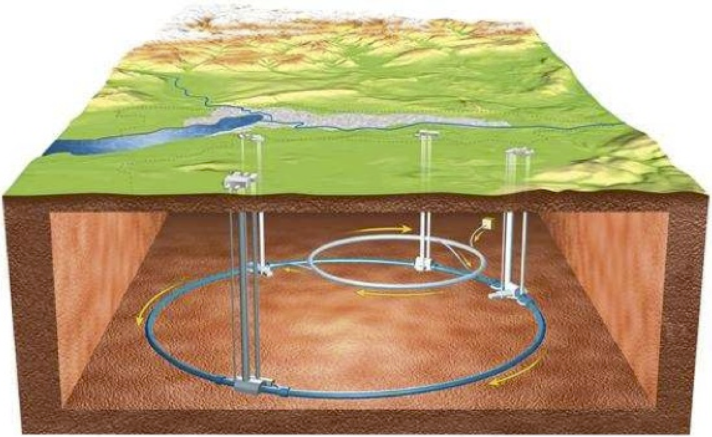
Periods and Cosmic Galois group

Francis Brown
All Souls College, Oxford
IHES, Bures-Sur-Yvette

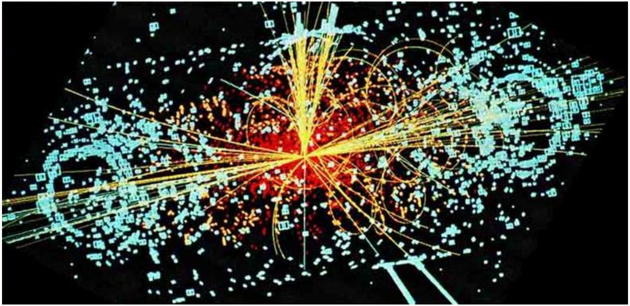
Premier Congrès de la SMF
7 Juin 2016

I. Particle physics

LHC



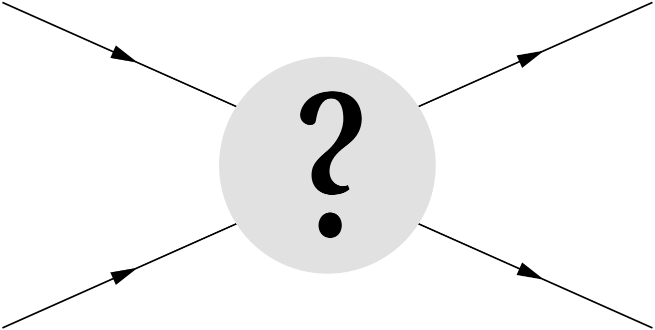
Collision of beam particles



Test the laws of physics by analysing particle tracks.

Perturbative Quantum Field theory

General framework describing fundamental forces and particles.



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Feynman *amplitude* is a complex probability assigned to G.

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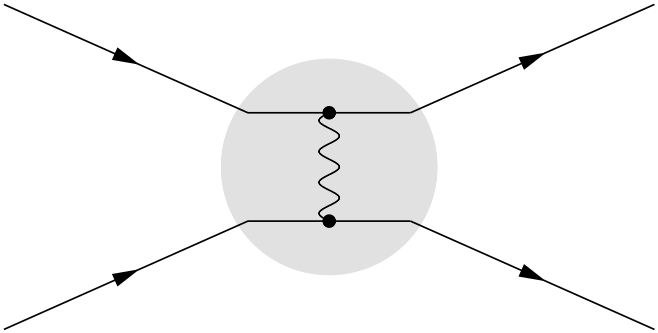

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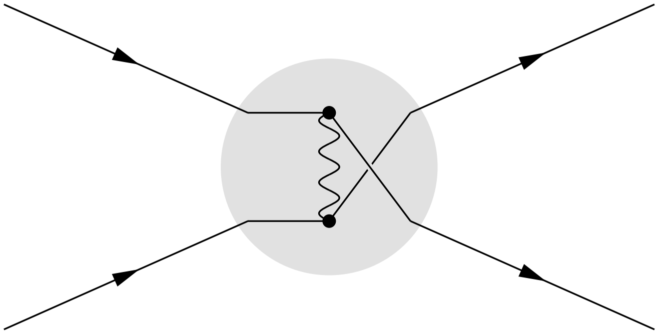

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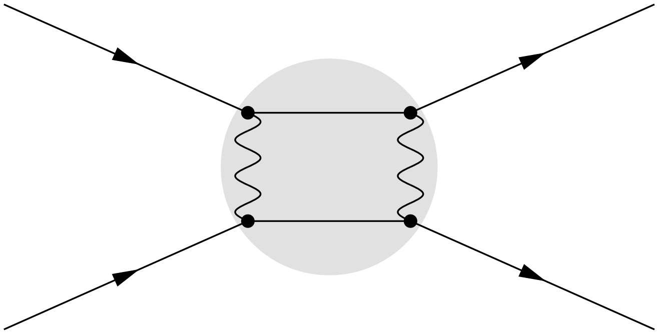

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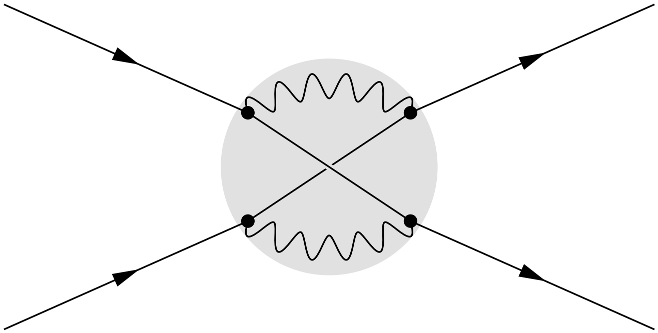

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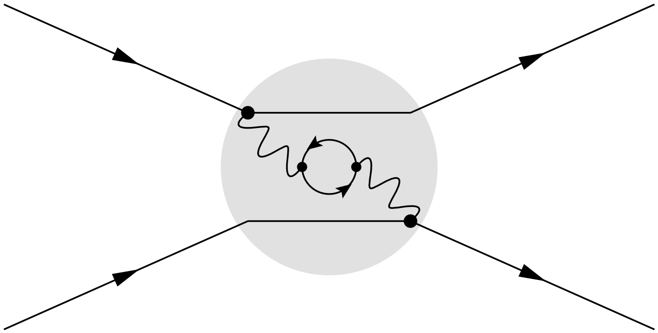

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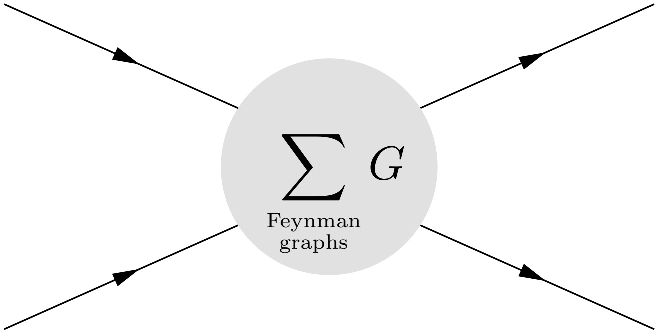

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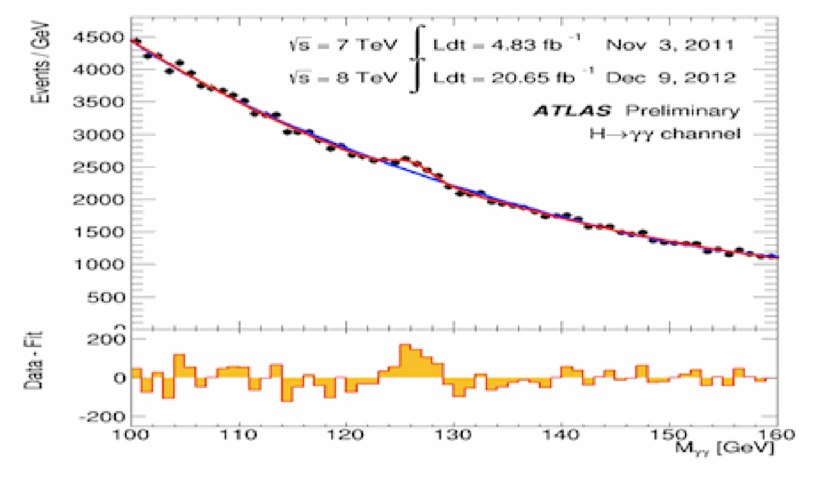
Higgs boson



The blue line (background) requires calculating a huge number of Feynman amplitudes.

II. Graphs and Numbers

Graph polynomials (Kirchhoff 1847)

Let $G = (V_G, E_G)$ be a connected graph. The *graph polynomial*

$$\Psi_G \in \mathbb{Z}[\alpha_e, e \in E(G)]$$

is a sum over spanning trees T of G

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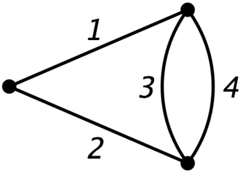
A tree $T \subset G$ is *spanning* if $V_T = V_G$.

Example



$\Psi_G = ?$

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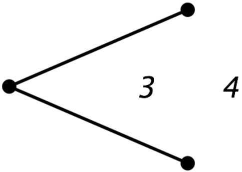

$$\Psi_G = \alpha_3 \alpha_4$$

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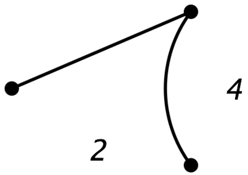

$$\Psi_G = \alpha_3\alpha_4 + \alpha_2\alpha_4$$

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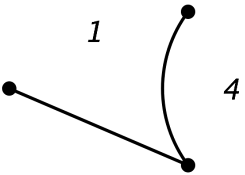

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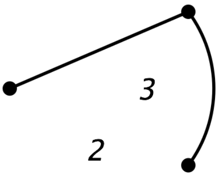

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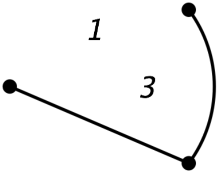

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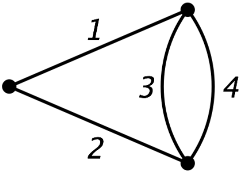

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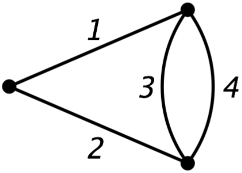

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Physically relevant graphs have vertices of degree ≤ 4. ('G in ϕ^4').

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$$\sigma = \{(\alpha_1 : \ldots : \alpha_{N_G}) \in \mathbb{P}^{N_G - 1}(\mathbb{R}) \text{ such that } \alpha_i \geq 0\}$$

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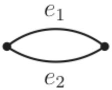

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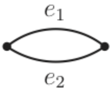

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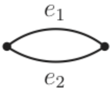

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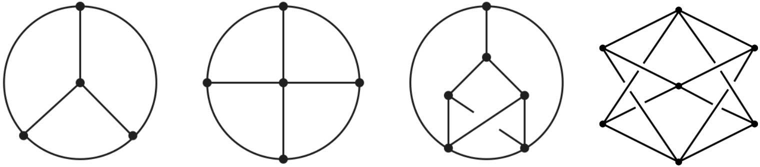
Compute the integral on the chart $\alpha_2 = 1$:

$$I_G = \int_\sigma \frac{\alpha_2 d\alpha_1 - \alpha_1 d\alpha_2}{(\alpha_1 + \alpha_2)^2} = \int_{\alpha_1 \geq 0} \frac{d\alpha_1}{(\alpha_1 + 1)^2} = 1$$

The Zoo



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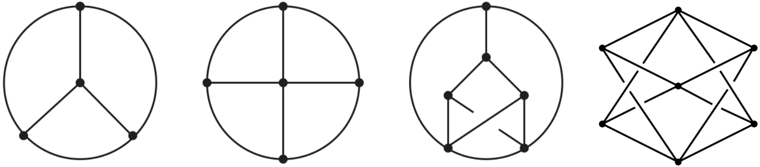

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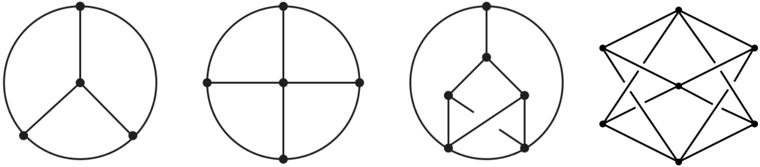

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Multiple Zeta Values, defined for $n_1, \ldots, n_{r-1} \geq 1$, and $n_r \geq 2$:

$$\zeta(n_1, \ldots, n_r) = \sum_{1 \leq k_1 < k_2 < \ldots < k_r} \frac{1}{k_1^{n_1} \ldots k_r^{n_r}} \quad \in \mathbb{R}$$

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Folklore conjecture 90's

The numbers I_G are $\mathbb{Q}$-linear combinations of multiple zeta values.

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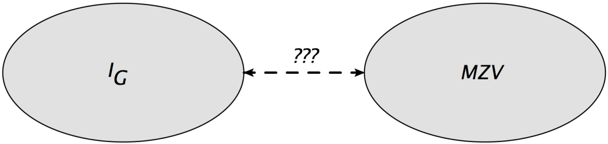

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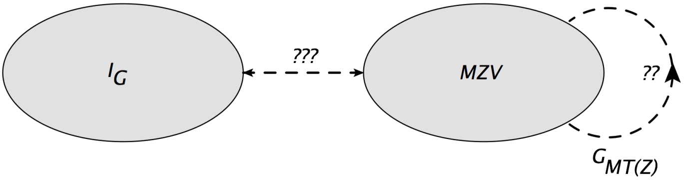

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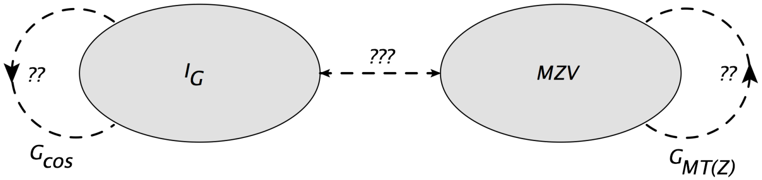

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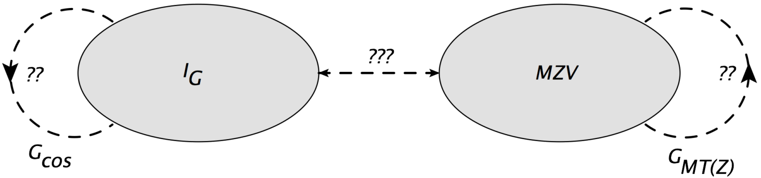

Cartier postulated the existence of a 'cosmic Galois group' (1998).

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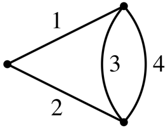

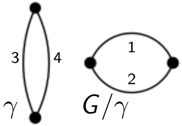

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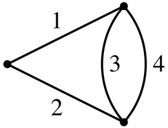

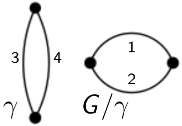

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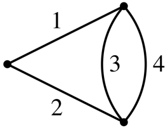

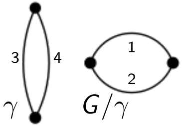

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Determines Ψ_G essentially uniquely.

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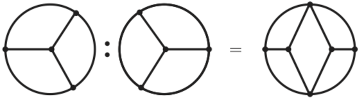

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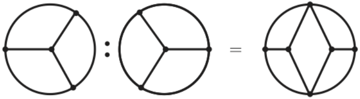

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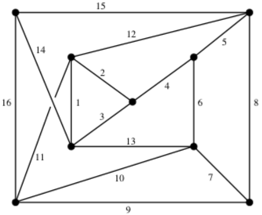
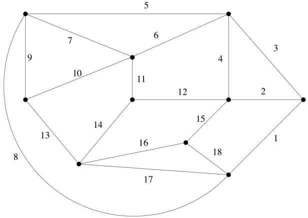
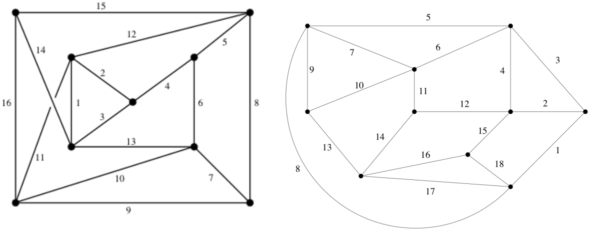
and planar duals, completion (Fourier transform), ...

Counterexamples







Ψ_G of degree 8 in 16 variables, 3785 terms.

Theorem: (B., Schnetz 2012)

The zero locus of Ψ_G is modular of weight 3, level 7.

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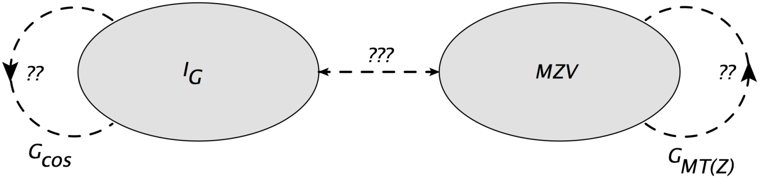

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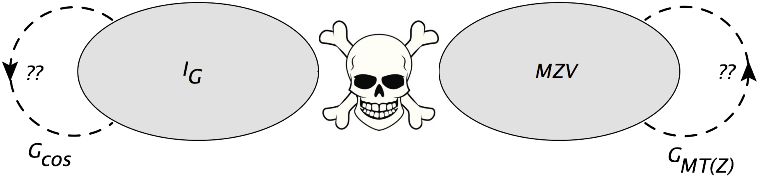

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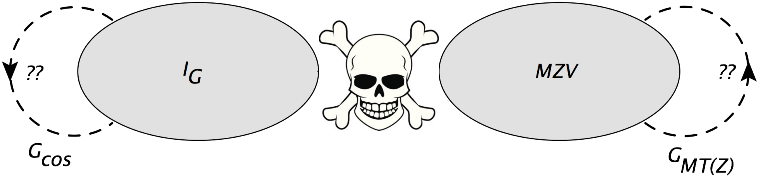

...........However, the group actions may yet survive.........

III. Periods

Definition (Kontsevich-Zagier)

Periods are complex numbers with real and imaginary parts of the form

$$I = \int_\sigma \frac{P}{Q}\, dx_1 \ldots dx_n$$

where $P, Q, \in \mathbb{Q}[x_1, \ldots, x_n]$, and σ finite union of sets $\{f_1, \ldots, f_N \geq 0\}$ with $f_i \in \mathbb{Q}[x_1, \ldots, x_n]$.

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Definition (Kontsevich-Zagier)

Periods are complex numbers with real and imaginary parts of the form

$$I = \int_\sigma \frac{P}{Q}\, dx_1 \dots dx_n$$

where $P, Q, \in \mathbb{Q}[x_1, \dots, x_n]$, and σ finite union of sets $\{f_1, \dots, f_N \geq 0\}$ with $f_i \in \mathbb{Q}[x_1, \dots, x_n]$.

$$\sqrt{2} = \int_{x^2 \leq 2} \frac{dx}{2}$$

$$\pi = \int_{x^2+y^2 \leq 1} dxdy$$

$$\log 2 = \int_{1 \leq x \leq 2} \frac{dx}{x}$$

Ring of periods

Periods form a ring:

$$\mathbb{Q} \subseteq \overline{\mathbb{Q}} \subset P \subset \mathbb{C}$$

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Non-periods?

$$e = \int_{x \leq 1} e^x \, dx$$

$$\gamma = \int_0^\infty \frac{e^{-x}}{e^{-x} - 1} - \frac{e^{-x}}{x} \, dx \, ?$$

Relations

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- Additivity in ω and σ:

$$\int_\sigma \omega_1 + \omega_2 = \int_\sigma \omega_1 + \int_\sigma \omega_2$$

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- Stokes:

$$\int_\sigma d\omega = \int_{\partial\sigma} \omega$$

Galois

Algebraic numbers are periods.

Can one extend Galois theory to periods?

Grothendieck, Kontsevich-Zagier, André, . . .

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Want a pro-algebraic group $\mathcal{G}$

$$\mathcal{G} \times P \longrightarrow P$$

which acts linearly on periods.

Example: MZV's

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Leibniz:

$$\zeta(2) = \int_{0 \le t_1 \le t_2 \le 1} \frac{dt_1}{1 - t_1} \frac{dt_2}{t_2}$$

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Multiple Zeta Values:

$$\zeta(n_1, \dots, n_r) = (-1)^r \int_{0 \le t_1 \le \cdots \le t_n \le 1} \frac{dt_1}{t_1 - \epsilon_1} \cdots \frac{dt_n}{t_n - \epsilon_n}$$

where $(\epsilon_1, \dots, \epsilon_n) = 10^{n_1 - 1} \dots 10^{n_r - 1}$.

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MZV's form a subring of the ring of periods:

$$\mathbb{Q} \subset \mathcal{Z} \subset P$$

Relations

In weight 4: generators $\zeta(4)$, $\zeta(1,3)$, $\zeta(2,2)$, and $\zeta(1,1,2)$.

Relations:

$$
\begin{aligned}
\zeta(2)^2 &= 2\zeta(2,2) + \zeta(4) \\
\zeta(2)^2 &= 4\zeta(1,3) + 2\zeta(2,2) \\
\zeta(1,3) + \zeta(4) &= 2\zeta(1,3) + \zeta(2,2) \\
2\zeta(1,1,2) + \zeta(2,2) + \zeta(1,4) &= 3\zeta(1,1,2)
\end{aligned}
$$

A Galois group of periods should respect these relations!

IV. Motivic Periods

(a Galois theory of periods for dummies)

Cohomology

X smooth affine over $\mathbb{Q}$.

- Algebraic de Rham cohomology:

$$H^n_{dR}(X;\mathbb{Q}) = \frac{\text{closed algebraic forms of degree n}}{\text{exact algebraic forms of degree n}}$$

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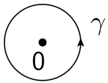
- Comparison theorem (de Rham, Grothendieck):

$$\begin{array}{ccc} H^n_{dR}(X;\mathbb{Q}) \otimes \mathbb{C} & \xrightarrow{\sim} & H^n_B(X) \otimes \mathbb{C} \\ \omega & \mapsto & (\gamma \mapsto \int_\gamma \omega) \end{array}$$

Example: $2i\pi$

$X = \mathbb{P}^1 \backslash \{0, \infty\}$. $X(\mathbb{C}) = \mathbb{C}^\times$.



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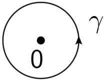

$$
\begin{aligned}
H^1_{dR}(X) &= \mathbb{Q}\left[\frac{dx}{x}\right] \\
H_1(X(\mathbb{C})) &= \mathbb{Q}[\gamma]
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$$

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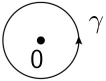

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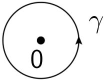

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$$\begin{array}{ccc} H^1_{dR}(X) \otimes \mathbb{C} & \xrightarrow{\sim} & H^1_B(X) \otimes \mathbb{C} \\ [\frac{dx}{x}] & \mapsto & 2i\pi\ [\gamma]^\vee \end{array}$$

1. $P^{\mathfrak{m}}$ the $\mathbb{Q}$-vector space spanned by symbols

$$[H^n(X), \omega, \gamma]$$

modulo an (elementary) equivalence relation, where $H^n(X)$ is a triplet $(H_B, H_{dR}, \int)$, with $H_B, H_{dR} \in \mathrm{Vec}_{\mathbb{Q}}$, $\int$ an isomorphism between their complexifications.

A ring of 'motivic' periods

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2. Ring structure (Künneth)

$$[V, \omega, \gamma] \otimes [W, \omega', \gamma'] = [V \otimes W, \omega \otimes \omega', \gamma \otimes \gamma']$$

- Period homomorphism

$$\begin{aligned}
\mathrm{per} : \mathcal{P}^{\mathfrak{m}} &\longrightarrow P \\
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- We gain the action of a pro-algebraic group

$$\mathcal{G}^{dR} \times \mathcal{P}^{\mathfrak{m}} \longrightarrow \mathcal{P}^{\mathfrak{m}}$$

It acts linearly on $H^n_{dR}(X)$:

$$g[H^n(X), \omega, \gamma] = [H^n(X), g\,\omega, \gamma]$$

A game

Given a period

$$I = \int_\gamma \omega$$

Try to express I as a period of cohomology

$$\omega \in H^n_{dR}(X) \qquad , \qquad \gamma \in H_n(X(\mathbb{C}))\ .$$

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Replace I by 'its' motivic version (NB choices!)

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The action of the group $\mathcal{G}^{dR}$ on $I^{\mathfrak{m}}$ generates a representation of a quotient of $\mathcal{G}^{dR}$. We can use group theory to define invariants, or to discover new relations.

Example: $2i\pi$

$X = \mathbb{P}^1 \backslash \{0, \infty\}$.

$$(2i\pi)^{\mathfrak{m}} := [H^1(X), [\tfrac{dx}{x}], [\gamma]]$$

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$g \in \mathcal{G}^{dR}$: $\qquad g(2i\pi)^{\mathfrak{m}} = \lambda_g (2i\pi)^{\mathfrak{m}}$

It spans a one-dimensional representation

$$\begin{array}{ccc} \mathcal{G}^{dR} & \longrightarrow & GL_1 \\ g & \mapsto & \lambda_g \end{array}$$

So $(2i\pi)^{\mathfrak{m}}$ is a motivic period of rank 1.

Example 2: logarithms

$$\log^{\mathfrak{m}}(2) \xmapsto{g} \lambda_g \log^{\mathfrak{m}}(2) + \nu_g$$

Equivalently

$$\begin{pmatrix} 1 & \log^{\mathfrak{m}}(2) \\ 0 & (2\pi i)^{\mathfrak{m}} \end{pmatrix} \mapsto \begin{pmatrix} 1 & \log^{\mathfrak{m}}(2) \\ 0 & (2\pi i)^{\mathfrak{m}} \end{pmatrix} \begin{pmatrix} 1 & \nu_g \\ 0 & \lambda_g \end{pmatrix}$$

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Hence a two-dimensional representation

$$\begin{array}{ccc} \mathcal{G}^{dR} & \longrightarrow & GL_2 \\ g & \mapsto & \begin{pmatrix} 1 & \nu_g \\ 0 & \lambda_g \end{pmatrix} \end{array}$$

So $\log^{\mathfrak{m}}(2)$ is of rank 2.

Example 3: Algebraic numbers

Let $\alpha \in \mathbb{C}$ algebraic. There exists canonical $\alpha^{\mathfrak{m}}$, whence

$$\overline{\mathbb{Q}} \quad \subset \quad \mathcal{P}^{\mathfrak{m}}$$

$\mathcal{G}^{dR}$ acts on $\overline{\mathbb{Q}}$ via a pro-algebraic quotient $\mathcal{A}_{\overline{\mathbb{Q}}}$:

$$\mathcal{A}_{\overline{\mathbb{Q}}}(\mathbb{C}) \cong \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \ .$$

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The *rank* of $\alpha^{\mathfrak{m}}$ is the dimension of the vector space spanned by the conjugates of α.

Period conjecture

Weak variant of Grothendieck's period conjecture.

$$\mathrm{per} : \ \mathcal{P}^{\mathfrak{m}} \longrightarrow \mathbb{C} \qquad \text{is injective}$$

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Means that $I^{\mathfrak{m}}$ does not depend on choices, and the action of $\mathcal{G}^{dR}$ on $\mathcal{P}^{\mathfrak{m}}$ can be transported onto the ring of periods itself.

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Via group and Hodge theory, we can define many new invariants:

...,rank, unipotency degree, weight, Hodge numbers, degree,.....

Unip. degree	Examples
0	Pure periods: π, elliptic integrals, ...
	(classical)
1	Periods of simple extensions: $\log 2, \zeta(2n+1), \ldots$
	(values of *L*-functions)
≥ 2	Multiple periods: MZV's, Feynman amplitudes, ...
	(unknown)
	— *Unexplored territory* —

V. Applications and questions

Multiple Zeta Values

Theorem (B. 2012)

There exist motivic versions of multiple zeta values

$$\zeta^{\mathfrak{m}}(n_1, \ldots, n_r) \in \mathcal{P}^{\mathfrak{m}}$$

for all $n_1, \ldots, n_r \geq 1$, $n_r \geq 2$, whose periods are $\zeta(n_1, \ldots, n_r)$. They satisfy the 'standard' relations. The ring $\mathcal{Z}^{\mathfrak{m}}$ generated by the $\zeta^{\mathfrak{m}}$ is stable under the group $\mathcal{G}^{dR}$.

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Not known if they satisfy more relations.

Think of $\mathcal{Z}^{\mathfrak{m}}$ as a Galois extension of $\mathbb{Q}$ with group

$$\mathcal{G}^{MZV} \times \mathcal{Z}^{\mathfrak{m}} \to \mathcal{Z}^{\mathfrak{m}}$$

Theorem (B. 2012)

$$\mathcal{G}^{MZV} = U^{MZV} \rtimes \mathbb{G}_m$$

where U^{MZV} pro-unipotent. Its graded Lie algebra is free on generators in degrees -3,-5,-7, ...

$$\sigma_3, \sigma_5, \sigma_7, \ldots \ .$$

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Hence $\mathcal{G}^{MZV} = G_{MT(Z)}$.

The σ_{2n+1} act via

$$\sigma_{2n+1}\, \zeta^{\mathfrak{m}}(2m+1) = \delta_{n,m}$$

Think of σ_3 as 'differentiation with respect to $\zeta^{\mathfrak{m}}(3)$', etc

Cosmic Galois group

Theorem (B. 2014, using Bloch-Esnault-Kreimer 2005)

There exist canonical 'motivic' Feynman amplitudes

$$I_G^{\mathfrak{m}} \in \mathcal{P}^{\mathfrak{m}}$$

for any convergent G, whose period is I_G.

Not all expected relations are known.

Many new invariants to amplitudes (weights, rank, etc).

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Many new invariants to amplitudes (weights, rank, etc).

Define a cosmic Galois group

$$\mathcal{G}_{cos} := \mathcal{G}^{dR}/K$$

where K is the subgroup acting trivially on all $I_G^{\mathfrak{m}}$.

Conjecture of Panzer and Schnetz

Extraordinary Conjecture

The vector space generated by the $I_G^{\mathfrak{m}}$, for G convergent in ϕ^4, is stable under the action of $\mathcal{G}_{cos}$.

Verified in every known example ~ 250 cases.

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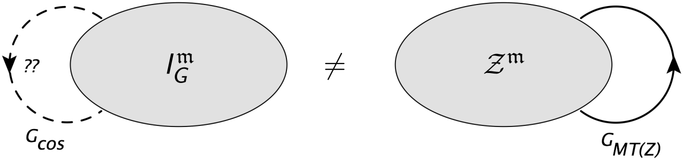
Very strong constraint on the possible amplitudes. Enables one to constrain I_G 'in advance' from smaller graphs.

Related to partial factorisation property of graph polynomials.

Conclusion

'Motivic' version of Cartier's dream:



QFT $\qquad\qquad$ $\mathbb{P}^1 \backslash \{0, 1, \infty\}$

"Motivic operad" $\qquad\qquad$ "Motivic π_1"

The two pictures look very similar, but are subtly different.

Q: What is the mathematical and geometric framework to describe amplitudes in quantum field theories?