A class of non-holomorphic modular forms

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Modular forms are everywhere
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Two motivations

1. Do there exist modular forms which correspond to mixed motives? Today, mixed Tate motives over $\mathbb{Z}$ mainly.

2. String theory. Genus one closed superstring amplitudes.

$$\text{Graph } G \quad \longrightarrow \quad \text{Modular-invariant function}$$

   Unknown class of functions. A few known examples. (..., M. Green, Vanhove, ..., Zagier, Zerbini, ...).

Goal: define a class of non-holomorphic modular forms.

I. General framework

Definitions I

Let

$$\mathfrak{H} = \{z : \mathrm{Im}\, z > 0\}$$

$$z = x + iy \quad , \quad q = e^{2i\pi z}.$$

For simplicity, let

$$\Gamma = \mathrm{SL}_2(\mathbb{Z}) \ .$$

Definition

A real analytic function $f : \mathfrak{H} \to \mathbb{C}$ is *modular of weights* (r, s) if

$$f(\gamma z) = (cz + d)^r (c\bar{z} + d)^s f(z)$$

for all

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \in \quad \Gamma \ .$$

Definitions II

Definition

Let $\mathcal{M}_{r,s}$ be the $\mathbb{C}$-vector space of functions $f : \mathfrak{H} \to \mathbb{C}$ which are real analytic modular of weights (r, s), such that

$$f(q) \quad \in \quad \mathbb{C}[[q, \overline{q}]][\mathbb{L}^{\pm}]$$

where

$$\mathbb{L} := \log |q| = i\pi(z - \overline{z}) = -2\pi y \ .$$

There is an $N \in \mathbb{N}$

$$f(q) = \sum_{k=-N}^{N} \sum_{m,n \geq 0} a_{m,n}^{(k)} \mathbb{L}^{k} q^{m} \overline{q}^{n}$$

where $a_{m,n}^{(k)} \in \mathbb{C}$.

Definitions III

Let

$$\mathcal{M} = \bigoplus_{r,s} \mathcal{M}_{r,s}$$

Then $\mathcal{M}$ is a bigraded algebra. Let

$$w = r + s \qquad \text{and} \qquad h = r - s \ .$$

Call w the *total weight*. Take w, h even.

The *constant part of* f is

$$f^0 \quad := \quad \sum_k a_{0,0}^{(k)} \mathbb{L}^k \quad \in \quad \mathbb{C}[\mathbb{L}^{\pm}]$$

Trivial examples

- $\mathbb{L} \in \mathcal{M}_{-1,-1}$
- For any holomorphic modular form

$$f \in M_{2n}$$

then $f \in \mathcal{M}_{2n,0}$ and $\overline{f} \in \mathcal{M}_{0,2n}$, e.g.,

$$\mathbb{G}_{2k} = -\frac{b_{2k}}{4k} + \sum_{n \geq 1} \sigma_{2k-1}(n) q^n \qquad k \geq 2$$

- The function $\mathbb{G}_2$ is not modular, but

$$\mathbb{G}_2^* = \mathbb{G}^2 - \frac{1}{4\mathbb{L}} \ \in \ \mathcal{M}_{2,0}$$

is an 'almost holomorphic' modular form.

Differential operators

Define

$$\partial_r = (z - \overline{z})\frac{\partial}{\partial z} + r \quad , \quad \overline{\partial}_s = (\overline{z} - z)\frac{\partial}{\partial \overline{z}} + s \ .$$

They preserve modularity

$$\partial_r : \mathcal{M}_{r,s} \longrightarrow \mathcal{M}_{r+1,s-1}$$

$$\overline{\partial}_s : \mathcal{M}_{r,s} \longrightarrow \mathcal{M}_{r-1,s+1}$$

Write

$$\partial = \sum_r \partial_r \qquad , \qquad \overline{\partial} = \sum_s \overline{\partial}_s \ .$$

$\mathfrak{sl}_2$-action

These operators generate an $\mathfrak{sl}_2$:

$$[h, \partial] = 2\partial \qquad , \qquad [h, \overline{\partial}] = -2\overline{\partial} \qquad , \qquad [\partial, \overline{\partial}] = h$$

where $h : \mathcal{M}_{r,s} \to \mathcal{M}_{r,s}$ is multiplication by $(r - s)$.

Bigraded Laplace operator

Define an operator

$$\Delta_{r,s} : \mathcal{M}_{r,s} \longrightarrow \mathcal{M}_{r,s}$$

by

$$\begin{aligned} \Delta_{r,s} &= -\overline{\partial}_{s-1}\partial_r + r(s-1) \\ &= -\partial_{r-1}\overline{\partial}_s + s(r-1) \end{aligned}$$

Then

$$\Delta_{0,0} = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

is the Laplace-Beltrami operator. Write

$$\Delta = \sum_{r,s} \Delta_{r,s}$$

Petersson inner product

Subspace of cuspidal functions (no constant part)

$$\mathcal{S}_{r,s} \subset \mathcal{M}_{r,s} \ .$$

Let $\mathcal{D}$ be a fundamental domain for Γ.

$$\begin{aligned} \mathcal{M}_{r,s} \times \mathcal{S}_{n-s,n-r} &\longrightarrow \mathbb{C} \\ \langle f, g \rangle &= \int_{\mathcal{D}} f(z)\overline{g(z)}\, y^n \frac{dxdy}{y^2} \end{aligned}$$

Special cases:

$$\langle f, g \rangle : \mathcal{M}_{r,s} \times S_{r-s} \longrightarrow \mathbb{C}$$

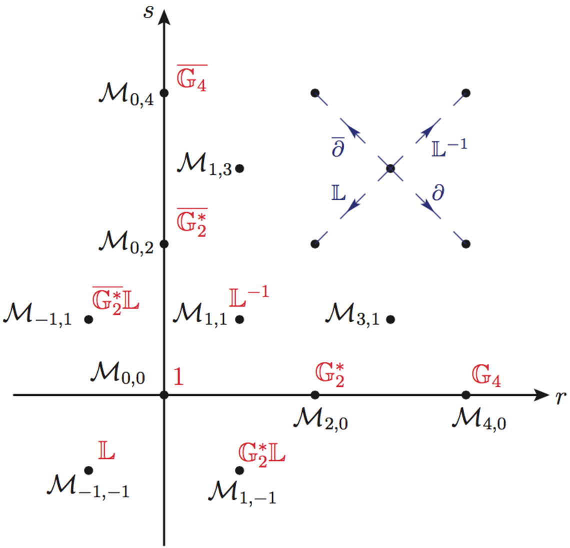
$$\langle f, \overline{g} \rangle : \mathcal{M}_{r,s} \times \overline{S}_{s-r} \longrightarrow \mathbb{C}$$

where $S_{2n} =$ holomorphic cusp forms.

holomorphic (Sturm)/antiholomorphic projection

$$p = p^h + p^a \quad : \quad \mathcal{M}_{r,s} \longrightarrow S_{r-s} \oplus \overline{S}_{s-r}$$

Picture of $\mathcal{M}$



II. Primitives and obstructions

Iterated primitives

Goal

Construct new elements from old by solving

$$\partial_r F = f \qquad \text{for} \quad F \in \mathcal{M}_{r,s}$$

This equation can't always be solved. Obstructions:

1. *Combinatorial*. Not all f admit a primitive $F \in \mathbb{C}[[q, \overline{q}]][\mathbb{L}^{\pm}]$. For example, f cannot contain any terms: $\mathbb{L}^{-r}\overline{q}^{n}$.
2. *Modularity*. Suppose $\partial F = f$ has a solution $F \in \mathbb{C}[[q, \overline{q}]][\mathbb{L}^{\pm}]$. Then F is not necessarily modular.
3. (*Arithmetic*. F will have transcendental coefficients.)

Orthogonality to cusp forms

Theorem

Suppose that $\partial F = f$ admits a solution $F \in \mathcal{M}_{r,s}$. Then

$$\langle f, g \rangle = 0 \qquad \text{for all } g \in S_{r-s+2} \ .$$

Equivalently, the holomorphic projection vanishes

$$p^h(f) = 0 \ .$$

Idea of proof: Stokes' theorem

$$\int_{\mathcal{D}} \mathbb{L}^{r+1} f(z)\overline{g(z)}\frac{dxdy}{y^2} = 4\pi^2 \int_{\partial\mathcal{D}} F(z)\overline{g(z)}d\overline{z}$$

The right-hand side vanishes by modularity.

Corollary

Let $f \in S$ be a non-zero holomorphic cusp form. Then

$$\partial F = \mathbb{L}^k f$$

has no modular solutions $F \in \mathcal{M}$.

'Cusp forms have no modular primitives'

But! this leaves open the possibility that

$$\partial F = \mathbb{L}^k \mathbb{G}_{2n+2}$$

might have a solution. Indeed, it does.

Out of this crack of light, a vast landscape will unfold!

Example: real analytic Eisenstein series

Definition

For $r, s \geq 0$ and $r + s = w > 0$

$$\mathcal{E}_{r,s}(z) = \frac{w!}{(2i\pi)^{w+2}} \frac{1}{2} \sum_{m,n \neq 0,0} \frac{\mathbb{L}}{(mz+n)^{r+1}(m\overline{z}+n)^{s+1}} \, .$$

These are the unique solutions to

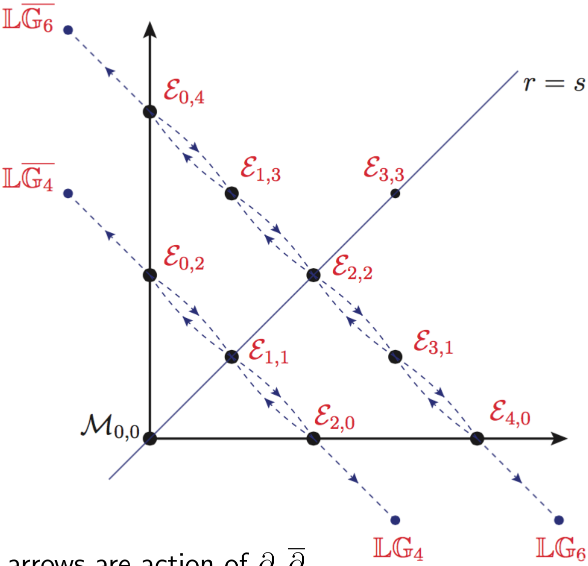
$$\begin{array}{rcll} \partial \mathcal{E}_{w,0} & = & \mathbb{L}\mathbb{G}_{w+2} & \\ \partial \mathcal{E}_{r,s} & = & (r+1)\mathcal{E}_{r+1,s-1} & \qquad s \geq 1 \end{array}$$

and

$$\begin{array}{rcll} \overline{\partial} \mathcal{E}_{0,w} & = & \mathbb{L}\overline{\mathbb{G}}_{w+2} & \\ \overline{\partial} \mathcal{E}_{r,s} & = & (s+1)\mathcal{E}_{r-1,s+1} & \qquad r \geq 1 \end{array}$$

such that coefficient of $\mathbb{L}^{-w/2}$ in $\mathcal{E}_{w/2,w/2}$ vanishes.

Picture



Dotted arrows are action of $\partial, \overline{\partial}$.

Properties

Eigenfunctions of the Laplacian:

$$\Delta \mathcal{E}_{r,s} = -w\, \mathcal{E}_{r,s}$$

Orthogonal to cusp forms

$$p(\mathcal{E}_{r,s}) = 0$$

Constant part involves odd zeta values:

$$\mathcal{E}^0_{r,s} = a\,\mathbb{L} + a'\zeta(2n+1)\,\mathbb{L}^{-2n}$$

where $a, a' \in \mathbb{Q}$.

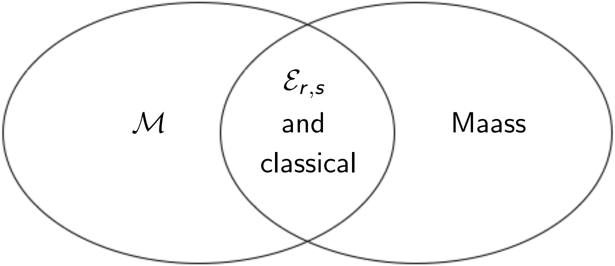
(Corresponds to $0 \to \mathbb{Q}(2n) \to E \to \mathbb{Q}(-1) \to 0$)

Relation with Maass waveforms

Theorem

If $f \in \mathcal{M}$ is an eigenfunction of the Laplacian, it is a linear combination with coeffs. in $\mathbb{C}[\mathbb{L}^{\pm}]$ of eigenfunctions of the form:

- real analytic Eisenstein series $\mathcal{E}_{r,s}$,
- an almost holomorphic modular form,
- an almost antiholomorphic modular form.



III. Modular iterated integrals

Now solve equations like

$$\partial F = \mathbb{L}^{s+1} f \mathcal{E}_{r,s} + \mathbb{L} g$$

where f and g holomorphic modular forms. Use the solutions F to generate new primitives, and so on, recursively.

This generates a huge space $\mathcal{MI} \subset \mathcal{M}$ of *modular iterated integrals*. It is filtered by the length: $\mathcal{MI}_k$.

Recall orthogonality condition $p^h(\partial F) = 0$. Implies

$$g = \sum_h -\langle f \mathcal{E}_{r,s}, h \rangle h \ ,$$

sum over basis of Hecke cusp eigenforms. By Rankin-Selberg, the length 2 elements of $\mathcal{MI}_2$ have coefficients involving $L(f \otimes h, n)$.

Vector bundles and equivariance

Define

$$V_{2n} = \bigoplus_{r+s=2n} X^r Y^s \mathbb{Q}$$

It is equipped with a right-action of SL_2:

$$(X, Y)\big|_\gamma = (aX + bY, cX + dY) \ .$$

Definition

A function $f : \mathfrak{H} \to V_{2n} \otimes \mathbb{C}$ is equivariant if

$$f(\gamma z)\big|_\gamma = f(z) \qquad \text{for all } \gamma \in \Gamma$$

Key point: Equivariant f can be uniquely written

$$f(z) \ = \sum_{r+s=2n} f_{r,s}(z)\,(X - zY)^r (X - \bar{z}Y)^s$$

where $f_{r,s} : \mathfrak{H} \to \mathbb{C}$ modular of weights (r, s)

Single-valued periods

We want to construct equivariant sections of V_{2n}.

Idea: apply the single-valued machine.

Example:

$$\log z = \int_1^z \frac{dt}{t}$$

It is a multi-valued function on $\mathbb{C}^\times$. Analytic continuation around 0 gives a discontinuity

$$\log z \quad \mapsto \quad \log z + 2\pi i \ .$$

Since $2\pi i$ is imaginary, the function

$$\log |z| \quad = \quad \mathrm{Re} \log z$$

is single-valued. It is the *single-valued version* of $\log z$.

Example

$$F(z) = 2\pi i \int_z^{\vec{1}_\infty} \mathbb{G}_{2n+2}(\tau)(X-\tau Y)^{2n} d\tau$$

Is not quite equivariant:

$$F(\gamma z)\big|_\gamma - F(z) = C_\gamma(X,Y) \qquad \text{for all } \gamma \in \Gamma$$

C_γ is the Eisenstein cocycle:

$$C_\gamma = a\,\zeta(2n+1)\,Y^{2n}\Big|_{\gamma - id} + (2\pi i)\,e^0_{2n}(\gamma)$$

where $a \in \pi^{-2n}\mathbb{Q}$, and $e^0_{2n}(\gamma) \in \mathbb{Q}[X,Y]$. Set

$$\mathcal{E}(z) \quad = \quad 2\,\mathrm{Re}\,F(z) - 2\,a\,\zeta(2n+1)Y^{2n}\ .$$

It is equivariant, and its coefficients are the $\mathcal{E}_{r,s}$.

Double Eisenstein integrals

For any $f \in M_{2n+2}$, define

$$\underline{f}(z) = 2\pi i\, f(z)(X - zY)^{2n} dz\ .$$

It is equivariant.

$$F^{2a,2b}(z) \;=\; \mathrm{Im} \int_z^{\vec{1}_\infty} \underline{\mathbb{G}_{2a}}\underline{\mathbb{G}_{2b}} - \mathrm{Re}\Big(\int_z^{\vec{1}_\infty} \underline{\mathbb{G}_{2a}} \Big) \times \int_z^{\vec{1}_\infty} \underline{\mathbb{G}_{2b}} + \int_z^{\vec{1}_\infty} \underline{f} + \int_z^{\vec{1}_\infty} \underline{\overline{g}} + c$$

is equivariant for suitable choices of modular forms f, g and constant c. Lowest weight modular component solves

$$\partial F = \mathbb{L}\mathbb{G}_{2a}\mathcal{E}_{2b-2,0} + \mathbb{L}f$$

Orthogonality relations

Orthogonality to cusp forms uniquely determines f:

$$\langle f, h\rangle + \langle \mathbb{G}_{2a}\mathcal{E}_{2b-2,0}, h\rangle = 0 \quad \text{for all } h \in S_{2a+2b-2}$$

By Rankin-Selberg, the right-hand term is proportional to

$$L(h, 2a-1)L(h, 2a+2b-2)$$

if h eigenform. Manin period relations implies we can construct combinations in which all cusp forms drop out. Example:

$$9(F^{4,10} - F^{10,4}) + 14(F^{6,8} - F^{8,6})$$

is an iterated integral involving Eisenstein series only. Want to generalise: space $\mathcal{MI}^E \subset \mathcal{MI}$ of *modular Eisenstein integrals*.

IV. General construction

A Lie algebra

Let $\mathrm{Lie}(\mathsf{a},\mathsf{b})$ be the free Lie algebra on two generators a,b. It has a right-action of SL_2.

There exist derivations (Tsunogai) for all $n \geq -1$

$$\begin{aligned} \varepsilon_{2n}(b) &= -\mathrm{ad}(b)^{2n}(a) \\ \varepsilon_{2n}([a,b]) &= 0 \end{aligned}$$

They generate a Lie algebra $\mathfrak{u}^{\mathrm{geom}} \subset \mathrm{Der}\,\mathrm{Lie}(\mathsf{a},\mathsf{b})$.

It corresponds to a group scheme $\mathcal{U}^{\mathrm{geom}}$.

Satisfy many relations, for example (Pollack):

$$[\varepsilon_{10},\varepsilon_4] - 3[\varepsilon_8,\varepsilon_6] = 0$$

Let

$$\omega = -\mathrm{ad}(\varepsilon_0)\frac{dq}{q} + \sum_{n\geq 1} \frac{2}{(2n)!}\varepsilon_{2n+2}\mathbb{G}_{2n+2}(q)\frac{dq}{q}$$

Then the generating series of iterated integrals

$$J(z) = 1 + \int_z^{\vec{1}_\infty} \omega + \int_z^{\vec{1}_\infty} \omega\omega + \dots$$

solves the differential equation (version of KZB)

$$dJ = \omega J \ .$$

It satisfies for all $\gamma \in \Gamma$:

$$J(\gamma z)\big|_\gamma = J(z)\mathcal{G}_\gamma$$

for some non-abelian cocycle

$$G_\gamma \in Z^1(\Gamma, \mathcal{U}^{\mathrm{geom}}(\mathbb{C}))$$

Theorem

Exist $b \in \mathcal{U}^{\mathrm{geom}}(\mathbb{C})$, and $\phi \in (\mathrm{Aut}\,\mathcal{U}^{\mathrm{geom}})^{\mathrm{SL}_2}(\mathbb{C})$ such that

$$b\big|_{\gamma}^{-1} \phi(\overline{\mathcal{G}}_{\gamma})\, b = \mathcal{G}_{\gamma} \qquad\qquad \text{for all } \gamma \in \Gamma\ .$$

Definition

$$J^{\mathrm{ev}}(z) = J(z) b^{-1} \phi(\overline{J(z)})^{-1}$$

Notice that the holomorphic differential equation is unchanged:

$$\frac{\partial J^{\mathrm{ev}}}{\partial z} = \omega J^{\mathrm{ev}}\ .$$

Theorem

The series J^{ev} is modular equivariant

$$J^{\mathrm{ev}}(\gamma z)\big|_{\gamma} = J^{\mathrm{ev}}(z) \ .$$

The coefficients of J^{ev} are equivariant real analytic functions on $\mathfrak{H}$. They generate a space of modular forms

$$\mathcal{MI}^{E} \subset \mathcal{M}$$

which only involve iterated integrals of Eisenstein series.

Main theorem

We have constructed explicitly a space of modular forms

$$\mathcal{MI}^E \subset \mathcal{M} \ .$$

Theorem

$\mathcal{MI}^E$ is an algebra, with modular weights (r, s) for $r, s \geq 0$.

(Expansion).
Every $f \in \mathcal{MI}^E$ admits an expansion

$$f \quad \in \quad \mathcal{Z}^{sv}[[q, \overline{q}]][\mathbb{L}^{\pm}]$$

where $\mathcal{Z}^{sv}$ is the ring of single-valued multiple zeta values.

(Length filtration).
It admits a filtration by length $\mathcal{MI}^E_k \subset \mathcal{MI}^E$. In length one, $\mathcal{MI}^E_1$ is generated by the functions $\mathcal{E}_{r,s}$.

Main theorem continued

(Differential structure).

$$\partial \mathcal{MI}^E_k \quad \subset \quad \mathcal{MI}^E_k \; + \; E[\mathbb{L}] \times \mathcal{MI}^E_{k-1}$$

$$\overline{\partial} \mathcal{MI}^E_k \quad \subset \quad \mathcal{MI}^E_k \; + \; \overline{E}[\mathbb{L}] \times \mathcal{MI}^E_{k-1}$$

where E is the space of holomorphic Eisenstein series.

(Weight grading).
The space $\mathcal{MI}^E$ has a grading:

$$\deg_M \mathbb{L} = 2 \quad \text{and} \quad \deg_M \mathcal{E}_{r,s} = 2\ .$$

(Finiteness).
The space $\mathrm{gr}^M_k \mathcal{MI}^E \cap \mathcal{M}_{r,s}$ of bounded modular weight and bounded M-degree is finite dimensional. An element is uniquely determined by finitely many terms in its $q, \overline{q}$ expansion.

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Main theorem continued

(Structure).
There is a non-canonical algebra isomorphism

$$lw(\mathcal{O}(\mathcal{U}^{\mathrm{geom}})) \xrightarrow{\sim} (\mathcal{MI}^E)_{\bullet,0}$$

(Laplace equation).
Every element of $\mathcal{MI}^E$ satisfies an inhomogeneous Laplace equation with eigenvalue $-w$.

Further properties (in progress):

1. (Double shuffle). The Lie algebra $\mathfrak{u}^{\mathrm{geom}}$ embeds into a space of polar solutions to linearised double shuffle equations.
2. (Galois action). The ring $\mathcal{O}(\mathcal{U}^{\mathrm{geom}})$ admits an action of the motivic Galois group of mixed Tate motives over $\mathbb{Z}$.

Conclusion

1. Bigraded space $\mathcal{M} = \bigoplus_{r,s} \mathcal{M}_{r,s}$ of real-analytic modular forms. Differential structure $\partial, \overline{\partial}$.
2. Large subspace $\mathcal{MI} \subset \mathcal{M}$ of iterated primitives of holomorphic modular forms. Correspond to non-trivial extensions of pure motives. Finiteness: uniquely determined by a finitely many coefficients in the $q, \overline{q}$-expansion.
3. Explicit construction $\mathcal{MI}^E \subset \mathcal{MI}$ of iterated primitives of Eisenstein series. Correspond to mixed Tate motives over $\mathbb{Z}$.
4. Closed genus one superstring amplitudes should lie in the component of $\mathcal{MI}^E[\mathbb{L}^{\pm}]$ of modular weights $(0, 0)$. Explains conjectural properties, plus existence of many relations.