] + [W_n/\!\!/I] = -\mathbb{L}^2(\mathbb{L}-1)^{n-2} ,$$

where O denotes any outer edge of W_n (on the rim of the wheel), and I denotes any internal edge or spoke. The combinatorial reason for this is not clear.

Remark 51. *The polynomials w_n should have equivariant versions with respect to the symmetry group of W_n. Computing these explicitly would be relevant to computing the full cohomology of the graph hypersurface complement of W_n and, what one ultimately wants: the action of the motivic Galois group.*

The first few values of the polynomials w_n are as follows:

$$\begin{aligned}
w_3 &= \mathbb{L}^2(\mathbb{L}^3+\mathbb{L}-1)\\
w_4 &= \mathbb{L}^2(\mathbb{L}^5+3\mathbb{L}^3-6\mathbb{L}^2+4\mathbb{L}-1)\\
w_5 &= \mathbb{L}^2(\mathbb{L}^7+6\mathbb{L}^5-15\mathbb{L}^4+16\mathbb{L}^3-11\mathbb{L}^2+5\mathbb{L}-1)\\
w_6 &= \mathbb{L}^2(\mathbb{L}^9+10\mathbb{L}^7-29\mathbb{L}^6+37\mathbb{L}^5-33\mathbb{L}^4+26\mathbb{L}^3-16\mathbb{L}^2+6\mathbb{L}-1)\\
w_7 &= \mathbb{L}^2(\mathbb{L}^{11}+15\mathbb{L}^9-49\mathbb{L}^8+71\mathbb{L}^7-70\mathbb{L}^6+64\mathbb{L}^5-57\mathbb{L}^4+42\mathbb{L}^3-22\mathbb{L}^2+7\mathbb{L}-1)
\end{aligned}$$

Note that the wheels W_n are the unique infinite family of graphs whose residue can be calculated, namely: $I_{W_n} = \binom{2n-1}{n-1}\zeta(2n-3)$ for $n \geq 3$. One of the main results of [5] is that

$$H^{2n-1}_c(\mathbb{P}^{2n-1}\backslash X_{W_n}) \cong \mathbb{Q}(-2) \tag{25}$$

and that $H^{2n-1}(\mathbb{P}^{2n-1}\backslash X_{W_n})$ is generated by the class of the integrand of I_G. It would be interesting to relate their proof to the above computation which gives $c_2(W_n) = -1$.

5.6. **Example 2: Zig-zag graphs.** The second application of the previous results is to compute the classes $[Z_n]$ for all n, where Z_n denotes the family of zig-zag graphs with n loops pictured below (left). Let $\overline{Z}_n$ denote the family of graphs obtained by doubling the edge '2' as shown on the right. Note that $Z_3 = W_3$.

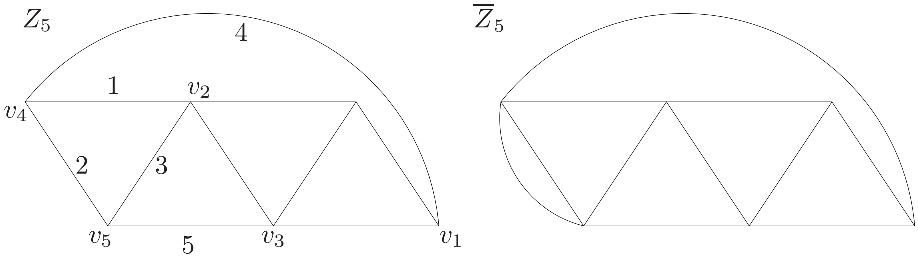


FIGURE 7. The zig-zag graphs Z_n.

The graphs Z_n are primitive-divergent graphs in ϕ^4 theory for all $n \geq 3$. Let $z_0 = 0$, $z_1 = \mathbb{L} + 1$, $z_2 = \mathbb{L}^3$, and $z_n = [Z_n]$ for all $n \geq 3$. Likewise, set $\overline{z}_0 = 1$, $\overline{z}_1 = \mathbb{L}^2$, $\overline{z}_2 = \mathbb{L}^4 + \mathbb{L}^3 - \mathbb{L}^2$, and $\overline{z}_n = [\overline{Z}_n]$ for all $n \geq 3$. Denote the corresponding generating series by $Z(t)$ and $\overline{Z}(t)$.

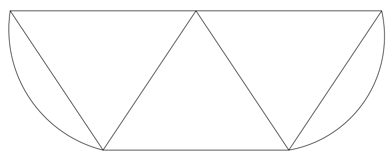
Lemma 52. *A straightforward application of the series-parallel operations gives*

$$\overline{z}_n = (\mathbb{L} - 2) z_n + (\mathbb{L} - 1)\mathbb{L}^2 \overline{z}_{n-2} + \mathbb{L}\overline{z}_{n-1} + \mathbb{L}^{2n-1} \quad n \geq 1 \tag{26}$$

Proof. If e_1, e_2 denote the two doubled edges, then this follows from the parallel reduction (11) on noting that $\overline{Z}_n \backslash e_1 \cong Z_n$, $\overline{Z}_n \backslash e_1 /\!\!/ e_2 \cong \overline{Z}_{n-1}$, and that $\overline{Z}_n \backslash \{e_1, e_2\}$ is isomorphic to the graph obtained from Z_{n-2} by subdividing two edges, whose class is $\mathbb{L}^2 z_{n-2}$ by two applications of (10). □

We next want to compute recursion relations for the numbers z_n by considering the split vertex shown above in the figure (left). Let us set $\widehat{z}_n = 0$ for $n < 3$, $\widehat{z}_n = \langle Z_n \rangle_{v_4}$ for all $n \geq 3$, and let $\widehat{Z}(t)$ be the corresponding generating series. Let ZB_n denote the family of graphs depicted below with n vertices. A trivial argument along the lines of lemma 47 shows that $[ZB_n] = b_n$, with generating series B.

ZB_5



Lemma 53. *The recurrence relation given in theorem* 38 *translates as:*

$$z_n + (\mathbb{L} - \mathbb{L}^2)(\overline{z}_{n-2} - \mathbb{L}^2 b_{n-3}) + (\mathbb{L} - 1) z_{n-1} = (\mathbb{L}^5 - \mathbb{L}^4)\widehat{z}_{n-1} + \mathbb{L}^{2n-4}(\mathbb{L}^3 + \mathbb{L} - 1) \tag{27}$$

for $n \geq 2$. The recurrence relation of corollary 44 *yields the relation*

$$\mathbb{L}\widehat{z}_n = (\mathbb{L}^3 - \mathbb{L}^2)\widehat{z}_{n-1} + \overline{z}_{n-2} + \mathbb{L} b_{n-2} - \mathbb{L}^{2n-4}, \qquad n \geq 2 \tag{28}$$

Proof. Let $G' = Z_n$, and apply theorem 38 to G' with the edge numbering shown above. Then $G \cong Z_{n-1}$, $G'\backslash 2/\!\!/\{4,5\} \cong \overline{Z}_{n-2}$, and $G\backslash\{1,3\}/\!\!/2$ is isomorphic to the graph obtained from ZB_{n-3} by subdividing two edges. It follows from (10) that $[\Psi_{G\backslash\{1,3\}/\!\!/2}] = \mathbb{L}^2 b_{n-3}$, which yields the first equation. The second equation follows from corollary 44, since $G'\backslash\{1,2\}/\!\!/\{4,5\}$ is isomorphic to the graph obtained from ZB_{n-2} by subdividing one edge, whose polynomial is $\mathbb{L}b_{n-2}$ by (10). □

Equations (26), (27), (28) imply the following identities of generating series:

$$[1-\mathbb{L}t+\mathbb{L}^2(1-\mathbb{L})t^2]\,\overline{Z}-(\mathbb{L}-2)Z-\mathbb{L}\,R = 1+(2-\mathbb{L})t$$

$$[\mathbb{L}^3-(\mathbb{L}^5-\mathbb{L}^4)t]\,\widehat{Z}-\mathbb{L}^2t^2\big(\overline{Z}+\mathbb{L}B\big)+R = t$$

$$[1+(\mathbb{L}-1)t]Z-(\mathbb{L}^5-\mathbb{L}^4)t\widehat{Z}+(\mathbb{L}-\mathbb{L}^2)t^2(\overline{Z}-\mathbb{L}^2tB)-(\mathbb{L}^3+\mathbb{L}-1)tR = (\mathbb{L}+1)t$$

in three unknowns, Z, $\overline{Z}$, and $\widehat{Z}$, where $R = R(t) = \frac{t}{(1-\mathbb{L}^2t)}$. These equations are easily solved using the expression for B (20). In particular, we obtain an explicit formula for the generating series for the zig-zag graphs:

$$\frac{1}{1-\mathbb{L}^2t} - \frac{\mathbb{M}^2\mathbb{L}^2t^3P(\mathbb{L},\mathbb{L}t)}{\big(1-\mathbb{M}\mathbb{L}t-\mathbb{M}\mathbb{L}^2t^2\big)^2\big(1-t-\mathbb{M}^2\mathbb{L}t^2\big)} \tag{29}$$

where $\mathbb{M} = \mathbb{L}-1$ and the polynomial P is defined by:

$$\begin{aligned} P(x,y) &= x(x-1)^3y^5+(x-1)(2x^3-3x^2+x+1)y^4 \\ &+ (x-1)(x^3-3x^2+2x+1)y^3-(3x^3-3x^2+2)y^2 \\ &- (x^3-x^2+1)y+x^2+x+1 \end{aligned}$$

The coefficient of t^n in (29) is z_n for $n \geq 3$ (see below). This is to our knowledge the only explicit formula for the class in the Grothendieck ring of a family of primitive-divergent graphs in ϕ^4. From this formula one obtains:

Corollary 54. *Let $c_i(Z_n)$ denote the coefficient of $\mathbb{L}^i$ in $[Z_n]$. Then $c_2(Z_n) = -1$, $c_{2n-1}(Z_n) = 1$ and $c_{2n-2}(Z_n) = 0$ for all n. The outermost non-trivial terms are $c_3(Z_3) = 1$, and $c_3(Z_n) = 8-n$ for all $n \geq 4$, and $c_{2n-3}(W_n) = 2n-5$ for $n \geq 3$.*

In the case of the zig-zags, the analogous result to (25) was proved for $n \geq 5$ by Doryn in his thesis [11]. It states that $\mathrm{gr}^W_{min}H^{2n-1}_c(\mathbb{P}^{2n-1}\backslash X_{Z_n}) \cong \mathbb{Q}(-2)$, which should again be related to our proof that $c_2(Z_n) = -1$.

For small n, we have:

$$\begin{aligned} z_3 &= \mathbb{L}^2(\mathbb{L}^3+\mathbb{L}-1) \\ z_4 &= \mathbb{L}^2(\mathbb{L}^5+3\mathbb{L}^3-6\mathbb{L}^2+4\mathbb{L}-1) \\ z_5 &= \mathbb{L}^2(\mathbb{L}^7+5\mathbb{L}^5-10\mathbb{L}^4+7\mathbb{L}^3-4\mathbb{L}^2+3\mathbb{L}-1) \\ z_6 &= \mathbb{L}^2(\mathbb{L}^9+7\mathbb{L}^7-12\mathbb{L}^6-2\mathbb{L}^5+16\mathbb{L}^4-12\mathbb{L}^3+2\mathbb{L}^2+2\mathbb{L}-1) \\ z_7 &= \mathbb{L}^2(\mathbb{L}^{11}+9\mathbb{L}^9-13\mathbb{L}^8-18\mathbb{L}^7+55\mathbb{L}^6-58\mathbb{L}^5+41\mathbb{L}^4-23\mathbb{L}^3+7\mathbb{L}^2+\mathbb{L}-1) \end{aligned}$$

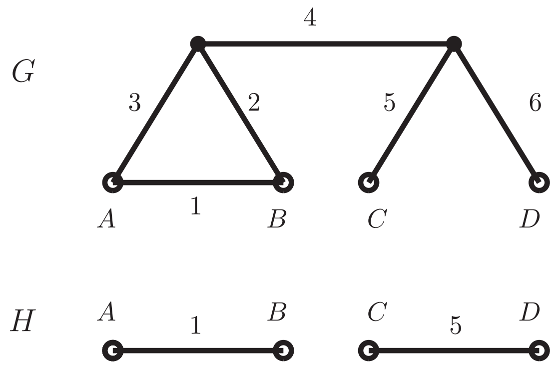
Note that explicit results for the zig-zag periods were conjectured in [3]. Remarkably they are a rational multiple of $\zeta(2n-3)$.

6. Non-Tate counter-examples at 8 loops

We use the denominator-reduction method to derive some non-Tate counter-examples to Kontsevich's conjecture at 8 and 9 loops.

6.1. Combinatorial reductions. In order to compute the c_2-invariant of an 8-loop graph G, we proceed in two simpler steps. The following lemmas will be applied to the main counterexample, depicted in figure 8 below.

Suppose that G is any connected graph with the shape depicted below, where the white vertices A, B, C, D may have anything attached to them. Let H be the minor obtained from G by deleting the edges 2 and 4, and contracting 3 and 6.



Lemma 55. *Let G, H be as above. Then* $D^6_G(1,2,3,4,5,6) = \pm\Psi^{1,5}_H \Psi^5_{H,1}$.

Proof. The proof is by direct computation of resultants, using the identities between Dodgson polynomials which follow from the existence of local stars and triangles. Since the edges $\{1,2,3\}$ form a triangle, we know from example 13 that

$$^5\Psi_G(1,2,3,4,5) = \pm\Psi^{123,345}_G \Psi^{4,5}_{G\backslash 2/\!\!/\{1,3\}} \ .$$

Since $\{2,3,4\}$ forms a three-valent vertex, we have $\Psi^{123,345} = \Psi^{1,5}_{G\backslash\{2,4\}/\!\!/3}$ by the last equation in §2.4, (3). By contraction-deletion, this last term is also $\Psi^{14,45}_{G\backslash 2/\!\!/3}$, giving

$$^5\Psi_G(1,2,3,4,5) = \pm\Psi^{14,45}_{G_2} \Psi^{4,5}_{G_2,1} \ ,$$

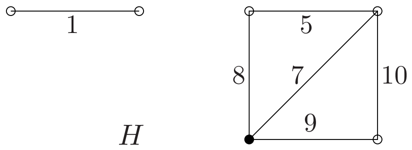
where G_2 is the minor $G\backslash 2/\!\!/3$ with the induced numbering of its edges. Now take the resultant with respect to edge 6. Since $\{4,5,6\}$ forms a three-valent vertex in G_2, it follows that $\Psi^{146,456}_{G_2} = 0$ by the vanishing property for vertices. Thus we have

$$[\Psi^{14,45}_{G_2}, \Psi^{4,5}_{G_2,1}]_6 = \pm\Psi^{14,45}_{G_2,6} \Psi^{46,56}_{G_2,1} \ .$$

Again, since $\{4,5,6\}$ is a three-valent vertex, $\Psi^{46,56}_{G_2,1} = \Psi^{46,56}_{G_2/\!\!/1} = \Psi^{45}_{G_2/\!\!/\{1,6\}} = \Psi^5_{G_2\backslash 4/\!\!/\{1,6\}}$, where the first and third equality are contraction-deletion relations. We have:

$$[\Psi^{14,45}_{G_2}, \Psi^{4,5}_{G_2,1}]_6 = \pm\Psi^{1,5}_{G_2\backslash 4/\!\!/6} \Psi^5_{G_2\backslash 4/\!\!/6,1}$$

The left-hand side is equal to $\pm D^6_G(1,2,3,4,5,6)$ by definition, and the minor $G_2\backslash 4/\!\!/6$ is exactly H, which completes the proof. □



Lemma 56. *Now let H be a graph with the general shape depicted above. The denominator reduction, applied five times to $\Psi_H^{1,5}\Psi_{H,1}^5$ with respect to the edges $7,8,9,10$ is $\pm\Psi_A^{15,78}\Psi_B$, where $A = H\backslash\{10\}/\!\!/9$ and $B = H\backslash\{5,7,9\}/\!\!/\{1,8,10\}$.*

Proof. By the second Dodgson identity, $[\Psi_H^{1,5},\Psi_{H,1}^5]_7 = \pm\Psi_H^{57,15}\Psi_{H,1}^{5,7}$. Applying the first Dodgson identity, we then get $[\Psi_H^{57,15},\Psi_{H,1}^{5,7}]_8 = \pm\Psi_H^{15,78}\Psi_{H,1}^{58,57}$. Now,

$$[\Psi_H^{15,78},\Psi_{H,1}^{58,57}]_9 = -\Psi_{H,9}^{15,78}\Psi_{H,1}^{579,589},$$

by definition of the resultant, using the fact that $\Psi_H^{159,789} = 0$, by the vanishing property for vertices applied to the three-valent vertex 7,8,9. Once more, by the vanishing property applied to the triangle $7,9,10$, we have $\Psi_{H,9X}^{15,78} = 0$ where X denotes the edge 10, and therefore

$$[\Psi_{H,9}^{15,78},\Psi_{H,1}^{579,589}]_{10} = \Psi_{H,9}^{15X,78X}\Psi_{H,1X}^{579,589}\ .$$

By contraction-deletion, the first factor is $\Psi_A^{15,78}$, and the second is $\Psi_{H'}^{7,8}$ where $H' = H\backslash\{5,9\}/\!\!/\{1,10\}$. In this latter graph, $7,8$ form a 2-valent vertex, and so $\Psi_{H'}^{7,8} = \Psi_{H',8}^{7} = \Psi_{H'\backslash 7/\!\!/8} = \Psi_B$. □

6.2. **An eight-loop counter-example.** Let G_8 be the eight-loop primitive-divergent ϕ^4 graph with vertices numbered $1,\ldots,9$ and (ordered) edges $e_1,\ldots,e_{16}$ defined by

$$34, 14, 13, 12, 27, 25, 58, 78, 89, 59, 49, 47, 35, 36, 67, 69\ , \tag{30}$$

where ij denotes an edge connecting vertices i and j.

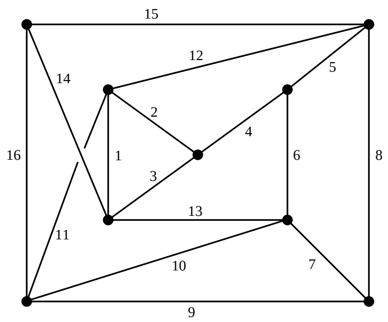


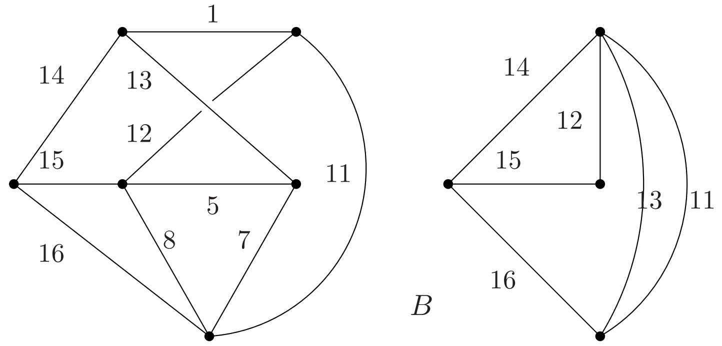
FIGURE 8. The graph G_8

This graph is isomorphic to $P_{8,37}$ minus vertex 3 or 5 in the census [17]. It has 3785 spanning trees. The first six edges form precisely the configuration depicted in lemma 55, and we can subsequently apply lemma 56 to reduce the next four edges. A further reduction with respect to edge 11 gives the following corollary.

Corollary 57. *Let G_8 be the 8-loop graph defined by* (30)*. Then*

$$D^{11}_{G_8}(e_1,\ldots,e_{11}) = \det\begin{pmatrix} \Psi^{15,78}_{A\backslash 11} & \Psi_{B\backslash 11} \\ \Psi^{15,78}_{A/\!\!/11} & \Psi_{B/\!\!/11}\end{pmatrix} ,$$

where A, B are depicted below.



A B

The polynomial $D^{11}_{G_8}(e_1,\ldots,e_{11})$ is irreducible, so to proceed further in the reduction, observe that A and B have a common minor $\gamma = B\backslash\{11\}/\!\!/\{12,13\}$ which is the sunset graph on 2 vertices and 3 edges $14, 15, 16$. Its graph polynomial is

$$\Psi_\gamma = \alpha_{14}\alpha_{15} + \alpha_{15}\alpha_{16} + \alpha_{14}\alpha_{16} \ .$$

By direct computation, one verifies that

$$\begin{aligned} \Psi^{15,78}_{A\backslash 11} &= -\alpha_{13}\alpha_{15} \\ \Psi^{15,78}_{A/\!\!/11} &= \alpha_{12}(\Psi_\gamma + \alpha_{13}\alpha_{16}) \\ \Psi_{B\backslash 11} &= \Psi_\gamma + \alpha_{12}\alpha_{13} + \alpha_{16}\alpha_{12} + \alpha_{14}\alpha_{12} + \alpha_{15}\alpha_{13} + \alpha_{14}\alpha_{13} \\ \Psi_{B/\!\!/11} &= \alpha_{13}(\Psi_\gamma + \alpha_{16}\alpha_{12} + \alpha_{14}\alpha_{12}) \ . \end{aligned} \tag{31}$$

By theorem 29, $c_2(G_8)_q \equiv -[D^{11}_{G_8}(e_1,\ldots,e_{11})]_q \mod q$. We can eliminate a further variable by exploiting the homogeneity of $D^{11}_{G_8}$ (or Ψ_{G_8}). The affine complement of the zero locus of a homogenous polynomial F admits a $\mathbb{G}_m$ action by scalar diagonal multiplication of the coordinates. For any coordinate α_e, we therefore have

$$[F]_q = [F,\alpha_e]_q + (q-1)[F,\alpha_e - 1]_q$$

Lemma 58. *$[D^{11}_{G_8},\alpha_{16}]_q$ is a polynomial in q.*

Proof. By inspection of (31), setting $\alpha_{16} = 0$ in the definition of $D^{11}_{G_8}$ causes the terms $\alpha_{14}\alpha_{15}$ to factor out. The other factor is of degree at most one in α_{14} and α_{15}, and by a simple application of lemma 16 is therefore a polynomial in q. □

We will henceforth work on the hyperplane $\alpha_{16} = 1$. Now we may scale α_{12} and α_{13} by Ψ_γ, which has the effect of replacing $D^{11}_{G_8}$ with $\widetilde{D}$ given by formally setting Ψ_γ to be 1 and $\alpha_{12}\alpha_{13}$ to be $\alpha_{12}\alpha_{13}\Psi_\gamma$ in the previous equations. Since this transformation is an isomorphism on the complement of $V(\Psi_\gamma)$, we have

$$[D^{11}_{G_8}]_q - [\Psi_\gamma, D^{11}_{G_8}]_q = [\widetilde{D}]_q - [\Psi_\gamma,\widetilde{D}]_q \ . \tag{32}$$

Lemma 59. *$[\widetilde{D}]_q$ and $[\Psi_\gamma, D^{11}_{G_8}]_q$ are constant modulo q.*

Proof. By inspection of (31), it is clear that the determinant $\widetilde{D}$ is of degree one in the variables α_{14} and α_{15}. Applying lemma 16 *(i)* twice, it follows that the class of $\widetilde{D}$ modulo q is equal to the class modulo q of its coefficient of $\alpha_{14}\alpha_{15}$, and this is $\alpha_{12}\alpha_{13}(\alpha_{13}\alpha_{12}+\alpha_{13}+\alpha_{12})$, which gives a polynomial in q. Likewise, a straightforward calculation using (31) shows that the intersection $V(\Psi_\gamma, D^{11}_{G_8})$ is union of intersections of hypersurfaces of degree at most 2 and linear in every variable, which can be treated using lemma 16 with components of small degree. □

It remains to compute $[\Psi_\gamma,\widetilde{D}]_q$, which is given mod q by the resultant $[\Psi_\gamma,\widetilde{D}]_{14}$. Explicitly, it is the polynomial:

$$\begin{aligned}(33)\qquad &\alpha_{12}+\alpha_{12}\alpha_{15}+\alpha_{13}\alpha_{12}^2+\alpha_{12}^2+\alpha_{13}\alpha_{12}+\alpha_{15}\alpha_{13}\alpha_{12}\\ &+\alpha_{13}^2\alpha_{15}+\alpha_{13}^2\alpha_{15}^2+\alpha_{13}^2\alpha_{15}\alpha_{12}+\alpha_{13}^2\alpha_{15}^2\alpha_{12}+\alpha_{15}^2\alpha_{13}\alpha_{12}\ .\end{aligned}$$

A final innocuous change of variables $\alpha_{13}\mapsto\alpha_{13}/(\alpha_{15}+1)$ can be handled as in the previous case (32) and reduces this equation to degree 4. Setting $a=\alpha_{13}+1$, $b=\alpha_{12}+1$, $c=\alpha_{15}$ leads to the equation

$$J=a^2bc-ab-ac^2-ac+b^2c+ab^2+abc^2-abc$$

which defines a singular surface in $\mathbb{A}^3$. In conclusion

$$(34)\qquad c_2(G_8)\equiv c-[J]_q \mod q$$

for some constant $c\in\mathbb{Z}$. Chasing the constant terms in the above gives $c=2$. Note that G has vertex-width 4 (realised by a different ordering on the edges from the one given above). See also [18] for the complete computer-reduction of a graph in the same completion class as this one. The proof that this is a counter-example continues in §7, where we study the point counting function of $V(J)$ in detail.

6.3. **A planar counter-example.** Consider the planar graph G_9 with nine loops and eighteen edges below. It is primitive-divergent and in ϕ^4 theory.

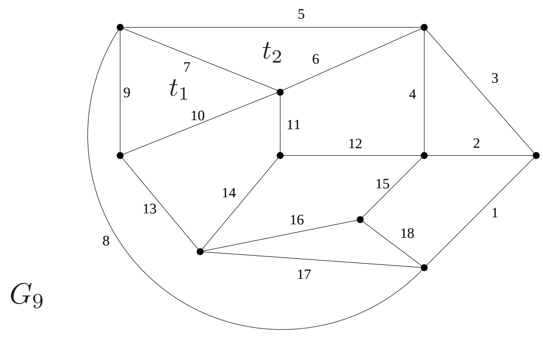


FIGURE 9. A planar counter-example to Kontsevich's conjecture, with vertex width 4 (for the edge-ordering shown).

It contains a double triangle (t_1 and t_2), bounded by edges $5,6,7,9,10$. By applying a double-triangle reduction, the c_2-invariant of this graph is equal to the c_2-invariant of a non-planar graph G'_9 at 8 loops. One verifies that the completion class of G'_9 is the same as that of G_8. Thus, accepting the completion conjecture, we have $c_2(G_9)_q\equiv 2-[J]_q \mod q$ also. In any case, a computer reduction of G'_9 (yielding a different quartic from J) confirms this prediction.

7. A singular K3 surface

Consider the homogeneous polynomial of degree four

$$F = b(a+c)(ac+bd) - ad(b+c)(c+d) \tag{35}$$

which satisfies $F|_{d=1} = J$. One easily checks that it has six singular points

$$e_1 = (0:0:0:1) \quad e_2 = (0:0:1:0) \quad e_3 = (0:1:0:0)$$

$$e_4 = (1:0:0:0) \quad e_5 = (0:0:-1:1) \quad e_6 = (1:1:-1:1)$$

which are all of du Val type. Its minimal desingularization is obtained by blowing up the six points $e_1, \ldots, e_6$ and is therefore a K3 surface X. Since the Hodge numbers of a K3 satisfy $h^{1,1} = 20$, and $h^{0,2} = h^{2,0} = 1$, both X and $V(F) \subset \mathbb{P}^3$ are not of Tate type and we can already conclude by (34) that the graph G_8 is a counterexample to Kontsevich's conjecture by (34).

7.1. **The Picard lattice.** We determine the Picard lattice of X as follows. It follows by inspection of F that the following lines lie on X.

$$\begin{array}{ll}
\ell_1 : \ c = d = 0 & \ell_8 : \ c = b + d = 0 \\
\ell_2 : \ b = d = 0 & \ell_9 : \ b = c + d = 0 \\
\ell_3 : \ a = d = 0 & \ell_{10} : \ a - b = c + d = 0 \\
\ell_4 : \ b = c = 0 & \ell_{11} : \ a = b = d \\
\ell_5 : \ a = c = 0 & \ell_{12} : \ a = b = -c \\
\ell_6 : \ a = b = 0 & \ell_{13} : \ a = -c = d \\
\ell_7 : \ a + c = d = 0 & \ell_{14} : \ a - d = b + c = 0
\end{array} \tag{36}$$

Let $\ell_{15}, \ldots, \ell_{20}$ denote the six exceptional divisors lying above the points $e_1, \ldots, e_6$. Since these rational curves have self-intersection -2, one easily deduces the following intersection matrix, where the rows and columns correspond to $\ell_1, \ldots, \ell_{20}$.

$$\begin{pmatrix}
-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
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0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -2
\end{pmatrix}$$

It has determinant -7.

Since 7 is prime, the lines $\ell_1, \ldots, \ell_{20}$ span the full Néron-Severi group. In particular, the rank of X is 20 and so it defines a singular K3 surface. Since $\mathbb{Q}(\sqrt{-7})$ has class number 1, X corresponds to the unique singular K3 in the Shioda-Inose classification [20] with discriminant -7. Now consider the elliptic curve $E = E_{49A1}$ with complex multiplication by $\mathbb{Q}(\sqrt{-7})$ which is given by the affine model:

$$y^2 + xy = x^3 - x^2 - 2x - 1 \ .$$

The results of [20] imply that the graph of the complex multiplication in $E \times E$ gives rise to a decomposition of $\mathrm{Sym}^2 H^1(E)$ into two pieces, one of which is $H^2_{tr}(X)$. The results of Livné [15] allow one to conclude that the weight 3 modular form corresponding to $H^2_{tr}(X)$ is given by the symmetric square of the modular form of E. It is given explicitly by the following cusp form of weight 3 and level 7:

$$\big(\eta(z)\eta(z^7)\big)^3 \tag{37}$$

where η denotes the Dedekind eta function (first entry of Table 2 in [19]).

Remark 60. *Consider Ramanujan's double theta function:*

$$\theta(r,s) = \sum_{n=-\infty}^{\infty} r^{n(n+1)/2} s^{n(n-1)/2}$$

and write $\theta_{a,b}(q) = \theta(-q^a, -q^b)$*. Then, following* [16]*, set*

$$\begin{aligned} f_{49}(q) &= \theta_{7,14}(q)^3 \big[q\,\theta_{21,28}(q) + q^2\theta_{14,35}(q) - q^4\theta_{7,42}(q)\big] \\ &= q + q^2 - q^4 - 3q^8 - 3q^9 + 4q^{11} - q^{16} - 3q^{18} + 4q^{22} + 8q^{23} + \ldots \end{aligned}$$

which spans the one-dimensional space of newforms of level 49 and weight 2 (see also [21]*). If* a_{p^n} *denotes the coefficient of* q^{p^n} *in* $f_{49}(q)$*, one knows that the number of points of* E *over* $\mathbb{F}_{p^n}$ *is* $p^n + 1 - a_{p^n}$*. One can show that:*

$$a_p = \begin{cases} 0 & \text{if} \quad p \equiv 0,3,5,6 \mod 7 \ , \\ \pm a & \text{where } 4p = a^2 + 7b^2 \text{ where } a,b \in \mathbb{Z}, \text{ if } p \equiv 1,2,4 \mod 7 \ . \end{cases} \tag{38}$$

Let b_{p^n} denote the coefficient of z^{p^n} in (37). Modulo p, we simply have

$$a^2_{p^n} \equiv b_{p^n} \mod p \ . \tag{39}$$

Theorem 61. *Let* G_8 *be the 8-loop non-planar graph defined in §6.2, and figure 8. Then the number of points of the affine graph hypersurface* X_{G_8} *over* $\mathbb{F}_{p^n}$ *satisfies:*

$$[X_{G_8}]_{p^n} \equiv -a^2_{p^n} p^{2n} \equiv -b_{p^n} p^{2n} \pmod{p^{2n+1}} \ . \tag{40}$$

Proof. Let $q = p^n$. We have $[X_{G_8}] \equiv c_2(G_8)\, q^2 \mod q^3$. Equation (34) states that $c_2(G_8) \equiv 2 - [J]_q \mod q$. Passing to the homogeneous version (35), one verifies that $[J]_q \equiv 2 - [F]_q \mod q$. Finally, the above discussion and equation (39) shows that $[F]_q \equiv -a^2_q \equiv -b_q \mod p$. Therefore $[X_{G_8}]$ modulo $q^2 p$ is given by (40). □

Consider the product of all finite fields $\mathbb{F}_p$ where p is prime:

$$A = \mathbb{F}_2 \times \mathbb{F}_3 \times \mathbb{F}_5 \times \ldots$$

and define the total c_2-invariant of a graph G to be

$$\widetilde{c}_2(G) = (c_2(G)_2, c_2(G)_3, c_2(G)_5, \ldots) \in A \ ,$$

where we identify $\mathbb{Z}/p\mathbb{Z}$ with $\mathbb{F}_p$. Let $\pi : \mathbb{Z} \to A$ denote the map whose p^{th} component is $n \mapsto n \mod p$. It follows from (38) that

$$\widetilde{c}_2(G_8) \notin \pi(K)$$

for all bounded sets $K \subset \mathbb{Z}$, since in the opposite case, all primes p congruent to $1, 2, 4 \mod 7$ would satisfy $4p \in -K + \{7\,b^2 : b \in \mathbb{Z}\}$. Since K is finite, this would contradict the prime number theorem. Therefore $c_2(G_8)$ is not (quasi-)constant, and therefore the graph G_8 is a counter-example to Kontsevich's conjecture in the strongest possible sense.

Assuming the completion conjecture 35, or by the double-triangle theorem and the computer calculation in [18], the graph G_9 of §6.3 has exactly the same property, and yields a planar counter-example at 9 loops.

7.2. **Discussion.** The prevalence of multiple zeta values in Feynman integral computations at low loop orders led Kontsevich to conjecture that the Euler characteristics of graph hypersurfaces were of mixed Tate type. This was shown to be generically false by Belkale and Brosnan, but despite this cautionary result, the following questions about the arithmetic nature of ϕ^4 theory remained open:

(1) Even though general graphs have non-Tate Euler characteristics, it could be that graphs coming from physically relevant theories are still of Tate type (the counter-examples of [2] have unphysical vertex degrees).
(2) It could be that the counter-examples occur at very high loop order rendering them physically less relevant.
(3) Failing (1) and (2), it could still be the case that planar graphs have Tate Euler characteristics, i.e., all non-Tate counter-examples can be characterized by having a high genus or crossing number.
(4) Even though the Euler characteristics are non-Tate, it could be that the piece of the graph motive which carries the period is always mixed Tate.

Our counter-examples show that (1), (2) and (3) are false. Point (4) is more subtle. However, it follows from the original interpretation of the denominator reduction in [8] that the c_2-invariant of a graph should correspond to the 'framing' on M_G, i.e., the smallest subquotient motive of M_G which is spanned by the integrand of (3). This makes it very probable that (4) is false too. In this case, one is led to expect the residue I_G (equation (3)) to be transcendental over the ring generated by multiple zeta values over $\mathbb{Q}$. Indeed, a likely candidate for the periods of the counter-examples G_8, G_9 might come from the periods of the motivic fundamental group of the elliptic curve E_{49A1} with punctures.

Finally, it should be emphasized that the residues I_G of primitive graphs in ϕ^4 are renormalization-scheme independent, and universal in the sense that any quantum field theory in 4 space-time dimensions will only affect the numerator, and not the denominator, of the corresponding parametric integral representation (barring infra-red divergences). Since the motive M_G only depends on the denominators, one can reasonably expect that such non-mixed Tate phenomena will propagate into most renormalizable massless quantum field theories with a four-valent vertex at sufficiently high loop orders.

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