Periods and Cosmic Galois group

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I. Particle physics
Collision of beam particles

Test the laws of physics by analysing particle tracks.
Perturbative Quantum Field theory

General framework describing fundamental forces and particles.

Every Feynman graph $G$ represents a possible particle interaction. Feynman *amplitude* is a complex probability assigned to $G$. 
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Every Feynman graph $G$ represents a possible particle interaction. Feynman *amplitude* is a complex probability assigned to $G$. 
The blue line (background) requires calculating a huge number of Feynman amplitudes.
II. Graphs and Numbers
Let $G = (V_G, E_G)$ be a connected graph. The graph polynomial

$$
\Psi_G \in \mathbb{Z}[\alpha_e, e \in E(G)]
$$

is a sum over spanning trees $T$ of $G$

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A tree $T \subset G$ is spanning if $V_T = V_G$. 
Example

In general, $\Psi_G$ is homogeneous of degree $h_G$ (‘loop number’).

$$\deg \Psi_G = h_G = \#E(G)$$

Physically relevant graphs have vertices of degree $\leq 4$. (‘$G$ in $\phi_4$’).

$\Psi_G = ?$
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Physically relevant graphs have vertices of degree \( \leq 4 \). (‘\( G \) in \( \phi^4 \)’).
For convergence, assume

1. \( N_G = 2h_G \)
2. \( N_\gamma > 2h_\gamma \) for all \( \gamma \subseteq G \).
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The \textit{residue} is the convergent integral

$$I_G = \int_\sigma \frac{\Omega_G}{\Psi_G^2} \quad \in \quad \mathbb{R}$$
Feynman integrals

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The residue is the convergent integral

$$I_G = \int_\sigma \frac{\Omega_G}{\Psi^2_G} \in \mathbb{R}$$

where

$$\Omega_G = \sum_{i=1}^{N_G} (-1)^i \alpha_i d\alpha_1 \wedge \ldots \wedge \hat{d}\alpha_i \wedge \ldots d\alpha_{N_G}$$
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where

$$\Omega_G = \sum_{i=1}^{N_G} (-1)^i \alpha_i d\alpha_1 \wedge \ldots \wedge \hat{d}\alpha_i \wedge \ldots d\alpha_{N_G}$$

$$\sigma = \{(\alpha_1 : \ldots : \alpha_{N_G}) \in \mathbb{P}^{N_G-1}(\mathbb{R}) \text{ such that } \alpha_i \geq 0\}$$
We obtain a map

\[ I : \{ \text{convergent graphs in } \phi^4 \} \rightarrow \mathbb{R} \]
Graphs and numbers

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Example:

![Diagram](image)
We obtain a map

\[ I : \{ \text{convergent graphs in } \phi^4 \} \longrightarrow \mathbb{R} \]

Example:

\[ \Psi_G = \alpha_1 + \alpha_2 \]
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Example:

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Compute the integral on the chart \( \alpha_2 = 1 \):

\[ I_G = \int_\sigma \frac{\alpha_2 d\alpha_1 - \alpha_1 d\alpha_2}{(\alpha_1 + \alpha_2)^2} = \int_{\alpha_1 \geq 0} \frac{d\alpha_1}{(\alpha_1 + 1)^2} = 1 \]
The Zoo

\[ I_G : \quad 6\zeta(3) \quad 20\zeta(5) \quad 36\zeta(3)^2 \quad N_{3,5} \]
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$I_G : \quad 6\zeta(3) \quad 20\zeta(5) \quad 36\zeta(3)^2 \quad N_{3,5}$

$$N_{3,5} = \frac{27}{5}\zeta(5, 3) + \frac{45}{4}\zeta(5)\zeta(3) - \frac{261}{20}\zeta(8)$$
Multiple Zeta Values, defined for $n_1, \ldots, n_{r-1} \geq 1$, and $n_r \geq 2$:

$$\zeta(n_1, \ldots, n_r) = \sum_{1 \leq k_1 < k_2 < \ldots < k_r} \frac{1}{k_1^{n_1} \cdots k_r^{n_r}} \in \mathbb{R}$$
The numbers $I_G$ are $\mathbb{Q}$-linear combinations of multiple zeta values.
Folklore conjecture 90’s

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Cartier’s dream:
Main problem

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Cartier’s dream:

$I_G$ $\rightarrow$ MZV

$G_{MT(Z)}$
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Cartier’s dream:

$G_{cos}$

$G_{MT(Z)}$
Main problem

Folklore conjecture 90’s

The numbers $I_G$ are $\mathbb{Q}$-linear combinations of multiple zeta values.

Cartier’s dream:

Cartier postulated the existence of a ‘cosmic Galois group’ (1998).
Contraction-Deletion:

\[ \Psi_G = \alpha_e \Psi_{G \backslash e} + \Psi_{G / e} \]
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2. **Partial factorisation:**

\[ \Psi_G = \Psi_\gamma \Psi_{G/\gamma} + R_{\gamma,G} \]
**Properties I**

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\[ \Psi_G = (\alpha_3 + \alpha_4)(\alpha_1 + \alpha_2) + \alpha_3\alpha_4 \]

\[ \Psi_\gamma \]

\[ \Psi_{G/\gamma} \]

\[ R_{\gamma, G} \]
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\[ \Psi_G = (\alpha_3 + \alpha_4)(\alpha_1 + \alpha_2) + \alpha_3 \alpha_4 \]

Determines \( \Psi_G \) essentially uniquely.
The graph polynomial is a determinant

$$\psi_G = \det(L_G)$$

where $L_G$ is the reduced graph Laplacian matrix.
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Many identities between $I_G$. For example:

$$I_{G_1} I_{G_2} = I_{G_1:G_2}.$$
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Many identities between \( I_G \). For example:

\[ I_{G_1} I_{G_2} = I_{G_1:G_2}. \]

and planar duals, completion (Fourier transform), …
Counterexamples

Theorem: (B., Schnetz 2012)

The zero locus of $\Psi_G$ is modular of weight 3, level 7.
\(\Psi_G\) of degree 8 in 16 variables, 3785 terms.

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$I_G$ should be algebraically independent from multiple zeta values!!
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The folklore conjecture is likely to be false. The nature of the numbers $I_G$ is unknown.
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\[ \text{However, the group actions may yet survive} \ldots \]
III. Periods
Definition (Kontsevich-Zagier)

Periods are complex numbers with real and imaginary parts of the form

\[ I = \int_{\sigma} \frac{P}{Q} \, dx_1 \ldots dx_n \]

where \( P, Q, \in \mathbb{Q}[x_1, \ldots, x_n] \), and \( \sigma \) finite union of sets \( \{ f_1, \ldots, f_N \geq 0 \} \) with \( f_i \in \mathbb{Q}[x_1, \ldots, x_n] \).
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\[ \pi = \int_{x^2 + y^2 \leq 1} \, dx \, dy \]

\[ \log 2 = \int_{1 \leq x \leq 2} \frac{dx}{x} \]
Periods form a ring:

\[ \mathbb{Q} \subseteq \overline{\mathbb{Q}} \subseteq P \subseteq \mathbb{C} \]
Ring of periods

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Non-periods?

\[ e = \int_{x \leq 1} e^x \, dx \]

\[ \gamma = \int_{0}^{\infty} \frac{e^{-x}}{e^{-x} - 1} - \frac{e^{-x}}{x} \, dx \]
Additivity in $\omega$ and $\sigma$:

$$\int \sigma_1 \omega_1 + \omega_2 = \int \sigma_1 \omega_1 + \int \sigma_2 \omega_2$$

Algebraic changes of variables

$$\int f^* \sigma \omega = \int \sigma f^* \omega$$

Stokes:

$$\int \sigma d \omega = \int \partial \sigma \omega$$
Additivity in $\omega$ and $\sigma$:

$$\int_{\sigma} \omega_1 + \omega_2 = \int_{\sigma} \omega_1 + \int_{\sigma} \omega_2$$
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• Algebraic changes of variables

$$\int_{f*\sigma} \omega = \int_\sigma f^* \omega$$
• Additivity in $\omega$ and $\sigma$:

\[ \int_{\sigma} (\omega_1 + \omega_2) = \int_{\sigma} \omega_1 + \int_{\sigma} \omega_2 \]

• Algebraic changes of variables

\[ \int_{f^*\sigma} \omega = \int_{\sigma} f^*\omega \]

• Stokes:

\[ \int_{\sigma} d\omega = \int_{\partial\sigma} \omega \]
Algebraic numbers are periods.

Can one extend Galois theory to periods?

Grothendieck, Kontsevich-Zagier, André, …
Algebraic numbers are periods.

Can one extend Galois theory to periods?

Grothendieck, Kontsevich-Zagier, André, . . .

Want a pro-algebraic group $\mathcal{G}$

$$\mathcal{G} \times P \longrightarrow P$$

which acts linearly on periods.
Example: MZV’s

\[ \zeta(2) = \int_{0}^{1} \frac{1}{t_1} dt_1 - \int_{t_1}^{t_2} \frac{1}{t_2} dt_2 \]

**Multiple Zeta Values:**

\[ \zeta(n_1, \ldots, n_r) = (-1)^r \int_{0}^{1} \cdots \int_{t_{n_1} \leq \cdots \leq t_n \leq 1} dt_1 \cdots dt_n \]

where \((\epsilon_1, \ldots, \epsilon_n) = 10^{n_1-1} \cdots 10^{n_r-1} \).

MZV’s form a subring of the ring of periods: \( \mathbb{Q} \subset \mathbb{Z} \subset \mathbb{P} \).
Example: MZV’s

Leibniz:

$$\zeta(2) = \int_{0 \leq t_1 \leq t_2 \leq 1} \frac{dt_1}{1 - t_1} \frac{dt_2}{t_2}$$
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MZV’s form a subring of the ring of periods:

\[ \mathbb{Q} \subset \mathbb{Z} \subset P \]
In weight 4: generators $\zeta(4)$, $\zeta(1, 3)$, $\zeta(2, 2)$, and $\zeta(1, 1, 2)$.

Relations:

\[
\begin{align*}
\zeta(2)^2 &= 2\zeta(2, 2) + \zeta(4) \\
\zeta(2)^2 &= 4\zeta(1, 3) + 2\zeta(2, 2) \\
\zeta(1, 3) + \zeta(4) &= 2\zeta(1, 3) + \zeta(2, 2) \\
2\zeta(1, 1, 2) + \zeta(2, 2) + \zeta(1, 4) &= 3\zeta(1, 1, 2)
\end{align*}
\]

A Galois group of periods should respect these relations!
IV. Motivic Periods

(a Galois theory of periods for dummies)
$X$ smooth affine over $\mathbb{Q}$.

- Algebraic de Rham cohomology:

$$H^n_{dR}(X; \mathbb{Q}) = \frac{\text{closed algebraic forms of degree } n}{\text{exact algebraic forms of degree } n}$$
Cohomology

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- Betti (singular) cohomology:

$$H^{n}_{B}(X) = H^{n}(X(\mathbb{C}))^\vee$$
X smooth affine over \(\mathbb{Q}\).

- **Algebraic de Rham cohomology:**
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  \]

- **Betti (singular) cohomology:**
  \[
  H^n_B(X) = H_n(X(\mathbb{C}))^\vee
  \]

- **Comparison theorem (de Rham, Grothendieck):**
  \[
  H^n_{dR}(X; \mathbb{Q}) \otimes \mathbb{C} \xrightarrow{\sim} H^n_B(X) \otimes \mathbb{C}
  \]
  \[
  \omega \mapsto (\gamma \mapsto \int_\gamma \omega)
  \]
Example: $2i\pi$

\[ X = \mathbb{P}^1 \setminus \{0, \infty\}. \quad X(\mathbb{C}) = \mathbb{C}^\times. \]

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Example: $2i\pi$

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$$\gamma$$

$$H^1_{dR}(X) = \mathbb{Q}\left[\frac{dx}{x}\right]$$

$$H_1(X(\mathbb{C})) = \mathbb{Q}[\gamma]$$
Example: $2i\pi$

$X = \mathbb{P}^1 \setminus \{0, \infty\}$. $X(\mathbb{C}) = \mathbb{C}^\times$.

Period:

$$
\int_\gamma \frac{dx}{x} = 2i\pi
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$H_1(X(\mathbb{C})) = \mathbb{Q}[\gamma]$

Period:

$$\int_{\gamma} \frac{dx}{x} = 2i\pi$$

$H^1_{dR}(X) \otimes \mathbb{C} \sim \to H^1_B(X) \otimes \mathbb{C}$

$[\frac{dx}{x}] \mapsto 2i\pi \ [\gamma]^\vee$
A ring of ‘motivic’ periods

$P^m$ the $\mathbb{Q}$-vector space spanned by symbols

$[H^n(X), \omega, \gamma]$ modulo an (elementary) equivalence relation, where $H^n(X)$ is a triplet $(H_B, H_{dR}, \int)$, with $H_B, H_{dR} \in \text{Vec}_\mathbb{Q}$, $\int$ an isomorphism between their complexifications.
A ring of ‘motivic’ periods

1. $P^m$ the $\mathbb{Q}$-vector space spanned by symbols

$$[H^n(X), \omega, \gamma]$$

modulo an (elementary) equivalence relation, where $H^n(X)$ is a triplet $(H_B, H_{dR}, \int)$, with $H_B, H_{dR} \in \text{Vec}_\mathbb{Q}$, $\int$ an isomorphism between their complexifications.

2. Ring structure (Künneth)

$$[V, \omega, \gamma] \otimes [W, \omega', \gamma'] = [V \otimes W, \omega \otimes \omega', \gamma \otimes \gamma']$$
Period homomorphism

\[ \text{per} : \mathcal{P}^m \rightarrow P \]

\[ [H^n(X), \omega, \gamma] \mapsto \int_\gamma \omega \]
Period homomorphism

\[
\text{per} : \mathcal{P}^m \longrightarrow \mathcal{P}
\]

\[
[H^n(X), \omega, \gamma] \mapsto \int_\gamma \omega
\]

We gain the action of a pro-algebraic group

\[
\mathcal{G}^{dR} \times \mathcal{P}^m \longrightarrow \mathcal{P}^m
\]

It acts linearly on \( H^n_{dR}(X) \):

\[
g[H^n(X), \omega, \gamma] = [H^n(X), g \omega, \gamma]
\]
Given a period

$$I = \int_\gamma \omega$$

Try to express $I$ as a period of cohomology

$$\omega \in H^n_{dR}(X), \quad \gamma \in H_n(X(\mathbb{C})).$$
Given a period

\[ I = \int_\gamma \omega \]

Try to express \( I \) as a period of cohomology

\[ \omega \in H^n_{dR}(X), \quad \gamma \in H_n(X(\mathbb{C})) \]

Replace \( I \) by ‘its’ motivic version (NB choices!)

\[ I^m = [H^n(X), \omega, \gamma] \in \mathcal{P}^m \]
Given a period

\[ I = \int_\gamma \omega \]

Try to express \( I \) as a period of cohomology

\[ \omega \in H^n_{dR}(X), \quad \gamma \in H_n(X(\mathbb{C})) . \]

Replace \( I \) by ‘its’ motivic version (NB choices!)

\[ I^m = [H^n(X), \omega, \gamma] \in P^m \]

The action of the group \( G^{dR} \) on \( I^m \) generates a representation of a quotient of \( G^{dR} \). We can use group theory to define invariants, or to discover new relations.
Example: $2i\pi$

\[ X = \mathbb{P}^1 \backslash \{ 0, \infty \}. \]

\[ (2i\pi)^m := [H^1(X), [\frac{dx}{x}], [\gamma]] \]
Example: $2i\pi$

\[ X = \mathbb{P}^1 \setminus \{0, \infty\}. \]

\[
(2i\pi)^m := [H^1(X), \left[ \frac{dx}{x} \right], [\gamma]]
\]

\[
g \in G^{dR} : \quad g(2i\pi)^m = \lambda_g (2i\pi)^m
\]
Example: $2i\pi$

$X = \mathbb{P}^1 \backslash \{0, \infty\}$.

$$(2i\pi)^m := [H^1(X), [\frac{dx}{x}], [\gamma]]$$

$g \in G^{dR} : g(2i\pi)^m = \lambda_g (2i\pi)^m$

It spans a one-dimensional representation

$$G^{dR} \longrightarrow GL_1$$

$g \mapsto \lambda_g$

So $(2i\pi)^m$ is a motivic period of rank 1.
Example 2: logarithms

\[
\log^m(2) \xrightarrow{\mathbf{g}} \lambda_g \log^m(2) + \nu_g
\]

Equivalently

\[
\begin{pmatrix}
1 & \log^m(2) \\
0 & (2\pi i)^m
\end{pmatrix} \xrightarrow{\mathbf{g}} \begin{pmatrix}
1 & \log^m(2) \\
0 & (2\pi i)^m
\end{pmatrix} \begin{pmatrix}
1 & \nu_g \\
0 & \lambda_g
\end{pmatrix}
\]
Example 2: logarithms

\[ \log^m(2) \xrightarrow{g} \lambda_g \log^m(2) + \nu_g \]

Equivalently

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\begin{pmatrix}
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\xrightarrow{g}
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1 & \log^m(2) \\
0 & (2\pi i)^m
\end{pmatrix}
\begin{pmatrix}
1 & \nu_g \\
0 & \lambda_g
\end{pmatrix}
\]

Hence a two-dimensional representation

\[ \mathcal{G}^{dR} \longrightarrow GL_2 \]

\[ g \quad \mapsto \quad \begin{pmatrix}
1 & \nu_g \\
0 & \lambda_g
\end{pmatrix} \]

So \( \log^m(2) \) is of rank 2.
Example 3: Algebraic numbers

Let $\alpha \in \mathbb{C}$ algebraic. There exists canonical $\alpha^m$, whence

$$
\overline{\mathbb{Q}} \subset \mathcal{P}^m
$$

$\mathcal{G}^{dR}$ acts on $\overline{\mathbb{Q}}$ via a pro-algebraic quotient $\mathcal{A}_{\overline{\mathbb{Q}}}$:

$$
\mathcal{A}_{\overline{\mathbb{Q}}} (\mathbb{C}) \cong \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) .
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$$A_{\overline{\mathbb{Q}}}(\mathbb{C}) \cong Gal(\overline{\mathbb{Q}}/\mathbb{Q}) .$$

The *rank* of $\alpha^m$ is the dimension of the vector space spanned by the conjugates of $\alpha$. 
Weak variant of Grothendieck’s period conjecture.

\[
\text{per} : \mathcal{P}^m \longrightarrow \mathbb{C} \quad \text{is injective}
\]
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Means that \( I^m \) does not depend on choices, and the action of \( \mathcal{G}^{dR} \) on \( \mathcal{P}^m \) can be transported onto the ring of periods itself.
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Means that \( I^m \) does not depend on choices, and the action of \( G^{dR} \) on \( \mathcal{P}^m \) can be transported onto the ring of periods itself.

Via group and Hodge theory, we can define many new invariants:
...
rank, unipotency degree, weight, Hodge numbers, degree,.....
<table>
<thead>
<tr>
<th>Unip. degree</th>
<th>Examples</th>
</tr>
</thead>
</table>
| 0            | Pure periods: $\pi$, elliptic integrals, \ldots  
               | (classical) |
| 1            | Periods of simple extensions: $\log 2, \zeta(2n+1), \ldots$  
               | (values of $L$-functions) |
| $\geq 2$     | Multiple periods: MZV’s, Feynman amplitudes, \ldots  
               | (unknown) |
               | —– Unexplored territory —– |
V. Applications and questions
There exist motivic versions of multiple zeta values \( \zeta_m^{(n_1, \ldots, n_r)} \in \mathbb{P}_m \) for all \( n_1, \ldots, n_r \geq 1, \ n_r \geq 2 \), whose periods are \( \zeta^{(n_1, \ldots, n_r)} \). They satisfy the 'standard' relations. The ring \( \mathbb{Z}_m \) generated by \( \zeta_m \) is stable under the group \( G_d \). Not known if they satisfy more relations. Think of \( \mathbb{Z}_m \) as a Galois extension of \( \mathbb{Q} \) with group \( G_MZV \times \mathbb{Z}_m \rightarrow \mathbb{Z}_m \).
Theorem (B. 2012)

There exist motivic versions of multiple zeta values

\[ \zeta^m(n_1, \ldots, n_r) \in \mathcal{P}^m \]

for all \( n_1, \ldots, n_r \geq 1, n_r \geq 2 \), whose periods are \( \zeta(n_1, \ldots, n_r) \). They satisfy the ‘standard’ relations. The ring \( \mathcal{Z}^m \) generated by the \( \zeta^m \) is stable under the group \( G^{dR} \).
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Think of \( \mathcal{Z}^m \) as a Galois extension of \( \mathbb{Q} \) with group

\[ \mathcal{G}^{MZV} \times \mathcal{Z}^m \rightarrow \mathcal{Z}^m \]
Theorem (B. 2012) $G_{MZV} = U_{MZV} \rtimes G_m$ where $U_{MZV}$ is pro-unipotent. Its graded Lie algebra is free on generators in degrees $-3, -5, -7, \ldots$. Hence $G_{MZV} = G_{MT}(\mathbb{Z})$.

Think of $\sigma_{2n+1}$ as 'differentiation with respect to $\zeta_m(2m+1)$' etc.
Theorem (B. 2012)

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The \( \sigma_{2n+1} \) act via

\[ \sigma_{2n+1} \zeta^m(2m + 1) = \delta_{n,m} \]

Think of \( \sigma_3 \) as ‘differentiation with respect to \( \zeta^m(3) \)’, etc

There exist canonical ‘motivic’ Feynman amplitudes $I_m G \in \mathbb{P}^m$ for any convergent $G$, whose period is $I_G$.

Not all expected relations are known.

Many new invariants to amplitudes (weights, rank, etc).

Define a cosmic Galois group $\mathbb{G}_{\text{cos}} = G_{dR}/K$ where $K$ is the subgroup acting trivially on all $I_m G$. 

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The vector space generated by the $I^m_G$, for $G$ convergent in $\phi^4$, is stable under the action of $G_{cos}$.

Verified in every known example $\sim 250$ cases.
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Related to partial factorisation property of graph polynomials.
‘Motivic’ version of Cartier’s dream:

\[ \mathcal{I}_G^m \neq \mathcal{Z}^m \]

\( G_{\text{cos}} \)

\( G_{\text{MT(Z)}} \)

QFT

\( \mathbb{P}^1 \setminus \{0, 1, \infty\} \)

“Motivic operad”

“Motivic \( \pi_1 \)”

The two pictures look very similar, but are subtly different.

Q: What is the mathematical and geometric framework to describe amplitudes in quantum field theories?