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Quantum Field Theory and Arithmetic

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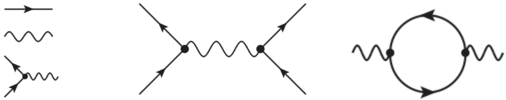
Perturbative Quantum Field Theory

A Quantum Field Theory describes the interactions between fundamental particles. Interactions are represented by Feynman graphs, built out of certain types of edges and vertices:

QED :



To every graph G in the theory, one associates an *amplitude*

$$G - -- > I_G(q_i, m_e)$$

which is a function of particle momenta q_i, and particle masses m_e. When all masses are zero, and all but one q_i vanish, e.g.,

$$I_G \sim \text{ number} \times |q|^{-2}$$

We get a map from graphs to some very interesting numbers. A *huge* effort goes into computing the quantities $I_G(q_i, m_e)$.

The graph polynomial

Let G be a connected graph with no self-edges. A graph is deemed to be 'physical' if all its vertices have degree at most 4, written

$$G \in \phi^4$$

Definition (Kirchhoff 1847)

The *graph polynomial* $\Psi_G \in \mathbb{Z}[\alpha_e, e \in E(G)]$ is defined by

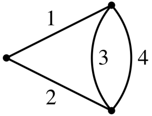
$$\Psi_G = \sum_{T \subset G} \prod_{e \notin T} \alpha_e$$

where the sum is over all spanning trees T of G.

A subgraph $T \subset G$ is a *spanning tree* if it is a tree (it is connected, and simply connected), and if it spans G. This means that it meets every vertex of G, or $V(T) = V(G)$.

Example

Consider the following graph with 4 edges, and 3 vertices.



Its spanning trees are the subgraphs given by the sets of edges

$$\{1,2\} \qquad \{1,3\} \qquad \{1,4\} \qquad \{2,3\} \qquad \{2,4\}$$

The graph polynomial is therefore

$$\Psi_G = \alpha_3\alpha_4 + \alpha_2\alpha_4 + \alpha_2\alpha_3 + \alpha_1\alpha_4 + \alpha_1\alpha_3$$

In general, Ψ_G is homogeneous of degree h_G, the first Betti number of G (also known as the 'loop number' of G).

$$\deg \Psi_G = h_G \qquad N_G = \#E(G)$$

Feynman integrals

In order to get convergent integrals, we shall assume that

- G is *overall log-divergent*: $N_G = 2h_G$
- G is *primitive*: $N_\gamma > 2h_\gamma$ for all $\gamma \subsetneq G$.

The Feynman amplitude is defined by the convergent integral

$$I_G = \int_\sigma \frac{\Omega_G}{\Psi_G^2} \in \mathbb{R}$$

where

$$\Omega_G = \sum_{i=1}^{N_G} (-1)^i \alpha_i d\alpha_1 \wedge \ldots \wedge \widehat{d\alpha_i} \wedge \ldots d\alpha_{N_G}$$

and the domain of integration σ is the real coordinate simplex

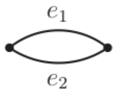
$$\sigma = \{(\alpha_1 : \ldots : \alpha_{N_G}) \in \mathbb{P}^{N_G-1}(\mathbb{R}) \text{ such that } \alpha_i \geq 0\}$$

The result is that we get a map from graphs to numbers:

$$I : \{\text{Primitive, log-divergent graphs in } \phi^4\} \longrightarrow \mathbb{R}$$

The whole problem is to try to understand this map.

Trivial example: consider the graph

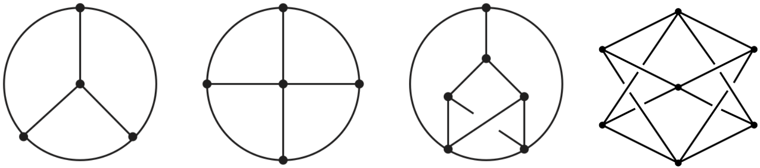


It satisfies $\Psi_G = \alpha_1 + \alpha_2$. We can compute the integral on the affine chart $\alpha_2 = 1$ in $\mathbb{P}^1$, where it reduces to

$$I_G = \int_0^\infty \frac{d\alpha_1}{(\alpha_1 + 1)^2} = 1$$

The Zoo

Some selected examples of primitive, log-divergent graphs in ϕ^4 theory, at $3, 4, 5$ and 6 loops, and their amplitudes:



I_G : $6\zeta(3)$ $20\zeta(5)$ $36\zeta(3)^2$ $N_{3,5}$

The number $N_{3,5}$ is given by

$$N_{3,5} = \frac{27}{5}\zeta(5,3) + \frac{45}{4}\zeta(5)\zeta(3) - \frac{261}{20}\zeta(8)$$

The amplitudes I_G are very hard to compute: in the first example, Ψ_G is of degree 3 in 6 variables, and has 16 terms.

Multiple Zeta Values

Let $n_1, \ldots, n_{r-1}$ be integers ≥ 1, and let $n_r \geq 2$. Euler defined

$$\zeta(n_1, \ldots, n_r) = \sum_{1 \leq k_1 < k_2 < \ldots < k_r} \frac{1}{k_1^{n_1} \ldots k_r^{n_r}} \quad \in \mathbb{R}$$

When $r = 1$, these are values of the Riemann zeta function $\zeta(n)$.

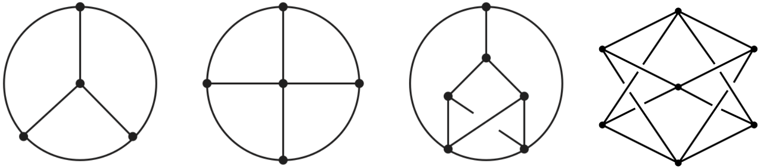
The *weight* is the quantity $n_1 + \ldots + n_r$.

Multiple zeta values form an algebra: for example

$$\zeta(n_1)\zeta(n_2) = \zeta(n_1, n_2) + \zeta(n_2, n_1) + \zeta(n_1 + n_2)$$

In general, the product of any two MZV's is equal to a sum of MZV's of the same total weight.

The Zoo II



I_G :	$6\zeta(3)$	$20\zeta(5)$	$36\zeta(3)^2$	$N_{3,5}$
weight :	3	5	6	8
$N_G - 3$:	3	5	7	9

One can show that the weight is bounded above by $N_G - 3$. The last two examples have *weight drop*: the true weight is strictly lower than the expected weight.

Main folklore conjecture

In the 90's, Broadhurst and Kreimer made very extensive computations of I_G, and found that for all graphs G for which I_G can be computed (e.g. $h_G \leq 6$), it is *numerically* an MZV.

Folklore conjecture

The numbers I_G are $\mathbb{Q}$-linear combinations of multiple zeta values.

Analogies with 2-dimensional Quantum Field Theories, and deformation quantization. Cartier's Cosmic Galois group.[1]

First, in the positive direction, I proved in 2009 that

Theorem 1

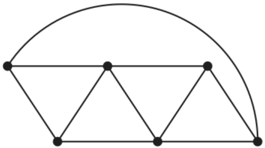
The conjecture is true for all graphs of 'vertex width' $vw(G) \leq 3$.

[1] c.f., Cartier, Connes-Marcolli, Kontsevich,. . .

An explicit family: the zig-zag graphs

Consider the following family of zig-zag graphs Z_n with n loops:

Z_5



In 1995 Broadhurst and Kreimer made the following conjecture:

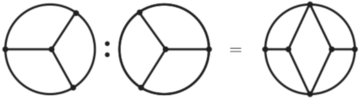
Theorem (with O. Schnetz 2012)

$$I_{Z_n} = 4\frac{(2n-2)!}{n!(n-1)!}\left(1 - \frac{1-(-1)^n}{2^{2n-3}}\right)\zeta(2n-3)\ .$$

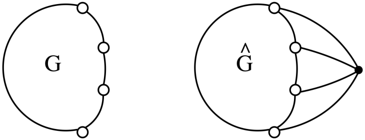
- This is the only infinite family of primitive graphs in ϕ^4 whose amplitude is known, or even conjectured.
- The proof uses a theorem of Zagier on $\zeta(2,\ldots,2,3,2,\ldots,2)$
- Experimentally, the zig-zag graphs are the only graphs in ϕ^4 whose amplitudes can be written as single zeta values.

Some Identities

1. (Partial multiplication law). When G_1 and G_2 each have two trivalent vertices connected by an edge, we can form the two-vertex join $G_1 : G_2$. Then $I_{G_1:G_2} = I_{G_1} I_{G_2}$.



2. (Completion) Every non-trivial primitive log-divergent graph G in ϕ^4 theory has exactly 4 trivalent vertices. Let $\widehat{G}$ be the graph obtained by connecting them to a single new vertex.



If $\widehat{G}_1 \cong \widehat{G}_2$ then $I_{G_1} = I_{G_2}$.

Counting points over finite fields (Kontsevich)

The *affine graph hypersurface* is the zero locus

$$X_G = V(\Psi_G) \subset \mathbb{A}^{N_G}$$

of the graph polynomial. It is usually irreducible (e.g., when G is primitive log-divergent), and is highly singular in general.

Let p be a prime, and $q = p^n$. Since Ψ_G has integer coefficients, we can consider the point-counting function

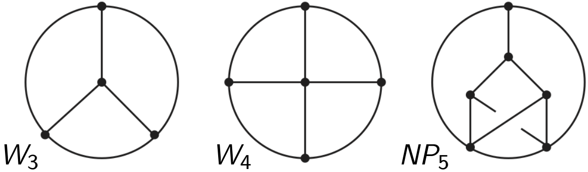
$$[G]_q : q \mapsto \#X(\mathbb{F}_q)$$

It is a function from the set of prime powers q to $\mathbb{N}$.

Conjecture (Kontsevich 1997)

The point counting function $[G]_q$ is a polynomial in q.

Consider the following three graphs:



Their point-counting functions over finite fields $\mathbb{F}_q$:

Graph	I_G	$[G]_q$
W_3	$6\zeta(3)$	$q^5 - q^3 - q^2$
W_4	$20\zeta(5)$	$q^7 + 3q^5 - 6q^4 + 4q^3 - q^2$
NP_5	$36\zeta(3)^2$	$q^9 + 4q^7 - 7q^6 + 3q^5$

The idea is that the amplitude I_G, and the point-counting function $[G]_q$ should be different aspects of the same object, called the 'motive' of G, denoted $mot(G)$.

We have the following heuristic picture:

$$\begin{array}{ccccc} I_G & \leftarrow -- & mot(G) & -- \rightarrow & [G]_q \\ \in & & \in & & \in \\ MZV & \leftarrow -- & MT(\mathbb{Z}) & -- \rightarrow & \mathbb{Z}[q] \end{array}$$

$MT(\mathbb{Z})$ denotes the category of mixed Tate motives[2] over $\mathbb{Z}$. The dashed arrows going to the left are the Hodge realization (period map), the arrows to the right the ℓ-adic realization.

This is just an analogy. We cannot *a priori* infer information about I_G from $[G]_q$ and vice-versa.

[2] Levine, Deligne-Goncharov building on Voevodsky, Hanamura, Levine, Bloch, Beilinson, Soulé, Borel, ...

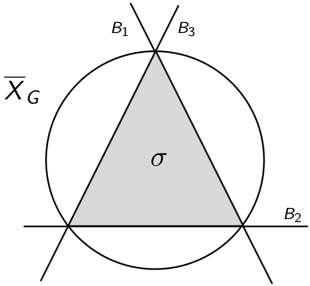
The graph motive (Bloch-Esnault-Kreimer 2007)

Recall that

$$I_G = \int_\sigma \omega_G \qquad \text{where} \qquad \omega_G = \frac{\Omega_G}{\Psi_G^2}$$

How to interpret this as a period? Consider the graph hypersurface, and coordinate hyperplanes in projective space:

$$\overline{X}_G = V(\Psi_G) \subset \mathbb{P}^{N_G - 1} \quad , \quad B_i = V(\alpha_i) \subset \mathbb{P}^{N_G - 1}$$



$$\omega_G \in \Omega^{N_G-1}(\mathbb{P}^{N_G-1} \backslash \overline{X}_G) \qquad \text{and} \qquad \partial\sigma \subset B = \cup_i B_i \ .$$

The graph motive (II)

The naive ‘motive’ (or rather, mixed Hodge structure) is

$$H^{N_G-1}(\mathbb{P}^{N_G-1}\backslash\overline{X}_G, B\backslash(B\cap\overline{X_G})$$

However, in reality, the domain of integration σ meets the singular locus $\overline{X}_G$ so we must do some blow-ups. B-E-K construct an explicit local resolution of singularities $\pi : P \to \mathbb{P}^{N_G-1}$ and define

$$mot(G) = H^{N_G-1}(P\backslash\widetilde{\overline{X}}_G, \widetilde{B}\backslash(\widetilde{B}\cap\widetilde{\overline{X}}_G))$$

Theorem (Bloch-Esnault-Kreimer 2007)

The Feynman amplitude I_G is a period of $mot(G)$

NB. The point-counting function depends on $\sum_i(-1)^i H^i_c(\overline{X}_G)$. The amplitude I_G depends on $mot(G)$. Not the same motive!

The status of the point-counting problem

Kontsevich's conjecture ($[G]_q$ is polynomial in q) is therefore a (rough) analogue of the folklore conjecture (I_G is an MZV).

Theorem (Stembridge 1998)

The conjecture is true for all graphs with ≤ 12 edges.

The following result came as a great surprise:

Theorem (Belkale-Brosnan 2003)

The conjecture is false in general. In fact, $[G]_q$ is of 'general type'.

The proof uses Mnëv's universality theorem in a very clever way. Counter-examples constructed via their proof would give graphs with huge numbers of edges, i.e., highly unphysical graphs.

Theorem (2009), effective version with Schnetz (2011)

The conjecture is true for all graphs of vertex width ≤ 3.

In reality, the point-counting function $[G]_q$ contains a lot of junk. We observe that for our graphs G, and prime powers q

$$[G]_q \equiv 0 \mod q^2 .$$

In particular, for each q there exists $c_2(G)_q \in \mathbb{Z}/\mathbb{Z}q$ such that

$$[G]_q \equiv q^2 c_2(G)_q \mod q^3$$

Definition

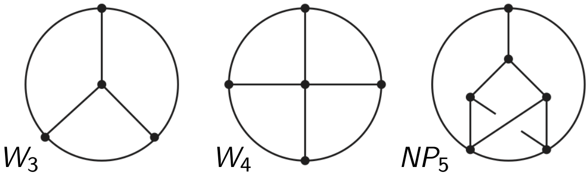
The c_2-invariant of a graph G is the element

$$c_2(G) = (c_2(G)_q) \in \prod_q \mathbb{Z}/\mathbb{Z}q$$

If $[G]_q$ is a polynomial in q, $c_2(G)_q$ is just the coefficient of q^2 in $[G]_q$, so there is a constant $M \in \mathbb{Z}$ such that $c_2(G)_q \equiv M \mod q$.

Point-counting examples II

Consider the following three graphs:



Graph	I_G	$[G]_q$	$c_2(G)_q$	Wt drop
W_3	$6\zeta(3)$	$q^5 - q^3 - q^2$	$-1 \mod q$	No
W_4	$20\zeta(5)$	$q^7 + 3q^5 - 6q^4 + 4q^3 - q^2$	$-1 \mod q$	No
NP_5	$36\zeta(3)^2$	$q^9 + 4q^7 - 7q^6 + 3q^5$	$0 \mod q$	Yes

It turns out that only the coefficient of q^2 contains the relevant information about the amplitude I_G.

Completion conjecture

All the relevant qualitative information about the amplitude I_G is contained in the c_2-invariant. The philosophy is that

$$I_{G_1} = I_{G_2} \quad \overset{Conj.}{\Longrightarrow} \quad c_2(G_1) = c_2(G_2)$$

In particular, we get the following very concrete conjecture:

Completion conjecture

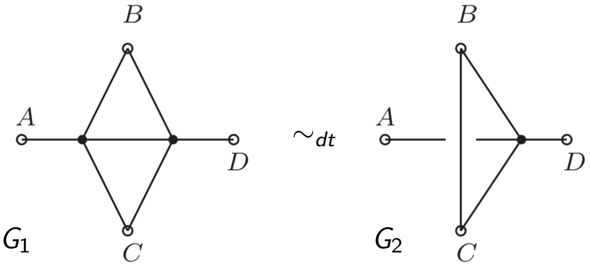
If G_1 and G_2 have isomorphic completions ($\widehat{G}_1 \cong \widehat{G}_2$) then

$$c_2(G_1)_q \equiv c_2(G_2)_q \quad \text{for all } q$$

The c_2-invariants have many nice combinatorial properties which makes them easy to compute.

$$c_2(G) = 0 \quad \longleftrightarrow \quad G \text{ has weight-drop}$$

Example of an identity (with K. Yeats). Double-triangle reduction

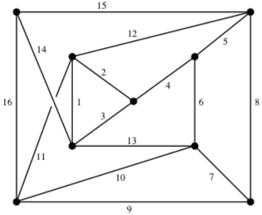


If $G_1 \sim_{dt} G_2$ are related as above then $c_2(G_1) = c_2(G_2)$.

This identity, and others, enable us to compute the c_2-invariant of graph hypersurfaces in terms of hypersurfaces of smaller and smaller dimensions. Idea: fiber in curves of genus 0 and use Chevalley-Warning theorem.

Modular counter-example

Let G_8 be the following graph with 8 loops. It is primitive log-divergent, in ϕ^4 theory (and has $vw(G) = 4$).



Theorem (with O. Schnetz, 2012) Uses Shioda-Inosé, Serre, Livné,. . . , Schütt

$$c_2(G_8)_p \equiv a_p \mod p$$

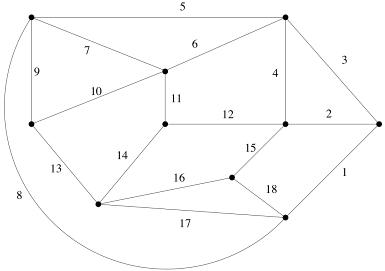
where a_p are the Fourier coefficients of the modular form $\big(\eta(z)\eta(z^7)\big)^3$ of weight 3 and level 7.

Recall $\eta(z) = z^{\frac{1}{24}} \prod_{n\geq 1}(1 - z^n)$ is the Dedekind eta function.

A planar counter-example

We can show by the prime number theorem that the a_p are highly non-constant. Therefore G_8 cannot be a polynomial in q.

The following counter-example at 9 loops has the same c_2:



and shows that Kontsevich's conjecture is false even for *planar* ϕ^4 theory. But could it still be the case that $mot(G)$ is mixed-Tate?

From point-counts to cohomology

Theorem (with D. Doryn, 2013) Uses Motivic CW theorem of Bloch-Esnault-Levine

The cohomology class of the Feynman integrand

$$\left[\frac{\Omega_{G_8}}{\Psi^2_{G_8}}\right] \in \mathrm{gr}^W_{24} H^{15}_{dR}(\mathbb{P}^{15}\backslash \overline{X}_{G_8})$$

is of Hodge type $(13, 11)$. In particular, it is not Tate.

Recall that if M mixed Tate then M_{dR} of type (n, n) only.

Corollary

The I_G cannot factor through a category of mixed Tate motives.

A variant of Grothendieck's standard transcendence conjecture for periods implies that I_{G_8} is not in the ring of multiple zeta values.

All variants of the folklore conjecture are completely false!

More modular counter-examples

With O. Schnetz, we computed $c_2(G)_p$ for the first 13 primes for all $\sim 10,000$ graphs up to 10 loops. The following modular 'hits':

weight	2	3	4	5	6	7	8
level	11	$\boxed{7}_8$	$\boxed{5}_8$	$\boxed{4}_9$	$\boxed{3}_8$	$\boxed{3}_9$	$\boxed{2}_{10}$
	14	$\boxed{8}_8$	$\boxed{6}_9$	7	$\boxed{4}_9$	7	3
	15	11	$\boxed{7}_{10}$	8	5	8	$\boxed{5}_{10}$
	17	$\boxed{12}_9$	8	11	6	11	6
	19	15	9	12	$\boxed{7}_9$	15	7
	20	15	10	15	8	15	8
	21	16	12	15	9	16	8
	24	19	$\boxed{13}_9$	19	$\boxed{10}_{10}$	19	9
	26	20	$\vdots$	20	10	20	10
	26	20	$\boxed{17}_{10}$	20	10	20	12

The subscript is the first loop order it occurs. Only the first example is proved. No modular forms of weight 2?

Conclusion

All versions of the folklore conjecture are true for small graphs (graphs with vertex-width ≤ 3) but completely false in general.

Even when we take the sum over all graphs (the physically meaningful quantity), there is nothing for the modular counter-examples to 'cancel with'. They remain in physical answer.

The numbers I_G coming from physics go beyond the realm of multiple zeta values, but are nonetheless highly constrained (no $\zeta(2)$, no modular forms of weight 2, ...). The situation is much more complex, and interesting, than anyone imagined.

What is the class of motives that Quantum Field Theory chooses?

Conjectural Trichotomy

For all graphs up to 10 loops it appears that we have 3 classes:

1. (Vanishing) $c_2(G) = 0$. These graphs should have weight-drop and contribute to the Quantum Field Theory in a special way. This class contains all non-primitive graphs.
2. (Tame) $c_2(G) = -1$. We found that all graphs in this class are equivalent, modulo completion and double-triangle reduction to *a single graph*, the wheel with 3 spokes:

3. (Wild) $c_2(G)$ is non-constant. These start at 7 loops, contains all the modular examples and most are unknown. These are all counter-examples to the point-counting conjecture.

Furthermore, we expect that class 2 give a precise and strict subspace of MZV's, in which, e.g., no $\zeta(2n)$'s (or $\zeta(a, b)$'s) occur.

Motivic versus Cosmic Galois group

Let $\mathcal{Z}$ denote the ring of MZV's. We should think of $\mathcal{Z}/\mathbb{Q}$ as a 'Galois' extension of transcendental numbers[3] with a pro-algebraic Galois group $Gal(MT(\mathbb{Z}))$. Its Lie algebra is free, with one generator in every odd degree corresponding to $\zeta(2n+1)$.

Let P be the ring spanned by Feynman amplitudes I_G for $G \in \phi^4$. The counter-examples suggest $P \not\subset \mathcal{Z}$ but also

$$P \cap \mathcal{Z} \subsetneq \mathcal{Z}$$

i.e., only special linear combinations of MZV's occur as amplitudes.

Miracle: Experimentally, $P \cap \mathcal{Z}$ is preserved by G^{MZV}

Holy grail would be a formula for the action of G^{MZV} in terms of graphs. Generators of Lie algebra $\leftrightarrow$ the zig-zag graphs Z_n.

Is P closed under the action of a bigger, 'cosmic' Galois group?

[3] Grothendieck, André, Deligne, Ihara, Kontsevich-Zagier, Goncharov, ...