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Single-valued hyperlogarithms and unipotent differential equations.

ABSTRACT. Hyperlogarithms are iterated integrals with singularities in a finite set $\Sigma \subset \mathbb{P}^1(\mathbb{C})$, and generalise the classical polylogarithms. We define a universal differential algebra of hyperlogarithms $\mathcal{HL}_\Sigma$, and study its algebraic and differential structure. We show it is the unipotent closure of the ring of regular functions on $\mathbb{P}^1(\mathbb{C})\backslash\Sigma$ and reflects the structure of solutions to any unipotent differential equation on $\mathbb{P}^1(\mathbb{C})\backslash\Sigma$. By taking linear combinations of products of hyperlogarithms and their complex conjugates, we construct a differential algebra of functions isomorphic to $\mathcal{HL}_\Sigma$ which has any prescribed unipotent monodromy. This is related to the Riemann-Hilbert problem. We obtain, in particular, a canonical differential algebra of single-valued hyperlogarithms which contains all generalisations of the Bloch-Wigner dilogarithm. This has applications in number theory and mathematical physics.

CONTENTS

1. INTRODUCTION

Let $\sigma_0, \ldots, \sigma_N$ be distinct complex numbers. Hyperlogarithms are solutions to a certain class of linear differential equations on $\mathbb{C}$ with regular singularities in the points σ_i. More precisely, if $X = \{x_0, \ldots, x_N\}$ is a set of non-commuting indeterminates, then for each word $w = x_{i_1} \ldots x_{i_n} \in X^*$, the *hyperlogarithm* $L_w(z)$ is the iterated integral ([Ch1])

$$L_{x_{i_1}\ldots x_{i_n}}(z) = \int_\gamma \frac{dt}{t-\sigma_{i_1}} \circ \ldots \circ \frac{dt}{t-\sigma_{i_n}}, \tag{1.1}$$

where γ is a path in $\mathbb{P}^1(\mathbb{C})\backslash\{\sigma_0, \ldots, \sigma_N\}$ beginning at some fixed point and terminating at z. These are multi-valued holomorphic functions on $\mathbb{P}^1(\mathbb{C})\backslash\{\sigma_0, \ldots, \sigma_N, \infty\}$, and were first studied by Kummer and Poincaré [K, P]. Later, Lappo-Danielevski used these functions to obtain a partial solution to the inverse monodromy, or Riemann-Hilbert, problem [L-D]. Hyperlogarithms are closely related to hypergeometric functions [Ca2, G-L] and have many special properties; they are known to satisfy functional equations, for example, but these are still very poorly understood ([A2, We, Ga, Z2]).

The present work is motivated by some new applications of the classical polylogarithms $\mathrm{Li}_n(z)$ which have emerged recently [Ha, Oe]. These functions are hyperlogarithms with singularities $\sigma_0 = 0$, $\sigma_1 = 1$, and correspond to the family of words $-x_0^{n-1}x_1$. They are defined by the series

$$\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n} \qquad \text{for } |z| < 1,$$

and extend to multi-valued functions on $\mathbb{P}^1(\mathbb{C})\backslash\{0, 1, \infty\}$. The *Bloch-Wigner dilogarithm* is a single-valued version of $\mathrm{Li}_2(z)$ defined by the expression

$$\mathrm{Im}\big(\mathrm{Li}_2(z) + \log|z|\log(1-z)\big). \tag{1.2}$$

It extends to a real-valued continuous function on the whole of $\mathbb{P}^1(\mathbb{C})$, and has important uses in many branches of mathematics: for example, it allows one to compute Borel's regulator on the algebraic K-group $K_3(\mathbb{C})$, volumes of hyperbolic manifolds in dimension 3, and special values of Artin L-functions at 2. Single-valued versions of the functions $\mathrm{Li}_n(z)$ are conjectured to play a similar role in higher K-theory. One of our main aims in this paper is to construct explicit single-valued versions for all hyperlogarithms and study their algebraic and differential relations. This generalises the results announced in [Br1], and is the main step in constructing a corresponding theory of multiple polylogarithms in many variables [Br2].

Hyperlogarithms have also resurfaced recently in the explicit construction of unipotent variations of mixed Hodge structure as part of the study of the motivic fundamental group of $\mathbb{P}^1(\mathbb{C})\backslash\Sigma$, where Σ consists of $0, \infty$ and roots of unity ([B-D, D-G, G1, Rac, W1]). In the special case $\Sigma = \{0, 1, \infty\}$, the set of hyperlogarithms coincides with the set of multiple polylogarithms in one variable. Their regularised values at 1 are multiple zeta values $\zeta(w)$, and have been studied extensively [Ca1, G2, Ho]. Furthermore, the generating series

$$Z^1(X) = \sum_{w \in \{x_0, x_1\}^*} \zeta(w)\, w,$$

is none other than Drinfeld's associator for the Knizhnik-Zamolodchikov equation. This series, and its generalisations, appear naturally in the monodromy computations of the hyperlogarithm functions studied here.

There is further motivation for studying hyperlogarithms with arbitrary singularities in quantum field theory, where hyperlogarithms (or 'harmonic polylogarithms') naturally arise in the evaluation of Feynman integrals (see *e.g.* [R-V]).

Let $\Sigma = \{\sigma_0, \ldots, \sigma_N, \infty\}$, and let $D = \mathbb{P}^1(\mathbb{C})\backslash\Sigma$ denote the punctured complex line with universal covering $p : \widehat{D} \to D$. If U is a simply-connected open dense subset of D, then hyperlogarithms $\{L_w(z) : w \in X^*\}$ are the unique family of holomorphic functions satisfying the recursive differential equations:

$$\frac{\partial}{\partial z} L_{x_i w}(z) = \frac{L_w(z)}{z - \sigma_i} \qquad z \in U,$$

such that $L_e(z) = 1$, $L_{x_0^n}(z) = \frac{1}{n!}\log^n(z - \sigma_0)$ for all $n \in \mathbb{N}$, and $L_w(z) \to 0$ as $z \to 0$ for all other words w. We also write $L_w(z)$ for the analytic continuation of these functions to $\widehat{D}$. We denote the ring of regular functions on D by

$$\mathcal{O}_\Sigma = \mathbb{C}\Big[z, \Big(\frac{1}{z - \sigma_i}\Big)_{0 \le i \le N}\Big],$$

and let L_Σ be the free $\mathcal{O}_\Sigma$-module spanned by all the hyperlogarithms $L_w(z)$, $w \in X^*$. The shuffle product formula for iterated integrals ([Ch1]) implies that L_Σ is a differential algebra, and it does not depend, up to isomorphism, on the choice of branch of the logarithm $L_{x_0}(z) = \log(z - \sigma_0)$.

Let $\mathbb{C}\langle X\rangle$ be the free tensor algebra on X, *i.e.* the complex vector space generated by X^*. One can define the shuffle product $ш : \mathbb{C}\langle X\rangle \times \mathbb{C}\langle X\rangle \to \mathbb{C}\langle X\rangle$, which makes $\mathbb{C}\langle X\rangle$ into a commutative polynomial algebra. We define the *universal algebra of hyperlogarithms* to be the vector space

$$\mathcal{HL}_\Sigma = \mathcal{O}_\Sigma \otimes \mathbb{C}\langle X\rangle,$$

equipped with the shuffle product, and define a linear map $\partial : \mathcal{HL}_\Sigma \to \mathcal{HL}_\Sigma$ which is a derivation for the shuffle product. This makes $\mathcal{HL}_\Sigma$ into a commutative differential algebra whose differential structure is described by the following theorem.

Theorem 1.1. *Every differential $\mathcal{O}_\Sigma$-subalgebra of $\mathcal{HL}_\Sigma$ is differentially simple. Its ring of constants is $\mathbb{C}$, and every element possesses a primitive in $\mathcal{HL}_\Sigma$.*

Recall that a differential algebra (R, ∂) is differentially simple if every ideal I such that $\partial I \subset I$ is 0 or the ring R itself. It follows that the field of fractions of $\mathcal{HL}_\Sigma$ is an infinite union of Picard-Vessiot extensions of $\mathbb{C}(z)$. We then define a *realisation of hyperlogarithms* to be any differential $\mathcal{O}_\Sigma$-algebra M, and a non-zero morphism

$$\rho : \mathcal{HL}_\Sigma \longrightarrow M.$$

Any such map is necessarily injective, since its kernel is a differential ideal, which is zero by the previous theorem. The *holomorphic realisation of hyperlogarithms* is the map

$$\begin{array}{ccc} \mathcal{HL}_\Sigma & \longrightarrow & L_\Sigma \\ w & \mapsto & L_w(z), \end{array}$$

which is therefore an isomorphism. Using a theorem due to Radford, one can show that $\mathcal{HL}_\Sigma$ is a commutative polynomial algebra, and write down an explicit transcendence basis. This gives a complete description of the relations between the functions $L_w(z)$.

One can show, in fact, that $\mathcal{HL}_\Sigma$ naturally carries the structure of a differential graded Hopf algebra. On the other hand, we show that $\mathcal{HL}_\Sigma$ can naturally be defined as the universal solution of unipotent differential equations with singularities in Σ. Suppose that A is any differential algebra which is finitely generated over $\mathcal{O}_\Sigma$. We say that A is unipotent if its group of $\mathcal{O}_\Sigma$-differential automorphisms is unipotent (along with some other conditions). We show that any such A embeds into $\mathcal{HL}_\Sigma$, and deduce that $\mathcal{HL}_\Sigma$ is the unipotent closure of $\mathcal{O}_\Sigma$:

$$\mathcal{HL}_\Sigma = \varinjlim A, \tag{1.3}$$

where A ranges over the set of all finite unipotent extensions of $\mathcal{O}_\Sigma$. The algebra $\mathcal{HL}_\Sigma$ therefore reflects the structure of the solutions to any unipotent differential equation on D. Equation (1.3) also implies that the differential Galois group $\mathrm{Gal}(\mathcal{HL}_\Sigma/\mathcal{O}_\Sigma)$ is pro-unipotent, since it is the projective limit of the Galois groups of all unipotent extensions of $\mathcal{O}_\Sigma$. We show that this group has a very simple description as the set of group-like elements in the complete Hopf algebra $\mathbb{C}\langle\langle X\rangle\rangle$. Its Lie algebra is just the pro-Lie algebra of primitive elements in $\mathbb{C}\langle\langle X\rangle\rangle$.

We next turn to the question of constructing single-valued versions of hyperlogarithms. These are constructed by taking linear combinations of products of hyperlogarithms and their complex conjugates. Let $z \mapsto \overline{z}$ denote complex conjugation, and write $\overline{\mathcal{O}}_\Sigma$ for the conjugate of the algebra $\mathcal{O}_\Sigma$. Let $\overline{L}_\Sigma$ be the $\overline{\mathcal{O}}_\Sigma$-span of the functions $\overline{L_w}(z)$, and let $\overline{\mathcal{HL}}_\Sigma$

be the conjugate of the universal algebra of hyperlogarithms, with differential we denote $\overline{\partial}$. We define the *algebra of generalised hyperlogarithms* to be the $\mathcal{O}_\Sigma \otimes \overline{\mathcal{O}}_\Sigma \cong \mathcal{O}_\Sigma\overline{\mathcal{O}}_\Sigma$-algebra $\mathcal{HL}_\Sigma \otimes \overline{\mathcal{HL}}_\Sigma$ equipped with both differentials ∂, $\overline{\partial}$. There is a generalised realisation:

$$\begin{array}{rcl} \mathcal{HL}_\Sigma \otimes \overline{\mathcal{HL}}_\Sigma & \longrightarrow & L_\Sigma \overline{L}_\Sigma \\ w \otimes w' & \mapsto & L_w(z)\overline{L}_{w'}(z). \end{array}$$

It follows from theorem 1.1 that $\mathcal{HL}_\Sigma \otimes \overline{\mathcal{HL}}_\Sigma$ is differentially simple, and thus that the functions $L_w(z)\overline{L}_{w'}(z)$ are linearly independent over $\mathcal{O}_\Sigma\overline{\mathcal{O}}_\Sigma$. By applying a combinatorial argument to an explicit monodromy computation, it is possible to take linear combinations of functions $L_w(z)\overline{L}_{w'}(z)$ in such a way as to eliminate their monodromy.

Theorem 1.2. *Let $\mathcal{U}_\Sigma \subset L_\Sigma\overline{L}_\Sigma$ denote the set of all single-valued functions in $L_\Sigma\overline{L}_\Sigma$. There is an explicit realisation of hyperlogarithms which induces*

$$\mathcal{HL}_\Sigma \otimes \overline{\mathcal{O}}_\Sigma \xrightarrow{\sim} \mathcal{U}_\Sigma.$$

In other words, we can associate to each word $w \in X^*$ a canonical single-valued hyperlogarithm $\mathcal{L}_w(z)$, which is an explicit linear combination of functions $L_\eta(z)L_{\eta'}(\overline{z})$. These functions are defined on D, as opposed to on the universal covering $\widehat{D}$. They form the unique family of single-valued functions satisfying

$$\frac{\partial}{\partial z}\mathcal{L}_{x_k w}(z) = \frac{\mathcal{L}_w(z)}{z-\sigma_k}$$

such that $\mathcal{L}_e(z) = 1$, $\mathcal{L}_{x_0^n}(z) = \frac{1}{n!}\log^n |z-\sigma_0|^2$, and $\lim_{z\to 0}\mathcal{L}_w(z) = 0$ for all words w not of the form x_0^n. It follows automatically that the functions $\mathcal{L}_w(z)$ are linearly independent over $\mathcal{O}_\Sigma\overline{\mathcal{O}}_\Sigma$ and satisfy the shuffle relations. The theorem also implies that every single-valued version of hyperlogarithms can be written as a unique linear combination of the functions $\mathcal{L}_w(z)$. The functions $\mathcal{L}_w(z)$ also satisfy an explicit differential equation with respect to $\partial/\partial\overline{z}$ with singularities in Σ. This yields, in particular, every possible generalisation of the Bloch-Wigner dilogarithm (1.2).

By generalising the construction in the previous theorem, we can in fact construct a realisation of hyperlogarithms

$$\mathcal{HL}_\Sigma \longrightarrow M \subset L_\Sigma\overline{L}_\Sigma$$

whose monodromy representation is arbitrary, provided that it is unipotent (theorem 7.4). This gives a correspondence between certain realisations of $\mathcal{HL}_\Sigma$ and pro-unipotent representations of the fundamental group of $\mathbb{P}^1(\mathbb{C})\backslash\Sigma$, which is related to the Riemann-Hilbert problem.

Equation (1.3) and the previous theorem imply that the algebra $\mathcal{U}_\Sigma$ is the universal single-valued solution to unipotent differential equations on $\mathbb{P}^1(\mathbb{C})\backslash\Sigma$:

Corollary 1.3. *Every unipotent differential equation on $\mathbb{P}^1(\mathbb{C})\backslash\Sigma$ has an explicit single-valued solution in terms of the functions $\mathcal{L}_w(z)$.*

Using this result, we show how functional equations for hyperlogarithms in one variable can be linearised by working in $\mathcal{U}_\Sigma$, for some suitable set Σ. This can be interpreted as a functoriality statement about the algebras $\mathcal{U}_\Sigma$. Another (future) application of the corollary is the construction of single-valued solutions to the Knizhnik-Zamalodchikov equation, for example.

The work in this paper is related to Chen's general theory of iterated integrals and his study of the fundamental group of certain differentiable manifolds. One should be able to generalise the constructions in this paper to this more general setting (*c.f.* [W1]). Indeed, a Picard-Vessiot theory of iterated integrals is suggested in [Ch2], though the complex analytic aspect is not treated there. It would also be interesting to compare the alternative approach to constructing single-valued polylogarithms via the period matrices of variations of mixed Hodge structures ([G1], [Ram]), which have been computed explicitly in some special cases, and Dupont's rather different construction ([Du]).

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2. Hyperlogarithms.

We recall the definition of hyperlogarithm functions and some of their analytic properties. See Lappo-Danilevskii [L-D] or Gonzalez-Lorca ([G-L]) for futher detail.

2.1. **A formal differential equation.** If $N \geq 1$, let $X = \{x_0, \ldots, x_N\}$ be an alphabet with $N+1$ letters, and let X^* be the free non-commutative monoid generated by X, *i.e.* the set of all words w in the symbols x_k, along with the empty word e. Let $\mathbb{C}\langle\langle X\rangle\rangle$ be the algebra of non-commutative formal power series in X^*, with coefficients in $\mathbb{C}$. We fix any injective map of sets $j : X \hookrightarrow \mathbb{C}$, and set $\sigma_0 = j(x_0)$, $\ldots$, $\sigma_N = j(x_N)$. Let Σ denote the set $j(X) \cup \{\infty\}$, and denote by $D = \mathbb{P}^1(\mathbb{C})\backslash\Sigma$ the complex plane with the points σ_k removed. We will frequently write Σ as a shorthand for the map $j : X \hookrightarrow \mathbb{C}$; alternatively, one can simply regard Σ as a set with the ordering induced from j.

Consider the following differential equation:

$$\frac{\partial}{\partial z} F(z) = \sum_{i=0}^{N} \frac{x_i}{z-\sigma_i} F(z), \tag{2.1}$$

which is an equation of Fuchs type, whose singularities are simple poles in Σ. Let $F(z)$ be a solution on D taking values in $\mathbb{C}\langle\langle X\rangle\rangle$. If we write

$$F(z) = \sum_{w \in X^*} F_w(z) w,$$

then (2.1) is equivalent to the system of equations

$$\frac{\partial}{\partial z} F_{x_k w}(z) = \frac{F_w(z)}{z-\sigma_k}, \tag{2.2}$$

for all $0 \leq k \leq N$ and all $w \in X^*$.

2.2. **Existence and uniqueness of holomorphic solutions.** One can construct explicit holomorphic solutions $L_w(z)$ to (2.2) on a certain domain U obtained by cutting $\mathbb{C}$. These functions extend by analytic continuation to multi-valued functions on the punctured plane D, and can equivalently be regarded as holomorphic functions on a universal covering space $p : \widehat{D} \to D$. Since no confusion arises, we shall always denote these functions by the same symbol $L_w(z)$.

For each $0 \leq k \leq N$, choose a closed half-line $\ell(\sigma_k) \subset \mathbb{C}$ starting at σ_k such that no two intersect. Let $U = \mathbb{C}\backslash \bigcup_{\sigma_k \in \Sigma} \ell(\sigma_k)$ be the simply-connected open subset of $\mathbb{C}$ obtained by cutting along these half-lines. Fix a branch of the logarithm $\log(z-\sigma_0)$ on $\mathbb{C}\backslash\ell(\sigma_0)$.

Theorem 2.1. *([G-L].) Equation* (2.1) *has a unique solution* $L(z)$ *on* U *such that*

$$L(z) = f_0(z) \exp(x_0 \log(z-\sigma_0)),$$

where $f_0(z)$ is a holomorphic function on $\mathbb{C}\backslash\bigcup_{k\neq 0}\ell(\sigma_k)$ which satisfies $f_0(\sigma_0)=1$. We write this $L(z)\sim(z-\sigma_0)^{x_0}$ as $z\to\sigma_0$. Furthermore, every solution of (2.1) which is holomorphic on D can be written

$$L(z)\,C,$$

where $C\in\mathbb{C}\langle\langle X\rangle\rangle$ is a constant series (i.e. depending only on Σ, and not on z).

Proof. (Sketch). We first define functions $L_w(z)$ which satisfy (2.2) for all words $w\in X^*$ not ending in the letter x_0. For such words, the limiting condition is just $\lim_{z\to\sigma_0}L_w(z)=0$. If we write $w=x_0^{n_r}x_{i_r}x_0^{n_{r-1}}x_{i_{r-1}}\dots x_0^{n_1}x_{i_1}$, where $1\leq i_1,\dots,i_r\leq N$, then $L_w(z)$ is defined in a neighbourhood of σ_0 by the formula

$$\sum_{1\leq m_1<\cdots<m_r}\frac{(-1)^r}{m_1^{n_1+1}\dots m_r^{n_r+1}}\Big(\frac{z-\sigma_0}{\sigma_{i_1}-\sigma_0}\Big)^{m_1}\Big(\frac{z-\sigma_0}{\sigma_{i_2}-\sigma_0}\Big)^{m_2-m_1}\cdots\Big(\frac{z-\sigma_0}{\sigma_{i_r}-\sigma_0}\Big)^{m_r-m_{r-1}}\tag{2.3}$$

which converges absolutely for $|z-\sigma_0|<\inf\{|\sigma_{i_1}-\sigma_0|,\dots,|\sigma_{i_r}-\sigma_0|\}$. One can easily check that this defines a family of holomorphic functions satisfying the equations (2.2) in this open ball, and that the boundary condition is trivially satisfied. The functions $L_w(z)$ extend analytically to the whole of U by the recursive integral formula

$$L_{x_kw}(z)=\int_0^z\frac{L_w(t)}{t-\sigma_k}\,dt,\tag{2.4}$$

which is valid for $0\leq k\leq N$. One can then write down an explicit formula for the function $f_0(z)$ in such a way that $L(z)=f_0(z)\exp(x_0\log(z-\sigma_0))$ satisfies equation (2.1), where $\log(z-\sigma_0)$ is the given branch of the logarithm on $\mathbb{C}\backslash\ell(\sigma_0)$ (see [G-L], page 3). The coefficients of $f_0(z)$ are linear combinations of the functions $L_w(z)$ just defined, and it follows that $f_0(z)$ is holomorphic in a neighbourhood of σ_0.[1]

In order to prove the uniqueness, suppose $K(z)$ is any other solution of (2.1) which is holomorphic on U. The series $L(z)$ defined above is invertible, as its leading coefficient is the constant function 1. We may therefore consider the function $F(z)=L(z)^{-1}K(z)$. On differentiating the equation $K(z)=L(z)F(z)$, we obtain by (2.1)

$$\sum_{i=0}^N\frac{x_i}{z-\sigma_i}K(z)=\sum_{i=0}^N\frac{x_i}{z-\sigma_i}L(z)F(z)+L(z)F'(z),$$

and therefore $L(z)F'(z)=0$. Since $L(z)$ is invertible, $F'(z)=0$, and so $F(z)$ is a constant series C. □

Remark 2.2. The functions $L_w(z)$ are hyperlogarithms. Clearly $L_e(z)=1$, and for $n\in\mathbb{N}$,

$$\begin{aligned}L_{x_i^n}(z)&=\frac{1}{n!}\log^n\Big(\frac{z-\sigma_i}{\sigma_0-\sigma_i}\Big)\quad\text{if}\quad i\geq 1,\\ L_{x_0^n}(z)&=\frac{1}{n!}\log^n(z-\sigma_0).\end{aligned}$$

Note that $L_{x_0^n}(z)$ depends on the choice of branch of $\log(z-\sigma_0)$ which was fixed previously, but that the functions $L_{x_i^n}(z)$ do not. They are the unique branches which satisfy the limiting condition $L_{x_i^n}(\sigma_0)=0$.

In the case when $\sigma_0=0$ and $\sigma_1=1$, and $D=\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}$, then we have $L_{x_0^{m-1}x_1}(z)=-\mathrm{Li}_m(z)$, where $\mathrm{Li}_m(z)=\sum_{n\geq 1}(z/n)^m$ is the classical m^{th} polylogarithm function. For all $w\in X^*$, $L_w(z)=(-1)^r\mathrm{Li}_w(z)$ where $\mathrm{Li}_w(z)$ is a multiple polylogarithm in one variable ([Br1]), and r is the number of occurences of x_1 in w. The difference in

[1] Another way to prove this is to define the functions $L_{wx_0^n}(z)$ directly via the shuffle relations (remark 2.5) and then show that $f_0(z)=L(z)\exp(-x_0\log(z-\sigma_0))$ is holomorphic at $z=\sigma_0$. See also remark 6.4.

sign comes about because it is traditional to consider iterated integrals of the 1-forms dz/z and $dz/1-z$ in the case $\Sigma = \{0,1,\infty\}$, whereas in the more general context it is more natural to take $dz/z-\sigma_i$.

Let us fix a branch of the logarithm $\log(z-\sigma_k)$ on $\mathbb{C}\backslash\ell(\sigma_k)$ for each $1\leq k\leq N$. By translating the variable z, we obtain a solution to (2.1) corresponding to each singularity:

Corollary 2.3. *For every $1\leq k\leq N$, there exists a unique solution $L^{\sigma_k}(z)$ of equation* (2.1) *on U such that*

$$L^{\sigma_k}(z) = f_k(z)\exp(x_k\log(z-\sigma_k)),$$

where $f_k(z)$ is holomorphic on $\mathbb{C}\backslash\bigcup_{i\neq k}\ell(\sigma_i)$ and satisfies $f_k(\sigma_k)=1$.

Likewise, by the change of variables $z'-\sigma_0=(z-\sigma_0)^{-1}$, one obtains a solution $L^{\infty}(z)$ to (2.1) on U which corresponds to the point at infinity. If we fix any branch of $\log z$ on U, then one can show that $L^{\infty}(z)\sim\exp(x_\infty\log z)$ where $x_\infty=\sum_{0\leq k\leq N}x_k$.

2.3. **Shuffle relations and the algebra L_Σ.** Let $\mathbb{C}\langle X\rangle$ denote the free non-commutative $\mathbb{C}-$algebra generated by the symbols $x_i\in X$. The *shuffle product* $ш : \mathbb{C}\langle X\rangle\otimes\mathbb{C}\langle X\rangle\rightarrow\mathbb{C}\langle X\rangle$, is defined recursively for each $w\in X^*$, $i,j\in\{0,1,\ldots,N\}$ as follows:

$$\begin{aligned} e \,ш\, w &= w \,ш\, e = e, \\ x_i w \,ш\, x_j w' &= x_i(w \,ш\, x_j w') + x_j(x_i w \,ш\, w'), \end{aligned} \tag{2.5}$$

The function $w\mapsto L_w(z)$, viewed as a linear map on $\mathbb{C}\langle X\rangle$, is a homomorphism for the shuffle product, *i.e.*

$$L_w(z)L_{w'}(z) = L_{w\,ш\,w'}(z) \tag{2.6}$$

for all $w,w'\in\mathbb{C}\langle X\rangle$, $z\in U$. This follows from equations (2.2), and the form of the limiting conditions at σ_0 which define $L(z)$, and can be checked by induction. Shuffle relations are in fact a general property of any suitably-defined iterated integrals ([Ch1]). Let

$$\mathcal{O}_\Sigma = \mathbb{C}\Big[z, \Big(\frac{1}{z-\sigma_i}\Big)_{\sigma_i\in\Sigma}\Big] \tag{2.7}$$

denote the ring of regular functions on D.

Definition 2.4. Let L_Σ be the free $\mathcal{O}_\Sigma$-module generated by the functions $L_w(z)$.

It follows from the shuffle relations (2.6) that L_Σ is closed under multiplication, and is therefore a differential algebra for the derivation $\partial/\partial z$. It does not depend, up to isomorphism, on the choice of branches of the logarithm.

Remark 2.5. Let X_c^* denote the set of words which do not end in x_0, and write $\mathbb{C}\langle X_c\rangle\subset\mathbb{C}\langle X\rangle$ for the subalgebra they generate. Any word in X^* can be written as a linear combination of shuffles of the words x_0^n with words $\eta\in X_c^*$. It follows that one could first define $L_w(z)$ explicitly for words $w\in X_c^*$ by (2.3), and extend the definition to all words by setting $L_{x_0}(z)=\log(z-\sigma_0)$ and demanding that $L_w(z)$ satisfy the shuffle relations. The holomorphy of $f_0(z)=L(z)\exp(-x_0L_{x_0}(z))$ is then equivalent to the identity

$$\sum_{i=0}^{n}\frac{(-1)^i}{i!}wx_0^{n-i}\,ш\,x_0^i\equiv 0 \mod \mathbb{C}\langle X_c^*\rangle \qquad \text{for all } w\in X_c^*,$$

which is easily proved by induction on n. This gives another way to define the functions $L_w(z)$, and prove theorem 2.1.

3. The universal algebra of hyperlogarithms.

We define a differential algebra $\mathcal{HL}_\Sigma$ of abstract solutions to equation (2.1), and study its algebraic and differential properties. This generalises the notion of universal algebra of polylogarithms defined in [Br1]. Recall that the subscript Σ stands for $j : X \hookrightarrow \Sigma$.

3.1. **Definition of $\mathcal{HL}_\Sigma$.** For each $0 \leq k \leq N$, we define a truncation operator, which is a linear map $\partial_k : \mathbb{C}\langle X\rangle \to \mathbb{C}\langle X\rangle$, defined by the formula:

$$\partial_k(x_j w) = \delta_{kj} w$$

for all $w \in X^*$, and all $0 \leq j \leq N$, where δ_{ij} is the Kronecker symbol. It is easy to check by induction using (2.5) that the operators ∂_k are derivations for the shuffle product.

Remark 3.1. The shuffle product is in fact the unique product on $\mathbb{C}\langle X\rangle$ for which e is a unit, and the operators ∂_k are all derivations.

Definition 3.2. The *universal algebra of hyperlogarithms* is the differential $\mathcal{O}_\Sigma$-algebra

$$\mathcal{HL}_\Sigma = \mathcal{O}_\Sigma \otimes_{\mathbb{C}} \mathbb{C}\langle X\rangle,$$

with multiplication induced by the shuffle product, and equipped with the derivation

$$\partial = \frac{\partial}{\partial z} \otimes 1 + \sum_{\sigma_i \in \Sigma} \Big(\frac{1}{z-\sigma_i}\Big) \otimes \partial_i.$$

The map $w \mapsto (-1)^r w$ is an isomorphism $\mathcal{HL}_{\{0,1,\infty\}} \xrightarrow{\sim} \mathcal{PL}$, where $\mathcal{PL}$ is the universal algebra of multiple polylogarithms in one variable defined in [Br1], and r is the number of occurrences of the symbol x_1 in w.

3.2. **Algebraic structure of $\mathcal{HL}_\Sigma$.** It is well-known that the shuffle product is associative and commutative. The algebraic structure of $\mathbb{C}\langle X\rangle$ is conveniently described by the set of *Lyndon words* $\mathbf{Lyn}(X) \subset X^*$, which are defined as follows. We first define an order on X by demanding that $x_0 < x_1 < \ldots < x_N$ and extend this lexigraphically to X^*. Then $w = x_{i_1} \ldots x_{i_n} \in X^*$ is in $\mathbf{Lyn}(X)$ if and only if

$$x_{i_1} \ldots x_{i_r} \leq x_{i_{r+1}} \ldots x_{i_n} \quad \text{for all } 1 \leq r \leq n.$$

Theorem 3.3. *(Radford [Rad]) $\mathbb{C}\langle X\rangle$ equipped with the shuffle product is the free polynomial algebra over the set of Lyndon words.*

For each $w \in X^*$, define the *weight* of w, written $|w|$, to be the number of symbols occurring in w. This gives a filtration on $\mathbb{C}\langle X\rangle$, with respect to which the operators ∂_k are strictly decreasing. The induced filtration on $\mathcal{HL}_\Sigma$ is defined for $m \in \mathbb{N}$ by

$$\mathcal{HL}_\Sigma^{(m)} = \{\sum_{|w|\leq m} f_w(z) \otimes w : f_w(z) \in \mathcal{O}_\Sigma\},$$

and is preserved by the derivation ∂, *i.e.* $\partial\mathcal{HL}_\Sigma^{(m)} \subset \mathcal{HL}_\Sigma^{(m)}$. There is an induced grading on $\mathcal{HL}_\Sigma$, which we denote $\mathrm{gr}_m\mathcal{HL}_\Sigma$, which is spanned by the $(N+1)^m$ words of weight m. By Radford's theorem, the number of algebraically independent elements in $\mathrm{gr}_m\mathcal{HL}_\Sigma$ is given by the number of Lyndon words of weight m, which is a polynomial in N of degree m. The Lyndon words of weight 1 are $\{x_k\}_{0\leq k\leq N}$, and those of weight 2 are $\{x_i x_j\}_{0\leq i<j\leq N}$.

$\mathbb{C}\langle X\rangle$ has two natural graded Hopf algebra structures which are dual to each other (in the graded sense) ([Bo]). The first has non-commutative multiplication given by the concatenation product, and a cocommutative coproduct $\Delta : \mathbb{C}\langle X\rangle \to \mathbb{C}\langle X\rangle \otimes \mathbb{C}\langle X\rangle$ which is the unique coproduct for which the words x_i, for $0 \leq i \leq N$, are primitive:

$$\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i. \tag{3.1}$$

The counit $\varepsilon : \mathbb{C}\langle X\rangle \to \mathbb{C}$ is given by projection onto the coefficient of the trivial word e. In order to define the antipode, we first define the mirror map

$$\begin{array}{rcl} \mathbb{C}\langle X\rangle & \to & \mathbb{C}\langle X\rangle \\ w & \mapsto & \widetilde{w}, \end{array} \tag{3.2}$$

which is the unique linear map which reverses the order of letters in each word. It restricts to the unique antihomomorphism of monoids $X^* \to X^*$ which maps each x_k to x_k. The antipode $a : \mathbb{C}\langle X\rangle \to \mathbb{C}\langle X\rangle$ is then the linear map defined by $a(w) = (-1)^{|w|}\widetilde{w}$. This defines the first (cocommutative) Hopf algebra structure. The maps ε, $\Delta, \widetilde{\ }, a$ extend naturally to the completed ring $\mathbb{C}\langle\langle X\rangle\rangle$ and the universal algebra of hyperlogarithms $\mathcal{HL}_\Sigma$, and we shall denote them by the same symbol.

The second Hopf algebra structure is commutative, and has multiplication given by the shuffle product. The coproduct $\Gamma : \mathbb{C}\langle X\rangle \to \mathbb{C}\langle X\rangle$ is defined by

$$\Gamma(w) = \sum_{uv=w} u \otimes v. \tag{3.3}$$

The counit is given by ε above, and the antipode is the unique linear map which sends $w \mapsto (-1)^{|w|}w$.

Corollary 3.4. *$\mathcal{HL}_\Sigma$ is a commutative differential graded Hopf algebra.*

We shall frequently use the following characterisation of the shuffle product, which follows from the duality between the two Hopf algebra structures on $\mathbb{C}\langle X\rangle$.

Corollary 3.5. *The map $S : \mathbb{C}\langle X\rangle \to \mathbb{C}$ is a homomorphism for the shuffle product if and only if the series*

$$S = \sum_{w\in X^*} S(w)w$$

is group-like for the coproduct Δ, i.e., $\Delta S = S \otimes S$.

Remark 3.6. Let $S \in \mathbb{C}\langle\langle X\rangle\rangle$. Although S may not necessarily be group-like for Δ, we can replace it with a series which is, *i.e.* there is a unique series $S^\times \in \mathbb{C}\langle\langle X\rangle\rangle$ such that

$$\Delta S^\times = S^\times \otimes S^\times, \quad \text{and} \quad S^\times(w) = S(w) \quad \text{for all } w \in \mathbf{Lyn}(X).$$

3.3. **Differential structure of $\mathcal{HL}_\Sigma$.** Recall that a (commutative, unitary) differential ring (A, ∂) is called *differentially simple* if A is a simple module over its ring of differential operators $A[\partial]$. A differential ideal of A is an ideal $I \subset A$ such that $\partial I \subset I$. Since ideals in a ring correspond to modules over that ring, it follows that A is differentially simple if and only if it has no non-trivial differential ideals. An equivalent condition is that for every $\theta \in A$, there exists an operator $D_\theta \in A[\partial]$ such that $D_\theta\, \theta = 1$. A differentially simple ring can be regarded as the differential analogue of a field in commutative algebra.

Recall that the ring of constants of A is just the kernel of ∂.

Theorem 3.7. *Every differential $\mathcal{O}_\Sigma$-subalgebra $A \subset \mathcal{HL}_\Sigma$ is differentially simple. The ring of constants of $\mathcal{HL}_\Sigma$ is $\mathbb{C}$, and every element has a primitive: i.e. the following sequence of complex vector spaces is exact:*

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{HL}_\Sigma \xrightarrow{\partial} \mathcal{HL}_\Sigma \longrightarrow 0.$$

The first part of the theorem says that for every $\theta \in \mathcal{HL}_\Sigma$, there is a $D_\theta \in \mathcal{O}_\Sigma[\partial]$ such that $D_\theta\theta = 1$. We in fact prove the stronger result that D_θ can be taken to be in $\mathbb{C}[z, \partial]$. One can show (*e.g.* [Ko]), that the simplicity of $\mathcal{HL}_\Sigma$ implies that the field of fractions of $\mathcal{HL}_\Sigma$ is an infinite Picard-Vessiot extension of $(\mathbb{C}(z), \partial/\partial z)$ (§4).

Proof. We first show that the ring of constants in $\mathcal{HL}_\Sigma$ is $\mathbb{C}$. Let $0 \neq x \in \mathcal{HL}_\Sigma$ such that $\partial x = 0$. We write

$$x = \prod_{i=0}^{N} (z - \sigma_i)^{-n_i} (f_0 + \ldots + f_n z^n) \tag{3.4}$$

where the $n_i \in \mathbb{N}$ are minimal, and $f_k \in \mathbb{C}\langle X \rangle$ with $f_n \neq 0$. Then $\partial x = 0$ implies

$$\sum_{i=0}^{N} \frac{n_i}{(z-\sigma_i)} (f_0 + \ldots + f_n z^n) = (f_1 + \ldots + f_n n z^{n-1}) + \sum_{i=0}^{N} \frac{\partial_i}{(z-\sigma_i)} (f_0 + \ldots + f_n z^n).$$

Set $p(z) = f_0 + \ldots + f_n z^n$. Taking the residue at $z = \sigma_i$ for $0 \leq i \leq N$, we obtain

$$n_i \, p(\sigma_i) = \partial_i \, p(\sigma_i).$$

Since the operators ∂_i are strictly decreasing for the filtration on $\mathbb{C}\langle X \rangle$, this implies that

$$n_i \, p(\sigma_i) = 0$$

for all $0 \leq i \leq N$. If $p(\sigma_i) = 0$ then $(z - \sigma_i)$ divides $p(z)$, which contradicts the minimality of the choice of n_i. It follows that $p(\sigma_i) \neq 0$ and n_i is zero for all $0 \leq i \leq N$. We have

$$p'(z) + \sum_{i=0}^{N} \frac{\partial_i}{(z-\sigma_i)} p(z) = 0.$$

Taking the residue of this equation at $z = \infty$, we obtain

$$n f_n = -\sum_{i=0}^{N} \partial_i f_n.$$

The right hand side is of strictly lower weight than the left hand side, so we conclude that $n f_n = 0$, and thus $n = 0$. We have shown that $x = f_0 \in \mathbb{C}\langle X \rangle$. One easily deduces from the equation $\partial x = 0$ that $\partial_i f_0 = 0$ for $0 \leq i \leq N$ and thus $f_0 \in \mathbb{C}$. This proves that the ring of constants in $\mathcal{HL}_\Sigma$ is $\mathbb{C}$. As a consequence, it follows that if $x \in \mathcal{HL}_\Sigma$ satisfies $\partial^n x = 0$ for some $n \in \mathbb{N}$, then $x \in \mathbb{C}[z]$.

We now prove by induction that for every non-zero $\theta \in \mathcal{HL}_\Sigma$, there exists $D_\theta \in \mathbb{C}[z, \partial]$ such that $D_\theta \theta = 1$. If θ is of weight 0, then $\theta \in \mathcal{O}_\Sigma$, and we may write

$$\theta = \prod_{i=0}^{N} (z - \sigma_i)^{-n_i} (a_0 + \ldots + a_n z^n)$$

where $a_k \in \mathbb{C}$, the n_i are integers, and $a_n \neq 0$. We may clearly take

$$D_\theta = \frac{1}{a_n} \frac{\partial^n}{n!} \prod_{i=0}^{N} (z - \sigma_i)^{n_i},$$

which lies in $\mathbb{C}[z, \partial]$. Now suppose that we know the result to be true for all elements up to weight $m - 1$, and let $\theta \in \mathcal{HL}_\Sigma^{(m)}$, $|\theta| = m$. We may write

$$\theta = \prod_{i=0}^{N} (z - \sigma_i)^{-n_i} (f_0 + \ldots + f_n z^n),$$

where $n_i \in \mathbb{N}$, and $f_i \in \mathbb{C}\langle X \rangle$ satisfy $m = \max_i |f_i|$. Setting

$$D = \partial^{n+1} \prod_{i=0}^{N} (z - \sigma_i)^{n_i},$$

and observing that $\partial \equiv \partial/\partial z \pmod{\mathcal{HL}_\Sigma^{(m-1)}}$ on $\mathcal{HL}^{(m)}$, we obtain

$$\phi = D\theta \in \mathcal{HL}_\Sigma^{(m-1)}.$$

If $\phi = 0$, then $\prod_{i=0}^N (z-\sigma_i)^{n_i}\theta \in \mathbb{C}[z]$ by a remark made previously. This implies $\theta \in \mathcal{O}_\Sigma$, and $m = 0$, which is a contradiction. Therefore we must have had $\phi \neq 0$, and by induction hypothesis there exists an operator $D_\phi \in \mathbb{C}[z,\partial]$ such that $D_\phi \phi = 1$. We complete the induction step by setting $D_\theta = D_\phi D \in \mathbb{C}[z,\partial]$.

It remains to show by induction that every element in $\mathcal{HL}_\Sigma$ has a primitive. Suppose that $w \in \mathbb{C}\langle X\rangle$, $|w| = m$, and that we have constructed a primitive for all elements in $\mathcal{HL}_\Sigma$ of strictly lower weight. By decomposing $f(z)w$, where $f(z) \in \mathcal{O}_\Sigma$, into partial fractions, it suffices to find a primitive of

$$(z-\sigma_i)^n w$$

for all $n \in \mathbb{N}$, and all $0 \leq i \leq N$. If $n = -1$, then this is just $x_i w$. If $n \neq -1$, then there exists a primitive by integration by parts and by the induction hypothesis.

□

3.4. **The differential Galois group of $\mathcal{HL}_\Sigma$.** We define the differential Galois group of the extension $\mathcal{HL}_\Sigma/\mathcal{O}_\Sigma$ to be the group of differential $\mathcal{O}_\Sigma$-automorphisms of $\mathcal{HL}_\Sigma$[2]:

$$\mathrm{Gal}(\mathcal{HL}_\Sigma/\mathcal{O}_\Sigma) = \mathrm{Aut}_{\mathcal{O}_\Sigma[\partial]}\mathcal{HL}_\Sigma.$$

The following lemma describes the differential $\mathcal{O}_\Sigma$-endomorphisms of $\mathcal{HL}_\Sigma$.

Lemma 3.8. *Let $\phi : \mathcal{HL}_\Sigma \to \mathcal{HL}_\Sigma$ be an $\mathcal{O}_\Sigma$-linear map. The following are equivalent:*
i). ϕ is a differential homomorphism: i.e. $\phi\,\partial = \partial\,\phi$.
ii). There exists a series $P = \sum_{w\in X^*} P(w)w \in \mathbb{C}\langle\langle X\rangle\rangle$ such that

$$\phi(w) = \sum_{\eta\eta'=w} \eta P(\eta').$$

Proof. Suppose ϕ commutes with ∂. For each $w \in X^*$, set $P(w) = \varepsilon(\phi(w)) \in \mathcal{O}_\Sigma$, where ε is the counit defined in §3.2 which projects onto the coefficient of the trivial word e. *ii*) holds for the word e, since $\partial\phi(e) = 0$, and so $\phi(e) \in \mathbb{C}$ by theorem 3.7. Now suppose by induction that *ii*) holds for all words of weight $\leq n$, and let $|w| = n$. For all $0 \leq i \leq N$,

$$\begin{aligned}
\partial\phi(x_i w) &= \frac{\phi(w)}{z-\sigma_i} = \frac{1}{z-\sigma_i}\sum_{\eta\eta'=w}\eta P(\eta'),\\
\phi(x_i w) &= \sum_{\eta\eta'=w} x_i\eta P(\eta') + \varepsilon(\phi(x_i w))\\
&= \sum_{\eta\eta'=x_i w}\eta P(\eta').
\end{aligned}$$

The second line is obtained by taking the primitive of the first; since the ring of constants in $\mathcal{HL}_\Sigma$ is $\mathbb{C}$ by theorem 3.7, this shows that $P(x_i w) = \varepsilon(\phi(x_i w))$ is in fact in $\mathbb{C}$, and completes the induction. This proves *i*) $\Rightarrow$ *ii*). The converse follows from a very similar calculation. □

This lemma may be regarded as an algebraic interpretation of the uniqueness part of theorem 2.1; any two solutions of (2.1) differ by right multiplication by a constant series. In §3.5 we show how this constant series corresponds to the series P in the lemma above.

[2]Differential Galois groups are usually defined for the field of fractions of a differential ring. The notation is nonetheless justified because $\mathrm{Gal}(\mathcal{HL}_\Sigma/\mathcal{O}_\Sigma)$ is isomorphic to $\mathrm{Gal}(\mathrm{Frac}(\mathcal{HL}_\Sigma)/\mathbb{C}(z))$.

Remark 3.9. We can interpret *ii*) in terms of the Hopf algebra structure on $\mathcal{HL}_\Sigma$, namely

$$\phi = (\mathrm{id} \otimes P)\Gamma,$$

where Γ is the coproduct on $\mathcal{HL}_\Sigma$ defined in (3.3).

Proposition 3.10. *There is a canonical isomorphism of groups:*

$$\mathrm{Gal}(\mathcal{HL}_\Sigma/\mathcal{O}_\Sigma) \cong \{S \in \mathbb{C}\langle\langle X\rangle\rangle : \Delta S = S \otimes S\}$$

where the group law in the right hand side is given by multiplication (concatenation) of series. There is a natural bijection between each of these groups and $\mathrm{Hom}\,(\mathbf{Lyn}(X), \mathbb{C})$.

Proof. The first isomorphism is given by the previous lemma. To $\phi \in \mathrm{Gal}(\mathcal{HL}_\Sigma/\mathcal{O}_\Sigma)$, we associate the series

$$P = \sum_{w \in X^*} \varepsilon(\phi(w))w.$$

ϕ respects the shuffle product if and only if P is group-like for Δ, by corollary 3.5. This proves the first part. Now suppose $S \in \mathbb{C}\langle\langle X\rangle\rangle$ is any group-like series. By theorem 3.3, the set of Lyndon words form a transcendence basis for $\mathbb{C}\langle X\rangle$ equipped with the shuffle product. So S is uniquely determined by the map $\mathbf{Lyn}(X) \to \mathbb{C}$ given by $w \mapsto S(w)$. □

In the previous proposition, both groups have a natural filtration induced by the weight, and these filtrations are preserved by the isomorphism. Explicitly, let $n \in \mathbb{N}$, and let U_n denote the set of all endomorphisms of the vector space $\mathcal{HL}_\Sigma^{(n)}$ which commute with ∂ and ш. The proposition implies that

$$U_n \cong \{S \in \mathbb{C}\langle\langle X\rangle\rangle / X^{n+1}\mathbb{C}\langle\langle X\rangle\rangle : \Delta S = S \otimes S\}.$$

Any group-like series S satisfies $(S-1)^{n+1} \in X^{n+1}\mathbb{C}\langle\langle X\rangle\rangle$, and so U_n is a unipotent matrix group, which acts naturally on the vector space $\mathbb{C}[w : |w| \leq n]$ of dimension $((N+1)^{n+1}-1)/N$. The dimension of U_n is the number of Lyndon words of weight $\leq n$. Since any series $S \in \mathbb{C}\langle\langle X\rangle\rangle$ is the projective limit of its successive truncations, we have

$$\mathrm{Gal}(\mathcal{HL}_\Sigma/\mathcal{O}_\Sigma) \cong \varprojlim U_n,$$

and $\mathrm{Gal}(\mathcal{HL}_\Sigma/\mathcal{O}_\Sigma)$ is pro-unipotent. By the proposition, its Lie algebra is isomorphic to the set of primitive elements in $\mathbb{C}\langle\langle X\rangle\rangle$, *i.e.*

$$\{T \in \mathbb{C}\langle\langle X\rangle\rangle : \Delta(T) = T \otimes 1 + 1 \otimes T\}.$$

Remark 3.11. Note that the matrix groups U_n are complicated and have large codimension in the group of all automorphisms of $\mathcal{HL}_\Sigma^{(n)}$. It is therefore much more convenient to work with generating series of functions, which is the approach we have adopted here, rather than with matrices of functions (*c.f.* for example [Ram]). The previous proposition shows how to pass between these two equivalent points of view.

3.5. **Realisations of $\mathcal{HL}_\Sigma$ and hyperlogarithms.**

Definition 3.12. A *realisation of hyperlogarithms* is a differential $\mathcal{O}_\Sigma$-algebra A and a non-zero homomorphism $\rho : \mathcal{HL}_\Sigma \longrightarrow A$.

The kernel of ρ is a differential ideal in $\mathcal{HL}_\Sigma$, so theorem 3.7 implies that ρ is injective, and therefore $\{\rho(w) : w \in X^*\}$ is an $\mathcal{O}_\Sigma$-basis for the image of ρ. The *holomorphic realisation of hyperlogarithms* is the $\mathcal{O}_\Sigma$-linear map:

$$\begin{array}{rcl} \mathcal{HL}_\Sigma & \longrightarrow & L_\Sigma, \\ w & \mapsto & L_w(z), \end{array} \tag{3.5}$$

which is a map of differential algebras by equations (2.2) and (2.6). Theorem 3.7 implies that this is an isomorphism, and has the following corollary.

Corollary 3.13. *The functions $L_w(z)$ form an $\mathcal{O}_\Sigma$-basis for L_Σ. The only algebraic relations with coefficients in $\mathcal{O}_\Sigma$ between the functions $L_w(z)$ are given by the shuffle product. Every function $F(z) \in L_\Sigma$ has a primitive in L_Σ which is unique up to a constant.*

The $\mathbb{C}$-linear independence of hyperlogarithms is already known by a monodromy argument (*e.g.* [Ch2]). One writes down a linear relation between hyperlogarithms of minimal weight and applies monodromy operators to obtain a relation of strictly lower weight, and reach a contradiction. This kind of argument does not generalise. Consider any non-zero real-analytic solution $F(z) = \sum_{w \in X^*} F_w(z) w$ to equation (2.1). We do not suppose, therefore, that $\partial F / \partial \overline{z} = 0$. We can assume that the functions $F_w(z)$ satisfy the shuffle relations by replacing the series $F(z)$ with $F^\times(z)$ if necessary, by remark 3.6. Let F denote the $\mathcal{O}_\Sigma$-module generated by the coefficients $F_w(z)$. There is a realisation

$$\begin{array}{ccc} \mathcal{HL}_\Sigma & \overset{\rho_F}{\longrightarrow} & F, \\ w & \mapsto & F_w(z), \end{array}$$

which is an isomorphism. The corresponding analogue of corollary 3.10 holds automatically. Note that the linear independence of the functions $F_w(z)$ cannot be proved by a straightforward monodromy argument, since the monodromy may act trivially on some elements of F. Indeed, in §8 we shall construct a realisation which has trivial monodromy.

Remark 3.14. Taking $\Sigma = \{0, 1, \infty\}$, this argument proves the linear independence of the $p-$adic multiple polylogarithms in one variable which were constructed in [Fu].

Now suppose that we are given two realisations $\rho_1, \rho_2 : \mathcal{HL}_\Sigma \overset{\sim}{\rightarrow} F$. They differ by an automorphism $\rho_2^{-1}\rho_1 \in \mathrm{Gal}(\mathcal{HL}_\Sigma / \mathcal{O}_\Sigma)$. By proposition 3.10, any such automorphism can be identified with a group-like series $S \in \mathbb{C}\langle\langle X \rangle\rangle$.

In particular, each solution L^{σ_k} to equation (2.1) given by corollary 2.3 gives rise to a realisation $\rho_{\sigma_k} : \mathcal{HL}_\Sigma \overset{\sim}{\rightarrow} L_\Sigma$. Each pair of realisations therefore gives rise to a certain group-like series in $\mathbb{C}\langle\langle X \rangle\rangle$. These series will be studied in §5.

4. Unipotent differential algebras over $\mathcal{O}_\Sigma$

It was shown in §3.4 that the differential Galois group $\mathrm{Gal}(\mathrm{Frac}(\mathcal{HL}_\Sigma)/\mathbb{C}(z))$ is pro-unipotent. It follows from differential Galois theory that $\mathrm{Frac}(\mathcal{HL}_\Sigma)$ is an infinite union of unipotent Picard-Vessiot extensions of $\mathbb{C}(z)$. In this section, we prove from first principles the stronger statement that $\mathcal{HL}_\Sigma$ is the union of all unipotent extensions of $\mathcal{O}_\Sigma$. We recall some elementary facts from differential algebra and include the proofs for completeness.

4.1. **Modules with unipotent connections.** Let M be an $\mathcal{O}_\Sigma$-module of finite type, equipped with a connection defined over $\mathcal{O}_\Sigma$, *i.e.* a linear map $\nabla : M \to M$ satisfying

$$\nabla(fm) = f'm + f\,\nabla(m),$$

for all $f \in \mathcal{O}_\Sigma$, $m \in M$, which, with respect to some $\mathcal{O}_\Sigma$-basis of M, may be written

$$\nabla = \frac{\partial}{\partial z} + P,$$

where P is a matrix with entries in $\mathcal{O}_\Sigma$. We define the *solutions of* M to be the smallest $\mathcal{O}_\Sigma$-submodule of M closed under the operation of taking a primitive:

$$\nabla\theta \in \mathrm{Sol}(M) \quad \Rightarrow \quad \theta \in \mathrm{Sol}(M).$$

M is necessarily free over $\mathcal{O}_\Sigma$, since $\mathcal{O}_\Sigma$ is differentially simple.

Lemma 4.1. *The following are equivalent:*
i). There is a filtration by $\mathcal{O}_\Sigma$-submodules

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M,$$

such that $\nabla M_i \subset M_i$, and $(M_i/M_{i-1}, \nabla) \cong (\mathcal{O}_\Sigma, \partial/\partial z)$ for all $1 \leq i \leq n$.
ii). There is a basis of M with respect to which P is strictly upper triangular.
iii). M is spanned by its solutions, i.e. $\mathrm{Sol}(M) = M$.

Proof. $i) \Rightarrow ii)$. Let $\theta_1, \ldots, \theta_n$ be an $\mathcal{O}_\Sigma$-basis of M adapted to the filtration, *i.e.*

$$M_k = \theta_k \mathcal{O}_\Sigma \oplus M_{k-1}, \qquad 1 \leq k \leq n.$$

Then $\nabla(f\theta_k) = f'\theta_k + f\nabla\theta_k \equiv f'\theta_k \mod M_{k-1}$ for all $f \in \mathcal{O}_\Sigma$. It follows that $\nabla\theta_k \subset M_{k-1}$, and therefore P is strictly upper triangular with respect to the basis $\{\theta_1, \ldots, \theta_n\}$.

$ii) \Rightarrow iii)$. Let $\{\theta_0, \ldots, \theta_n\}$ be a basis of M with respect to which P is upper triangular. Write $\theta_{-1} = 0$. Suppose by induction that $\theta_i \in \mathrm{Sol}(M)$ for all $-1 \leq i \leq k$. But $\nabla\theta_{k+1} \in \theta_{-1}\mathcal{O}_\Sigma \oplus \ldots \oplus \theta_k\mathcal{O}_\Sigma \subset \mathrm{Sol}(M)$, which implies $\theta_{k+1} \in \mathrm{Sol}(M)$.

$iii) \Rightarrow i)$. Set $M_0 = 0$, and define $\mathcal{O}_\Sigma$-submodules of $\mathrm{Sol}(M)$ inductively as follows: $M_{i+1} = \mathcal{O}_\Sigma\{\theta \in \mathrm{Sol}(M) : \nabla\theta \in M_i\}$. Then there is a filtration

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = \mathrm{Sol}(M) = M$$

compatible with the action of ∇. By taking a suitable sub-filtration, we can ensure that the $\mathcal{O}_\Sigma$-rank of M_i/M_{i-1} is exactly 1. □

Definition 4.2. M is *unipotent* if it satisfies any of the above conditions.

4.2. **Unipotent algebras.** Let (A, ∂) be a finitely generated differential $\mathcal{O}_\Sigma$-algebra. We will suppose that A is generated by an $\mathcal{O}_\Sigma$-submodule $M \subset A$ of finite type such that $\partial M \subset M$. Then ∂ defines a connection on M, which we further assume to be defined over $\mathcal{O}_\Sigma$ in the sense of §4.1. Finally, we assume that the ring of constants $\{x \in A : \partial x = 0\}$ is $\mathbb{C}$. Let $\mathrm{Gal}(A/\mathcal{O}_\Sigma)$ denote the group of $\mathcal{O}_\Sigma$-linear differential automorphisms of A.

Definition 4.3. A is *unipotent* if M is unipotent.

Theorem 4.4. *The following are equivalent:*

i). A is unipotent.

ii). There is an $\mathcal{O}_\Sigma$-basis of M which is annihilated by an operator $D \in \mathcal{O}_\Sigma[\partial]$ which factorizes as a product of terms of the form ∂ and $(z-\sigma_k)$ for $0 \leq k \leq N$.

iii). The group $\mathrm{Gal}(A/\mathcal{O}_\Sigma)$ acts on M and is unipotent, and $A^{\mathrm{Gal}(A/\mathcal{O}_\Sigma)} = \mathcal{O}_\Sigma$.

iv). There is an embedding $i : A \hookrightarrow \mathcal{HL}_\Sigma$ of differential algebras.

When these hold, $\mathrm{Frac}(A)$ *is a Picard-Vessiot extension of* $\mathbb{C}(z)$*, and* $\mathrm{Gal}(A/\mathcal{O}_\Sigma) \cong \mathrm{Gal}(\mathrm{Frac}(A)/\mathbb{C}(z))$*. The embedding in iv) is unique up to an element in this group.*

Before proving the theorem we will need the following variant of a well-known lemma concerning extensions of differential fields by adjoining integrals.

Lemma 4.5. *Let (R,∂) be a differentially simple extension of $\mathcal{O}_\Sigma$. Consider the extension $R[y]$ formed by adding an integral, i.e. $\partial y = r \in R$. If $\partial x = r$ has no solution $x \in R$ then y is transcendent over R, and $R[y]$ is also differentially simple.*

Proof. Suppose $f(y) \in R[y]$ is a polynomial satisfied by y of minimal degree:

$$a_n y^n + a_{n-1} y^{n-1} + \ldots + a_0 = 0,$$

where $a_i \in R$, $a_n \neq 0$. Since R is differentially simple, we know there exists $\Delta \in R[\partial]$ such that $\Delta a_n = 1$. Applying this operator to the equation above, we may assume $a_n = 1$. On applying ∂ we obtain

$$(nr + \partial a_{n-1}) y^{n-1} + \ldots + (a_1 r + \partial a_0) = 0.$$

By minimality, this polynomial is identically 0, so $\partial x = r$ already has a solution $x = a_{n-1}/n \in R$. This proves the transcendence of y. Now suppose $I \subset R[y]$ is a non-trivial differential ideal, and let $f(y) \in I$ of minimal degree. Exactly the same computation as above shows that $\partial x = r$ must already have had a solution in R. It follows that $R[y]$ has no non-trivial differential ideals. □

Proof of theorem: *i)* $\Rightarrow$ *iv)*. Suppose that $A = \mathcal{O}_\Sigma[M]$, where M is unipotent. Since M is spanned by its solutions (lemma 3.1), we can find a sequence of algebras

$$\mathcal{O}_\Sigma = A_1 \subset A_2 \subset \ldots \subset A_n = A$$

formed by adding integrals, *i.e.* $A_i = A_{i-1}[y_i]$ where $\partial y_i = a_{i-1} \in A_{i-1}$, and $\partial x = a_{i-1}$ has no solution in A_{i-1} (this follows from the assumption that the ring of constants of A is $\mathbb{C}$) . Now suppose by induction that we have defined an injective map of differential $\mathcal{O}_\Sigma$-algebras $i_k : A_k \hookrightarrow \mathcal{HL}_\Sigma$ for some $k \geq 1$. By theorem 3.7, A_k is differentially simple, and $\mathcal{HL}_\Sigma$ is closed under taking primitives, so there exists a $\phi \in \mathcal{HL}_\Sigma$ such that

$$i_k(\partial y_{k+1}) = \partial \phi.$$

By lemma 4.5, y_{k+1} is transcendent over A_k, so if we set $i_{k+1}(y_{k+1}) = \phi$, the map $i_{k+1} : A_{k+1} \to \mathcal{HL}_\Sigma$ is a differential homomorphism. The kernel of i_{k+1} is a differential ideal in $A_k[y_{k+1}]$, which is differentially simple by lemma 4.5, so we deduce that i_{k+1} is injective. This proves *iv)*.

iv) $\Rightarrow$ *iii)*. If $A \subset \mathcal{HL}_\Sigma$, then A inherits a filtration given by adding integrals:

$$\mathcal{O}_\Sigma = A_0 \subsetneq \ldots \subsetneq A_n = A,$$

where $A_i = A_{i-1}[y_i]$ and $\partial y_i = a_{i-1} \in A_{i-1}$. One can check by induction that $\mathrm{Gal}(A/\mathcal{O}_\Sigma)$ respects the filtration and is unipotent, and one can construct elements $g \in \mathrm{Gal}(A/\mathcal{O}_\Sigma)$ such that $g(y_i) \neq y_i$. It follows that the set of fixed points of $\mathrm{Gal}(A/\mathcal{O}_\Sigma)$ is exactly $\mathcal{O}_\Sigma$.

iii) $\Rightarrow$ *ii)*. Let $\theta_0, \ldots, \theta_n$ be an $\mathcal{O}_\Sigma$-basis of M with respect to which the group $\mathrm{Gal}(A)$ acts by unipotent matrices, *i.e.* $\phi(\theta_{i+1}) - \theta_{i+1} \in \mathcal{O}_\Sigma[\theta_0, \ldots, \theta_i]$ for $0 \leq i \leq n-1$, and

$\phi(\theta_0) = \theta_0$ for all $\phi \in \mathrm{Gal}(A)$. This implies $\theta_0 \in A^{\mathrm{Gal}(A/\mathcal{O}_\Sigma)} = \mathcal{O}_\Sigma$, and therefore there exists an operator D_0 of the required form such that $D_0\theta_0 = 0$. Now assume by induction that there are operators $D_0, \ldots, D_k$ of the required type such that the product $D_k D_{k-1} \ldots D_0$ annihilates $\theta_0, \ldots, \theta_k$. For every $\phi \in \mathrm{Gal}(A)$, $D_k \ldots D_0(\phi(\theta_{k+1}) - \theta_{k+1}) = 0$, and therefore $D_k \ldots D_0\theta_{k+1}$ is invariant under ϕ. It follows that $D_k \ldots D_0\theta_{k+1} \in \mathcal{O}_\Sigma$, so there exists D_{k+1} of the required form such that $D_{k+1}D_k \ldots D_0$ annihilates $\theta_{k+1}, \ldots, \theta_0$. This completes the induction step.

ii) $\Rightarrow$ *i*). Let $\theta \in M$ satisfy $D\theta = 0$, where D is of the specified form. It is clear that $\theta \in \mathrm{Sol}(M)$. If a basis of M is annihilated by such an operator D, then $M \subset \mathrm{Sol}(M)$ and therefore $M = \mathrm{Sol}(M)$. It follows that M is unipotent, by lemma 4.5.

The uniqueness statement is equivalent to the fact that any two embeddings of A into $\mathcal{HL}_\Sigma$ have the same image. This follows by inspection of the proof that *i*) $\Rightarrow$ *iv*) above. □

The theorem implies that $\mathcal{HL}_\Sigma$ is characterized by the following property.

Corollary 4.6. *$\mathcal{HL}_\Sigma$ is the unipotent closure of $\mathcal{O}_\Sigma$, i.e.*

$$\mathcal{HL}_\Sigma = \lim_{\substack{\to \\ A}} A,$$

where A ranges over all finitely generated unipotent extensions of $\mathcal{O}_\Sigma$.

Finitely generated unipotent extensions can be defined over any differentially simple ring in a similar way. The corollary implies that $\mathcal{HL}_\Sigma$ is the unique smallest extension of $\mathcal{O}_\Sigma$ which has only trivial unipotent extensions.

Since any unipotent extension A of $\mathcal{O}_\Sigma$ embeds into $\mathcal{HL}_\Sigma$, it inherits many of the structural properties of $\mathcal{HL}_\Sigma$. For example, the number of $\mathcal{O}_\Sigma$-algebraically independent elements in A is bounded above by the number of Lyndon words (theorem 3.3) up to a certain weight n. The number n is given by the length of a maximal filtration on M (lemma 4.1 *i*)), or the degree of the underlying differential equation (theorem 4.4 *ii*)). Unipotent differential equations with singularities in Σ can therefore be completely understood in terms of hyperlogarithms (compare [A1]).

Remark 4.7. Picard-Vessiot extensions and a corresponding differential Galois theory can be defined for differentially simple rings instead of differential fields. Any unipotent extension of a differentially simple ring is differentially simple, by lemma 4.5, so the preceding theorem is just the unipotent part of such a Galois correspondence over $\mathcal{O}_\Sigma$. One has to be careful to exclude extensions of $\mathcal{O}_\Sigma$ such as $\mathcal{O}_\Sigma[1/\log(z-\sigma_i)]$ which are non-trivial but have a trivial automorphism group over $\mathcal{O}_\Sigma$.

5. Regularised values and zeta series.

5.1. **The series $Z^{\sigma_k}(X)$ and properties.** Let $0 \leq k \leq N$. By corollary 2.3, there is a unique solution $L^{\sigma_k}(z)$ of (2.1) which is asymptotically $\exp(x_k \log(z-\sigma_k))$ as z approaches σ_k, where $L^{\sigma_k}_{x_k}(z) = \log(z-\sigma_k)$ is the branch of the logarithm on the cut plane $\mathbb{C}\backslash \ell(\sigma_k)$ which was fixed in §2.2. The ratio of two such solutions is a constant series by the last part of theorem 2.1, or the last remark in §3.5. This series will depend on the initial choices of branches of the logarithms.

Definition 5.1. Let $\sigma_k \in \Sigma$. The *regularized zeta series* at σ_k is

$$Z^{\sigma_k}(X) = L^{\sigma_k}(z)^{-1} L(z) \in \mathbb{C}\langle\langle X \rangle\rangle,$$

where $z \in U$ is arbitrary. We will sometimes write $Z^{\sigma_k}(x_0, \ldots, x_N)$ instead of $Z^{\sigma_k}(X)$, and denote its coefficients by $\zeta^{\sigma_k}(w)$:

$$Z^{\sigma_k}(X) = \sum_{w \in X^*} \zeta^{\sigma_k}(w)\, w.$$

The zeta series describe the limiting behaviour of $L(z)$ near a singularity. We write $\log(z-\sigma_k)$ for $L^{\sigma_k}_{x_k}(z)$, the branch of logarithm that was fixed in §2.2.

Lemma 5.2. *For* $0 \leq k \leq N$, $\quad Z^{\sigma_k}(X) = \lim_{z \to \sigma_k} \exp(-x_k \log(z-\sigma_k))\, L(z)$.

Proof. By corollary 2.3,

$$L(z) = L^{\sigma_k}(z) Z^{\sigma_k}(X) \sim \exp(x_k \log(z-\sigma_k)) Z^{\sigma_k}(X), \tag{5.1}$$

as $z \to \sigma_k$ along any path in D, not necessarily contained in U.

□

Equation (5.1) implies that $L_w(z)$ has the following logarithmic expansion.

Corollary 5.3. *Let $w \in X^*$. For any $\alpha \in \mathbb{C}$, we can write*

$$L_w(z) = f_0(z) + f_1(z)\log(z-\alpha) + \ldots + f_m(z) \log^m(z-\alpha),$$

where $m \in \mathbb{N}$, and the $f_i(z)$ are holomorphic functions converging uniformly on the open ball centered at α of radius $\inf_{\sigma\in\Sigma, \sigma\neq\alpha}\{|\sigma-\alpha|\}$. If $\alpha \notin \Sigma$, then $m = 0$. If $\alpha = \sigma_k$, then $f_i(\alpha) \neq 0$ if and only if $w = x_k^i \eta$, where $\eta \in X^*$ and $\zeta^{\sigma_k}(\eta) \neq 0$.

In particular, $L_w(z)$ has a well-defined limit at σ_k for all words $w \in X^*$ which do not begin with x_k.

Definition 5.4. Let $w \in X^*$. The *regularized value* of $L_w(z)$ at α is defined to be $f_0(\alpha)$ in the preceding corollary, and is written $\mathrm{Reg}(L_w(z), \alpha)$.

If $\alpha \notin \Sigma$ this is just $L_w(\alpha)$. If w does not begin in the letter x_k, the preceding corollary implies that $\mathrm{Reg}(L_w(z), \sigma_k) = \lim_{z\to\sigma_k} L_w(z)$. It follows from (5.1) that

$$Z^{\sigma_k}(X) = \mathrm{Reg}\,(L(z), \sigma_k). \tag{5.2}$$

Remark 5.5. Fix any $\alpha \in \mathbb{C}$, and let P_α denote the differential ring $\mathbb{C}[\log(z-\alpha)]\,[[z-\alpha]]$. P_α contains $\mathcal{O}_\Sigma$, and is closed under the operation of taking primitives. By theorem 2.1, there exists a realisation of hyperlogarithms

$$\mathcal{HL}_\Sigma \longrightarrow P_\alpha,$$

which yields a formal logarithmic expansion similar to corollary 5.3. The regularisation map $P_\alpha \to \mathbb{C}$ is just the map which projects a series onto its constant term. Logarithmic expansions of this kind have been studied in detail in [BdeM] in the case $\Sigma = \{0, 1, \infty\}$.

By the remark after corollary 3.12, the series $Z^{\sigma_k}(X)$ are group-like:

$$\Delta Z^{\sigma_k}(X) = Z^{\sigma_k}(X) \otimes Z^{\sigma_k}(X), \qquad \text{for } \; 0 \leq k \leq N. \tag{5.3}$$

Equivalently, the coefficients $\zeta^{\sigma_k}(w)$ satisfy the shuffle relations (this follows from corollary 3.5, or from the fact that Reg is a homomorphism for multiplication of functions). In particular, $Z^{\sigma_k}(X)$ is an invertible element in $\mathbb{C}\langle\langle X\rangle\rangle$, with leading coefficient 1. Since group-like elements in any Hopf algebra are inverted by the antipode, we deduce that

$$\widetilde{Z^{\sigma_k}}(-x_0,\ldots,-x_N)Z^{\sigma_k}(x_0,\ldots,x_N) = 1, \qquad \text{for } \; 0 \leq k \leq N. \tag{5.4}$$

where $\widetilde{\ }$ denotes the mirror map (3.2). This can also be proved analytically ([G-L]).

5.2. **Explicit computation of $Z^{\sigma_k}(X)$.**

Lemma 5.6. *Let $1 \leq k \leq N$. Then $w \mapsto \zeta^{\sigma_k}(w)$ extends to the unique linear function on $\mathbb{C}\langle X\rangle$ which is a homomorphism for the shuffle product and satisfies:*

$$\begin{aligned} \zeta^{\sigma_k}(x_k) &= -L^{\sigma_k}_{x_k}(\sigma_0), \\ \zeta^{\sigma_k}(x_0) &= L_{x_0}(\sigma_k), \\ \zeta^{\sigma_k}(w) &= \lim_{\substack{z\to\sigma_k \\ z\in U}} L_w(z), \qquad \textit{for all } w \notin X^*x_0 \cup x_k X^*. \end{aligned}$$

Proof. By the remark after definition 5.4, $\zeta^{\sigma_k}(w) = \mathrm{Reg}(L_w(z),\sigma_k) = \lim_{z\to\sigma_k} L_w(z)$ for all words w not beginning in x_k. The formulae for $\zeta^{\sigma_k}(x_k)$ and $\zeta^{\sigma_k}(x_0)$ follow on taking the regularised values of the expressions for $L_{x_k}(z)$ and $L_{x_0}(z)$ given in remark 2.2. For example, lemma 5.2 gives

$$\zeta^{\sigma_k}(x_k) = -L^{\sigma_k}_{x_k}(z) + L_{x_k}(z)$$

for all $z \in U$, and this is equal to $-L^{\sigma_k}_{x_k}(\sigma_0)$, since $L_{x_k}(z)$ vanishes at σ_0 by definition. One can check that every $w \in X^*$ can be written uniquely as a linear combination of shuffle products of the words x_0, x_k, and η; where η neither begins in x_k, nor ends in x_0. This proves the uniqueness. □

Remark 5.7. In the case $\sigma_0 = 0$, $\sigma_1 = 1$, and $U = \mathbb{C}\backslash((-\infty,0]\cup[1,\infty))$ it is customary to take $L^{\sigma_1}_{x_1}(z) = \log(1-z)$, and $L_{x_0}(z) = \log(z)$, for the principal branch of logarithm which vanishes at 1. It follows that the zeta series at 1 is uniquely determined by the equations $\zeta^1(x_1) = \zeta^1(x_0) = 0$, and $\zeta^1(w) = \lim_{z\to 1} L_w(z)$ for all $w \in x_0X^*x_1$.

It therefore suffices to compute $L_w(\sigma_k)$ for all words w which neither begin in x_k, nor end in x_0. The following argument, which is essentially due to Lappo-Danilevskii, enables us to write such $L_w(\sigma_k)$ as an absolutely convergent power series.

Lemma 5.8. *Let $\alpha \in \mathbb{C}\backslash\Sigma$. Set $L^{(\alpha)}_e = 1$. For each word $w = x_{i_r}\ldots x_{i_1} \in X^*$, we set*

$$L^{(\alpha)}_w(z) = \sum_{1\leq m_1<\cdots<m_r} \frac{(-1)^r}{m_1\ldots m_r}\Big(\frac{z-\alpha}{\sigma_{i_1}-\alpha}\Big)^{m_1}\Big(\frac{z-\alpha}{\sigma_{i_2}-\alpha}\Big)^{m_2-m_1}\cdots\Big(\frac{z-\alpha}{\sigma_{i_r}-\alpha}\Big)^{m_r-m_{r-1}},$$

which converges absolutely on compacta in $\{z : |z-\alpha| < R\}$, where $R = \inf_{0\leq i\leq N}|\sigma_i-\alpha|$. If w does not begin in x_k, and $|\sigma_k - \alpha| = R$, then it also converges for $z = \sigma_k$. The generating series

$$L^{(\alpha)}(z) = \sum_{w\in X^*} wL^{(\alpha)}_w(z)$$

is then the unique solution to (2.1) *satisfying $L^{(\alpha)}(\alpha) = 1$.*

Proof. The convergence of the series $L^{(\alpha)}_w(z)$ is clear. One checks without difficulty that $L^{(\alpha)}(z)$ satisfies equations (2.2), and that $L^{(\alpha)}_w(z)$ vanishes at $z = \alpha$ for all $w \neq e$. □

Now we can always find a sequence of points $\alpha_1, \ldots, \alpha_n \in \mathbb{C}\backslash\Sigma$ such that

$$\begin{aligned} |\alpha_1 - \sigma_0| &< \inf_{i\neq 0} |\sigma_i - \sigma_0|, \\ |\alpha_{\ell+1} - \alpha_\ell| &< \inf_{i} |\sigma_i - \alpha_\ell| \quad \text{for each} \quad 0 \leq \ell \leq n-1, \\ |\alpha_n - \sigma_k| &< \inf_{i\neq k} |\sigma_i - \sigma_k|. \end{aligned}$$

Then, by repeated application of the uniqueness part of theorem 1.1,

$$L(z) = L^{(\alpha_1)}(z)\, L(\alpha_1) = \ldots = L^{(\alpha_n)}(z)\, L^{(\alpha_{n-1})}(\alpha_n) \ldots L^{(\alpha_1)}(\alpha_2)\, L(\alpha_1).$$

We therefore have an explicit decomposition for every $w \in X^*$ of the form:

$$L_w(z) = \sum_{\eta_n \ldots \eta_1 = w} L^{(\alpha_n)}_{\eta_n}(z)\, L^{(\alpha_{n-1})}_{\eta_{n-1}}(\alpha_n) \ldots L^{(\alpha_2)}_{\eta_2}(\alpha_1)\, L_{\eta_1}(\alpha_1),$$

where the sum is over all possible factorisations of w. If $w \notin x_k X^* \cup X^* x_0$, then η_n can never begin in x_k, and η_1 can never end in x_0, and we may take the limit:

$$\zeta^{\sigma_k}(w) = \lim_{z\to\sigma_k} L_w(z) = \sum_{\eta_n \ldots \eta_1 = w} L^{(\alpha_n)}_{\eta_n}(\sigma_k)\, L^{(\alpha_{n-1})}_{\eta_{n-1}}(\alpha_n) \ldots L^{(\alpha_2)}_{\eta_2}(\alpha_1)\, L_{\eta_1}(\alpha_1).$$

The series $L^{(\alpha_n)}_{\eta_n}(\sigma_k)$,..., $L^{(\alpha_2)}_{\eta_2}(\alpha_1)$ are all absolutely convergent, by the previous lemma. Likewise, $L_{\eta_1}(\alpha_1)$ is given by the absolutely convergent series (2.3).

This gives the required explicit formula for $\zeta^{\sigma_k}(w)$ in terms of absolutely convergent series. Note that, although $Z^{\sigma_k}(X)$ depends on the initial choices of branches of the logarithms used to define $L(z)$ and $L^{\sigma_k}(z)$, it does not depend on the homotopy class of the path given by the sequence of points $\alpha_1, \ldots, \alpha_n$.

Remark 5.9. We can compute $L^{\sigma_k}(\alpha)$ by a similar method for all k. Another approach to computing $Z^{\sigma_k}(X)$ would therefore be to choose any point $\alpha \notin \Sigma$ and apply the formula $L^{\sigma_k}(\alpha)^{-1} L(\alpha) = Z^{\sigma_k}(X)$.

5.3. **Examples.** By definition 5.1, $Z^{\sigma_0}(X) = 1$. Lemma 5.6 implies that $\zeta^{\sigma_k}(x_i) = L_{x_i}(\sigma_k)$ for $i \neq k$, so we have

$$Z^{\sigma_k}(X) = 1 + \log(\sigma_k - \sigma_0)\, x_0 + \sum_{1\leq i\neq k} \log\Big(\frac{\sigma_k - \sigma_i}{\sigma_0 - \sigma_i}\Big) x_i - \log(\sigma_0 - \sigma_k) x_k + \ldots,$$

where the omitted terms consist of words of length ≥ 2. This follows from remark 2.2, bearing in mind that the x_0 and x_k terms in the above expression refer to different branches of the logarithm from the others. The case $\Sigma = \{0, 1, \infty\}$ is of special interest since the series Z^1 is precisely the Drinfeld associator for the K-Z equation ([G-L]). By remark 5.7, the coefficients of x_0 and x_1 vanish, and one can show that this series can be written

$$Z^1(X) = 1 - \zeta(2)[x_0, x_1] + \zeta(3)([[x_0, x_1], x_1] - [x_0, [x_0, x_1]]) + \ldots,$$

where $\zeta(n)$ are values of the Riemann zeta function, and $[w_1, w_2] = w_1 w_2 - w_1 w_2$ is the ordinary Lie bracket on $\mathbb{C}\langle X\rangle$. The higher coefficients of this series are multiple zeta values, which have been much studied (see *e.g.* [Ca1]). The study of algebraic relations between these numbers is an open problem. Likewise, when Σ consists of 0, ∞ and roots of unity, the numbers $\zeta^{\sigma_k}(w)$ (also known as coloured multiple polylogarithms) arise as periods of mixed Tate motives in algebraic geometry ([G1], [Rac]), and in the evaluation of Feynman diagrams in mathematical physics ([R-V]). In the general case, however, there is no canonical way to choose branches of logarithms $L^{\sigma_k}_{x_k}(z)$, so the corresponding generalised multiple zeta values $\zeta^{\sigma_k}(w)$ depend on these choices, as remarked above.

6. Computation of the Monodromy.

Given any realisation of $\mathcal{HL}_\Sigma$ as real-analytic functions, we obtain a representation of the fundamental group of D on $\mathcal{HL}_\Sigma$ by the action of monodromy. We compute this representation explicitly for the realisation L_Σ in terms of regularised zeta series. This amounts to an explicit determination of the monodromy of hyperlogarithms $L_w(z)$.

6.1. **The monodromy representation of a realisation of hyperlogarithms.** Let $C^\infty(D,\mathbb{C})$ denote the algebra of multi-valued real-analytic functions on the punctured plane D. This is a differential algebra with respect to the two operators $\partial/\partial z$ and $\partial/\partial \overline{z}$. If we fix $z_0 \in D$, the fundamental group $\pi_1(D,z_0)$ is the free non-commutative group on generators $\{\gamma_0,\ldots,\gamma_N\}$ where γ_i denotes a loop from z_0 winding once around the point σ_i in the positive direction. For each $0 \leq k \leq N$, we write

$$\mathcal{M}_{\sigma_k} : C^\infty(D,\mathbb{C}) \rightarrow C^\infty(D,\mathbb{C})$$

for the monodromy operator given by analytic continuation of functions around the path γ_k. The map $\mathcal{M}_{\sigma_k}$ is a homomorphism of algebras and commutes with $\partial/\partial z$ and $\partial/\partial \overline{z}$. Now consider any isomorphic realisation of hyperlogarithms

$$\rho_A : \mathcal{HL}_\Sigma \xrightarrow{\sim} A,$$

where A is a $\partial/\partial z$-differential $\mathcal{O}_\Sigma$-subalgebra of $C^\infty(D,\mathbb{C})$. The isomorphism ρ_A gives rise to an isomorphism of differential automorphism groups $\mathrm{Gal}(A/\mathcal{O}_\Sigma) \cong \mathrm{Gal}(\mathcal{HL}_\Sigma/\mathcal{O}_\Sigma)$.

Definition 6.1. A is *closed under the action of monodromy* if, for all $0 \leq k \leq N$, $\mathcal{M}_{\sigma_k} A \subset A$. In this case, the restriction of the action of monodromy to A gives rise to a representation

$$\begin{array}{rcl} \pi_1(D,z_0) & \longrightarrow & \mathrm{Gal}(A/\mathcal{O}_\Sigma), \\ \gamma_i & \mapsto & \mathcal{M}_{\sigma_i}|_A. \end{array}$$

The *monodromy of* A is defined to be the representation

$$\mathcal{M}_A : \pi_1(D,z_0) \rightarrow \mathrm{Gal}(\mathcal{HL}_\Sigma/\mathcal{O}_\Sigma),$$

obtained on identifying $\mathrm{Gal}(A/\mathcal{O}_\Sigma)$ with $\mathrm{Gal}(\mathcal{HL}_\Sigma/\mathcal{O}_\Sigma)$ via ρ_A.

The monodromy representation is completely determined by the image of the generators $\gamma_0,\ldots,\gamma_N$ under the action of $\mathcal{M}_A$. By proposition 3.10, these can be identified with $N+1$ series $A_0,\ldots,A_N \in \mathbb{C}\langle\langle X\rangle\rangle$ which satisfy $\Delta A_k = A_k \otimes A_k$ for $0 \leq k \leq N$.

Remark 6.2. If $F(z) = \sum_{w\in X^*} \rho_A(w)w$ is the generating series for ρ_A, then $\mathcal{M}_{\sigma_k}F(z)$ and $F(z)$ are both solutions to (2.1). By the uniqueness part of theorem 2.1,

$$\mathcal{M}_{\sigma_k}F(z) = F(z)A_k(\overline{z}), \qquad \text{for} \quad 0 \leq k \leq N,$$

where $A_k(\overline{z})$ are series which depend only upon $\overline{z}$. The realisation A is closed under the action of monodromy if and only if $\partial A_k(\overline{z})/\partial\overline{z} = 0$ for all $0 \leq k \leq N$, in which case $A_k(\overline{z}) = A_k \in \mathbb{C}\langle\langle X\rangle\rangle$, and the series coincide with those defined above.

We will show that any realisation of $\mathcal{HL}_\Sigma$ which is closed under the action of monodromy can in fact be identified with an algebra of hyperlogarithms and their complex conjugates (theorem 7.4).

6.2. **Computation of the monodromy of L_Σ.**

Proposition 6.3. *Let $0 \leq k \leq N$. The monodromy operator $\mathcal{M}_{\sigma_k}$ acts as follows:*

$$\mathcal{M}_{\sigma_k} L(z) = L(z)\,(Z^{\sigma_k}(X))^{-1} e^{2\pi i x_k} Z^{\sigma_k}(X).$$

Proof. By theorem 2.1, $L(z) = f_0(z)\exp(x_0 \log(z-\sigma_0))$, where $f_0(z)$ is holomorphic at σ_0. It follows immediately that $\mathcal{M}_{\sigma_0} L(z) = L(z)\exp(2\pi i x_0)$. Likewise, by corollary 2.3, $\mathcal{M}_{\sigma_k} L^{\sigma_k}(z) = L^{\sigma_k}(z)\exp(2\pi i x_k)$. By regarding the series $Z^{\sigma_k}(X)$ (definition 5.1) as a constant function on the universal cover $\widehat{D}$ of D, and viewing the equation $L(z) = L^{\sigma_k}(z) Z^{\sigma_k}_X$ as an equation on $\widehat{D}$, it follows that

$$\mathcal{M}_{\sigma_k} L(z) = \mathcal{M}_{\sigma_k} L^{\sigma_k}(z)\, Z^{\sigma_k}(X) = L^{\sigma_k}(z)\, e^{2\pi i x_k} Z^{\sigma_k}(X) = L(z)\,(Z^{\sigma_k}(X))^{-1} e^{2\pi i x_k} Z^{\sigma_k}(X).$$

□

See also [HPV] for a similar computation of the monodromy in the case $\Sigma = \{0, 1, \infty\}$.

Remark 6.4. The statement for $k = 0$ is equivalent to the holomorphy of the function $f_0(z)$ in theorem 2.1, whose proof can be obtained a different way as follows. We know that $\mathcal{M}_{\sigma_0} L(z) = L(z) M$ for some group-like series $M = \sum_{w \in X^*} M(w)\, w \in \mathbb{C}\langle\langle X \rangle\rangle$. For each $1 \neq w \in X^*$ not ending in x_0, we showed that $L_w(z)$ is holomorphic at σ_0, and so $M(w) = 0$. Since M satisfies the shuffle relations, it is completely determined by $M(x_0) = 2i\pi$, the monodromy of $\log(z - \sigma_0)$. It follows that $\mathcal{M}_{\sigma_k} L(z) = L(z)\exp(2i\pi x_0)$, which proves that $f_0(z) = L(z)\exp(-x_0 \log(z - \sigma_0))$ has trivial monodromy at σ_0, and therefore extends to a holomorphic function at σ_0.

The operators $\mathcal{M}_{\sigma_k}$ are unipotent, and it is immediate that

$$(Z^{\sigma_k}(X))^{-1} e^{2\pi i x_k} Z^{\sigma_k}(X) = 1 + 2i\pi x_k + \ldots,$$

where the omitted terms are words of length ≥ 2.

Corollary 6.5. *If $w \in X^*$ of weight $|w| = n$, then*

$$\begin{aligned}(\mathcal{M}_{\sigma_k} - \mathrm{id}) L_{w x_j}(z) &\equiv 0 \mod \mathcal{HL}^{(n-1)} \qquad \textit{if} \quad j \neq k,\\ (\mathcal{M}_{\sigma_k} - \mathrm{id}) L_{w x_k}(z) &\equiv 2\pi i L_w(z) \mod \mathcal{HL}^{(n-1)}.\end{aligned}$$

One can show using the previous corollary that the canonical monodromy representation $\mathcal{M}_{L_\Sigma} : \pi_1(D, z_0) \to \mathrm{Gal}\,(\mathcal{HL}_\Sigma)$ is an injective homomorphism with Zariski-dense image. This is equivalent to a theorem due to Chen ([Ch2], see also [A1]).

7. A variant of the Riemann-Hilbert problem.

7.1. Unipotent differential equations and the Riemann-Hilbert correspondence. Consider the differential equation

$$Y'(z) = \sum_{i=0}^{N} \frac{Q_i}{z-\sigma_i} Y(z), \tag{7.1}$$

where $Q_i \in M_n(\mathbb{C})$. It has $m \leq n$ linearly independent solutions $Y_1, \ldots, Y_m$ which are holomorphic, multi-valued functions on D. If S denotes the complex vector space spanned by these solutions, one can check that the action of monodromy gives rise to a representation

$$\pi_1(D) \rightarrow \mathrm{End}_{\mathbb{C}}(S) \cong M_m(\mathbb{C}).$$

Let $M_0, \ldots, M_N$ denote the images of the generators $\gamma_0, \ldots, \gamma_N$ of $\pi_1(D)$ in $\mathrm{End}_{\mathbb{C}}(S)$ (§6.1). One version of the Riemann-Hilbert problem asks whether every such representation occurs in this manner (see [B] for a survey of this problem's interesting history). Lappo-Danielevskii obtained a partial and elementary solution using hyperlogarithms ([L-D], [A1]), which may be summarized as follows. Consider the linear map

$$\begin{array}{rcl} \Phi : \mathbb{C}\langle X\rangle & \rightarrow & M_n(\mathbb{C}), \\ x_{i_1}\ldots x_{i_n} & \mapsto & Q_{i_1}\ldots Q_{i_n}. \end{array} \tag{7.2}$$

If we make some assumptions about convergence, then Φ extends to $C\langle\langle X\rangle\rangle$ and we can define

$$Y(z) = \sum_{w\in X^*} L_w(z)\Phi(w) = \Phi(L_X(z)). \tag{7.3}$$

The columns of $Y(z)$ form m linearly independent solutions to the differential equation above. By proposition 6.4 and corollary 6.5,

$$\begin{array}{rcl} \mathcal{M}_{\gamma_k} Y(z) & = & Y(z)\Phi(Z(X)^{-1}\exp(2i\pi x_k)Z(X)), \\ & = & Y(z)(I + 2i\pi Q_k + \ldots). \end{array}$$

The map $Q_k \mapsto M_k = I + 2i\pi Q_k + \ldots$ is analytic, and its derivative at the origin is $2\pi i$ times the identity. It follows that it is a diffeomorphism in the neighbourhood of I, so every representation of $\pi_1(D)$ such that the matrices $M_0, \ldots, M_N$ are sufficiently close to the identity will arise in this way. One can show, in particular, that every set of unipotent matrices $M_0, \ldots, M_N$ can occur as the monodromy of the differential equation above ([A1]). See [ENS] for a detailed account of various other approaches to this problem.

The problem we address here is slightly different. We wish to find real analytic functions which resemble hyperlogarithms as closely as possible in their analytic and algebraic properties, but whose monodromy is as general as possible. This means finding a solution to (2.1) with prescribed, unipotent monodromy, in such a way as to ensure that the coefficients satisfy the shuffle relations and no other relations. This amounts to a correspondence between realisations of $\mathcal{HL}_\Sigma$ and monodromy representations of the fundamental group of D. More precisely, suppose that $\rho : \mathcal{HL}_\Sigma \xrightarrow{\sim} A$ is an isomorphic realisation of hyperlogarithms, where $A \subset C^\infty(D, \mathbb{C})$. Then ρ is determined by a series $S_A(\overline{z})$ satisfying

$$\sum_{w\in X^*} \rho(w)\, w = L_X(z) S_A(\overline{z}). \tag{7.4}$$

This follows essentially from the uniqueness part of theorem 2.1. It therefore suffices to find an explicit series $S_A(\overline{z})$ which has the required monodromy, which is essentially the explicit Riemann-Hilbert problem in the unipotent case. The shuffle relations for $\rho(w)$

are equivalent to proving that $S_A(\overline{z})$ is group-like, and the linear independence of the modified hyperlogarithms $\rho(w)$ follows immediately from theorem 3.7.

7.2. **Generalised hyperlogarithms.** Let

$$\overline{\mathcal{O}}_\Sigma = \mathbb{C}[\overline{z}, \Big(\frac{1}{\overline{z}-\overline{\sigma}_i}\Big)_{\sigma_i\in\Sigma}]$$

denote the ring of anti-regular functions on $\mathbb{P}^1(\mathbb{C})\backslash\Sigma$, and let $\overline{L}_\Sigma$ be the $\overline{\mathcal{O}}_\Sigma$-algebra spanned by the functions $\{\overline{L_w}(z),\ w\in X^*\}$. Consider the algebra $\overline{\mathcal{HL}}_\Sigma = \overline{\mathcal{O}}_\Sigma \otimes_{\mathbb{C}} \mathbb{C}\langle X\rangle$ equipped with the derivation

$$\overline{\partial} = \frac{\partial}{\partial\overline{z}}\otimes 1 + \sum_{\sigma_i\in\Sigma}\Big(\frac{1}{\overline{z}-\overline{\sigma}_i}\Big)\otimes\partial_i.$$

We define the *algebra of generalised hyperlogarithms* to be the $\mathcal{O}_\Sigma\otimes\overline{\mathcal{O}}_\Sigma\cong\mathcal{O}_\Sigma\overline{\mathcal{O}}_\Sigma$-algebra

$$\mathcal{HL}_\Sigma\otimes\overline{\mathcal{HL}}_\Sigma. \tag{7.5}$$

This is a bigraded differential algebra equipped with two commuting derivations ∂ and $\overline{\partial}$.

Theorem 7.1. *Every $\mathcal{O}_\Sigma\overline{\mathcal{O}}_\Sigma[\partial,\overline{\partial}]$-subalgebra of $\mathcal{HL}_\Sigma\otimes\overline{\mathcal{HL}}_\Sigma$ is differentially simple. The ring of constants of $\mathcal{HL}_\Sigma\otimes\overline{\mathcal{HL}}_\Sigma$ is $\mathbb{C}$, and for every pair of elements $p_1,p_2\in\mathcal{HL}_\Sigma\otimes\overline{\mathcal{HL}}_\Sigma$ such that $\overline{\partial}p_1=\partial p_2$, there is a $P\in\mathcal{HL}_\Sigma\otimes\overline{\mathcal{HL}}_\Sigma$ such that $\partial P=p_1$ and $\overline{\partial}P=p_2$.*

Proof. The theorem follows from theorem 3.7, since the tensor product of two differentially simple algebras with commuting differentials is differentially simple, *i.e.*, there are no non-trivial ideals $I\subset\mathcal{HL}_\Sigma\otimes\overline{\mathcal{HL}}_\Sigma$ such that $\partial I\subset I$ and $\overline{\partial}I\subset I$. Equivalently, by the remarks preceding theorem 3.7, $\mathcal{HL}_\Sigma\otimes\overline{\mathcal{HL}}_\Sigma$ is simple over the ring of operators $\mathcal{O}_\Sigma\overline{\mathcal{O}}_\Sigma[\partial,\overline{\partial}]$. For the remainder of the theorem, we tensor the exact sequence of theorem 3.7 with $\overline{\mathcal{HL}}_\Sigma$ to give an exact sequence

$$0\longrightarrow\overline{\mathcal{HL}}_\Sigma\longrightarrow\mathcal{HL}_\Sigma\otimes\overline{\mathcal{HL}}_\Sigma\overset{\partial\otimes 1}{\longrightarrow}\mathcal{HL}_\Sigma\otimes\overline{\mathcal{HL}}_\Sigma\longrightarrow 0.$$

This shows that the ring of constants of $\partial,\overline{\partial}$ on $\mathcal{HL}_\Sigma\otimes\overline{\mathcal{HL}}_\Sigma$ is just the ring of constants of $\overline{\partial}$ on $\overline{\mathcal{HL}}_\Sigma$, which is $\mathbb{C}$ by theorem 3.7. Now let $p_1,p_2\in\mathcal{HL}_\Sigma\otimes\overline{\mathcal{HL}}_\Sigma$ such that $\overline{\partial}p_1=\partial p_2$. By the sequence above, we can find $q\in\mathcal{HL}_\Sigma\otimes\overline{\mathcal{HL}}_\Sigma$ such that $\partial q=p_1$. Since $\overline{\partial}p_1-\partial p_2=0$, we have $\overline{\partial}q-p_2\in\ker\partial\otimes 1$. The exact sequence above implies that $\overline{\partial}q-p_2\in\overline{\mathcal{HL}}_\Sigma$. Since every element in $\overline{\mathcal{HL}}_\Sigma$ has a $\overline{\partial}$-primitive (theorem 3.7), there is a $q'\in\overline{\mathcal{HL}}_\Sigma$ such that $\overline{\partial}q'=\overline{\partial}q-p_2$. The element $P=q-q'$ therefore satisfies $\partial P=p_1$, $\overline{\partial}P=p_2$. ☐

There is a canonical $\mathcal{O}_\Sigma\overline{\mathcal{O}}_\Sigma[\partial,\overline{\partial}]$-linear map of differential algebras

$$\begin{aligned}\mathcal{HL}_\Sigma\otimes\overline{\mathcal{HL}}_\Sigma &\longrightarrow L_\Sigma\overline{L}_\Sigma,\\ w\otimes w' &\mapsto L_w(z)\overline{L_{w'}}(z),\end{aligned}\tag{7.6}$$

which is an isomorphism by the previous theorem.

Corollary 7.2. *The functions $\{L_w(z)\overline{L_{w'}}(z) : w,w'\in X^*\}$ are linearly independent over $\mathcal{O}_\Sigma\overline{\mathcal{O}}_\Sigma$. All algebraic relations they satisfy are consequences of the shuffle product.*

If $F\in L_\Sigma\overline{L}_\Sigma$ satisfies $\partial F/\partial z=0$, then clearly $F\in\overline{L}_\Sigma$. Suppose now that $F\notin\overline{\mathcal{O}}_\Sigma$. By corollary 6.5, F has non-trivial monodromy around some σ_k for $0\leq k\leq N$. This proves the following:

Lemma 7.3. *Let $F\in L_\Sigma\overline{L}_\Sigma$. Then $F\in\overline{\mathcal{O}}_\Sigma$ if and only if $\partial F/\partial z=0$ and*

$$\mathcal{M}_{\sigma_k}F=F\qquad\textit{for all}\quad 0\leq k\leq N.$$

This implies that $\partial F/\partial z$ and $(\mathcal{M}_{\sigma_k}F)_{\sigma_k\in\Sigma}$ determine $F\in L_\Sigma\overline{L}_\Sigma$ up to an element in $\overline{\mathcal{O}}_\Sigma$.

7.3. **Hyperlogarithms with arbitrary monodromy.** Let A be an $\mathcal{O}_\Sigma[\partial]$-subalgebra of $L_\Sigma\overline{L}_\Sigma$, and consider an isomorphic realisation of hyperlogarithms $\rho_A : \mathcal{HL}_\Sigma \xrightarrow{\sim} A$ which is closed under the action of monodromy (definition 6.1). Let $\{A_0,\ldots,A_N\} \in \mathbb{C}\langle\langle X\rangle\rangle$ denote the images of the generators $\gamma_0,\ldots,\gamma_N$ of $\pi_1(D,z_0)$ under the monodromy representation (§6.1). This defines a map

$$\rho_A \mapsto \{A_0,\ldots,A_N\}$$

from any such realisation of hyperlogarithms to the set of all N-tuples of group-like series. The following theorem gives a canonical inverse to this map. It is in fact equivalent to an explicit version of the classical Riemann-Hilbert correspondence in the unipotent case, as discussed in §7.1.

Theorem 7.4. *Let $A_0,\ldots,A_N \in \mathbb{C}\langle\langle X\rangle\rangle$ such that $\Delta A_k = A_k \otimes A_k$ for $0 \leq k \leq N$. Then there exists a canonical realisation of hyperlogarithms $A \subset L_\Sigma\overline{L}_\Sigma$:*

$$\rho_A : \mathcal{HL}_\Sigma \xrightarrow{\sim} A,$$

whose monodromy representation is given by the series $\{A_0,\ldots,A_N\}$. If $A, A' \subset L_\Sigma\overline{L}_\Sigma$ are two such realisations, then $A \otimes \overline{\mathcal{O}}_\Sigma = A' \otimes \overline{\mathcal{O}}_\Sigma$.

Remark 7.5. The generating series of A can be written

$$\mathcal{L}_A(z) = L_X(z)\widetilde{\overline{L_{X'}}}(z)\overline{L_Y}(z),$$

where $X' = \{x'_0,\ldots,x'_N\}$ is an alphabet which can be written explicitly in terms of $X = \{x_0,\ldots,x_N\}$, and $Y = \{y_0,\ldots,y_N\}$ is an alphabet which can be written explicitly in terms of X and the series $A_0,\ldots,A_N$. Since $A_0,\ldots,A_N$ are group-like, we can write

$$A_k = 1 + \sum_{j=0}^{N} a_{kj}x_j + \ldots \qquad \text{for} \quad 0 \leq k \leq N,$$

where the omitted terms are words of weight ≥ 2. We will see that for all $0 \leq k \leq N$,

$$\begin{aligned} x'_k &= x_k + \ldots \in \mathbb{C}\langle\langle X\rangle\rangle, \\ y_k &= \frac{1}{2\pi i}\sum_{j=0}^{N} a_{kj}x_j + \ldots \in \mathbb{C}\langle\langle X\rangle\rangle, \end{aligned}$$

where the omitted terms are of weight ≥ 2. These series are primitive, *i.e.*,

$$\Delta x'_k = x'_k \otimes 1 + 1 \otimes x'_k \quad \text{and} \quad \Delta y_k = y_k \otimes 1 + 1 \otimes y_k \quad \text{for all} \quad 0 \leq k \leq N.$$

Their coefficients can be computed algorithmically in terms of the coefficients of $Z^{\sigma_j}(X)$ and A_j.

7.4. **Proof of theorem 7.4.**

Lemma 7.6. *Let $U = \{u_0,\ldots,u_N\}$, $V = \{v_0,\ldots,v_N\}$ be two alphabets. Consider two sets of series $S_i \in \mathbb{C}\langle\langle U\rangle\rangle$, and $T_i \in \mathbb{C}\langle\langle V\rangle\rangle$ for $0 \leq i \leq N$ with equal constant terms:*

$$\varepsilon(S_i) = \varepsilon(T_i) \qquad \textit{for} \quad 0 \leq i \leq N.$$

If we write

$$S_i = \varepsilon(S_i) + \sum_{j=0}^{N} S_{ij}u_j + \ldots,$$

$$T_i = \varepsilon(T_i) + \sum_{j=0}^{N} T_{ij}v_j + \ldots,$$

and if we suppose that the matrix $S = (S)_{ij}$ is invertible, then there is a unique map $\phi : \mathbb{C}\langle\langle U\rangle\rangle \to \mathbb{C}\langle\langle V\rangle\rangle$ such that $\phi(S_i) = T_i$ for $0 \leq i \leq N$. It follows that

$$\phi(u_i) = \sum_{j=0}^{N} \Phi_{ij} v_j + \ldots ,$$

where $\Phi_{ij} = \sum_k (S^{-1})_{ik} T_{kj}$ and the higher order coefficients are explicitly calculable in terms of the coefficients of the series S_i and T_i. If, furthermore, the series S_i, T_i are group-like, then the series $\phi(u_i)$ are primitive for $0 \leq i \leq N$.

The proof is by a straightforward induction, and gives an algorithm for computing the coefficients of ϕ, weight by weight. We omit the details (see also [Ch1]). For the last part, observe that if S_i, T_i are group-like, then ϕ defines a map of Hopf algebras, and therefore commutes with Δ. Since the series u_i are primitive, so too must be $\phi(u_i)$, for $0 \leq i \leq N$.

We first construct an alphabet $X' = \{x'_0, \ldots, x'_N\}$ in terms of X such that the function

$$\mathcal{L}_U(z) = L_X(z)\widetilde{\overline{L_{X'}}}(z)$$

is single-valued. By proposition 6.3, the monodromy operators $\mathcal{M}_{\sigma_k}$ act as follows:

$$\mathcal{M}_{\sigma_k}\mathcal{L}_U(z) = L_X(z) M_{\sigma_k} \widetilde{\overline{L_{X'}}}(z),$$

where, for all $0 \leq k \leq N$,

$$M_{\sigma_k} = Z^{\sigma_k}(X)^{-1}\, e^{2i\pi x_k}\, Z^{\sigma_k}(X)\, \widetilde{\overline{Z^{\sigma_k}}}(X')^{-1}\, e^{-2\pi i x'_k}\, \widetilde{\overline{Z^{\sigma_k}}}(X').$$

Since $L_X(z)$ and $\widetilde{\overline{L_{X'}}}(z)$ are invertible, the uniformity of $\mathcal{L}_U(z)$ is equivalent to the following set of equations:

$$\widetilde{\overline{Z^{\sigma_k}}}(X')\, x'_k\, \widetilde{\overline{Z^{\sigma_k}}}(X')^{-1} = Z^{\sigma_k}(X)^{-1}\, x_k\, Z^{\sigma_k}(X) \qquad \text{for} \quad 0 \leq k \leq N. \tag{7.7}$$

By the previous lemma, there is a unique, explicitly computable solution:

$$x'_k = x_k + \ldots \in \mathbb{C}\langle\langle X\rangle\rangle \qquad \text{for} \quad 0 \leq k \leq N,$$

where the omitted words are of length ≥ 2. Both sides of equation (7.7) are group-like by (5.3), so we deduce that x'_k is primitive. The coefficients of this series are in the field generated by $2\pi i$ and the coefficients of the zeta series $Z^{\sigma_k}(X)$.

Definition 7.7. The generating series of *uniform hyperlogarithms* is the series

$$\mathcal{L}_X(z) = L_X(z)\widetilde{\overline{L_{X'}}}(z), \tag{7.8}$$

where X' is the solution to the equations (7.7) above.

We will now compute the alphabet $Y = \{y_0, \ldots, y_N\}$. If we set

$$\mathcal{L}_A(z) = L_X(z)\widetilde{\overline{L_{X'}}}(z)\overline{L_Y}(z) = \mathcal{L}_U(z)\overline{L_Y}(z), \tag{7.9}$$

then we must check that $\mathcal{M}_{\sigma_k}\mathcal{L}_A(z) = \mathcal{L}_A(z)A_k$ for $0 \leq k \leq N$. By proposition 6.3, this is equivalent to the equations

$$\overline{Z^{\sigma_k}}(Y)^{-1} e^{2\pi i y_k}\, \overline{Z^{\sigma_k}}(Y) = A_k \qquad \text{for} \quad 0 \leq k \leq N. \tag{7.10}$$

By assumption, the series A_k are group-like for Δ, so we may write

$$A_k = 1 + \sum_{j=0}^{N} a_{kj} x_j + \ldots \qquad \text{for} \quad 0 \leq k \leq N,$$

where the omitted terms are words of weight ≥ 2. It follows that we may apply the previous lemma to find an explicit solution Y to (7.10) in the form of the series

$$y_k = \frac{1}{2\pi i}\sum_{j=0}^{N} a_{kj}x_j + \ldots \in \mathbb{C}\langle\langle Y\rangle\rangle,$$

which is necessarily primitive. We have proved that the series $\mathcal{L}_A(z)$ defined by (7.9) has the required monodromy, and satisfies the necessary differential equations with respect to $\partial/\partial z$.

It remains to prove the shuffle relations for the functions just constructed. This is equivalent, by corollary 3.5, to verifying that $\mathcal{L}_A(z) = L_X(z)\overline{\widetilde{L_{X'}}(z)}\overline{L_Y(z)}$ is group-like for Δ. The series $L_X(z)$ is group-like by construction. Since the series $x'_k \in \mathbb{C}\langle\langle X\rangle\rangle$ are all primitive elements, the map $\phi : x_k \mapsto x'_k$ for $0 \leq k \leq N$ commutes with Δ, and it follows that $L_{X'}(z) = \phi(L_X(z))$ is also group-like. Since Δ is cocommutative, it commutes with the mirror map, and $\widetilde{L_{X'}}(z)$ is also group-like. A similar argument proves that $L_Y(z)$ is group-like too. Since the product of group-like series is group-like (Δ is a homomorphism for the concatenation product, by §3.2), it follows that $\mathcal{L}_A(z)$ is group-like.

To show the uniqueness, consider two series $\mathcal{L}_A(z), \mathcal{L}'_A(z)$ satisfying (2.1) whose monodromy is given by the series A_k, $0 \leq k \leq N$. Then $F(z) = (\mathcal{L}_A(z))^{-1}\mathcal{L}'_A(z)$ satisfies

$$\frac{\partial}{\partial z}F(z) = 0 \quad \text{and} \quad \mathcal{M}_{\sigma_k}F(z) = F(z), \qquad 0 \leq k \leq N.$$

By lemma 7.3, $F(z) \in \overline{\mathcal{O}}_\Sigma$ which implies $A \otimes \overline{\mathcal{O}}_\Sigma = A' \otimes \overline{\mathcal{O}}_\Sigma$. This completes the proof.

Corollary 7.8. *$\mathcal{L}_A(z)$ is the unique real-analytic solution to (2.1) such that*

$$\begin{aligned} \mathcal{M}_k\mathcal{L}_A(z) &= \mathcal{L}_A(z)\,A_k \qquad \textit{for} \quad 0 \leq k \leq N,\\ \mathcal{L}_A(z) &\sim \exp(x_0\log|z-\sigma_0|^2 + y_0\,\overline{\log(z-\sigma_0)}) \qquad \textit{as} \quad z \to \sigma_0, \end{aligned}$$

where $y_0 = (2i\pi)^{-1}\log A_0$.

Proof. The asymptotic condition at σ_0 follows from the definition of $\mathcal{L}_A(z)$ and theorem 2.1. The expression for y_0 follows since $\exp(2i\pi y_0) = A_0$, by (7.10). The leading term of A_0 is 1 since it is group-like by assumption. The uniqueness follows from lemma 7.3. □

8. Single-valued hyperlogarithms and functoriality

In the previous section we defined a series $\mathcal{L}_X(z)$ corresponding to the trivial monodromy representation $A_k = 1$ for all $0 \leq k \leq N$. We write

$$\mathcal{L}_X(z) = \sum_{w \in X^*} \mathcal{L}_w(z)\, w,$$

and define $\mathcal{U}_\Sigma$ to be the $\mathcal{O}_\Sigma$-algebra generated by the functions $\mathcal{L}_w(z)$. The *uniform* or *single-valued realisation of* $\mathcal{HL}_\Sigma$ is the realisation

$$\begin{array}{rcl} \rho_\Sigma : \mathcal{HL}_\Sigma & \overset{\sim}{\longrightarrow} & \mathcal{U}_\Sigma, \\ w & \mapsto & \mathcal{L}_w(z). \end{array} \tag{8.1}$$

Theorem 8.1. *The series $\mathcal{L}_X(z)$ is single-valued, and is the unique solution to the following differential equations where x_i' are defined by* (7.7)*:*

$$\begin{array}{rcl} \dfrac{\partial}{\partial z}\mathcal{L}_X(z) & = & \Big(\displaystyle\sum_{i=0}^{N} \frac{x_i}{z-\sigma_i}\Big)\mathcal{L}_X(z), \\ \dfrac{\partial}{\partial \overline{z}}\mathcal{L}_X(z) & = & \mathcal{L}_X(z)\Big(\displaystyle\sum_{i=0}^{N} \frac{x_i'}{\overline{z}-\overline{\sigma}_i}\Big), \\ \mathcal{L}_X(z) & \sim & e^{x_0 \log(|z-\sigma_0|^2)} \qquad as\ z \to \sigma_0. \end{array}$$

The functions $\mathcal{L}_w(z)$ are linearly independent over $\mathcal{O}_\Sigma\overline{\mathcal{O}}_\Sigma$ and satisfy the shuffle relations. Every element in $\mathcal{U}_\Sigma$ has a primitive with respect to $\partial/\partial z$, and every single-valued function $F(z) \in L_\Sigma\overline{L}_\Sigma$ can be written as a unique $\mathcal{O}_\Sigma\overline{\mathcal{O}}_\Sigma$-linear combination of functions $\mathcal{L}_w(z)$.

Proof. The first part follows from theorem 7.4. It only remains to check that

$$\mathcal{U}_\Sigma \otimes \overline{\mathcal{O}}_\Sigma = \{F \in L_\Sigma\overline{L}_\Sigma : \mathcal{M}_{\sigma_k}F = F \quad \text{for} \quad 0 \leq k \leq N\}.$$

Let $F \in L_\Sigma\overline{L}_\Sigma$ be a uniform function. There is an operator $\mathcal{D} \in \mathbb{C}[z,\partial]$ of the form given in theorem 4.4 *ii*), such that $\mathcal{D}F(z) = 0$. By theorem 3.7, we can solve this equation in $\mathcal{U}_\Sigma$ by taking uniform primitives. By lemma 7.3, uniform primitives with respect to ∂ in $L_\Sigma\overline{L}_\Sigma$ are unique up to a function in $\overline{\mathcal{O}}_\Sigma$. This proves that $F \in \mathcal{U}_\Sigma \otimes \overline{\mathcal{O}}_\Sigma$. □

The following corollary follows immediately from lemma 7.3.

Corollary 8.2. *Any single-valued solution to the differential equation*

$$\frac{\partial}{\partial z}F(z) = \Big(\sum_{i=0}^{N} \frac{x_i}{z-\sigma_i}\Big)F(z),$$

satisifies $F(z) = \mathcal{L}_X(z)\,C$, where $C \in \overline{\mathcal{O}}_\Sigma\langle\langle X\rangle\rangle$.

The series $\mathcal{L}_X(z)$, and hence the algebra $\mathcal{U}_\Sigma$, is the universal single-valued solution to unipotent differential equations on $\mathbb{P}^1(\mathbb{C})\backslash\Sigma$. If A is any unipotent extension of $\mathcal{O}_\Sigma$ in the sense of theorem 4.4, then the embedding $A \hookrightarrow \mathcal{HL}_\Sigma \cong \mathcal{U}_\Sigma$ given by that theorem gives rise to a realisation of A in terms of unipotent functions. This can be made explicit:

Corollary 8.3. *Let $Q_i \in M_n(\mathbb{C})$ be strictly upper triangular matrices, for $0 \leq i \leq N$. Then equation* (7.1) *has an explicit single-valued solution*

$$Y(z) = \sum_{w \in X^*} \mathcal{L}_w(z)\,\Phi(w) = \Phi(\mathcal{L}_X(z)),$$

where Φ is defined in (7.2) *and is 0 for all but a finite number of words $w \in X^*$.*

Remark 8.4. It would be interesting to construct explicit single-valued versions of more general types of differential equations. One can immediately obtain a larger class of single-valued functions by composing $\mathcal{L}_w(z)$ with other single-valued functions (*e.g.*, exp), or by considering series given by corollary 8.3 which converge but are not necessarily finite.

8.1. **Functoriality of $\mathcal{HL}_\Sigma$ and functional equations.** It is well-known that hyperlogarithms satisfy complicated functional equations in many variables ([Ga], [Le], [W1], [Z2]), which are difficult to construct, and poorly understood. It is more natural to consider single-valued versions of these equations since they are cleaner, *i.e.*, there are no lower-order parasite terms arising from the monodromy, and one does not have to keep track of the choices of branches.

When one considers the whole algebra of single-valued hyperlogarithms $\mathcal{U}_\Sigma$, for varying Σ, it becomes possible to find all linear relations between single-valued hyperlogarithms $\mathcal{L}_{w_i}(f_j(z))$ where $w_i \in X^*$, and f_j are rational functions of z. The following proposition summarizes all such relations by a simple functoriality property.

Proposition 8.5. *Let $f(z) \in \mathbb{C}(z)$. Let $\Sigma, \Sigma' \subset \mathbb{P}^1(\mathbb{C})$ be any finite subsets containing ∞ such that $f^{-1}(\Sigma) \subset \Sigma'$. The natural map $\mathcal{O}_\Sigma \to \mathcal{O}_{\Sigma'}$ induced by composition by f extends to a canonical morphism of $\mathbb{C}$-algebras $\phi : \mathcal{HL}_\Sigma \to \mathcal{HL}_{\Sigma'}$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{HL}_\Sigma & \xrightarrow{\phi} & \mathcal{HL}_{\Sigma'} \\ \downarrow & & \downarrow \\ \mathcal{U}_\Sigma & \xrightarrow{f_*} & \mathcal{U}_{\Sigma'}, \end{array}$$

where the vertical maps are the uniform realisation maps (8.1)*, and the map $f_* : \mathcal{U}_\Sigma \to \mathcal{U}_{\Sigma'}$ is given by $\mathcal{L}_w(z) \mapsto \mathcal{L}_w(f(z))$.*

Proof. (Compare [HPV]). The generating series $\mathcal{L}_X(f(z))$ is single-valued, and satisfies

$$\frac{\partial}{\partial z}\mathcal{L}_X(f(z)) = \Omega_f\, \mathcal{L}_X(f(z))$$

where

$$\Omega_f = \sum_{i=0}^{N} \frac{x_i\, f'(z)}{f(z) - \sigma_i} = \frac{\partial}{\partial z} \sum_{i=0}^{N} x_i \log(f(z) - \sigma_i).$$

By assumption, Σ' contains the poles of f (and therefore f'), and the zeros of $f(z) - \sigma_i$, so $f'(z)/(f(z) - \sigma_i) \in \mathcal{O}_{\Sigma'}$, and we can decompose Ω_f into partial fractions:

$$\Omega_f = \sum_{i=0}^{N'} \frac{u_i}{z - \sigma'_i},$$

where u_i are linear combinations of $x_i \in X$, and $\Sigma' = \{\sigma'_0, \ldots, \sigma'_{N'}, \infty\}$. It follows that $\mathcal{L}_{\{u_0,\ldots,u_{N'}\}}(z)$ and $\mathcal{L}_X(f(z))$ are single-valued solutions to the same differential equation, and therefore differ by a series $C \in \overline{\mathcal{O}}_{\Sigma'}\langle\langle X \rangle\rangle$, by corollary 8.2. Since f and the u_i have no dependence on $\overline{z}$ one can check that $C \in \mathbb{C}\langle\langle X \rangle\rangle$. It follows that $\mathcal{L}_X(f(z)) = \mathcal{L}_{\{u_0,\ldots,u_{N'}\}}(z)\, C$, which proves that $\mathcal{L}_w(f(z)) \in \mathcal{U}_{\Sigma'}$ for all $w \in X^*$ as required. Alternatively, one can define ϕ inductively using the formula $\partial \phi_f(x_i w) = f'(z)/(f(z) - \sigma_i)\phi_f(w)$, and by taking uniform primitives. □

Remark 8.6. It follows that the algebras $\mathcal{U}_\Sigma$ define a sheaf of single-valued real-analytic functions on $\mathbb{P}^1(\mathbb{C})$ for the Zariski topology.

The proposition implies that the $\mathbb{C}(z)$-algebra $\bigcup_\Sigma \mathcal{U}_\Sigma \otimes \mathbb{C}(z)$, which is the universal single-valued solution of all unipotent differential equations on $\mathbb{P}^1(\mathbb{C})$, linearizes all functional equations of hyperlogarithms in one variable. To illustrate, let η be a primitive n^{th} root of unity, and let $\sigma'_0 = 0$, $\sigma'_i = \eta^i$ for $1 \leq i \leq n$. We set $X = \{x_0, x_1\}$ and define $X \to \Sigma = \{0, 1, \infty\}$ as usual; and set $U = \{u_0, \ldots, u_n\}$ with the obvious map $U \to \Sigma' := \{0, \eta^i, \infty\}$. If we consider the map $f(z) = z^n$, then, in the notations of the proof of proposition 8.5, we have

$$\Omega_f = nz^{n-1}\Big(\frac{x_0}{z^n} + \frac{x_1}{z^n - 1}\Big) = \frac{nx_0}{z} + \sum_{i=1}^n \frac{x_1}{z - \eta^i}.$$

Therefore $u_0 = nx_0$, and $u_i = x_1$ for $i \geq 1$, and one obtains $C = 1$. In this case, one can verify that ϕ is the unique map which commutes with the concatenation product and satisfies $\phi(x_0) = nu_0$ and $\phi(x_1) = \sum_{i=1}^n u_i$. This implies the distribution relation

$$\mathcal{L}_{x_0^k x_1}(z^n) = n^k \sum_{i=1}^n \mathcal{L}_{u_0^k u_i}(z),$$

and shows how the functions $\mathrm{Li}_k(z^n)$ are linearised in the larger algebra $\mathcal{U}_{\Sigma'}$. In general, it is much more convenient to work with the linearised versions, since their algebraic relations are completely understood by §3.

There are other interesting examples given by finite subgroups of the group of automorphisms of $\mathbb{P}^1(\mathbb{C})$, which is isomorphic to $\mathrm{PSL}_2(\mathbb{C})$. For example, if $\Sigma = \{0, 1, \infty\}$, then there is an action of the symmetric group $\mathfrak{S}_3$ on $\mathcal{U}_\Sigma$ which is generated by the maps $z \mapsto 1 - z$ and $z \mapsto z/z - 1$. One easily checks, with the obvious notation, that

$$\begin{aligned} \mathcal{L}_{x_0,x_1}(1-z) &= \mathcal{L}_{-x_1,-x_0}(z)\, Z^1(X)\widetilde{Z^1}(X'), \\ \mathcal{L}_{x_0,x_1}\Big(\frac{z}{z-1}\Big) &= \mathcal{L}_{x_0,\, x_0+x_1}(z), \end{aligned}$$

where X' is the alphabet defined in §7.4 (see also [Br1]). A similar expression was obtained in [HPV] for ordinary multiple polylogarithms in one variable (see also [U]). Note that the series $Z^1(X)\widetilde{Z^1}(X')$ is the regularised value of $\mathcal{L}_X(z)$ at 1 and is sparse, *i.e.*, much cancellation occurs. Likewise, if Σ consists of 0 and the set of n^{th} roots of unity, then the maps $z \mapsto e^{2i\pi/n}z$, $z \mapsto z^{-1}$ define an action of the dihedral group of order $2n$ on $\mathcal{U}_\Sigma \cong \mathcal{HL}_\Sigma$. The other finite subgroups of $\mathrm{PSL}_2(\mathbb{C})$ are isomorphic to the symmetry groups of the tetrahedron, octahedron, and icosahedron, which all give rise to actions on suitable algebras of single-valued hyperlogarithms (*c.f.* [We]).

If we now fix a function $\mathcal{L}_w(z)$ and vary the rational arguments $f(z)$, then the proposition can also be used to deduce the existence of functional equations for $\mathcal{L}_w(z)$. Suppose that we are given finite sets Σ, $\Sigma' \subset \mathbb{P}^1(\mathbb{C})$ which contain ∞ and a number of rational functions $f_j(z) \in \mathbb{C}(z)$ such that $f_j^{-1}(\Sigma) \subset \Sigma'$. The proposition states that the functions $\mathcal{L}_w(f_j(z))$ can all be written as explicit elements in the finite-dimensional vector space $V \subset \mathcal{U}_{\Sigma'}$ consisting of functions of weight $\leq |w|$. Since we have an explicit basis for this vector space, and since we know all the algebraic relations between elements in the basis (theorem 3.3), this enables us to write down all possible relations between the $\mathcal{L}_w(f_j(z))$ simply by doing linear algebra. In particular, if the number of such functions f_j exceeds the dimension of the space V then we can immediately deduce the existence of a functional relation (*c.f.* there are other criteria for the existence of functional equations for polylogarithms due to Zagier and Wojtkowiak ([Le])).

These ideas are discussed in greater detail in the multi-variable case in [Br2].

8.2. **Examples.** We have

$$\mathcal{L}_{x_0^n}(z) = \frac{1}{n!}\log^n|z-\sigma_0|^2, \quad \text{and} \quad \mathcal{L}_{x_i^n}(z) = \frac{1}{n!}\log^n\Big|\frac{z-\sigma_i}{\sigma_0-\sigma_i}\Big|^2 \quad \text{for} \quad 1\leq i\leq N.$$

Now let $\sigma_0 = 0$, $\sigma_1 = 1$, and $D = \mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}$. One can check that the formula (7.8) for the two infinite families of words $\{x_0^{n-1}x_1\}$ and $\{x_1x_0^{n-2}x_1\}$ does not involve zeta values: *i.e.*, for all such w one has

$$\mathcal{L}_w(z) = \sum_{u\widetilde{v}=w} \mathrm{Li}_u(z)\mathrm{Li}_v(\overline{z}).$$

In the first case one obtains single-valued versions of the classical polylogarithms which, on applying the shuffle relations, can be written in the equivalent form:

$$\mathcal{L}_{x_0^{n-1}x_1}(z) = -\mathrm{Li}_n(z) + \sum_{k=0}^{n-1}(-1)^{n-k}\Big(\frac{2^k}{k!}\Big)\log^k|z|\mathrm{Li}_{n-k}(\overline{z}). \tag{8.2}$$

These functions first appeared in [Ram] and are related to various other single-valued versions of the classical polylogarithms ([W2], [Z1]) in [Br1]. In the second case, one obtains:

$$\mathcal{L}_{x_1x_0^{n-2}x_1}(z) = 2\,\mathrm{Re}\,(\mathrm{Li}_{x_1x_0^{n-2}x_1}(z)) + \sum_{\substack{i+j=n,\\ i\geq 1,\, j\geq 1}} \mathrm{Li}_{x_1x_0^{i-1}}(z)\,\mathrm{Li}_{x_1x_0^{j-1}}(\overline{z}),$$

which, using the shuffle relations, can equivalently be written:

$$\mathcal{L}_{x_1x_0^{n-2}x_1}(z) = 2\,\mathrm{Re}\,(\mathrm{Li}_{x_1x_0^{n-2}x_1}(z)) + \sum_{\substack{i+j+k=n,\\ j,\, k\geq 1}} (-1)^{j+k}\Big(\frac{2^i}{i!}\Big)\log^i|z|\,\mathrm{Li}_j(z)\,\mathrm{Li}_k(\overline{z}). \tag{8.3}$$

Similar expressions can be obtained explicitly for any such infinite family of words using the methods of §7.4. In general, the formulae obtained will involve multiple zeta values $\zeta^1(w)$. The procedure works equally well for any set of singularities Σ.

9. Bibliography

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