SINGLE-VALUED PERIODS AND MULTIPLE ZETA VALUES

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Abstract. The values at 1 of single-valued multiple polylogarithms span a certain subalgebra of multiple zeta values. In this paper, the properties of this algebra are studied from the point of view of motivic periods.

1. Introduction

The goal of this paper is to study a special class of multiple zeta values which occur as the values at 1 of single-valued multiple polylogarithms. The latter were defined in [8] and generalize the Bloch-Wigner dilogarithm

$$D(z) = \mathrm{Im}(\mathrm{Li}_2(z) + \log|z|\log(1-z)) \tag{1.1}$$

which is a single-valued version of $\mathrm{Li}_2(z)$, to the case of all multiple polylogarithms in one variable. These are defined for any integers $n_1, \ldots, n_r \geq 1$ by

$$\mathrm{Li}_{n_1,\ldots,n_r}(z) = \sum_{0<k_1<\ldots<k_r} \frac{z^{k_r}}{n_1^{k_1}\ldots n_r^{k_r}}$$

and are iterated integrals on $\mathbb{P}^1\backslash\{0,1,\infty\}$ obtained by integrating along the straight line path from 0 to 1 along the real axis. In the convergent case $n_r \geq 2$, their values at one are precisely Euler's multiple zeta values

$$\zeta(n_1, \ldots, n_r) = \sum_{0<k_1<\ldots<k_r} \frac{1}{n_1^{k_1}\ldots n_r^{k_r}} \ . \tag{1.2}$$

The values at one of the single-valued multiple polylogarithms define an interesting sub-class of multiple zeta values, which we denote by

$$\zeta_{\mathrm{sv}}(n_1, \ldots, n_r) \in \mathbb{R} \ . \tag{1.3}$$

They satisfy $\zeta_{\mathrm{sv}}(2) = D(1) = 0$, as one immediately sees from (1.1). These numbers are, in a precise sense, the values of iterated integrals on $\mathbb{P}^1\backslash\{0,1,\infty\}$ which are obtained by integrating from 0 to 1 *independently of all choices of path*.

The numbers (1.3) have recently found several applications in physics, in:

1. O. Schnetz' theory of graphical functions for Feynman amplitudes [21]
2. The coefficients of the closed super-string tree-level amplitude [23]
3. Wrapping functions[1] in $N = 4$ super Yang-Mills [19],

as well as in [16, 15], and also in mathematics as the coefficients of Deligne's associator. A general theme seems to be that a large class of (but not all) Feynman integrals in 4-dimensional renormalisable quantum field theories lie in the subspace of single-valued multiple zeta values. This raises an interesting possibility of replacing general amplitudes with their single-valued versions (see §3), which should lead to considerable simplifications.

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[1] I was informed by D. Volin that [19] equation (63) is a single-valued MZV to order g^{18}.

1.1. **Contents.** In [7], motivic multiple zeta values $\zeta^{\mathfrak{m}}(n_1,\ldots,n_r)$ were defined as elements of a certain graded algebra $\mathcal{H}$, equipped with a period homomorphism

$$\mathrm{per}:\mathcal{H}\longrightarrow\mathbb{C}$$

which maps $\zeta^{\mathfrak{m}}(n_1,\ldots,n_r)$ to $\zeta(n_1,\ldots,n_r)$. In this paper, motivic versions of the single-valued numbers (1.3), denoted $\zeta^{\mathfrak{m}}_{\mathrm{sv}}(n_1,\ldots,n_r)$, are defined. They generate a subalgebra $\mathcal{H}^{\mathrm{sv}}\subset\mathcal{H}$. Its main properties can be summarized as follows.

Theorem 1.1. *There is a natural homomorphism $\mathcal{H}\to\mathcal{H}^{\mathrm{sv}}$ which sends $\zeta^{\mathfrak{m}}(n_1,\ldots,n_r)$ to $\zeta^{\mathfrak{m}}_{\mathrm{sv}}(n_1,\ldots,n_r)$. In particular, the $\zeta^{\mathfrak{m}}_{\mathrm{sv}}(n_1,\ldots,n_r)$ satisfy all motivic relations for multiple zeta values, together with the relation $\zeta^{\mathfrak{m}}_{\mathrm{sv}}(2)=0$.*

The algebra $\mathcal{H}^{\mathrm{sv}}$ is isomorphic to the polynomial algebra generated by

$$\zeta^{\mathfrak{m}}_{\mathrm{sv}}(n_1,\ldots,n_r)$$

where $n_i\in\{2,3\}$ and $(n_1,\ldots,n_r)$ is a Lyndon word (for the ordering $3<2$) of odd weight. Furthermore, $\mathcal{H}^{\mathrm{sv}}$ is preserved under the action of the motivic Galois group.

In particular, the numbers $\zeta_{\mathrm{sv}}(n_1,\ldots,n_r)$ satisfy the same double shuffle and associator relations as usual multiple zeta values, and many more relations besides: the space $\mathcal{H}^{\mathrm{sv}}$ is much smaller than $\mathcal{H}$ (§7.4). By way of example:

$$\begin{aligned}\zeta_{\mathrm{sv}}(2n+1) &= 2\,\zeta(2n+1)\qquad\text{for all } n\geq 1\\ \zeta_{\mathrm{sv}}(5,3) &= 14\,\zeta(3)\zeta(5)\\ \zeta_{\mathrm{sv}}(3,5,3) &= 2\,\zeta(3,5,3)-2\,\zeta(3)\zeta(3,5)-10\,\zeta(3)^2\zeta(5)\end{aligned}$$

The reader who is only interested in the single-valued multiple zeta values and not their motivic versions can turn directly to §5 for an elementary definition (which only uses the Ihara action §4.2), and §7.4 for enumerative properties and examples.

1.2. **Motivic periods.** Whilst writing this paper, it seemed a good opportunity to clarify certain concepts relating to motivic multiple zeta values. There are two conflicting notions of motivic multiple zeta values in the literature, one due to Goncharov [17] (for which the motivic version of $\zeta(2)$ vanishes), via the concept of framed objects in mixed Tate categories, and another for which the motivic version of $\zeta(2)$ is non-zero [7], later simplified by Deligne [13]. It can be paraphrased as follows:

Definition 1.2. Let $\mathcal{M}$ be a Tannakian category of motives, with two fiber functors ω_{dR},ω_B. A *motivic period* is an element of the affine ring of the torsor of periods

$$\mathcal{P}^{\mathfrak{m}}=\mathcal{O}(\mathrm{Isom}_{\mathcal{M}}(\omega_{dR},\omega_B))\ .$$

Given a motive $M\in\mathcal{M}$, and classes $\eta\in\omega_{dR}(M)$, $X\in\omega_B(M)^\vee$, the *motivic period* [13] associated to this data is the function on $\mathrm{Isom}_{\mathcal{M}}(\omega_{dR},\omega_B))$ defined by

$$[M,\eta,X]^{\mathfrak{m}}:=\phi\mapsto\langle\phi(\eta),X\rangle$$

This definition is nothing other than the standard construction of the ring of functions on the Tannaka groupoid, and appears in a similar form in [2], §23.5. However, it is the interpretation and application of this concept which is of interest here; in particular the idea that one can sometimes deduce results about periods from their motivic versions and vice-versa (see e.g., [7], §4.1). The ring of motivic periods is a bitorsor over the Tannaka groups $(G_{\omega_{dR}},G_{\omega_B})$ and thus gives rise to a Galois theory of motivic periods. In this paper, we only consider the special case where $\mathcal{M}=\mathcal{MT}(\mathbb{Z})$ is the category of mixed Tate motives over $\mathbb{Z}$: the generalization to other categories of mixed Tate motives [14] is relatively straightforward if one replaces ω_{dR} with the canonical fiber functor, and bears in mind that there can be several different Betti realizations.

In a similar vein, one can replace ω_{dR}, ω_B with any pair of fiber functors, to obtain various different notions of motivic period. One can consider the ring of *de Rham periods* $\mathcal{P}^{\mathfrak{dr}}$, where we replace $\mathfrak{m} = (\omega_{dR}, \omega_B)$ with $\mathfrak{dr} = (\omega_{dR}, \omega_{dR})$, and a weaker notion of *unipotent de Rham periods* $\mathcal{P}^{\mathfrak{a}}$ which are their restriction to the unipotent radical $U_{\omega_{dR}}$ of the Tannaka group $G_{\omega_{dR}}$. The latter are precisely the 'framed objects' studied in [5], [6], [17]. Although there is no direct period (integration) map for de Rham motivic periods, we construct a related notion in §3 in the case $\mathcal{M} = \mathcal{MT}(\mathbb{Z})$, which we call the single-valued motivic period. It gives a well-defined homomorphism from unipotent de Rham periods to motivic periods

$$\mathrm{sv}^{\mathfrak{m}} : \mathcal{P}^{\mathfrak{a}} \longrightarrow \mathcal{P}^{\mathfrak{m}}$$

Composing with the period map attaches a complex number to de Rham periods. This gives a transcendental pairing between a de Rham cohomology class and a de Rham homology class. In the case of $\mathcal{MT}(\mathbb{Z})$, the numbers one obtains are precisely the single-valued multiple zeta values (1.3). Since the definition of $\mathrm{sv}^{\mathfrak{m}}$ requires nothing more than complex conjugation and the weight grading, it comes perhaps as a surprise that this map is already so intricate in this special case (see, for example (7.4)).

Section 2 consists of generalities on motivic and de Rham periods, and section 3 defines the motivic single-valued map $\mathrm{sv}^{\mathfrak{m}}$. The remainder of the paper applies this construction to the case of the motivic fundamental group of $\mathbb{P}^1 \backslash \{0, 1, \infty\}$. Section 4 consists of reminders, and §5 aims to give a completely elementary definition of ζ_{sv}. Section 6 defines the motivic versions $\zeta^{\mathfrak{m}}_{\mathrm{sv}}$ and §6.3 reconstructs the single-valued multiple polylogarithms from the point of view of the unipotent fundamental group. Section 7 applies the main theorem of [7] to deduce structural results about $\mathcal{H}^{\mathrm{sv}}$.

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1.2.1. *Conventions.* All tensor products are over $\mathbb{Q}$ unless expressly stated otherwise.

2. Generalities on periods and mixed Tate motives

See [14], §2 for the background material on mixed Tate motives required in this section. Much of what follows applies to any category of mixed Tate motives over a number field, provided that one replaces the de Rham fiber functor with the canonical fiber functor $\omega_n(M) = \mathrm{Hom}_{\mathcal{MT}}(\mathbb{Q}(-n), \mathrm{gr}^W_{2n} M)$ (see [14], §1.1).

2.1. **Mixed Tate motives over $\mathbb{Z}$.** Let $\mathcal{M} = \mathcal{MT}(\mathbb{Z})$ denote the Tannakian category of mixed Tate motives over $\mathbb{Z}$ [14]. Its canonical fiber functor is equal to the fiber functor ω_{dR} given by the de Rham realization, and it is equipped with a fiber functor ω_B given by the Betti realization with respect to the unique embedding $\mathbb{Q} \hookrightarrow \mathbb{C}$. Let G_{dR} and G_B denote the corresponding Tannaka groups. They are affine group schemes over $\mathbb{Q}$. We shall mainly focus on G_{dR}.

The action of G_{dR} on $\mathbb{Q}(-1) \in \mathcal{M}$ defines a map $G_{dR} \to \mathbb{G}_m$ whose kernel is denoted by $\mathcal{U}_{dR}$. It is a pro-unipotent affine group scheme over $\mathbb{Q}$. Furthermore, since ω_{dR} is graded, G_{dR} admits a decomposition as a semi-direct product ([14], §2.1)

$$G_{dR} \cong \mathcal{U}_{dR} \rtimes \mathbb{G}_m \ . \tag{2.1}$$

A mixed Tate motive $M \in \mathcal{M}$ can be represented by a finite-dimensional graded $\mathbb{Q}$-vector space $M_{dR} = \omega_{dR}(M)$ equipped with an action of $\mathcal{U}_{dR}$ which is compatible with the grading. We shall write $(M_{dR})_n$ for the component in degree n, i.e., $(M_{dR})_n = (W_{2n} \cap F^n)M_{dR}$. The Betti realization of M, denoted $M_B = \omega_B(M)$, is a finite-dimensional $\mathbb{Q}$-vector space equipped with an increasing filtration $W_\bullet M_B$.

The two are related by a canonical comparison isomorphism

$$\mathrm{comp}_{B,dR} : M_{dR} \otimes_{\mathbb{Q}} \mathbb{C} \overset{\sim}{\longrightarrow} M_B \otimes_{\mathbb{Q}} \mathbb{C} \tag{2.2}$$

which can be computed by integrating differential forms. We shall often use the fact that $\mathbb{Q}(0) \in \mathcal{M}$ has rational periods, i.e., that

$$\mathrm{comp}_{B,dR} : \mathbb{Q}(0)_{dR} \overset{\sim}{\longrightarrow} \mathbb{Q}(0)_B \ . \tag{2.3}$$

In general, given any pair of fiber functors ω_1, ω_2 on $\mathcal{M}$, let

$$\mathfrak{P}_{\omega_1,\omega_2} = \mathrm{Isom}(\omega_1, \omega_2)$$

denote the set of isomorphisms of fiber functors from ω_1 to ω_2. It is a scheme over $\mathbb{Q}$, and is a bitorsor over $(G_{\omega_1}, G_{\omega_2})$, where $G_{\omega_i} = \mathfrak{P}_{\omega_i,\omega_i}$ is the Tannaka group scheme relative to ω_i, for $i = 1, 2$. The comparison map defines a complex point

$$\mathrm{comp}_{B,dR} \in \mathfrak{P}_{\omega_{dR},\omega_B}(\mathbb{C}) \ .$$

2.2. **Motivic periods.** There are two conflicting notions of motivic multiple zeta values in the literature, one due to [17] and the other due to [7]. One can reconcile the two definitions with minimal damage to existing terminology as follows.

Definition 2.1. Let ω_1, ω_2 be two fiber functors on $\mathcal{M}$. Let $M \in \mathrm{Ind}(\mathcal{M})$ and let $\eta \in \omega_1(M)$, and $X \in \omega_2(M)^\vee$. A *motivic period* of M of type (ω_1, ω_2)

$$[M, \eta, X]^{\omega_1,\omega_2} \in \mathcal{O}(\mathfrak{P}_{\omega_1,\omega_2}) \tag{2.4}$$

is the function $\mathfrak{P}_{\omega_1,\omega_2} \to \mathbb{Q}$ defined by $\phi \mapsto \langle \phi(\eta), X\rangle = \langle \eta, {}^t\phi(X)\rangle$.

One can clearly extend definition 2.1 to other Tannakian categories (of motives), but only $\mathcal{M} = \mathcal{MT}(\mathbb{Z})$ will be considered in this paper.

Definition 2.1 in the case $(\omega_1, \omega_2) = (\omega_{dR}, \omega_B)$ is due to Deligne [13], and simplifies the definition in [7]. Since this is the case of primary interest for us, we shall call (2.4) a *motivic period*, and denote the pair (ω_{dR}, ω_B) simply by $\mathfrak{m}$. We shall also consider the case $(\omega_1, \omega_2) = (\omega_{dR}, \omega_{dR})$. We shall call the corresponding period (2.4) a *de Rham period*, and denote the pair $(\omega_{dR}, \omega_{dR})$ by $\mathfrak{dr}$.

Definition 2.2. Let $M \in \mathcal{M}$ and let ω_1, ω_2 be a pair of fiber functors as above. We shall denote the space of all motivic periods of type (ω_1, ω_2) by

$$\mathcal{P}^{\omega_1,\omega_2} = \mathcal{O}(\mathfrak{P}_{\omega_1,\omega_2}) \ , \tag{2.5}$$

and we shall write $\mathcal{P}^{\omega_1,\omega_2}(M)$ for the $\mathbb{Q}$-subspace of $\mathcal{P}^{\omega_1,\omega_2}$ spanned by the motivic periods of M of type (ω_1, ω_2).

It follows from (2.5) that the set of all motivic periods forms an algebra over $\mathbb{Q}$. The set of fiber functors on $\mathcal{M}$ form a groupoid with respect to composition

$$\mathfrak{P}_{\omega_1,\omega_2} \times \mathfrak{P}_{\omega_2,\omega_3} \to \mathfrak{P}_{\omega_1,\omega_3}$$

for any three fiber functors $\omega_1, \omega_2, \omega_3$. Dualizing, we obtain a coalgebroid structure on spaces of motivic periods:

$$\mathcal{P}^{\omega_1,\omega_3} \longrightarrow \mathcal{P}^{\omega_1,\omega_2} \otimes \mathcal{P}^{\omega_2,\omega_3} \ , \tag{2.6}$$

which, in the case $\omega_1 = \omega_2 = \omega_{dR}$, and $\omega_3 = \omega_B$ becomes a coaction:

$$\Delta_{\mathfrak{dr},\mathfrak{m}} : \mathcal{P}^{\mathfrak{m}} \longrightarrow \mathcal{P}^{\mathfrak{dr}} \otimes \mathcal{P}^{\mathfrak{m}} \ . \tag{2.7}$$

By the definition of the Tannaka group,

$$G_{dR} = \mathrm{Spec}\,(\mathcal{P}^{\mathfrak{dr}}) \ , \tag{2.8}$$

and so (2.7) defines in particular an action of $G_{dR}(\mathbb{Q})$ on the space of motivic periods $\mathcal{P}^{\mathfrak{m}}$. Since ω_{dR} is graded, the (left) action of $\mathbb{G}_m \subset G_{dR}$ corresponds to a grading on $\mathcal{P}^{\omega_{dR},\omega}$ for any fiber functor ω. We shall sometimes call the degree the weight, in keeping with standard terminology for multiple zeta values. It is one half of the Hodge-theoretic weight, i.e., $\mathbb{Q}(-n)$ has weight n.

The notions of motivic and de Rham periods are fundamentally different. In the case of motivic periods, pairing with the element (2.2) defines the period homomorphism

$$\mathrm{per} : \mathcal{P}^{\mathfrak{m}} \longrightarrow \mathbb{C} \ . \tag{2.9}$$

The point $1 \in G_{dR}$ defines a map $\mathcal{P}^{\mathfrak{dr}} \to \mathbb{Q}$. The period map *per se* is not available for de Rham periods, although we shall define a substitute in §3.

2.3. **Formulae.** By the main construction of Tannaka theory ([10], §4.7), motivic periods are spanned by symbols $[M, \eta, X]^{\omega_1,\omega_2}$, where $M \in \mathcal{M}$, $\eta \in \omega_1(M)$, and $X \in \omega_2(M)^\vee$, modulo the equivalence relation generated by

$$[M_1, \eta_1, X_1]^{\omega_1,\omega_2} \sim [M_2, \eta_2, X_2]^{\omega_1,\omega_2} \tag{2.10}$$

for every morphism $\rho : M_1 \to M_2$ such that $\eta_2 = \omega_1(\rho)\eta_1$, and $X_1 = \omega_2(\rho)^t X_2$.

The multiplication on motivic periods is given concretely by the formula:

$$\begin{array}{rcl} \mathcal{P}^{\omega_1,\omega_2}(M_1) \times \mathcal{P}^{\omega_1,\omega_2}(M_2) & \longrightarrow & \mathcal{P}^{\omega_1,\omega_2}(M_1 \otimes M_2) \\ {[M_1, \eta_1, X_1]^{\omega_1,\omega_2} \times [M_2, \eta_2, X_2]^{\omega_1,\omega_2}} & = & [M_1 \otimes M_2, \eta_1 \otimes \eta_2, X_1 \otimes X_2]^{\omega_1,\omega_2} \ . \end{array} \tag{2.11}$$

In particular, if M is an algebra object in $\mathrm{Ind}\,(\mathcal{M})$, then $\mathcal{P}^{\omega_1,\omega_2}(M)$ is a commutative ring, and its spectrum is an affine scheme over $\mathbb{Q}$.

Given three fiber functors $\omega_1, \omega_2, \omega_3$, the Hopf algebroid structure (2.6) can be computed explicitly by the usual coproduct formula for endomorphisms

$$\begin{array}{rcl} \Delta_{\omega_1,\omega_2;\omega_2,\omega_3} : \mathcal{P}^{\omega_1,\omega_3}(M) & \longrightarrow & \mathcal{P}^{\omega_1,\omega_2}(M) \otimes \mathcal{P}^{\omega_2,\omega_3}(M) \\ {[M, \eta, X]^{\omega_1,\omega_3}} & \mapsto & \sum_v [M, \eta, v^\vee]^{\omega_1,\omega_2} \otimes [M, v, X]^{\omega_2,\omega_3} \end{array} \tag{2.12}$$

where $\{v\}$ is a basis of $\omega_2(M)$ and $\{v^\vee\}$ is the dual basis. The previous formula does not depend on the choice of basis. In the case $\omega_1 = \omega_2 = \omega_{dR}$, equation (2.12) gives the following formula for the weight of a motivic period:

$$\deg\,[M, \eta, X]^{\omega_{dR},\omega} = m \ , \tag{2.13}$$

whenever $\eta \in (M_{dR})_m$ has degree m, and $\omega_3 = \omega$ is any fiber functor.

Finally, given a motivic period $[M, \eta, X]^{\mathfrak{m}} \in \mathcal{P}^{\mathfrak{m}}$, its period is given by

$$\mathrm{per}([M, \eta, X]^{\mathfrak{m}}) = \langle \mathrm{comp}_{B,dR}(\eta), X \rangle \in \mathbb{C} \ . \tag{2.14}$$

In principle it can always be computed by integrating a differential form representing η along a topological cycle representing X.

2.4. **Unipotent de Rham periods.** There is yet another notion of de Rham period which is obtained by restricting to the unipotent radical $U_{dR} \subset G_{dR}$.

Definition 2.3. Let $v \in M_{dR}$, and $f \in M_{dR}^{\vee}$. A *unipotent de Rham period* is the image of $[M, v, f]^{\mathfrak{dr}}$ under the map $\mathcal{O}(G_{dR}) \to \mathcal{O}(U_{dR})$. Denote it by

$$[M, v, f]^{\mathfrak{a}} \in \mathcal{O}(U_{dR})$$

and denote the ring of unipotent de Rham periods by

$$\mathcal{P}^{\mathfrak{a}} \cong \mathcal{O}(U_{dR}) \ . \tag{2.15}$$

Unipotent de Rham periods are equivalent to the notion of framed objects in mixed Tate categories considered, for example, in ([17], §2). There is a natural map

$$\pi_{\mathfrak{a},\mathfrak{dr}} : \mathcal{P}^{\mathfrak{dr}} \longrightarrow \mathcal{P}^{\mathfrak{a}} \ ,$$

and hence, by taking $\omega_1 = \omega_2 = \omega_{dR}$ and $\omega_3 = \omega_B$ and restricting the left-hand factor of the right-hand side of (2.12) to $\mathcal{P}^{\mathfrak{a}}$, we obtain a coaction

$$\Delta_{\mathfrak{m},\mathfrak{a}} : \mathcal{P}^{\mathfrak{m}} \quad \longrightarrow \quad \mathcal{P}^{\mathfrak{a}} \otimes \mathcal{P}^{\mathfrak{m}} \tag{2.16}$$

The action of $\mathbb{G}_m$ by conjugation gives $\mathcal{O}(U_{dR})$ a grading. A (non-zero) unipotent de Rham period $[M, v_m, f_n]^{\mathfrak{a}}$ is homogeneous of degree

$$\deg\, [M, v_m, f_n]^{\mathfrak{a}} = m - n \tag{2.17}$$

whenever $v_m \in (M_{dR})_m$, and $f_n \in ((M_{dR})_n)^{\vee}$. Note that the formula only agrees with (2.13) when $n = 0$. Since $\mathcal{O}(U_{dR})$ has weights ≥ 0, $[M, v_m, f_n]^{\mathfrak{a}}$ vanishes if $m < n$. With these definitions, the coaction (2.16) is homogeneous in the weight.

Remark 2.4. In [17], it is assumed that one framing, namely f_n, is in degree zero. This defines a smaller space of de Rham periods for a given motive M, and the corresponding coproduct formula requires an extra Tate twist in the left-hand factor.

2.5. **Motives generated by motivic periods.** It is very useful to think of a space of motivic periods $\mathcal{P}^{\mathfrak{m}}(M)$ as a motive in its own right.

Definition 2.5. Let $\xi \in \mathcal{P}^{\mathfrak{m}}$ be a motivic period. Let $M(\xi)_{dR}$ denote the graded $\mathcal{O}(U_{dR})$-comodule it generates via the coaction (2.16).

By the Tannakian formalism, this is the de Rham realization of a motive we denote by $M(\xi) \in \mathcal{M}$. Define *the motive generated by the motivic period* ξ to be $M(\xi)$.

Lemma 2.6. *For any $\xi \in \mathcal{P}^{\mathfrak{m}}$, ξ is a motivic period of $M(\xi)$.*

Proof. If we represent ξ by a triple $[M, \eta, X]^{\mathfrak{m}}$, then the de Rham orbit $G_{dR}\eta$ defines a submotive $M^1 \subset M$ such that $M^1_{dR} = G_{dR}\eta$. We have an equivalence

$$[M^1, \eta, X^1]^{\mathfrak{m}} \xrightarrow{\sim} [M, \eta, X]^{\mathfrak{m}} = \xi$$

where X^1 is the image of X in $(M^1_B)^{\vee}$. Now define M^2 to be the quotient motive of M^1 whose de Rham realization is $M^1_{dR}/(\mathfrak{P}_{dR,B}X^1)^{\perp}$. Then

$$[M^1, \eta, X^1]^{\mathfrak{m}} \xrightarrow{\sim} [M^2, \eta_2, X^1]^{\mathfrak{m}}$$

are equivalent, where η_2 is the image of η in M^2_{dR}. In particular, ξ is a motivic period of M^2. The de Rham realization of M_2 is exactly

$$\frac{G_{dR}\eta}{G_{dR}\eta \cap (\mathfrak{P}_{dR,B}X)^{\perp}}$$

which is isomorphic to the G_{dR}-module generated by the function $[M, \eta, X]^{\mathfrak{m}} \in \mathcal{P}^{\mathfrak{m}}$. Therefore $M^2_{dR} = M(\xi)_{dR}$ and hence $M^2 = M(\xi)$. □

Thus $M(\xi)$ is the smallest subquotient motive M' of M such that $\xi \in \mathcal{P}^m(M')$.

2.6. **Geometric periods.** The notions of de Rham and motivic periods can be related to each other via the following algebra of geometric periods.

Definition 2.7. Let $\mathcal{P}^{\mathfrak{m},+} \subset \mathcal{P}^{\mathfrak{m}}$ be the largest graded subalgebra of $\mathcal{P}^{\mathfrak{m}}$ such that:
i). $\mathcal{P}^{\mathfrak{m},+}$ has weights ≥ 0,
ii). $\mathcal{P}^{\mathfrak{m},+}$ is a comodule under $\mathcal{P}^{\mathfrak{a}}$, i.e.,

$$\Delta_{\mathfrak{a},\mathfrak{m}} : \mathcal{P}^{\mathfrak{m},+} \longrightarrow \mathcal{P}^{\mathfrak{a}} \otimes \mathcal{P}^{\mathfrak{m},+} \ . \tag{2.18}$$

Suppose that $M \in \mathcal{M}$ has positive weights, i.e., $W_{-1}M = 0$. Then

$$\mathcal{P}(M) \subset \mathcal{P}^{\mathfrak{m},+} \ .$$

Lemma 2.8. *The algebra $\mathcal{P}^{\mathfrak{m},+}$ is generated by the motivic periods of M, where M has non-negative weights ($W_{-1}M = 0$).*

Proof. The graded vector space $\mathcal{P}^{\mathfrak{m},+}$ is an $\mathcal{O}(U_{dR})$-comodule by (2.18). It is therefore the de Rham realization of an object $\mathbb{P} \in \mathrm{Ind}(\mathcal{M})$ which has weights ≥ 0 by 2.7, *i*). By lemma 2.6, every $\xi \in \mathcal{P}^{\mathfrak{m},+}$ is a motivic period of $\mathbb{P}$. □

It follows from lemma 2.8 that

$$\mathcal{P}^{\mathfrak{m},+}_0 \cong \mathbb{Q} \ . \tag{2.19}$$

This is because a motivic period of weight zero of a motive M satisfying $W_{-1}M = 0$, is equivalent to a period of $\mathbb{Q}(0)$, which is rational. Note that the isomorphism (2.19) uses $\mathrm{comp}_{B,dR}$ via (2.3). As a consequence, there is an augmentation map

$$\varepsilon : \mathcal{P}^{\mathfrak{m},+} \longrightarrow \mathbb{Q}$$

given by projection onto $\mathcal{P}^{\mathfrak{m},+}_0$. This defines a map

$$\pi_{\mathfrak{a},\mathfrak{m}+} : \mathcal{P}^{\mathfrak{m},+} \longrightarrow \mathcal{P}^{\mathfrak{a}} \tag{2.20}$$

by composing the coaction $\Delta_{\mathfrak{a},\mathfrak{m}} : \mathcal{P}^{\mathfrak{m},+} \longrightarrow \mathcal{P}^{\mathfrak{a}} \otimes \mathcal{P}^{\mathfrak{m},+}$ with ε. The map $\pi_{\mathfrak{a},\mathfrak{m}+}$ respects the weight gradings, and is an isomorphism in weight zero $\pi_{\mathfrak{a},\mathfrak{m}+} : \mathcal{P}^{\mathfrak{m},+}_0 \cong \mathcal{P}^{\mathfrak{a}}_0$.

The map $\pi_{\mathfrak{a},\mathfrak{m}+}$ can be computed another way. Let M satisfy $W_{-1}M = 0$. Then W_0M is a direct sum of copies of $\mathbb{Q}(0)$, which has rational periods (2.3). We have

$$\mathrm{gr}^W_0 M_{dR} = W_0 M_{dR} \overset{\mathrm{comp}_{B,dR}}{\longrightarrow} W_0 M_B \hookrightarrow M_B \ .$$

Since M_{dR} is graded, we can first apply the projection $M_{dR} \rightarrow \mathrm{gr}^W_0 M_{dR}$ and then apply the previous map. This defines a rational comparison morphism

$$c_0 : M_{dR} \longrightarrow M_B \ . \tag{2.21}$$

Then (2.20) is given by the formula

$$\begin{array}{rcl} \pi_{\mathfrak{a},\mathfrak{m}+} : \mathcal{P}^{\mathfrak{m}}(M) & \longrightarrow & \mathcal{P}^{\mathfrak{a}}(M) \\ {[M,\eta,X]^{\mathfrak{m}}} & \mapsto & [M,\eta,{}^t c_0(X)]^{\mathfrak{a}} \end{array} \tag{2.22}$$

for all $\eta \in M_{dR}, X \in M_B^\vee$. It only depends on the restriction of X to $W_0 M_B$.

2.7. **Example: the Lefschetz motive.** Let $M = H^1(\mathbb{G}_m) \cong \mathbb{Q}(-1)$. Then $M_{dR} = H^1_{dR}(\mathbb{G}_m;\mathbb{Q}) \cong \mathbb{Q}\,\omega_0$, and $M_B^\vee = H_1(\mathbb{C}^\times;\mathbb{Q}) = \mathbb{Q}\gamma_0$, where $\omega_0 = \frac{dz}{z}$, and γ_0 is a loop winding around 0 in the positive direction. Denote the Lefschetz motivic period by

$$\mathbb{L}^{\mathfrak{m}} = [M, [\omega_0], [\gamma_0]]^{\mathfrak{m}} \tag{2.23}$$

whose weight is one and whose period is

$$\mathrm{per}(\mathbb{L}^{\mathfrak{m}}) = \int_{\gamma_0} \omega_0 = 2i\pi .$$

The element $\mathbb{L}^{\mathfrak{m}}$ is invertible in $\mathcal{P}^{\mathfrak{m}}$. We use the notation $\mathbb{L}^{\mathfrak{m}}$ in order to avoid the rather ugly alternative $(2i\pi)^{\mathfrak{m}}$. Define the Lefschetz de Rham period by

$$\mathbb{L}^{\mathfrak{dr}} = [M, [\omega_0], [\omega_0]^\vee]^{\mathfrak{dr}} \ . \tag{2.24}$$

It is group-like for the coproduct on $\mathcal{O}(G_{dR})$: $\Delta_{\mathfrak{dr},\mathfrak{dr}}\mathbb{L}^{\mathfrak{dr}} = \mathbb{L}^{\mathfrak{dr}} \otimes \mathbb{L}^{\mathfrak{dr}}$. Since U_{dR} acts trivially on $\mathbb{Q}(-1)$, the unipotent de Rham Lefschetz period is trivial:

$$\mathbb{L}^{\mathfrak{a}} = \pi_{\mathfrak{a},\mathfrak{dr}}(\mathbb{L}^{\mathfrak{dr}}) = 1 \ . \tag{2.25}$$

By definition, $\mathbb{L}^{\mathfrak{dr}}$ can be viewed as a coordinate on $\mathbb{G}_m$, and

$$\mathbb{G}_m \cong \mathrm{Spec}\, \mathbb{Q}[(\mathbb{L}^{\mathfrak{dr}})^{-1}, \mathbb{L}^{\mathfrak{dr}}] \ . \tag{2.26}$$

On the other hand, $\mathrm{gr}_0^W M = 0$, so $c_0([\gamma_0]) = 0$ and therefore

$$\pi_{\mathfrak{a},\mathfrak{m}+}(\mathbb{L}^{\mathfrak{m}}) = [M, [\omega_0], c_0([\gamma_0])]^{\mathfrak{a}} = 0 \ . \tag{2.27}$$

By (2.12), the coaction $\Delta_{\mathfrak{dr},\mathfrak{m}} : \mathcal{P}^{\mathfrak{m}} \to \mathcal{P}^{\mathfrak{dr}} \otimes \mathcal{P}^{\mathfrak{m}}$ acts on the motivic Lefschetz period by $\Delta_{\mathfrak{dr},\mathfrak{m}}\mathbb{L}^{\mathfrak{m}} = \mathbb{L}^{\mathfrak{dr}} \otimes \mathbb{L}^{\mathfrak{m}}$. By (2.25) the coaction $\Delta_{\mathfrak{a},\mathfrak{m}} : \mathcal{P}^{\mathfrak{m}} \to \mathcal{P}^{\mathfrak{a}} \otimes \mathcal{P}^{\mathfrak{m}}$ satisfies

$$\Delta_{\mathfrak{a},\mathfrak{m}}(\mathbb{L}^{\mathfrak{m}}) = 1 \otimes \mathbb{L}^{\mathfrak{m}} \ . \tag{2.28}$$

2.8. **Structure of de Rham periods.** The fact that G_{dR} is a semi-direct product (2.1) implies that $G_{dR} \cong U_{dR} \times \mathbb{G}_m$ as schemes, and hence

$$\mathcal{P}^{\mathfrak{dr}} \cong \mathcal{P}^{\mathfrak{a}} \otimes \mathbb{Q}[(\mathbb{L}^{\mathfrak{dr}})^{-1}, \mathbb{L}^{\mathfrak{dr}}] \ . \tag{2.29}$$

The coaction of $\mathcal{P}^{\mathfrak{a}}$ on the right-hand side is given by the formula $\Delta_{\mathfrak{a},\mathfrak{dr}}(\mathbb{L}^{\mathfrak{dr}}) = 1 \otimes \mathbb{L}^{\mathfrak{dr}}$, by (2.25). Equivalently, the map $\mathbb{G}_m \to G_{dR}$ induces a projection

$$\pi_{\mathbb{L},\mathfrak{dr}} : \mathcal{P}^{\mathfrak{dr}} \longrightarrow \mathbb{Q}[(\mathbb{L}^{\mathfrak{dr}})^{-1}, \mathbb{L}^{\mathfrak{dr}}] \ , \tag{2.30}$$

or explicitly $\pi_{\mathbb{L},\mathfrak{dr}}\big([M,v,f]^{\mathfrak{dr}}\big) = f(v)\,(\mathbb{L}^{\mathfrak{dr}})^n$ if $v \in (M_{dR})_n$. The isomorphism (2.29) is then induced by composing the coaction $\Delta_{\mathfrak{a},\mathfrak{dr}} : \mathcal{P}^{\mathfrak{dr}} \to \mathcal{P}^{\mathfrak{a}} \otimes \mathcal{P}^{\mathfrak{dr}}$ with $\mathrm{id} \otimes \pi_{\mathbb{L},\mathfrak{dr}}$.

Remark 2.9. If $v \in (M_{dR})_m$ and $f \in ((M_{dR})_n)^\vee$ are of degrees $m > n$ respectively, then the image of $[M,v,f]^{\mathfrak{a}}$ under the implied section $\mathcal{P}^{\mathfrak{a}} \to \mathcal{P}^{\mathfrak{dr}}$ is $[M(n),v,f]$, where v now sits in degree $m-n$, and f in degree zero. The literature on framed mixed Tate objects essentially identifies $\mathcal{P}^{\mathfrak{a}}$ with its image $\mathcal{P}^{\mathfrak{a},0}$ in $\mathcal{P}^{\mathfrak{dr}}$.

2.9. **Structure of motivic periods.** Rather than using the canonical isomorphism of fiber functors $\mathrm{comp}_{dR,B}$, which is defined over $\mathbb{C}$, we prefer to choose rational isomorphisms, which are non-canonical.

Proposition 2.10. *There exists an isomorphism of fiber functors from ω_B to ω_{dR}.*

Proof. See the proof of proposition 8.10 in [11]. □

By choosing such an element $s \in \mathrm{Isom}(\omega_B, \omega_{dR})$, we obtain an isomorphism

$$\mathrm{Isom}(\omega_{dR}, \omega_B) \overset{\sim}{\longrightarrow} \mathrm{Isom}(\omega_{dR}, \omega_{dR}) \ . \tag{2.31}$$

Dually, this gives $s^t : \mathcal{P}^{\mathfrak{dr}} \overset{\sim}{\longrightarrow} \mathcal{P}^{\mathfrak{m}}$, and so (2.29) gives a non-canonical isomorphism

$$s^t : \mathcal{P}^{\mathfrak{a}} \otimes \mathbb{Q}[(\mathbb{L}^{\mathfrak{dr}})^{-1}, \mathbb{L}^{\mathfrak{dr}}] \overset{\sim}{\longrightarrow} \mathcal{P}^{\mathfrak{m}} \ .$$

By §2.7 we can assume that $s^t(\mathbb{L}^{\mathfrak{dr}}) = \mathbb{L}^{\mathfrak{m}}$, and write the previous isomorphism as

$$\mathcal{P}^{\mathfrak{m}} \cong \mathcal{P}^{\mathfrak{a}} \otimes \mathbb{Q}[(\mathbb{L}^{\mathfrak{m}})^{-1}, \mathbb{L}^{\mathfrak{m}}] \qquad \text{(depending on } s) \ . \tag{2.32}$$

It is compatible with the coaction $\Delta_{\mathfrak{a},\mathfrak{m}} : \mathcal{P}^{\mathfrak{m}} \to \mathcal{P}^{\mathfrak{a}} \otimes \mathcal{P}^{\mathfrak{m}}$, and the weight gradings.

Corollary 2.11. *There is a non-canonical decomposition*

$$\mathcal{P}^{\mathfrak{m},+} \cong \mathcal{P}^{\mathfrak{a}} \otimes \mathbb{Q}[\mathbb{L}^{\mathfrak{m}}] \ . \tag{2.33}$$

Proof. The decomposition (2.32) is induced by $(\mathrm{id} \otimes \pi_{\mathbb{L}^{\mathfrak{m}}}) \circ \Delta_{\mathfrak{a},\mathfrak{m}}$, where $\pi_{\mathbb{L}^{\mathfrak{m}}}$ is given by $(s^t)^{-1}$ followed by (2.30), and $\mathbb{L}^{\mathfrak{dr}} \mapsto \mathbb{L}^{\mathfrak{m}}$. Since $\mathcal{P}^{\mathfrak{m},+}$ has weights ≥ 0, and $\mathbb{L}^{\mathfrak{m}}$ has weight 1, the restriction of (2.32) to $\mathcal{P}^{\mathfrak{m},+}$ gives an injective map

$$\mathcal{P}^{\mathfrak{m},+} \longrightarrow \mathcal{P}^{\mathfrak{a}} \otimes \mathbb{Q}[\mathbb{L}^{\mathfrak{m}}] \ .$$

The image of $\mathcal{P}^{\mathfrak{a}} \otimes \mathbb{Q}[\mathbb{L}^{\mathfrak{m}}]$ in $\mathcal{P}^{\mathfrak{m}}$ has weights ≥ 0, and is G_{dR}-stable. Since $\mathcal{P}^{\mathfrak{m},+}$ is the largest subalgebra of $\mathcal{P}^{\mathfrak{m}}$ with this property, the previous map is an isomorphism. □

Sending $\mathbb{L}^{\mathfrak{m}}$ to zero in (2.33) gives back the map $\pi_{\mathfrak{a},\mathfrak{m}+} : \mathcal{P}^{\mathfrak{m},+} \to \mathcal{P}^{\mathfrak{a}}$.

2.10. **Real Frobenius.** Since there is a unique embedding from $\mathbb{Q}$ to $\mathbb{C}$, complex conjugation defines the real Frobenius $c : M_B \to M_B$. It induces an involution

$$\begin{array}{rcl} c : \mathcal{P}^{\mathfrak{m}} & \longrightarrow & \mathcal{P}^{\mathfrak{m}} \\ {[M, \eta, X]^{\mathfrak{m}}} & \mapsto & [M, \eta, c(X)]^{\mathfrak{m}} \end{array} \tag{2.34}$$

which is compatible, via the period homomorphism, with complex conjugation on $\mathbb{C}$. If $\Delta_{\mathfrak{a},\mathfrak{m}} : \mathcal{P}^{\mathfrak{m}} \to \mathcal{P}^{\mathfrak{a}} \otimes \mathcal{P}^{\mathfrak{m}}$ denotes the coaction, then clearly $\Delta_{\mathfrak{a},\mathfrak{m}} c = (\mathrm{id} \otimes c)\Delta_{\mathfrak{a},\mathfrak{m}}$. Since

$$c(\mathbb{L}^{\mathfrak{m}}) = -\mathbb{L}^{\mathfrak{m}}$$

it follows that c acts on a decomposition (2.33) by multiplying $(\mathbb{L}^{\mathfrak{m}})^n$ by $(-1)^n$.

Corollary 2.12. *If $\mathcal{P}^{\mathfrak{m},+}_{\mathbb{R}}$ (respectively $\mathcal{P}^{\mathfrak{m},+}_{i\mathbb{R}}$) denotes the subspace of $\mathcal{P}^{\mathfrak{m},+}$ of invariants (anti-invariants) of the map c, then we have*

$$\begin{array}{rrcl} & \mathcal{P}^{\mathfrak{m},+}_{i\mathbb{R}} & \cong & \mathcal{P}^{\mathfrak{m},+}_{\mathbb{R}} \mathbb{L}^{\mathfrak{m}} \\ \textit{and} & \mathcal{P}^{\mathfrak{m},+}_{\mathbb{R}} & \cong & \mathcal{P}^{\mathfrak{a}} \otimes \mathbb{Q}[(\mathbb{L}^{\mathfrak{m}})^2] \end{array}$$

with respect to some choice of decomposition (2.33)*.*

2.11. **Universal comparison map.** The identity map $\mathrm{id} : \mathcal{P}^{\mathfrak{m}} \to \mathcal{P}^{\mathfrak{m}}$ defines a canonical element in $(\mathrm{Spec}\, \mathcal{P}^{\mathfrak{m}})(\mathcal{P}^{\mathfrak{m}})$ which we denote by

$$\mathrm{comp}^{\mathfrak{m}}_{B,dR} \in \mathrm{Isom}_{\omega_{dR}, \omega_B}(\mathcal{P}^{\mathfrak{m}}) \ .$$

It reduces to the usual comparison map $\mathrm{comp}_{B,dR}$ after applying the period homomorphism to the coefficient ring $\mathcal{P}^{\mathfrak{m}}$. As a formula, it is given for $M \in \mathcal{M}$ by

$$\begin{array}{rcl} \mathrm{comp}^{\mathfrak{m}}_{B,dR} : M_{dR} & \longrightarrow & M_B \otimes \mathcal{P}^{\mathfrak{m}}(M) \\ \eta & \mapsto & \sum_x x \otimes [M, \eta, x^{\vee}]^{\mathfrak{m}} \end{array} \tag{2.35}$$

where the sum ranges over a basis $\{x\}$ of M_B, and $\{x^\vee\}$ is the dual basis. We can also write (2.36) as an isomorphism after tensoring with all motivic periods:

$$\mathrm{comp}^{\mathfrak{m}}_{B,dR} : M_{dR} \otimes \mathcal{P}^{\mathfrak{m}} \xrightarrow{\sim} M_B \otimes \mathcal{P}^{\mathfrak{m}} \ . \qquad (2.36)$$

In the other direction, we have a universal map

$$\mathrm{comp}^{\mathfrak{m}}_{dR,B} : M_B \longrightarrow M_{dR} \otimes \mathcal{P}^{\omega_B,\omega_{dR}}(M)$$

which is defined in a similar way. It will not be used here.

The universal comparison maps can be used to compare the action of the de Rham motivic Galois group G_{dR} with the action of the Betti Galois group G_B on $\mathcal{P}^{\mathfrak{m}}$.

3. Single-valued motivic periods

The single-valued period is an analogue of the period homomorphism for de Rham periods. First, we construct a well-defined map (the single-valued motivic period)

$$\mathrm{sv}^{\mathfrak{m}} : \mathcal{P}^{\mathfrak{a}} \longrightarrow \mathcal{P}^{\mathfrak{m},+}$$

and define the single-valued period to be

$$\mathrm{sv} : \mathcal{P}^{\mathfrak{a}} \longrightarrow \mathbb{C}$$

by composing with the usual period $\mathrm{per} : \mathcal{P}^{\mathfrak{m}} \to \mathbb{C}$. The map sv is similar to what is sometimes referred to as the 'real period' in the literature [18], §4. Since multiple zeta values are already real numbers, this terminology could lead to confusion, so we prefer not to use it. Note that the single-valued periods of a motive M are not in fact periods of M, but elements of the algebra generated by the periods of M.

In the latter half of the paper, we shall compute the single-valued versions of motivic multiple zeta values using the motivic fundamental group of $\mathbb{P}^1\backslash\{0,1,\infty\}$. More precisely, we compute the following map

$$\mathcal{P}^{\mathfrak{m},+} \overset{\pi_{\mathfrak{a},\mathfrak{m}^+}}{\longrightarrow} \mathcal{P}^{\mathfrak{a}} \overset{\mathrm{sv}^{\mathfrak{m}}}{\longrightarrow} \mathcal{P}^{\mathfrak{m},+}$$

on the subspace $\mathcal{H} \subset \mathcal{P}^{\mathfrak{m},+}$ of motivic multiple zeta values.

3.1. **Single-valued motivic periods.** The weight-grading on $\mathcal{P}^{\mathfrak{m}}$ is given by an action of $\mathbb{G}_m$, which we shall denote by τ. Thus $\tau(\lambda)$ is the map which in weight n acts via multiplication by λ^n, for any $\lambda \in \mathbb{Q}^\times$.

Definition 3.1. Let $\sigma : \mathcal{P}^{\mathfrak{m}} \longrightarrow \mathcal{P}^{\mathfrak{m}}$ be the involution

$$\sigma = \tau(-1)\, c \qquad (3.1)$$

where c is the real Frobenius of §2.10. For example, $\sigma(\mathbb{L}^{\mathfrak{m}}) = \mathbb{L}^{\mathfrak{m}}$.

Remark 3.2. If $\Delta_{\mathfrak{a},\mathfrak{m}} : \mathcal{P}^{\mathfrak{m}} \to \mathcal{P}^{\mathfrak{a}} \otimes \mathcal{P}^{\mathfrak{m}}$ denotes the coaction, then

$$\Delta_{\mathfrak{a},\mathfrak{m}}\sigma = (\overline{\sigma} \otimes \sigma) \circ \Delta_{\mathfrak{a},\mathfrak{m}}$$

where $\overline{\sigma} : \mathcal{P}^{\mathfrak{a}} \to \mathcal{P}^{\mathfrak{a}}$ is given by the action of $\tau(-1)$ on $\mathcal{P}^{\mathfrak{a}}$ by conjugation. In other words, $\overline{\sigma}$ acts by multiplication by $(-1)^n$ in degree n, where the degree is (2.17).

Consider the following affine scheme over $\mathbb{Q}$:

$$\mathbb{P} = \mathrm{Spec}\,(\mathcal{P}^{\mathfrak{m}}) \qquad \big(= \mathrm{Isom}(\omega_{dR}, \omega_B)\big) \ .$$

The coaction $\mathcal{P}^{\mathfrak{m}} \to \mathcal{P}^{\mathfrak{dr}} \otimes \mathcal{P}^{\mathfrak{m}}$ defines an action we denote by $\circ$:

$$\circ : G_{dR} \times \mathbb{P} \longrightarrow \mathbb{P}$$

and makes $\mathbb{P}$ a torsor over G_{dR} (by proposition 2.10). The maps $\mathrm{id}, \sigma : \mathcal{P}^{\mathfrak{m}} \to \mathcal{P}^{\mathfrak{m}}$ can be viewed as elements $\mathrm{id}, \sigma \in \mathbb{P}(\mathcal{P}^{\mathfrak{m}})$.

Definition 3.3. Define $\mathrm{sv}^{\mathfrak{m}}$ to be the unique element of $G_{dR}(\mathcal{P}^{\mathfrak{m}})$ such that

$$\mathrm{sv}^{\mathfrak{m}} \circ \sigma = \mathrm{id}\ . \tag{3.2}$$

Let us compute $\mathrm{sv}^{\mathfrak{m}}(\mathbb{L}^{\mathfrak{dr}})$. Recall that $\Delta_{\mathfrak{dr},\mathfrak{m}}\mathbb{L}^{\mathfrak{m}} = \mathbb{L}^{\mathfrak{dr}} \otimes \mathbb{L}^{\mathfrak{m}}$. Thus

$$\mathbb{L}^{\mathfrak{m}} = (\mathrm{sv}^{\mathfrak{m}} \circ \sigma)(\mathbb{L}^{\mathfrak{m}}) = \mu(\mathrm{sv}^{\mathfrak{m}} \otimes \sigma)(\mathbb{L}^{\mathfrak{dr}} \otimes \mathbb{L}^{\mathfrak{m}}) = \mathrm{sv}^{\mathfrak{m}}(\mathbb{L}^{\mathfrak{dr}})\sigma(\mathbb{L}^{\mathfrak{m}})$$

where μ denotes multiplication. Since $\sigma(\mathbb{L}^{\mathfrak{m}}) = \mathbb{L}^{\mathfrak{m}}$, we deduce that $\mathrm{sv}^{\mathfrak{m}}(\mathbb{L}^{\mathfrak{dr}}) = 1$. Therefore $\mathrm{sv}^{\mathfrak{m}}$ actually lies in the image of $U_{dR}(\mathcal{P}^{\mathfrak{m}})$ in $G_{dR}(\mathcal{P}^{\mathfrak{m}})$ and we can view it as a homomorphism $\mathrm{sv}^{\mathfrak{m}} : \mathcal{P}^{\mathfrak{a}} \longrightarrow \mathcal{P}^{\mathfrak{m}}$. Even more precisely, we have:

Proposition 3.4. *For all $g \in G_{dR}$, and $\xi \in \mathcal{P}^{\mathfrak{a}}$,*

$$\mathrm{sv}^{\mathfrak{m}}(c_g\xi) = g\, \mathrm{sv}^{\mathfrak{m}}(\xi) \tag{3.3}$$

where c_g denotes the action of $g \in G_{dR}$ on $\mathcal{P}^{\mathfrak{a}}$ by twisted conjugation

$$c_g(\xi) = g\, \xi\, \overline{g}^{-1}$$

where $\overline{g} = \tau(-1)\, g\, \tau(-1)$. In particular, $\mathrm{sv}^{\mathfrak{m}}$ defines a homomorphism

$$\mathrm{sv}^{\mathfrak{m}} : \mathcal{P}^{\mathfrak{a}} \longrightarrow \mathcal{P}^{\mathfrak{m},+} \tag{3.4}$$

which is homogeneous for the weight-gradings on both sides.

Proof. For any $g \in G_{dR}$, define $\mathrm{sv}^{\mathfrak{m}}_g : \mathcal{P}^{\mathfrak{a}} \to \mathcal{P}^{\mathfrak{m}}$ to be $\mathrm{sv}^{\mathfrak{m}}_g(x) = g\, \mathrm{sv}^{\mathfrak{m}}(x)$, and similarly define $\sigma_g, \mathrm{id}_g \in G_{dR}(\mathcal{P}^{\mathfrak{m}})$, where $\sigma_g(x) = g\, \sigma(x)$, $\mathrm{id}_g(x) = g\, \mathrm{id}(x)$, and the action of g is on the ring of coefficients $\mathcal{P}^{\mathfrak{m}}$. By the definition (3.2) of $\mathrm{sv}^{\mathfrak{m}}$, we have

$$\mathrm{sv}^{\mathfrak{m}}_g \circ \sigma_g = \mathrm{id}_g\ .$$

Clearly $\mathrm{id}_g = g$, but $\sigma_g = \overline{g} \circ \sigma$ by remark 3.2. Therefore

$$\mathrm{sv}^{\mathfrak{m}}_g \circ \overline{g} \circ \sigma = g\ .$$

Since $\mathbb{P}$ is a torsor over G_{dR}, this has the unique solution $\mathrm{sv}^{\mathfrak{m}}_g = g \circ \mathrm{sv}^{\mathfrak{m}} \circ \overline{g}^{\circ -1}$, which is precisely (3.3). Since the weight-grading on $\mathcal{P}^{\mathfrak{a}}$ is given by conjugation by g for $g \in \mathbb{G}_m$ and $\overline{g} = g$ for such g (because $\mathbb{G}_m$ is commutative), we deduce from (3.3) that $\mathrm{sv}^{\mathfrak{m}}$ is homogeneous in the weight. In particular, since $\mathcal{P}^{\mathfrak{a}}$ has weight ≥ 0, the image of $\mathrm{sv}^{\mathfrak{m}}$ has weight ≥ 0, is stable under G_{dR}, and hence is contained in $\mathcal{P}^{\mathfrak{m},+}$. □

The formula (3.3) can be translated into coactions as follows. Let

$$\mathcal{L}^{\mathfrak{a}} = \frac{\mathcal{P}^{\mathfrak{a}}_{>0}}{\mathcal{P}^{\mathfrak{a}}_{>0}\mathcal{P}^{\mathfrak{a}}_{>0}}$$

denote the Lie coalgebra of indecomposable elements of $\mathcal{P}^{\mathfrak{a}}$. Projecting from $\mathcal{P}^{\mathfrak{a}}_{>0}$ to $\mathcal{L}^{\mathfrak{a}}$ defines infinitesimal versions of the usual coactions (2.12)

$$\begin{array}{rcl} \delta : \mathcal{P}^{\mathfrak{m}} & \longrightarrow & \mathcal{L}^{\mathfrak{a}} \otimes \mathcal{P}^{\mathfrak{m}} \\ \delta_L : \mathcal{P}^{\mathfrak{a}} & \longrightarrow & \mathcal{L}^{\mathfrak{a}} \otimes \mathcal{P}^{\mathfrak{a}} \\ \delta_R : \mathcal{P}^{\mathfrak{a}} & \longrightarrow & \mathcal{P}^{\mathfrak{a}} \otimes \mathcal{L}^{\mathfrak{a}} \cong \mathcal{L}^{\mathfrak{a}} \otimes \mathcal{P}^{\mathfrak{a}} \end{array}$$

where δ_L, δ_R are obtained from the left and right coactions of $\mathcal{P}^{\mathfrak{a}}$ on itself. Then

$$\delta\, \mathrm{sv}^{\mathfrak{m}}(\xi) = \mathrm{sv}^{\mathfrak{m}}(\delta_L\xi) + (\overline{S} \otimes \mathrm{id})\, \mathrm{sv}^{\mathfrak{m}}(\delta_R\xi)\ , \tag{3.5}$$

where $\overline{S} : \mathcal{L}^{\mathfrak{a}} \to \mathcal{L}^{\mathfrak{a}}$ is multiplication by $(-1)^n$ in degree n followed by the infinitesimal version of the antipode $S : \mathcal{L}^{\mathfrak{a}} \to \mathcal{L}^{\mathfrak{a}}$.

Definition 3.5. Let $\mathcal{P}^{\mathrm{sv}} \subset \mathcal{P}^{\mathfrak{m},+}$ denote the image of the map $\mathrm{sv}^{\mathfrak{m}}$. We shall call it the ring of single-valued motivic periods.

3.2. **Properties of the single-valued motivic period.** For computations, it is convenient to trivialize the torsor $\mathbb{P}$ as follows. By proposition 2.10, we can choose an isomorphism of fiber functors $s' \in \mathrm{Isom}(\omega_B, \omega_{dR})$. It defines an isomorphism (2.31)

$$s : \mathcal{O}(G_{dR}) = \mathcal{P}^{\mathfrak{dr}} \xrightarrow{\sim} \mathcal{P}^{\mathfrak{m}} \ , \tag{3.6}$$

where $s = (s')^t$, which we view as a $\mathcal{P}^{\mathfrak{m}}$-valued point of G_{dR}, denoted $s \in G_{dR}(\mathcal{P}^{\mathfrak{m}})$. The action of the involution (3.1) on its coefficients will be denoted by ${}^{\sigma}$.

Then $\mathrm{sv}^{\mathfrak{m}} \in G_{dR}(\mathcal{P}^{\mathfrak{m}})$ can be computed via the expression

$$\mathrm{sv}^{\mathfrak{m}} = s \circ ({}^{\sigma}s)^{\circ -1} \tag{3.7}$$

where the inversion and multiplication $\circ$ take place in the group G_{dR}.

Remark 3.6. To check that (3.7) is well-defined, let $s_1', s_2' \in \mathrm{Isom}(\omega_B, \omega_{dR})(\mathbb{Q})$. Since the latter is a $(G_B, G_{dR})(\mathbb{Q})$-bitorsor, there exists an element $\rho' \in G_{dR}(\mathbb{Q})$ such that $s_2' = \rho' s_1'$. Transposing gives $s_2 = s_1 \circ \rho$, where ρ is the image of ρ' in $G_{dR}(\mathcal{P}^{\mathfrak{m}})$ via $\mathbb{Q} \subset \mathcal{P}^{\mathfrak{m}}$. In particular, ${}^{\sigma}\rho = \rho$ since its coefficients are rational of weight zero. Thus

$$s_2 \circ ({}^{\sigma}s_2)^{\circ -1} = s_1 \circ \rho \circ ({}^{\sigma}\rho)^{\circ -1} \circ ({}^{\sigma}s_1)^{\circ -1} = s_1 \circ ({}^{\sigma}s_1)^{\circ -1} \ ,$$

and (3.7) is well-defined, as expected.

Definition 3.7. Let $\mathcal{P}^{\mathfrak{m},0} \subset \mathcal{P}^{\mathfrak{m},+}$ denote the subring of motivic periods

$$\mathcal{P}^{\mathfrak{m},0} = \bigcap_s s(\mathcal{P}^{\mathfrak{a}})$$

where s ranges over maps $s : \mathcal{P}^{\mathfrak{a}} \to \mathcal{P}^{\mathfrak{m},+}$ induced by decompositions (2.33). Since $\pi_{\mathfrak{a},\mathfrak{m}+}$ is injective on the image of such an s, it is injective on $\mathcal{P}^{\mathfrak{m},0}$.

Lemma 3.8. *We have $\mathcal{P}^{\mathrm{sv}} \subset \mathcal{P}^{\mathfrak{m},0}$. In particular, $\pi_{\mathfrak{a},\mathfrak{m}+} : \mathcal{P}^{\mathrm{sv}} \to \mathcal{P}^{\mathfrak{a}}$ is injective.*

The compositum $\pi_{\mathfrak{a},\mathfrak{m}+}\mathrm{sv}^{\mathfrak{m}} : \mathcal{P}^{\mathfrak{a}} \to \mathcal{P}^{\mathfrak{a}}$ is given by the element

$$\mathrm{id} \circ \sigma^{\circ -1} \in U_{dR}(\mathcal{P}^{\mathfrak{a}}) \ . \tag{3.8}$$

Proof. A choice of isomorphism (2.32) defines a map $s : \mathcal{P}^{\mathfrak{a}} \to \mathcal{P}^{\mathfrak{m},+}$ (and hence an element $s \in U_{dR}(\mathcal{P}^{\mathfrak{m}})$) which we can use to compute $\mathrm{sv}^{\mathfrak{m}}$. By a similar argument to the discussion preceding (3.7), except that we work in U_{dR} instead of G_{dR}, we have $\mathrm{sv}^{\mathfrak{m}} = s \circ ({}^{\sigma}s)^{\circ -1}$. The coefficients of s, and a fortiori $\mathrm{sv}^{\mathfrak{m}}$, lie in the subspace $s(\mathcal{P}^{\mathfrak{a}}) \subset \mathcal{P}^{\mathfrak{m},+}$. This proves the first statement.

For the second statement, observe that $\pi_{\mathfrak{a},\mathfrak{m}+}s$ is the identity map on $\mathcal{P}^{\mathfrak{a}}$, and therefore $\pi_{\mathfrak{a},\mathfrak{m}+}\mathrm{sv}^{\mathfrak{m}} = \mathrm{id} \circ ({}^{\sigma}\mathrm{id})^{\circ -1}$, which gives exactly (3.8). □

In particular, the map $\pi_{\mathfrak{a},\mathfrak{m}+}\mathrm{sv}^{\mathfrak{m}} : \mathcal{P}^{\mathfrak{a}} \to \mathcal{P}^{\mathfrak{a}}$ is not the identity, and has a large kernel. The previous lemma will be used in §6.2 to determine the structure of $\mathcal{P}^{\mathrm{sv}}$.

3.3. **Formulae.** We can translate (3.7) into a formula as follows. Let H be any connected, commutative graded Hopf algebra, with coproduct $\Delta : H \to H \otimes H$. Denote its reduced coproduct by $\Delta^{(1)} = \Delta - 1 \otimes \mathrm{id} - \mathrm{id} \otimes 1$, and its iterated coproduct by

$$\Delta^{(n)} = (\mathrm{id} \otimes \Delta^{(n-1)})\Delta^{(1)} = (\Delta^{(n-1)} \otimes \mathrm{id})\Delta^{(1)}$$

for $n \geq 2$. By a version of Sweedler's notation we can write

$$\Delta^{(n-1)}(x) = \sum_{(x)} x^{(1)} \otimes \ldots \otimes x^{(n)}$$

where all elements $x^{(i)}$ have degree ≥ 1. In any such Hopf algebra, the antipode can be written $S = \sum_{n\geq 1}(-1)^n \mu_n \Delta^{(n)}$, where $\mu_n : H^{\otimes n} \to H$ is the n-fold multiplication, and $\Delta^{(0)}$ is the identity. In Sweedler's notation, this is

$$S(x) = \sum_{n\geq 1} (-1)^n \sum_{(x)} x^{(1)} \dots x^{(n)}$$

In order to apply the above, we must consider the Hopf algebra $\mathcal{P}^{\mathfrak{a},0}$ defined to be the image of $\mathcal{P}^{\mathfrak{a}}$ in $\mathcal{P}^{\mathfrak{dr}}$ via the isomorphism (2.29). It is a commutative graded Hopf algebra, spanned by objects $[M, v, f]^{\mathfrak{dr}}$ where f is in degree 0 by remark 2.9. A choice of homomorphism (3.6) gives a homomorphism $s : \mathcal{P}^{\mathfrak{a},0} \to \mathcal{P}^{\mathfrak{m}}$. Applying the previous remarks to the Hopf algebra $\mathcal{P}^{\mathfrak{a},0}$, we obtain

$$\begin{array}{rcl} \mathrm{sv}^{\mathfrak{m}} : \mathcal{P}^{\mathfrak{a},0} & \longrightarrow & \mathcal{P}^{\mathfrak{m}} \\ \mathrm{sv}^{\mathfrak{m}}(\xi) & = & s(\xi) + \widetilde{s}(\xi) + \sum_{(\xi)} s(\xi^{(1)}) \widetilde{s}(\xi^{(2)}) \end{array}$$

where we use the notation

$$\widetilde{s}(\xi) = \sum_{n\geq 1} \sum_{(\xi)} (-1)^n \, {}^{\sigma}s(\xi^{(1)}) \dots {}^{\sigma}s(\xi^{(n)}) \ .$$

This essentially follows from the formula (3.7), after replacing G_{dR} with U_{dR}.

Concretely, the motivic periods $s(\xi)$ can be computed as follows. If $M \in \mathcal{M}$, then the element s defines an isomorphism $s : M_B \to M_{dR}$. Let s^t be its transpose. Then if $v \in M_{dR}$ and $f \in ((M_{dR})_0)^\vee$ is of degree 0,

$$s[M, v, f]^{\mathfrak{a}} = [M, v, s^t f]^{\mathfrak{m}} \ .$$

In general we must first compose with $\mathcal{P}^{\mathfrak{a}} \overset{\sim}{\to} \mathcal{P}^{\mathfrak{a},0}$, which introduces a Tate twist by remark 2.9. It follows that the single-valued motivic periods of M are in fact products of motivic periods of Tate twists of M.

4. The motivic fundamental group of $\mathbb{P}^1 \backslash \{0, 1, \infty\}$

The main references for this section are [11], [14], [7]. A good introduction can be found in [12].

Let $X = \mathbb{P}^1 \backslash \{0, 1, \infty\}$, and let $\overrightarrow{1}_0, -\overrightarrow{1}_1$ denote the tangential base points on X given by the vector 1 at 0, and the vector -1 at 1. Denote the motivic fundamental torsor of paths on X by

$${}_0\Pi_1^{\mathfrak{m}} = \pi_1^{\mathfrak{m}}(X, \overrightarrow{1}_0, -\overrightarrow{1}_1) \ .$$

It is an affine scheme in the category $\mathcal{MT}(\mathbb{Z})$. This means that there is a finitely generated commutative algebra object $\mathcal{O}({}_0\Pi_1^{\mathfrak{m}}) \in \mathrm{Ind}\, \mathcal{MT}(\mathbb{Z})$, and $\omega({}_0\Pi_1^{\mathfrak{m}})$ is defined to be Spec of the commutative algebra $\omega(\mathcal{O}({}_0\Pi_1^{\mathfrak{m}}))$, for any fiber functor ω.

We shall denote the de Rham realization $\omega_{dR}({}_0\Pi_1^{\mathfrak{m}})$ of ${}_0\Pi_1^{\mathfrak{m}}$ simply by

$${}_0\Pi_1 = \mathrm{Spec}\, \mathcal{O}({}_0\Pi_1) \ ,$$

where $\mathcal{O}({}_0\Pi_1)$ is isomorphic to $H^0(B(\Omega^\bullet_{\log}(\mathbb{P}^1, \{0, 1, \infty\}; \mathbb{Q})))$, where B is the bar complex. Writing e^0 for $\frac{dz}{z}$ and e^1 for $\frac{dz}{1-z}$, we can identify the latter with the graded $\mathbb{Q}$-algebra

$$\mathcal{O}({}_0\Pi_1) \cong \mathbb{Q}\langle e^0, e^1 \rangle \ .$$

Its underlying vector space is spanned by the set of words w in the letters e^0, e^1, together with the empty word, and the multiplication is given by the shuffle product $ш : \mathbb{Q}\langle e^0, e^1\rangle \otimes \mathbb{Q}\langle e^0, e^1\rangle \to \mathbb{Q}\langle e^0, e^1\rangle$ which is defined recursively by

$$(e_i w) \,ш\, (e_j w') = e_i(w \,ш\, e_j w') + e_j(e_i w \,ш\, w')$$

for all words w, w' in $\{e_0, e_1\}$ and $i, j \in \{0, 1\}$. The empty word will be denoted by 1. It is the unit for the shuffle product: $1 \,ш\, w = w \,ш\, 1$ for all w.

The de Rham realization ${}_0\Pi_1$ is therefore isomorphic to $\operatorname{Spec} \mathbb{Q}\langle e^0, e^1\rangle$. It is the affine scheme over $\mathbb{Q}$ which to any commutative unitary $\mathbb{Q}$-algebra R associates the set of group-like formal power series in two non-commuting variables e_0 and e_1

$${}_0\Pi_1(R) = \{S \in R\langle\langle e_0, e_1\rangle\rangle^\times : \Delta S = S \widehat{\otimes} S\} \ .$$

Here, Δ is the completed coproduct $R\langle\langle e_0, e_1\rangle\rangle \to R\langle\langle e_0, e_1\rangle\rangle \widehat{\otimes}_R R\langle\langle e_0, e_1\rangle\rangle$ for which the elements e_0 and e_1 are primitive: $\Delta e_i = 1 \otimes e_i + e_i \otimes 1$ for $i = 0, 1$.

Since the bar complex is augmented, we have an augmentation map ${}_0\Pi_1 \to \mathbb{Q}$ which is the projection onto the empty word. Dually, this corresponds to an element denoted

$${}_01_1 \in {}_0\Pi_1(\mathbb{Q})$$

which is called the canonical de Rham path from $\vec{1}_0$ to $-\vec{1}_1$.

On the other hand, the Betti realization of ${}_0\Pi_1^{\mathfrak{m}}$ is the affine scheme over $\mathbb{Q}$ given by the Malĉev completion of the topological fundamental torsor of paths

$$\omega_B({}_0\Pi_1^{\mathfrak{m}}) \cong \pi_1^{un}(X(\mathbb{C}), \vec{1}_0, -\vec{1}_1) \ .$$

There is a natural map $\pi_1(X(\mathbb{C}), \vec{1}_0, -\vec{1}_1) \to \pi_1^{un}(X(\mathbb{C}), \vec{1}_0, -\vec{1}_1)(\mathbb{Q})$.

4.1. **Drinfeld's associator.** There is a canonical straight line path ('droit chemin')

$$\text{dch} \in \pi_1(X(\mathbb{C}), \vec{1}_0, -\vec{1}_1) \tag{4.1}$$

which therefore corresponds to an element in $\omega_B({}_0\Pi_1^{\mathfrak{m}})$. Via the isomorphism (2.2), it defines an element in ${}_0\Pi_1(\mathbb{C})$, which we denote by

$$Z(e_0, e_1) \in {}_0\Pi_1(\mathbb{C})$$

It is precisely Drinfeld's associator, and is given in low degrees by the formula

$$Z(e_0, e_1) = 1 + \zeta(2)[e_1, e_0] + \zeta(3)([e_1, [e_1, e_0]] + [e_0, [e_0, e_1]]) + \dots \tag{4.2}$$

In general, the coefficients are multiple zeta values. In fact, (4.2) is the non-commutative generating series of (shuffle-regularized) multiple zeta values

$$Z(e_0, e_1) = \sum_{w \in \{e_0, e_1\}^\times} \zeta(w)\, w \ .$$

The coefficient $\zeta(w)$ is given by the regularized iterated integral

$$\zeta(e_{a_1} \dots e_{a_n}) = \int_{\text{dch}} \omega_{a_1} \dots \omega_{a_n} \qquad \text{for } a_i \in \{0, 1\}$$

where $\omega_0 = \frac{dt}{t}$ and $\omega_1 = \frac{dt}{1-t}$, and the integration begins on the left. One shows that the $\zeta(w)$ are linear combinations of multiple zeta values (1.2) and that for $n_r \geq 2$,

$$\zeta(e_1 e_0^{n_1-1} e_1 e_0^{n_2-1} \dots e_1 e_0^{n_r-1}) = \zeta(n_1, \dots, n_r) \ .$$

4.2. **The Ihara action.** Since $\mathcal{O}({}_0\Pi_1)$ is the de Rham realization of an Ind-object in the category $\mathcal{MT}(\mathbb{Z})$, it inherits an action of the motivic Galois group

$$\mathcal{U}_{dR} \times {}_0\Pi_1 \longrightarrow {}_0\Pi_1 \ .$$

The action of $\mathcal{U}_{dR}$ on the element ${}_01_1 \in {}_0\Pi_1$ defines a map

$$g \mapsto g({}_01_1) : \mathcal{U}_{dR} \longrightarrow {}_0\Pi_1 \ ,$$

and one shows [14], §5.8, that the action of $\mathcal{U}_{dR}$ on ${}_0\Pi_1$ factors through a map

$$\circ : {}_0\Pi_1 \times {}_0\Pi_1 \longrightarrow {}_0\Pi_1 \tag{4.3}$$

which, on the level of formal power series, is given by the following formula

$$\begin{array}{rcl} R\langle\langle e_0, e_1\rangle\rangle^{\times} \times R\langle\langle e_0, e_1\rangle\rangle & \longrightarrow & R\langle\langle e_0, e_1\rangle\rangle \\ F(e_0,e_1) \circ G(e_0,e_1) & = & G(e_0, F(e_0,e_1)e_1F(e_0,e_1)^{-1})F(e_0,e_1) \end{array} \tag{4.4}$$

which was first considered by Y. Ihara. The action (4.3) makes ${}_0\Pi_1$ into a torsor over ${}_0\Pi_1$ for $\circ$. More prosaically, given two invertible formal power series G, H, one can solve $F \circ G = H$ for F recursively by writing equation (4.4) as

$$F = G(e_0, Fe_1F^{-1})^{-1}H \tag{4.5}$$

If all the coefficients in F of words of length $\leq N$ have been determined, then the coefficients of Fe_1F^{-1}, and hence $G(e_0, Fe_1F^{-1})$ are determined up to length $N+1$. Equation (4.5) determines the coefficients of F in length $N+1$. A similar recurrence based on the number of occurrences of e_1 in a word (the depth) sometimes allows one to write down closed formulae in low depth and in all weights (§7.4).

4.3. **Motivic multiple zeta values.** Let $\mathrm{dch}_B \in \omega_B({}_0\Pi_1^{\mathfrak{m}})(\mathbb{Q})$ denote the Betti image of the straight line path (4.1). It defines an element $\mathrm{dch}_B \in \omega_B({}_0\Pi_1^{\mathfrak{m}})^{\vee}$. Let w be any word in $\{e^0, e^1\}$. It defines an element $w \in \mathcal{O}({}_0\Pi_1) \cong \mathbb{Q}\langle e^0, e^1\rangle$, the de Rham realization of $\mathcal{O}({}_0\Pi_1^{\mathfrak{m}})$.

Definition 4.1. The *motivic multiple zeta value* $\zeta^{\mathfrak{m}}(w)$ is the motivic period

$$\zeta^{\mathfrak{m}}(w) = [\mathcal{O}({}_0\Pi_1^{\mathfrak{m}}), w, \mathrm{dch}_B]^{\mathfrak{m}} \ .$$

The *algebra of motivic multiple zeta values* $\mathcal{H} \subset \mathcal{P}^{\mathfrak{m}}$ is the graded $\mathbb{Q}$-algebra spanned by the $\zeta^{\mathfrak{m}}(w)$, i.e., the image of the map $w \mapsto \zeta^{\mathfrak{m}}(w) : \mathbb{Q}\langle e^0, e^1\rangle \to \mathcal{P}^{\mathfrak{m}}$.

Since $\mathcal{O}({}_0\Pi_1^{\mathfrak{m}})$ has weights ≥ 0, and is stable under U_{dR}, it follows that $\mathcal{H} \subset \mathcal{P}^{\mathfrak{m},+}$ by definition 2.7. Thus $\mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}_n$ is positively graded, and there is a natural map

$$\begin{array}{rcl} \mathbb{Q}\langle e^0, e^1\rangle & \longrightarrow & \mathcal{H} \\ w & \mapsto & \zeta^{\mathfrak{m}}(w) \ . \end{array} \tag{4.6}$$

which is a homomorphism for the shuffle product. The period map (2.9) yields

$$\begin{array}{rcl} \mathrm{per} : \mathcal{H} & \longrightarrow & \mathbb{R} \\ \zeta^{\mathfrak{m}}(w) & \mapsto & \zeta(w) \end{array} \tag{4.7}$$

and the periods of motivic multiple zeta values are the usual multiple zeta values.

There is a corresponding notion of unipotent de Rham multiple zeta value. Instead of dch_B, we now take a de Rham framing ${}_01_1 \in {}_0\Pi_1(\mathbb{Q}) \subset \mathcal{O}({}_0\Pi_1)^{\vee}$.

Definition 4.2. The *unipotent de Rham multiple zeta value* $\zeta^{\mathfrak{a}}(w)$ is

$$\zeta^{\mathfrak{a}}(w) = [\mathcal{O}({}_0\Pi_1^{\mathfrak{m}}), w, {}_01_1]^{\mathfrak{a}} \ .$$

The *algebra of unipotent de Rham multiple zeta values* $\mathcal{A} \subset \mathcal{P}^{\mathfrak{a}}$ is the graded $\mathbb{Q}$-algebra spanned by the $\zeta^{\mathfrak{a}}(w)$, i.e., the image of the map $w \mapsto \zeta^{\mathfrak{a}}(w) : \mathbb{Q}\langle e^0, e^1\rangle \to \mathcal{P}^{\mathfrak{a}}$.

Since $\mathcal{O}({}_0\Pi_1^{\mathfrak{m}})$ has non-negative weights, and because the de Rham image $Z(e_0, e_1)$ of dch has leading term 1, we verify that

$$ {}^t c_0(\mathrm{dch}_B) = {}_0 1_1 \ . $$

By equation (2.20), we deduce a surjective homomorphism

$$ \begin{array}{rcl} \pi_{\mathfrak{a},\mathfrak{m}+} : \mathcal{H} & \longrightarrow\!\!\!\!\rightarrow & \mathcal{A} \\ \zeta^{\mathfrak{m}}(w) & \mapsto & \zeta^{\mathfrak{a}}(w) \end{array} \qquad (4.8) $$

The motivic multiple zeta values $\zeta^{\mathfrak{m}}(w)$ were defined in [7], and simplified by Deligne [13]. The unipotent de Rham multiple zeta values $\zeta^{\mathfrak{a}}(w)$ are equivalent to the 'motivic multiple zeta values' considered in [17].

Remark 4.3. It is important to note that $\zeta^{\mathfrak{m}}(2) \neq 0$, whereas $\zeta^{\mathfrak{a}}(2) = 0$ ([7]).

The algebra $\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{A}_n$ is again positively graded, and is a commutative Hopf algebra by (2.6). We have a commutative diagram

$$ \begin{array}{ccc} \mathcal{H} & \longrightarrow & \mathcal{A} \otimes \mathcal{H} \\ \downarrow & & \downarrow \\ \mathcal{A} & \longrightarrow & \mathcal{A} \otimes \mathcal{A} \end{array} $$

Let us write $\mathbb{A} = \mathrm{Spec}\,(\mathcal{A})$, and $\mathbb{H} = \mathrm{Spec}\,(\mathcal{H})$. Then $\mathbb{A}$ is a pro-unipotent affine group scheme over $\mathbb{Q}$, which embeds in $\mathbb{H}$ via (4.8), and acts upon it on the left:

$$ \mathbb{A} \times \mathbb{H} \longrightarrow \mathbb{H} \ . \qquad (4.9) $$

4.4. **Compatibility with the Ihara action.** The fact that the action of the motivic Galois group factors through the Ihara action (§4.2) can be expressed by the following commutative diagram, where the maps $\mathbb{A} \hookrightarrow \mathbb{H} \hookrightarrow {}_0\Pi_1$ are induced by (4.8), (4.6):

$$ \begin{array}{ccc} \mathbb{A} \times \mathbb{H} & \longrightarrow & \mathbb{H} \\ \downarrow \quad \downarrow & & \downarrow \\ {}_0\Pi_1 \times {}_0\Pi_1 & \longrightarrow & {}_0\Pi_1 \end{array} $$

and the map along the bottom is the Ihara action $\circ : {}_0\Pi_1 \times {}_0\Pi_1 \rightarrow {}_0\Pi_1$. Dually, we have the following commutative diagram [7]:

$$ \begin{array}{ccc} \mathcal{O}({}_0\Pi_1) & \longrightarrow & \mathcal{O}({}_0\Pi_1) \otimes \mathcal{O}({}_0\Pi_1) \\ \downarrow & & \downarrow \\ \mathcal{H} & \longrightarrow & \mathcal{A} \otimes \mathcal{H} \end{array} $$

where the map along the top is the Ihara coaction, which can be effectively replaced [9], with an explicit formula which is due to Goncharov (who proved it for the unipotent de Rham periods $\zeta^{\mathfrak{a}}$, i.e. modulo $\zeta(2)$'s, but in fact gives the correct coaction for $\zeta^{\mathfrak{m}}$ also. See [9] for a direct and very short proof using the Ihara coaction).

4.5. **The motivic Drinfel'd associator.** In this section, all Hom's are in the category of commutative unitary $\mathbb{Q}$-algebras.

Definition 4.4. Define the *motivic version of the Drinfel'd associator* by

$$ Z^{\mathfrak{m}}(e_0, e_1) = \sum_{w \in \{e_0, e_1\}^{\times}} \zeta^{\mathfrak{m}}(w) w \qquad \in \qquad {}_0\Pi_1(\mathcal{H}) \subset \mathcal{H}\langle\langle e_0, e_1 \rangle\rangle $$

Define the *unipotent de Rham version of the Drinfel'd associator* by

$$ Z^{\mathfrak{a}}(e_0, e_1) = \sum_{w \in \{e_0, e_1\}^{\times}} \zeta^{\mathfrak{a}}(w) w \qquad \in \qquad {}_0\Pi_1(\mathcal{A}) \subset \mathcal{A}\langle\langle e_0, e_1 \rangle\rangle \ . $$

It is useful to view $Z^{\mathfrak{m}}$ as a morphism via the following general nonsense. For any commutative unitary ring R, we have an isomorphism

$$\mathrm{Hom}(\mathbb{Q}\langle e_0, e_1\rangle, R) \overset{\sim}{\longrightarrow} R\langle\langle e_0, e_1\rangle\rangle \ .$$

Via this isomorphism, we see that $Z^{\mathfrak{m}}$ is simply the image of the canonical map (4.6). Composing with the canonical map (4.6) gives a map:

$$\mathrm{Hom}(\mathcal{H}, R) \longrightarrow \mathrm{Hom}(\mathbb{Q}\langle e_0, e_1\rangle, R) \longrightarrow R\langle\langle e_0, e_1\rangle\rangle$$

which is simply another way to write $\mathbb{H}(R) \hookrightarrow {}_0\Pi_1(R)$. Setting $R = \mathcal{H}$, we can view the motivic Drinfel'd associator as the image of the identity map

$$\mathrm{id}_{\mathcal{H}} \in \mathrm{Hom}(\mathcal{H}, \mathcal{H}) \qquad \longrightarrow \qquad Z^{\mathfrak{m}} \in \mathcal{H}\langle\langle e_0, e_1\rangle\rangle \ . \tag{4.10}$$

The usual Drinfel'd associator is the image of the element

$$\mathrm{per} \in \mathrm{Hom}(\mathcal{H}, \mathbb{C}) \qquad \longrightarrow \qquad Z \in \mathbb{C}\langle\langle e_0, e_1\rangle\rangle \ .$$

The unipotent de Rham Drinfel'd associator is the image of the map (4.8):

$$\pi_{\mathfrak{a},\mathfrak{m}+} \in \mathrm{Hom}(\mathcal{H}, \mathcal{A}) \qquad \longrightarrow \qquad Z^{\mathfrak{a}} \in \mathcal{A}\langle\langle e_0, e_1\rangle\rangle \ .$$

4.6. **Decomposition with respect to $\zeta^{\mathfrak{m}}(2)$'s.**

Lemma 4.5. *We have*

$$\zeta^{\mathfrak{m}}(2) = -\frac{(\mathbb{L}^{\mathfrak{m}})^2}{24}$$

Proof. The action of U_{dR} on $\zeta^{\mathfrak{m}}(2)$ is trivial, by [7], §3.2. Since $\zeta^{\mathfrak{m}}(2)$ has weight 2, it is equal to a rational multiple of $(\mathbb{L}^{\mathfrak{m}})^2$. The rational multiple is determined by applying the period map and using Euler's formula $\zeta(2) = \pi^2/6$. □

The analogue of Euler's theorem is false for de Rham periods, since $\zeta^{\mathfrak{dr}}(2) = 0$.

Lemma 4.6. *([7], §2.3) There is a non-canonical isomorphism*

$$\mathcal{H} \cong \mathcal{A} \otimes \mathbb{Q}[\zeta^{\mathfrak{m}}(2)] \tag{4.11}$$

Proof. Since the motive $\mathcal{O}({}_0\Pi_1^{\mathfrak{m}})$ has weights ≥ 0, and is stable under U_{dR}, $\mathcal{H}$ is contained in$\mathcal{P}^{\mathfrak{m},+}$. Futhermore, since the path dch is invariant under complex conjugation we deduce that $\mathcal{H} \subset \mathcal{P}^{\mathfrak{m},+}_{\mathbb{R}}$. By corollary 2.12, there is an injective map

$$\mathcal{H} \longrightarrow \mathcal{P}^{\mathfrak{a}} \otimes \mathbb{Q}[(\mathbb{L}^{\mathfrak{m}})^2] \cong \mathcal{P}^{\mathfrak{a}} \otimes \mathbb{Q}[\zeta^{\mathfrak{m}}(2)] \ . \tag{4.12}$$

which is compatible with the $\mathcal{P}^{\mathfrak{a}}$-coaction. Since, by definition, $\pi_{\mathfrak{a},\mathfrak{m}+}(\mathcal{H}) = \mathcal{A}$, and because $\zeta^{\mathfrak{m}}(2) \in \mathcal{H}$, the image of (4.12) is equal to $\mathcal{A} \otimes \mathbb{Q}[\zeta^{\mathfrak{m}}(2)]$. □

A choice of decomposition (4.11) defines a homomorphism $Z^{\mathfrak{m}}_o : \mathcal{A} \to \mathcal{H}$, and via the augmentation on $\mathcal{P}^{\mathfrak{a}}$, a homomorphism $\gamma^{\mathfrak{m}} : \mathcal{H} \to \mathbb{Q}[\zeta^{\mathfrak{m}}(2)]$.

Corollary 4.7. *There exist elements $\gamma^{\mathfrak{m}} \in \mathbb{H}(\mathbb{Q}[\zeta^{\mathfrak{m}}(2)])$, and $Z^{\mathfrak{m}}_o \in \mathbb{A}(\mathcal{H})$ such that*

$$Z^{\mathfrak{m}} = Z^{\mathfrak{m}}_o \circ \gamma^{\mathfrak{m}} \ . \tag{4.13}$$

Proof. The map $\mathcal{H} \to \mathcal{A} \otimes \mathbb{Q}[\zeta^{\mathfrak{m}}(2)] \to \mathcal{H}$ is the identity, where the second map is $\mu(Z^{\mathfrak{m}}_o \otimes \mathrm{id})$ and μ denotes multiplication. This implies that $\mathrm{id}_{\mathcal{H}} = \mu(Z^{\mathfrak{m}}_o \otimes \gamma^{\mathfrak{m}})\Delta$, i.e., $\mathrm{id}_{\mathcal{H}}$ is the convolution product of $Z^{\mathfrak{m}}_o$ and $\gamma^{\mathfrak{m}}$. This is exactly (4.13), by (4.10). □

5. A class of multiple zeta values (elementary version)

The class of single-valued multiple zeta values is constructed in this section in a completely 'elementary' way, i.e., with no reference to motivic periods.

5.1. **Deligne's canonical associator.** Consider the continuous antilinear map

$$\begin{array}{rccc} (5.1) & \sigma : \mathbb{C}\langle\langle e_0, e_1\rangle\rangle & \longrightarrow & \mathbb{C}\langle\langle e_0, e_1\rangle\rangle \\ & \sigma(e_i) & \mapsto & -e_i \end{array}$$

which acts by complex conjugation on the coefficients of words. Let $Z \in \mathbb{R}\langle\langle e_0, e_1\rangle\rangle$ denote the Drinfeld associator (4.2).

Lemma 5.1. *There exists a unique element* $W \in \mathbb{R}\langle\langle e_0, e_1\rangle\rangle$ *such that*

$$W \circ {}^{\sigma}Z = Z \ . \tag{5.2}$$

Proof. By §4.2, the Ihara action is transitive and faithful. The equation (5.2) can be solved recursively using (4.5) and the comments which follow. □

The series W is Deligne's associator. We show in §6.1 that it is indeed an associator.

5.2. **Single-valued multiple polylogarithms.** We briefly recall the construction given in [8]. See §6.3 for a more conceptual derivation. The conventions for iterated integrals will be switched relative to [8] in order to remain compatible with the above. The generating series of multiple polylogarithms is

$$L_{e_0,e_1}(z) = \sum_{w\in\{e_0,e_1\}^\times} L_w(z) w \ ,$$

and is defined to be the unique solution to the K-Z equation

$$\frac{d}{dz} L_{e_0,e_1}(z) = L(z)\Big(\frac{e_0}{z} + \frac{e_1}{1-z}\Big)$$

which is equal to $h(z)\exp(e_0 \log(z))$ near the origin, where $h(z)$ is a holomorphic function at 0, where it takes the value 1.

Definition 5.2. There is a unique element $e_1' \in {}_0\Pi_1(\mathbb{R}) = \mathbb{R}\langle\langle e_0, e_1\rangle\rangle$ which satisfies the fixed-point equation:

$$Z(-e_0, -e_1')e_1' Z(-e_0, -e_1')^{-1} = Z(e_0, e_1) e_1 Z(e_0, e_1)^{-1} \ . \tag{5.3}$$

One can easily show that (5.3) can be solved recursively in the weight, and so e_1' does indeed exist and is unique [8].

The generating series of single-valued multiple polylogarithms was defined by

$$\mathcal{L}(z) = \widetilde{L}_{e_0,e_1'}(\overline{z}) L_{e_0,e_1}(z) \ , \tag{5.4}$$

where $\sim$ denotes reversal of words. Since the antipode in the Hopf algebra $\mathbb{C}\langle\langle e_0, e_1\rangle\rangle$ is given by $e_{i_1}\dots e_{i_n} \mapsto (-1)^n e_{i_n}\dots e_{i_1}$, and since $L(z)$ is group-like, we have

$$L_{e_0,e_1}(z)^{-1} = \widetilde{L}_{-e_0,-e_1}(z)$$

and therefore we can rewrite (5.4) as

$$\mathcal{L}(z) = (L_{-e_0,-e_1'}(\overline{z}))^{-1} L_{e_0,e_1}(z) \ ,$$

In [8] it was shown that the coefficients $\mathcal{L}_w(z)$ of w in the generating series $\mathcal{L}(z)$ are single-valued functions of z, are linearly independent over $\mathbb{C}$, and satisfy the same shuffle and differential equations (with respect to $\frac{\partial}{\partial z}$) as $L_w(z)$. The last two properties are obvious from (5.4). Their values at one are given by

$$\mathcal{L}(1) = (Z(-e_0, -e_1'))^{-1} Z(e_0, e_1) \ . \tag{5.5}$$

5.3. **Values of single-valued multiple polylogarithms at 1.** The values of single-valued multiple polylogarithms at 1 are exactly the coefficients of Deligne's associator.

Lemma 5.3. *Equation* (5.3) *has the unique solution* $e_1' = We_1W^{-1}$.

Proof. By the formula for the Ihara action (4.4), we have

$$W \circ {}^{\sigma}Z(e_0, e_1) = {}^{\sigma}Z(e_0, We_1W^{-1})W \tag{5.6}$$

Let $e_1' = We_1W^{-1}$, and write $Z' = Z(e_0, e_1')$, and ${}^{\sigma}Z' = Z(-e_0, -e_1')$. We have

$${}^{\sigma}Z' \overset{(5.6)}{=} (W \circ {}^{\sigma}Z)W^{-1} \overset{(5.2)}{=} ZW^{-1} \tag{5.7}$$

which implies that ${}^{\sigma}Z' e_1' \, {}^{\sigma}(Z')^{-1} = ZW^{-1}(We_1W^{-1})WZ^{-1} = Ze_1Z^{-1}$.

For the uniqueness, any solution to (5.3) is of the form $e_1' = Ae_1A^{-1}$ for some series A with leading coefficient 1, and (5.3) is just $(A \circ {}^{\sigma}Z)e_1(A \circ {}^{\sigma}Z)^{-1} = Ze_1Z^{-1}$. This readily implies that $A \circ {}^{\sigma}Z = Z$ and so $A = W$ by (5.2). □

By equation (5.5), $\mathcal{L}(1)$ is $({}^{\sigma}Z')^{-1}Z$, which is exactly W by (5.7).

Corollary 5.4. $\mathcal{L}(1) = W$.

6. The single-valued associator (motivic version)

6.1. **Single-valued motivic multiple zeta values.** Recall that $Z^{\mathfrak{m}} \in \mathbb{H}(\mathcal{H})$ is the motivic Drinfel'd associator (4.10), and that the action of σ (definition 3.1) on the ring $\mathcal{H}$ of coefficients of $\mathbb{H}(\mathcal{H})$ is denoted by ${}^{\sigma}$.

Lemma 6.1. *There exists a unique* $W^{\mathfrak{m}} \in \mathbb{A}(\mathcal{H})$ *such that*

$$W^{\mathfrak{m}} \circ {}^{\sigma}Z^{\mathfrak{m}} = Z^{\mathfrak{m}} \ . \tag{6.1}$$

Proof. Using a decomposition $Z^{\mathfrak{m}} = Z_o^{\mathfrak{m}} \circ \gamma^{\mathfrak{m}}$ (4.13), set $W^{\mathfrak{m}} = Z_o^{\mathfrak{m}} \circ ({}^{\sigma}Z_o^{\mathfrak{m}})^{\circ -1}$, where the inversion and multiplication take place in the group $\mathbb{A}(\mathcal{H})$. Since σ acts trivially on the coefficients of $\gamma^{\mathfrak{m}}$, it is independent of the chosen decomposition: replacing $(Z_o^{\mathfrak{m}}, \gamma^{\mathfrak{m}})$ with $(Z_o^{\mathfrak{m}} \circ h, h^{\circ -1} \circ \gamma^{\mathfrak{m}})$, where ${}^{\sigma}h = h$, gives rise to the same element.

Then $W^{\mathfrak{m}} \circ {}^{\sigma}Z_o^{\mathfrak{m}} = Z_o^{\mathfrak{m}}$, which implies (6.1). □

Via $\mathbb{A}(\mathcal{H}) = \mathrm{Hom}(\mathcal{A}, \mathcal{H})$, we view $W^{\mathfrak{m}}$ as an algebra morphism

$$W^{\mathfrak{m}} : \mathcal{A} \longrightarrow \mathcal{H}$$

Composing with $\mathbb{Q}\langle e^0, e^1 \rangle \to \mathcal{H} \to \mathcal{A}$ gives a map we denote by $W^{\mathfrak{m}} : \mathbb{Q}\langle e^0, e^1 \rangle \to \mathcal{H}$.

Definition 6.2. For every word $w \in \{e^0, e^1\}$, define the single-valued motivic multiple zeta value to be the image of w under the map $W^{\mathfrak{m}}$. Denote it by

$$\zeta^{\mathfrak{m}}_{\mathrm{sv}}(w) \in \mathcal{H} \ .$$

Let $\mathcal{H}^{\mathrm{sv}} \subset \mathcal{H}$ be the algebra spanned by the $\zeta^{\mathfrak{m}}_{\mathrm{sv}}(w)$.

The fact that the map $\mathbb{Q}\langle e^0, e^1 \rangle \to \mathcal{H}^{\mathrm{sv}}$ factors through $\mathcal{A}$ means the following.

Corollary 6.3. *The elements* $\zeta^{\mathfrak{m}}_{\mathrm{sv}}(w)$ *satisfy all the motivic relations between motivic multiple zeta values, together with the relation*

$$\zeta^{\mathfrak{m}}_{\mathrm{sv}}(2) = 0 \ .$$

In particular, the $\zeta^{\mathfrak{m}}_{\mathrm{sv}}(w)$ satisfy the usual double shuffle equations and associator relations. Note, however, that the map $\mathcal{A} \to \mathcal{H}^{\mathrm{sv}}$ is not injective, so the elements $\zeta^{\mathfrak{m}}_{\mathrm{sv}}$ satisfy many more relations than their non single-valued counterparts.

We can write the element $W^{\mathfrak{m}}$ as a generating series:

$$W^{\mathfrak{m}} = \sum_{w} \zeta^{\mathfrak{m}}_{\mathrm{sv}}(w)\, w$$

Its period $\mathrm{per}(W^{\mathfrak{m}})$ is precisely W defined in (5.2), which shows that W is an associator. Therefore, by corollary 5.4, the period of $\zeta^{\mathfrak{m}}_{\mathrm{sv}}(w)$ is given by the value at 1 of the corresponding single-valued multiple polylogarithm:

$$\mathrm{per}(\zeta^{\mathfrak{m}}_{\mathrm{sv}}(w)) = \mathcal{L}_w(1)\ . \tag{6.2}$$

6.2. **Structure of $\mathcal{H}^{\mathrm{sv}}$.** Let $\mathcal{A}^{\mathrm{sv}} \subset \mathcal{A}$ denote the image of $\mathcal{H}^{\mathrm{sv}}$ under the map $\pi_{\mathfrak{a},\mathfrak{m}+}$.

Lemma 6.4. *The map $\pi_{\mathfrak{a},\mathfrak{m}+} : \mathcal{H}^{\mathrm{sv}} \to \mathcal{A}^{\mathrm{sv}}$ is an isomorphism, and $\mathcal{H}^{\mathrm{sv}} \cong \mathcal{A}^{\mathrm{sv}}$ is the image of $\mathcal{A}$ under the homomorphism*

$$\mathrm{sv}_{\mathcal{A}} := \mathrm{id} \circ \sigma^{\circ -1} : \mathcal{A} \longrightarrow \mathcal{A} \tag{6.3}$$

where the multiplication $\circ$ and inverse take place in the group $\mathbb{A}(\mathcal{A})$.

Proof. This follows from lemma 3.8 since $\mathcal{H}^{\mathrm{sv}} \subset \mathcal{P}^{\mathrm{sv}}$. □

Denote the Lie coalgebra of indecomposable elements of $\mathcal{A}$ by

$$\mathcal{L} = \frac{\mathcal{A}_{>0}}{\mathcal{A}_{>0}\mathcal{A}_{>0}}\ .$$

Since $\mathcal{A}$ is a commutative, graded Hopf algebra, it follows from standard facts that $\mathcal{A}$ is isomorphic to the polynomial algebra generated by the elements of $\mathcal{L}$.

Proposition 6.5. *The algebra $\mathcal{H}^{\mathrm{sv}}$ of single-valued motivic multiple zeta values is isomorphic to the polynomial algebra generated by elements $\mathcal{L}^{\mathrm{odd}}$ of $\mathcal{L}$ of odd weight.*

Proof. By lemma 6.4, $\mathcal{H}^{\mathrm{sv}} \cong \mathrm{sv}_{\mathcal{A}}(\mathcal{A})$, where $\mathrm{sv}_{\mathcal{A}} = \mathrm{id} \circ \sigma^{\circ -1}$. Since $\mathrm{sv}_{\mathcal{A}}$ is a homomorphism, it defines a map $\mathrm{sv}_{\mathcal{L}} : \mathcal{L} \to \mathcal{L}$. Writing $\mathrm{sv}_{\mathcal{A}} = \mu(\mathrm{id} \otimes \sigma^{\circ -1})\Delta$, where $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is the coproduct, and μ is the multiplication on $\mathcal{A}$, we see that

$$\mathrm{sv}_{\mathcal{A}} \equiv \mathrm{id} - \sigma \quad \text{(modulo products)}$$

and therefore $\mathrm{sv}_{\mathcal{L}} = 2\,\pi^{\mathrm{odd}}$ where $\pi^{\mathrm{odd}} : \mathcal{L} \to \mathcal{L}^{\mathrm{odd}}$ is the projection onto the part of odd weight. It follows that $\mathcal{A}^{\mathrm{sv}}$ is multiplicatively generated by $\mathcal{L}^{\mathrm{odd}}$. □

6.3. **Single-valued multiple polylogarithms revisited.** We can re-derive the construction of the single-valued multiple polylogarithms of [8] and §5.2 as follows.

Consider $X = \mathbb{P}^1 \backslash \{0, 1, \infty\}$, with the base-points $\{\vec{1}_0, -\vec{1}_1, z\}$ for some $z \in X(\mathbb{C})$. Its de Rham fundamental groupoid consists of a copy of $\mathbb{Q}\langle e_0, e_1\rangle$ for each pair of these base-points. We shall only consider the copies ${}_0\Pi_0, {}_0\Pi_1, {}_0\Pi_z$, corresponding to the canonical de Rham paths between the base-points indicated by their subscripts.

Let *Aut* denote the group of automorphisms of $\pi_1(X, \{\vec{1}_0, -\vec{1}_1, z\})$ which preserves the copy of e_0 in ${}_0\Pi_0$ and e_1 in ${}_0\Pi_1$. Modifying [14], Proposition 5.9 accordingly, the action of *Aut* on the elements ${}_01_1, {}_01_z$ defines an injective map

$$\begin{array}{rcl} Aut & \hookrightarrow & {}_0\Pi_1 \times {}_0\Pi_z \\ a & \mapsto & (a_1, a_z) \end{array} \tag{6.4}$$

where for any $a \in Aut$, we write $a_z = a({}_01_z)$ and $a_1 = a({}_01_1)$. We leave to the reader the verification that this is an isomorphism. Since ${}_0\Pi_z$ is a left ${}_0\Pi_0$-torsor, we immediately deduce a formula for the generalized Ihara action

$$\begin{array}{rcl} Aut \times {}_0\Pi_z & \longrightarrow & {}_0\Pi_z \\ a \circ b & = & \langle a_1 \rangle_0(b)\,.\,a_z \end{array} \tag{6.5}$$

where $\langle a \rangle_0$ denotes the action of $a \in {}_0\Pi_1$ on ${}_0\Pi_0$ ([14], (5.9.4)). Concretely, this gives

$$\begin{array}{rcl} ({}_0\Pi_1 \times {}_0\Pi_z) \times ({}_0\Pi_1 \times {}_0\Pi_z) & \longrightarrow & ({}_0\Pi_1 \times {}_0\Pi_z) \\ (F_1, F_z) \;\circ\; (G_1, G_z) & = & (G_1(e_0, F_1e_1F_1^{-1})F_1, G_z(e_0, F_1e_1F_1^{-1})F_z) \end{array} \tag{6.6}$$

The action of ${}_0\Pi_1 \times {}_0\Pi_z$ on ${}_0\Pi_1$ factors through the usual Ihara action of ${}_0\Pi_1$ on ${}_0\Pi_1$.

Let us fix a path ch_z from $\overrightarrow{1}_0$ to z in $X(\mathbb{C})$. Its de Rham image in ${}_0\Pi_z(\mathbb{C})$ is exactly (some branch of) the generating series of multiple polylogarithms (§5.2)

$$\mathrm{ch}_z^{dR} = L(z) \;\in {}_0\Pi_z(\mathbb{C})$$

By the general single-valued principle, we seek an element $W = (W_1, W_z)$ in the group $Aut(\mathbb{C}) \cong {}_0\Pi_1(\mathbb{C}) \times {}_0\Pi_z(\mathbb{C})$ such that

$$W \circ ({}^{\sigma}Z, {}^{\sigma}L(z)) = (Z, L(z)) \ .$$

It has a solution since ${}_0\Pi_1 \times {}_0\Pi_z$ is a torsor over $Aut \cong {}_0\Pi_1 \times {}_0\Pi_z$. By the formula for the action (6.6), this is equivalent to the pair of equations

$$\begin{array}{rcl} {}^{\sigma}L_{e_0, W_1e_1W_1^{-1}}(z)W_z & = & L(z) \ . \\ {}^{\sigma}Z(e_0, W_1e_1W_1^{-1})W_1 & = & Z \ , \end{array} \tag{6.7}$$

and so $W_1 \circ {}^{\sigma}Z = Z$, and W_1 is equal to the element W defined in (5.2). As in §5.2, write $e_1' = We_1W^{-1}$. Therefore by (6.7) we deduce the following formula for W_z,

$$W_z = L^{-1}_{-e_0, -e_1'}(\overline{z})L(z)$$

It is independent of the choice of path ch_z, and is therefore single-valued. This gives another derivation of the construction in [8].

7. Generators for $\mathcal{H}^{\mathrm{sv}}$ and examples

Up to this point we have used no deep results about the category of mixed Tate motives, nor about the structure of motivic multiple zeta values.

7.1. **Periods of mixed Tate motives.** In [7], it was shown that

$$\mathcal{A} \cong \mathcal{O}(U_{dR}) = \mathcal{P}^{\mathfrak{a}} \ . \tag{7.1}$$

The following proposition, due to Deligne [13], is a more precise statement about periods of mixed Tate motives than the one stated in [7].

Proposition 7.1. [13] *Let $M \in \mathcal{MT}(\mathbb{Z})$ be a mixed Tate motive over $\mathbb{Z}$ with non-negative weights, i.e., $W_{-1}M = 0$. Let $\eta \in (M_{dR})_n$ and $X \in M_B^{\vee}$.*

i). If $c(X) = X$ then the motivic period $[M, \eta, X]^{\mathfrak{m}}$ is a rational linear combination of motivic multiple zeta values of weight n.

ii). If $c(X) = -X$ then the motivic period $[M, \eta, X]^{\mathfrak{m}}$ is a rational linear combination of motivic multiple zeta values of weight $n-1$, multiplied by $\mathbb{L}^{\mathfrak{m}}$.

Proof. By (7.1), and §4.6, $\mathcal{H} \cong \mathcal{P}^{\mathfrak{a}} \otimes \mathbb{Q}[(\mathbb{L}^{\mathfrak{m}})^2] \cong \mathcal{P}^{\mathfrak{m},+}_{\mathbb{R}}$. The result then follows immediately from the definitions of $\mathcal{P}^{\mathfrak{m},+}_{\mathbb{R}}$ and $\mathcal{P}^{\mathfrak{m},+}_{i\mathbb{R}}$, and corollary 2.12. □

The methods of [13] give an equivalent but slightly different proof of corollary 2.12.

7.2. **A model for $\mathcal{H}^{\rm sv}$.** Applying a choice of trivialization (2.32) and a choice of generators for $\mathcal{O}(U_{dR})$ to (7.1) gives a non-canonical isomorphism [7], §2.5:

$$\mathcal{H} \cong \mathcal{U} \otimes \mathbb{Q}[f_2] \tag{7.2}$$

such that the natural map $\pi_{\mathfrak{a},\mathfrak{m}+} : \mathcal{H} \to \mathcal{A}$ induces an isomorphism $U \cong \mathcal{A}$, and where

$$\mathcal{U} = \mathbb{Q}\langle f_3, f_5, f_7, \ldots \rangle$$

is the graded Hopf algebra cogenerated by one generator f_{2n+1} in every odd degree $2n+1 \geq 3$, equipped with the shuffle product and the deconcatenation coproduct

$$\Delta_{dec}(f_{i_1} \ldots f_{i_n}) = \sum_{k=0}^{n} f_{i_1} \ldots f_{i_k} \otimes f_{i_{k+1}} \ldots f_{i_n}$$

The element f_2 corresponds to $(\mathbb{L}^{\mathfrak{m}})^2$ and satisfies $\Delta(f_2) = 1 \otimes f_2$.

The map $\mathrm{sv} : \mathcal{A} \to \mathcal{A}$ defines a homomorphism

$$\begin{array}{rcl} \mathrm{sv} : \mathcal{U} & \longrightarrow & \mathcal{U} \\ w & \mapsto & \displaystyle\sum_{uv=w} u \,ш\, \widetilde{v} \end{array} \tag{7.3}$$

where $\widetilde{\ }$ denotes reversal of words. To see this, note that $\sigma : \mathcal{U} \to \mathcal{U}$ is the map $f_{2n+1} \mapsto -f_{2n+1}$, and the antipode S on $\mathcal{U}$ is given by $w \mapsto \sigma(\widetilde{w})$. By formula (6.3), we have $\mathrm{sv} = \mu(\mathrm{id} \otimes {}^{\sigma}S)\Delta_{dec}$, where μ is multiplication, which immediately gives (7.3).

Since $\pi_{\mathfrak{a},\mathfrak{m}+} : \mathcal{H}^{\rm sv} \cong \mathcal{A}^{\rm sv}$ by lemma 6.4, we conclude that

$$\mathcal{H}^{\rm sv} \cong \mathcal{A}^{\rm sv} \cong \mathcal{U}^{\rm sv}$$

where $\mathcal{U}^{\rm sv}$ is the image of the map (7.3). By way of example,

$$\mathrm{sv}(f_a) = 2f_a \quad , \quad \mathrm{sv}(f_a f_b) = 2(f_a f_b + f_b f_a)$$

$$\mathrm{sv}(f_a f_b f_c) = 2(f_a f_b f_c + f_a f_c f_b + f_c f_a f_b + f_c f_b f_a)$$

where a, b, c are odd integers ≥ 3. In general, we have the formula (3.5), which gives

$$\mathrm{sv}(f_a w f_b) = f_a\, \mathrm{sv}(w f_b) + f_b\, \mathrm{sv}(f_a w)$$

for any word $w \in \{f_{2n+1}\}$, and a, b odd integers ≥ 3. This follows immediately from (7.3), since, via the recursive definition of $ш$, we have

$$f_a u \,ш\, f_b \widetilde{v} = f_a(u \,ш\, f_b \widetilde{v}) + f_b(f_a u \,ш\, \widetilde{v}) \ .$$

7.3. **Hoffman-type generators for $\mathcal{H}^{\rm sv}$.** Let V be a finite ordered set. A Lyndon word in the elements of V is a word which is smaller in the lexicographic ordering than its strict right factors: if $w = uv$, then $w < v$ whenever u, v are non-empty.

In ([7], §8), the following theorem was proved.

Theorem 7.2. *The ring of motivic multiple zeta values $\mathcal{H}$ is generated by the Hoffman-Lyndon elements $\zeta^{\mathfrak{m}}(w)$ where w is a Lyndon word in the alphabet $\{2,3\}$, where $3 < 2$.*

It immediately follows from proposition 6.5 that

Corollary 7.3. *The ring of single-valued motivic multiple zeta values $\mathcal{H}^{\rm sv}$ is generated by the Hoffman-Lyndon elements*

$$\zeta^{\mathfrak{m}}_{\rm sv}(w)$$

where w is a Lyndon word of odd weight in the alphabet $\{2,3\}$, where $3 < 2$.

A Hoffman-Lyndon word of odd weight necessarily has an odd number of 3's. It follows from theorem 7.2 that the Poincaré series of $\mathcal{H}$ is given by

$$\sum_{n\geq 0} \dim \mathcal{H}_n \, t^n = \frac{1}{1-t^2-t^3} \ .$$

The dimensions $\ell_n = \dim \mathcal{L}_n$ of the Lie coalgebra $\mathcal{L}$ are determined by

$$\prod_{n\geq 1} (1-t^n)^{-\ell_n} = \frac{1}{1-t^2-t^3} \ .$$

The numbers ℓ_n can be interpreted either as the number of Lyndon words of weight n in $\{2,3\}$, where $3<2$, or as the number of Lyndon words of weight n in the alphabet $\{f_3 < f_5 < f_7, \ldots, \}$, via the isomorphism (7.2). By proposition 6.5,

Corollary 7.4. *The Poincaré series of* $\mathcal{H}^{\mathrm{sv}}$ *is given by*

$$\sum_{n\geq 0} \dim \mathcal{H}^{\mathrm{sv}}_n \, t^n = \prod_{n \,\mathrm{odd} \geq 1} (1-t^n)^{-\ell_n} \ .$$

7.4. **Examples.** For the convenience of the reader, we list the dimensions of the space of motivic multiple zeta values $\mathcal{H}$ and its version modulo products $\mathcal{L}$:

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\dim \mathcal{L}_N$	0	1	1	0	1	0	1	1	1	1	2	2	3	3	4	5	7	8	11	13
$\dim \mathcal{H}_N$	1	1	1	4	2	2	3	4	5	7	9	12	16	21	28	37	49	65	86	114

Next, their single-valued versions $\mathcal{H}^{\mathrm{sv}}$ and $\mathcal{L}^{\mathrm{sv}}$:

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\dim \mathcal{L}^{\mathrm{sv}}_N$	0	0	1	0	1	0	1	0	1	0	2	0	3	0	4	0	7	0	11	0
$\dim \mathcal{H}^{\mathrm{sv}}_N$	1	0	1	0	1	1	1	1	2	2	3	3	5	5	8	8	13	14	21	23

Note that $\dim_{\mathbb{Q}} \mathcal{H}^{\mathrm{sv}}_N$ happens to equal $\dim_{\mathbb{Q}} \mathcal{L}_{N+2}$ for $1 \leq N \leq 12$, which adds to the large supply of evidence for exercising caution when identifying integer sequences!

Below we list algebra generators for $\mathcal{H}^{\mathrm{sv}}_N$ for $1 \leq N \leq 14$. They were calculated by Oliver Schnetz using [21], which gives a very efficient way to compute (5.2) [22].

N	3	5	7	9	11	13
Generators of $\mathcal{H}_{\mathrm{sv}}$	$\zeta^{\mathfrak{m}}_{\mathrm{sv}}(3)$	$\zeta^{\mathfrak{m}}_{\mathrm{sv}}(5)$	$\zeta^{\mathfrak{m}}_{\mathrm{sv}}(7)$	$\zeta^{\mathfrak{m}}_{\mathrm{sv}}(9)$	$\zeta^{\mathfrak{m}}_{\mathrm{sv}}(11)$, $\zeta^{\mathfrak{m}}_{\mathrm{sv}}(3,5,3)$	$\zeta^{\mathfrak{m}}_{\mathrm{sv}}(13)$, $\zeta^{\mathfrak{m}}_{\mathrm{sv}}(5,3,5)$, $\zeta^{\mathfrak{m}}_{\mathrm{sv}}(3,7,3)$

Here, $\zeta^{\mathfrak{m}}_{\mathrm{sv}}(2n+1) = 2\,\zeta^{\mathfrak{m}}(2n+1)$ for all $n \geq 1$, and

$$\begin{aligned}
\zeta^{\mathfrak{m}}_{\mathrm{sv}}(3,5,3) &= 2\zeta^{\mathfrak{m}}(3,5,3) - 2\zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(3,5) - 10\zeta^{\mathfrak{m}}(3)^2\zeta^{\mathfrak{m}}(5) \\
\zeta^{\mathfrak{m}}_{\mathrm{sv}}(5,3,5) &= 2\zeta^{\mathfrak{m}}(5,3,5) - 22\zeta^{\mathfrak{m}}(5)\zeta^{\mathfrak{m}}(3,5) - 120\zeta^{\mathfrak{m}}(5)^2\zeta^{\mathfrak{m}}(3) \\
&\quad -10\zeta^{\mathfrak{m}}(5)\zeta^{\mathfrak{m}}(8) \\
\zeta^{\mathfrak{m}}_{\mathrm{sv}}(3,7,3) &= 2\zeta^{\mathfrak{m}}(3,7,3) - 2\zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(3,7) - 28\zeta^{\mathfrak{m}}(3)^2\zeta^{\mathfrak{m}}(7) \\
&\quad -24\zeta^{\mathfrak{m}}(5)\zeta^{\mathfrak{m}}(3,5) - 144\zeta^{\mathfrak{m}}(5)^2\zeta^{\mathfrak{m}}(3) - 12\zeta^{\mathfrak{m}}(5)\zeta^{\mathfrak{m}}(8) \ .
\end{aligned} \tag{7.4}$$

Remark 7.5. The generating series of unipotent de Rham multiple zetas in depth r is

$$Z_r(x_1,\ldots,x_r) = \sum_{n_1,\ldots,n_r\geq 1} \zeta^{\mathfrak{a}}(n_1,\ldots,n_r) x_1^{n_1-1}\ldots x_r^{n_r-1} \ . \tag{7.5}$$

Let $Z_r^{\rm sv}$ denote the corresponding single-valued version. Then using the methods of [9] we can verify that

$$\begin{array}{rcl} Z_1^{\rm sv} & = & Z_1 - {}^{\sigma}Z_1 \\ Z_2^{\rm sv} & \equiv & Z_2 - {}^{\sigma}Z_2 - 2Z_1 \,\underline{\circ}\, {}^{\sigma}Z_1 \\ Z_3^{\rm sv} & \equiv & Z_3 - {}^{\sigma}Z_3 - 2Z_1 \,\underline{\circ}\, {}^{\sigma}Z_2 - 2Z_1 \,\underline{\circ}\, (Z_1 \,\underline{\circ}\, {}^{\sigma}Z_1) \end{array}$$

where the equivalence sign means modulo $\zeta^{\mathfrak{m}}(2)$ and modulo terms of lower depth, and where $\underline{\circ}$ is the linearized Ihara operator defined in [9], §6. For example:

$$\begin{array}{rcl} f(x_1) \,\underline{\circ}\, g(x_1) & = & f(x_1)g(x_2) + f(x_2 - x_1)\big(g(x_1) - g(x_2)\big) \\ f(x_1) \,\underline{\circ}\, g(x_1, x_2) & = & f(x_1)g(x_2, x_3) + f(x_2 - x_1)(g(x_1, x_3) - g(x_2, x_3)) \\ & + & f(x_3 - x_2)(g(x_1, x_2) - g(x_1, x_3)) \end{array}$$

In particular, this confirms the formulae (7.4) in odd weights, modulo $\zeta^{\mathfrak{m}}(2)$.

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