A Taxonomy of 2d TQFTs

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October 28, 2013

1 Introduction

My goal in this talk is to explain several extensions of ordinary TQFTs in two dimensions and how they're related to one another. The most famous example is the 2d *fully extended* TQFT as defined by Lurie [Lur09b], but I'll also discuss some weaker objects from a few perspectives. An important tool is the 2d version of Lurie's *cobordism hypothesis*, which states that a framed 2d fully extended TQFT is completely determined by the object it assigns to the point, which is a fully dualisable object in the target 2-category. I'll discuss the kinds of objects – satisfying weaker hypotheses than full dualisability – which classify other types of 2d TQFT. The main reference for the talk will be section 4.2 of [Lur09b].

The main intermediate type of TQFT which we'll discuss is classified by a *Calabi-Yau* object in the target 2-category. We might call the theory this object classifies a *non-compact* or $1\frac{1}{2}$-*extended* theory. We'll also discuss how such theories are related to the *open* and *open-closed* theories described by Costello in [Cos07].

As well as Lurie's paper, there are details descriptions of extended 2d TQFTs in chapter two of the paper [BZN09] by Ben-Zvi and Nadler, and the thesis [Dav11] of Davidovitch.

2 Ordinary TFTs and Extended TFTs

We recall the following definition of an ordinary n-dimensional TQFT, following the notation of [Lur09b]:

Definition 2.1. A *framed* n-dimensional TQFT taking values in a symmetric monoidal category $\mathcal{C}$ is a symmetric monoidal functor

$$Z\colon \mathrm{Cob}^{\mathrm{fr}}(n) \to \mathcal{C}$$

where $\mathrm{Cob}^{\mathrm{fr}}(n)$ is the category of framed n-dimensional bordisms between closed $(n-1)$-manifolds with monoidal structure given by disjoint union. We similarly define *oriented* and *unoriented* TQFTs as functors from the appropriate bordism categories.

In most familiar examples, the target category $\mathcal{C}$ is a linear category, such as the category of vector spaces or chain complexes over a field, or a category of topological vector spaces with a sufficiently well-behaved tensor product. It will be useful, for now, to keep the category $\mathcal{C}$ arbitrary.

We'll describe some ways, in the case where $n = 2$, to extend this definition to one that assigns some meaningful data not only to closed 1-manifolds and 2d bordisms between them, but also to certain 1-manifolds *with boundary* and bordisms between *them*. An extended 2d theory is a particularly good way of doing this, with the cobordism hypothesis giving us a concrete way of constructing examples explicitly.

In order to describe extended theories we define a *2-category* of extended 2d bordisms. It will be important to work with the appropriate homotopical notion of 2-category: an $(\infty, 2)$-category, where one requires the 2-category axioms to hold only up to homotopy, but keeps track of the homotopies. In this talk we'll keep the definitions fairly imprecise, but one can make everything precise using one of several equivalent theories of higher categories, for instance the theory of complete n-fold Segal spaces. Lurie gives a construction of the bordism categories in this language in section 2.2 of [Lur09b], and another construction for $n = 2$ (as a bicategory) is given by Schommer-Pries in [SP11].

Definition 2.2. We define an $(\infty, 2)$-category $\mathrm{Bord}^{\mathrm{fr}}(n)$ as follows:

- **Objects** are framed 0-manifolds.
- **Morphisms** $X \to Y$ are framed 1-manifolds with boundary $X \sqcup \overline{Y}$, where $\overline{Y}$ is the manifold Y with opposite framing.
- **2-Morphisms** $f \Rightarrow g$ are framed 2-manifolds with corners, with boundary homotopic to the fibre product of f and g along their common boundary.
- **3-Morphisms** are framing preserving diffeomorphisms between 2-manifolds fixing the boundary.
- **4-Morphisms and higher** are higher isotopies between diffeomorphisms.

The category admits a symmetric monoidal structure via disjoint union.

Let's be more precise about what this means. By a *framing* we always mean a trivialisation of the stable tangent bundle up to isomorphism.

- 0-manifolds are disjoint unions of points. The point admits two framings, corresponding to the two components of $GL_n(\mathbb{R})$. We denote the two framed points by $+$ and $-$.
- 1-manifolds are disjoint unions of intervals and circles. The interval as a bordism $+ \to +$ admits an infinite family of framings indexed by $\mathbb{Z}$, where the framing "twists" n times. The circle also admits a $\mathbb{Z}$ worth of framings. Concretely these arise by gluing together two intervals, one with the trivial framing and one with a framing with n twists. Alternatively we can see this intrinsically; if we fix the framing ϕ_0 induced from the standard embedding $S^1 \to \mathbb{R}^2$, and choose any other 2-framing

  $$\phi_x \colon T_x S^1 \times \mathbb{R} \to \mathbb{R}^2,$$

  chosen to coincide with ϕ_0 at a basepoint, then $(\phi_0)_x^{-1}\phi_x$ describes a loop in $GL_2(\mathbb{R})$, i.e. an element of $\Omega GL_2(\mathbb{R}) \cong \mathbb{Z}$. The standard framing coming from an orientation on the circle corresponds to $2 \in \mathbb{Z}$. We'll denote the circle with framing corresponding to $n \in \mathbb{Z}$ by S^1_n.
- There are a few noteworthy framed 2-bordisms which we should comment on. Firstly, the pair of pants gives several framed bordisms: for any integers m and n the pairs of pants induces a framed bordism $S^1_n \sqcup S^1_m \to S^1_{n+m}$. There is a unique pair of pants that bounds a disc, namely S^1_0, hence the disc describes a bordism $S^1_0 \to \varnothing$. Finally, the so-called *whistle* describes a bordism from a circle to an interval, and can be described for any framing on the circle and interval (with the same number of twists).

  Any framed 2-bordism can be glued together out of these constituents and their reversed versions.

Now, let $\mathcal{C}$ be a symmetric monoidal $(\infty, 2)$-category.

Definition 2.3. A *fully extended* framed 2d TQFT taking values in $\mathcal{C}$ is a symmetric monoidal functor of $(\infty, 2)$-categories

$$Z\colon \mathrm{Bord}^{\mathrm{fr}}(2) \to \mathcal{C}.$$

Remark 2.4. A weaker notion than a fully extended TQFT is a *homological field theory*. From the 2-category $\mathrm{Bord}^{\mathrm{fr}}(2)$ one can produce a category $H_*(\mathrm{Bord}^{\mathrm{fr}}(2); \mathbb{Q})$ enriched in abelian groups by taking homology. A *homological conformal field theory* is a functor of Ab-enriched categories

$$H_*(\mathrm{Bord}^{\mathrm{fr}}(2); \mathbb{Q}) \to \mathrm{Ch}(k),$$

where $\mathrm{Ch}(k)$ is the category of chain complexes over a field k. If a chain complex valued extended TQFT factors through the homology, then it induces a homological conformal field theory.

There are two main families of target categories which are worth keeping in mind. The first directly extends the targets for ordinary TQFTs I mentioned above: we might take the 2-category of linear categories over a field, or more homotopically the $(\infty, 2)$-category of categories enriched in chain complexes. Alternatively, we could use *Morita categories*, which make sense in any context where we can talk about algebra and module objects.

Definition 2.5. The *Morita category* in a symmetric monoidal $(\infty, 1)$-category [1] $\mathcal{C}$ is the $(\infty, 2)$-category with

- **Objects** given by algebra objects in $\mathcal{C}$,
- **Morphisms** from A to B given by A, B-bimodule objects, with composition given by tensor product of modules over the appropriate algebra, and
- **2-Morphisms and higher** from M to N given by the (classifying space of) bimodule maps.

The tensor product in $\mathcal{C}$ makes this a symmetric monoidal $(\infty, 2)$-category.

If I just refer to a Morita category over a field k, I mean the Morita category in the homotopy category of chain complexes over k, with the derived tensor product.

In order to state the cobordism hypothesis, let's talk about full dualisability in $(\infty, 2)$-categories. From an $(\infty, 2)$-category we can produce an ordinary (strict) 2-category by considering 2-morphisms up to equivalence. This is all we'll need to talk about dualisability, so for the rest of this section we can take $\mathcal{C}$ to be a symmetric monoidal strict 2-category

Definition 2.6. An object $X \in \mathrm{Ob}\,\mathcal{C}$ is *dualisable* if there is another object $X^\vee \in \mathrm{Ob}\,\mathcal{C}$ and morphisms

$$\mathrm{ev}\colon X \otimes X^\vee \to 1$$
$$\mathrm{coev}\colon 1 \to X \otimes X^\vee$$

where 1 is the monoidal unit, such that $(\mathrm{ev} \otimes \mathrm{id}_X) \circ (\mathrm{id}_X \otimes \mathrm{coev}) = \mathrm{id}_X$ and $(\mathrm{id}_{X^\vee} \otimes \mathrm{ev}) \circ (\mathrm{coev} \otimes \mathrm{id}_{X^\vee}) = \mathrm{id}_{X^\vee}$.

Example 2.7. In a Morita category, every object A is dualisable with dual $A^\vee = A^{\mathrm{op}}$, the *opposite algebra*.

Definition 2.8. A morphism $f\colon X \to Y$ in $\mathcal{C}$ is *left dualisable* if there exists a morphism $f^\vee\colon Y \to X$ and 2-morphisms

$$\mathrm{ev}\colon f \circ g \Rightarrow \mathrm{id}_Y$$
$$\mathrm{coev}\colon \mathrm{id}_X \Rightarrow g \circ f$$

such that $(\mathrm{ev} \cdot \mathrm{id}_f) \circ (\mathrm{id}_g \cdot \mathrm{coev}) = \mathrm{id}_f$ and $(\mathrm{id}_g \cdot \mathrm{ev}) \circ (\mathrm{coev} \cdot \mathrm{id}_g) = \mathrm{id}_g$. *Right* dualisability is defined analogously, with $\mathrm{ev}\colon g \circ f \Rightarrow \mathrm{id}_X$ and $\mathrm{coev}\colon \mathrm{id}_Y \Rightarrow f \circ g$.

Remark 2.9. The first definition is actually a special case of the second. If $\mathcal{C}$ is a monoidal category, one can form a 2-category $B\mathcal{C}$ with one object, morphisms the objects of $\mathcal{C}$, 2-morphisms the morphisms of $\mathcal{C}$, and horizontal and vertical composition given by the tensor product and usual composition respectively. Dualisability of morphisms in $B\mathcal{C}$ corresponds to dualisability of objects in $\mathcal{C}$ (and *symmetric* monoidal means left and right duals coincide).

Definition 2.10. An object X of $\mathcal{C}$ is *fully dualisable* if it is dualisable, and every endomorphism $f\colon X \to X$ admits both left and right duals. In general, the fully dualisable objects of $\mathcal{C}$ and dualisable morphisms generate a subcategory, denoted $\mathcal{C}^{\mathrm{fd}}$.

There is a neat characterisation of fully dualisable objects in this situation (Proposition 4.2.3 of [Lur09b]):

[1] At least, any sufficiently nice such category, where $\mathcal{C}$ admits and the tensor product preserves certain colimits.

Proposition 2.11. An object X of $\mathcal{C}$ is fully dualisable if and only if it is dualisable, and the evaluation morphism admits both left and right duals.

We can now state the cobordism hypothesis in dimension 2. For $\mathcal{C}$ an $(\infty, 2)$-category, let $\widetilde{\mathcal{C}}$ denote its *maximal ∞-groupoid*, i.e. the $(\infty, 0)$-subcategory obtained from $\mathcal{C}$ by discarding all non-invertible 1- and 2-morphisms.

Theorem 2.12. There is an equivalence of categories

$$\mathrm{Hom}^{\otimes}(\mathrm{Bord}^{\mathrm{fr}}(2), \mathcal{C}) \to \widetilde{\mathcal{C}^{\mathrm{fd}}}$$

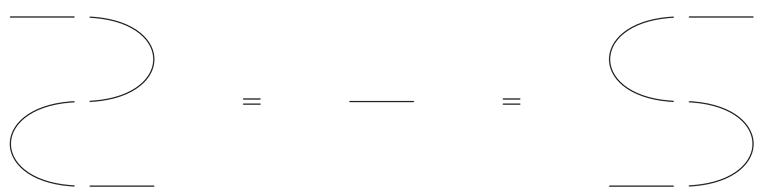
sending an extended TQFT Z to the object $Z(+)$.

I won't prove this, but In section 3 I'll describe a map one way: starting from a fully dualisable object of $\mathcal{C}$ I'll construct a 2d TQFT assigning that object to the point.

3 Constructing Z from $Z(+)$

Let $\mathcal{C}$ be a symmetric monoidal $(\infty, 2)$-category as before. I'll describe how to start from a fully dualisable object $X \in \mathrm{Ob}\, \widetilde{C}$ and produce a framed TQFT, in as concrete a way as possible.

- First, we evaluate the TQFT on objects. To the positive point $Z(+)$ we will assign the object X, and everything else will be determined from this choice.
- The positively framed interval is the identity morphism $+ \to +$, so maps to the identity morphism id_X. However, we can decompose the positively framed interval as a "zig-zag" gluing of two 1-bordisms as follows:



  Applying the functor Z yields an object $Z(-)$, and morphisms $Z(\subset)$ and $Z(\supset)$ satisfying the axioms of 2.6. Duals are unique, so $Z(-) = X^\vee$, $Z(\supset) = \mathrm{ev}_X$ and $Z(\subset) = \mathrm{coev}_X$.
- As Z is a monoidal functor, the TQFT is now determined on all objects: a 0-manifold consisting of n positive points and m negative points is assigned $X^{\otimes n} \otimes (X^\vee)^{\otimes m}$. The empty manifold is assigned the monoidal unit.
- The framed circle S^1_1 with framing coming from an orientation can be decomposed as a union of two framed intervals. It is therefore assigned the object $\mathrm{ev}_X \circ \mathrm{coev}_X \in \mathrm{Mor}(1, 1)$. We can also investigate what the theory assigns to the other circles by investigating what it assigns to intervals with twisted framings. The interval with a single twist in the framing is mapped to an automorphism $S\colon X \to X$. One can check that one can obtain S from ev^R, the right dual to ev by the composite

$$X \xrightarrow{1\otimes \mathrm{ev}^R} X \otimes X \otimes X^\vee \xrightarrow{\mathrm{swap}\otimes 1} X \otimes X \otimes X^\vee \xrightarrow{1\otimes \mathrm{ev}} X\ .$$

In the case where $\mathcal{C}$ is a Morita category, we can compute $Z(S^1)$ more explicitly by understanding the evaluation and coevaluation maps. An algebra A is naturally an A, A-bimodule, indeed, this realises the identity morphism id_A in the Morita category. An A, A-bimodule is the same thing as an $A \otimes A^\vee$ left- or right-module: this realises the evaluation and coevaluation morphisms. Composition is given by tensor product, so the map $\mathrm{ev}_A \circ \mathrm{coev}_A$ is realised by the k, k-bimodule

$$HH_*(A) = A \otimes_{A \otimes A^\vee} A,$$

the *Hochschild homology* of A. Further, we can investigate other framed circles. The circle S^1_0 receives the composite $\mathrm{ev}_A \otimes \mathrm{ev}_A^R$, or equivalently $\mathrm{ev}_A \circ (1 \otimes S) \circ \mathrm{coev}_A$. The right adjoint ev^R is assigned the linear dual A^* as a left $A \otimes A^\vee$-module, therefore the composite $\mathrm{ev}_A \otimes \mathrm{ev}_A^R$ is realised by the k, k-bimodule

$$HH^*(A) = A^* \otimes_{A \otimes A^\vee} A = \mathrm{End}_{A \otimes A^\vee}(A),$$

the *Hochschild cohomology* of A. We observe that is the theory Z assigning A to a point was actually *oriented* then the modules assigned to all the framed circles would coincide, and we'd have that $HH^*(A) \cong HH_*(A)$.

With a bit more work, one can go further and describe the 2-morphisms associated to 2-bordisms. I won't do this, but I'll at least mention that the pair of pants as a framed bordism $S^1_0 \sqcup S^1_0 \to S^1_0$ gives us a multiplication on the Hochschild cohomology, the pair of pants as a framed bordism $S^1_0 \sqcup S^1_1 \to S^1_1$ gives an action of the Hochschild cohomology on the Hochschild homology, and the bordism $S^1_0 \sqcup S^1_0 \to \varnothing$ given by capping a disc on a pair of pants gives a bilinear form on the Hochschild cohomology. These notions all agree with more classical descriptions of the same, for instance the product on cohomology agrees with the multiplication on Ext groups.

4 Calabi-Yau Objects

Suppose that instead of *framed* TQFTs, we were looking at *oriented* TQFTs. The group $SO(2)$ acts orientation preservingly on the circle by rotation, inducing an action of $SO(2)$ on the image of the circle under the functor Z, namely $\mathrm{Mor}(X, X)$. This allows us to make the following definition.

Definition 4.1. An object $X \in \mathrm{Ob}\,\mathcal{C}$ is called *Calabi-Yau* if it is dualisable, coev_X is right dual to ev_X, and the evaluation (counit) map $\eta\colon \mathrm{ev}_X \circ \mathrm{coev}_X \to \mathrm{id}_1$ in $\mathrm{Mor}_{\mathcal{C}}(1, 1)$ exhibiting this duality is $SO(2)$-invariant.

This is strictly weaker than full dualisability (in the oriented setting: $SO(2)$-invariance implies that if an inverse to ev_X exists then it must be given by coev_X. Indeed, without orientability one can compute the left and right inverses to ev_X in terms of coev_X and the Serre automorphism.)

Example 4.2. By way of example, what does this mean in the Morita category? Well, the claim about the right dual boils down to requiring that the Serre automorphism is trivialised, or equivalently that $\mathrm{coev} \cong \mathrm{ev}^R$. In the Morita category we already computed what these modules are: we're requiring that $A \cong A^*$, i.e. *self-duality* up to equivalence. The counit map is the *trace* map

$$HH^*(A) \to k,$$

which we require to be invariant for the circle action on Hochschild cohomology. The example motivating the name fits in to this context. Let $A = \Omega^*_{\mathrm{Dol}}(M)$: the Dolbeault complex of a complex manifold M. If M is *Calabi-Yau* then the self-duality requirement holds by Serre duality. In particular Hochschild homology and cohomology agree. One can show (and we'll discuss this later) that

$$HH_*(\Omega^*_{\mathrm{Dol}}(M)) \cong C^*(LM)$$

where LM is the free loop space of M. In the case where M is Calabi-Yau, one can realise the $SO(2)$-invariant trace map by pulling back under the inclusion of *constant* loops, then integrating using the Calabi-Yau structure.

Example 4.3. Another example justifying the name shows up when $\mathcal{C}$ is the 2-category of linear categories. The category $D^b\,\mathrm{Coh}(X)$ is a Calabi-Yau object if X is a smooth projective Calabi-Yau variety. This is discussed in [Cos07] in the context of A_∞-categories.

Like fully dualisable objects, Calabi-Yau objects also generate a kind of extended 2d TQFT, but with slightly less data. One can no longer construct the *cap*: the disc as a bordism $S^1 \to \varnothing$.

Definition 4.4. The $(\infty, 2)$-category of *non-compact* bordisms $\mathrm{Bord}^{\mathrm{nc}}(2)$ has the same objects and 1-morphisms as the category $\mathrm{Bord}^{\mathrm{or}}(2)$ of oriented bordisms, but a constraint on 2-morphisms: that every component of a 2-bordism must have non-empty incoming boundary.

Theorem 4.5. The map

$$\mathrm{Hom}^{\otimes}(\mathrm{Bord}^{\mathrm{nc}}(2), \mathcal{C}) \to \mathcal{C}^{\mathrm{CY}}$$

sending an extended TQFT Z to the object $Z(+)$ induces a bijection on equivalence classes.

Here $\mathcal{C}^{\mathrm{CY}}$ is the set of Calabi-Yau objects of $\mathcal{C}$. (I'm not quite confident enough to claim an equivalence of categories here.)

We'll sketch a construction of this bijection following Lurie and Costello.

5 Open and Open-Closed Theories

There's another approach to defining 2d topological field theories with boundary conditions, as described by Costello in [Cos07]. We define an $(\infty, 1)$-category $\mathcal{OC}$ of *open-closed bordisms* as follows:

- **Objects** are oriented 1-manifolds with boundary.
- **Morphisms** are oriented 2-bordisms, such that every connected component has non-empty incoming boundary.
- **2-Morphisms and higher** are oriented diffeomorphisms, isotopies etc between bordisms.

There is a full subcategory $\mathcal{O}$ of *open* bordisms consisting of only those objects which are unions of intervals, i.e. every component must have a non-empty boundary. There is also a natural *boundary* functor $\partial \colon \mathcal{OC} \to \mathrm{Bord}_1^{\mathrm{or}}$, sending an object J to its boundary ∂J. This describes an essential surjective functor of symmetric monoidal $(\infty, 1)$-categories.

Remark 5.1. Of course, we could *also* describe an $(\infty, 1)$-category of *closed* bordisms, using only those objects without boundary. This recovers the most classical notion of a bordism category: basically the category Cob_2, with higher morphisms between 2-bordisms given by diffeomorphisms etc as usual.

Remark 5.2. In [Cos07], Costello discusses open-closed theories with a fixed set of *branes* Λ. This means that we associate an additional piece of data to objects: an object is now an oriented 1-manifold I with a function from ∂I to Λ, and a morphism is a bordism compatible with these functions (so we think of the data of the function as assigning a brane to each "free" piece of the boundary of a 2-bordism.

Costello also described *d-twisted* versions of $\mathcal{O}$ and $\mathcal{OC}$. From our description of $\mathcal{OC}$ we can construct a symmetric monoidal *dg-category*, with the same objects as above, and $\mathrm{Hom}(I, J)$ given by the complex of chains on the moduli space of Riemann surfaces whose underlying 2-manifold is a 2-bordism from I to J as above. This can be modified by taking instead chains with coefficients in a local system, specifically the d^{th} tensor power of the determinant local system. This won't play a role in my talk, but Aron will discuss it in more detail in his talk next week.

The category of open-closed bordisms arises naturally from the forgetful functor $\mathrm{Bord}_1^{\mathrm{or}} \to \mathrm{Bord}_2^{\mathrm{nc}}$ via a *Grothendieck construction*. This is realised by the following theorem ([Lur09b] 3.3.28).

Theorem 5.3. For a fixed symmetric monoidal $(\infty, 1)$-category $\mathcal{C}$, there is an equivalence between the following data:

1. Essentially surjective symmetric monoidal functors of $(\infty, 2)$-categories $\mathcal{C} \to \mathcal{D}$.
2. Symmetric monoidal cocartesian fibrations $\mathcal{B} \to \mathcal{C}$.

The map from 1. to 2. can be concretely described. Starting from a functor $F\colon \mathcal{C} \to \mathcal{D}$ we perform a *Grothendieck construction* to produce $\mathcal{B} = \mathrm{Groth}(F)$, defined as the $(\infty, 1)$-category whose objects are pairs (X, η), where $X \in \mathrm{Ob}\,\mathcal{C}$ and $\eta\colon 1 \to F(X)$, and whose morphisms are given by the (classifying spaces of) 2-morphisms in $\mathcal{D}$ from $\eta_1\colon 1 \to F(X_1)$ to $\eta_2\colon 1 \to F(X_2)$. This comes equipped with a forgetful functor to $\mathcal{C}$ which is a symmetric monoidal cocartesian fibration [2].

This equivalence interchanges the natural inclusion functor $\mathrm{Bord}_1^{\mathrm{or}} \to \mathrm{Bord}_2^{\mathrm{nc}}$ and the boundary map $\mathcal{OC} \to \mathrm{Bord}_1^{\mathrm{or}}$. Indeed, performing the Grothendieck construction on this functor we find the category whose objects are oriented 1-manifolds with boundary, and whose morphisms are the 2-morphisms in $\mathrm{Bord}_2^{\mathrm{nc}}$, which gives another description of $\mathcal{OC}$.

Now, let $\mathcal{C}$ be a symmetric monoidal $(\infty, 1)$-category. We can now describe open and open-closed field theories.

Definition 5.4. An *open-closed field theory* with target $\mathcal{C}$ is a symmetric monoidal functor $Z\colon \mathcal{OC} \to \mathcal{C}$ of $(\infty, 1)$-categories. Similarly, an *open field theory* is a symmetric monoidal functor $Z\colon \mathcal{O} \to \mathcal{C}$.

Clearly there is a *restriction* functor from open-closed theories to open theories, given by precomposing with the inclusion $i\colon \mathcal{O} \hookrightarrow \mathcal{OC}$: we denote this functor i^*. More interestingly, it is possible to go the other way, by performing *left Kan extension* along the inclusion, constructing a left adjoint functor $i_!$ to i^*. In general, the Kan extension of a functor $F\colon \mathcal{C} \to X$ along a functor $\Phi\colon \mathcal{C} \to \mathcal{D}$ is a universal lift to $G\colon \mathcal{D} \to X$ and natural transformation $G \circ \Phi \to F$, which can be built as a certain colimit. Lurie describes the analogous theory for $(\infty, 1)$-categories in [Lur09a].

Costello describes this functor by describing a general construction: given a functor $F\colon \mathcal{C} \to \mathcal{D}$ of symmetric monoidal dg-categories he describes a functor $F_! = \mathcal{D} \otimes^L_{\mathcal{C}}\colon \mathcal{C}\text{-mod} \to \mathcal{D}\text{-mod}$ by the usual tensor product of modules applied to a suitable flat resolution of objects, given by a Bar construction. He then gives explicit descriptions of $\mathcal{OC}$ and $\mathcal{O}$ as symmetric monoidal dg-categories, so that TQFTs are described as modules for these categories. The hard step is showing that the open-closed theories thus obtained are *split*: that is, there are natural maps

$$Z(I) \otimes Z(J) \to Z(I \sqcup J),$$

and one has to show that they are equivalences in the target category. Costello does this by constructing explicit models for $\mathcal{O}$ and $\mathcal{OC}$ (quasi-isomorphic symmetric monoidal dg-complexes) and computing the relevant natural maps directly in this category.

Remark 5.5. Realising $\mathcal{OC}$ as the Grothendieck construction applied to the inclusion $\iota\colon\ \mathrm{Bord}_1^{\mathrm{or}} \to \mathrm{Bord}_2^{\mathrm{nc}}$ gives us another description of an open-closed theory, via the equivalence 5.3. Let $Z\colon\ \mathrm{Bord}_2^{\mathrm{nc}} \to \mathcal{C}$ be a non-compact TQFT as in section 4. By precomposing with the inclusion $\iota\colon\ \mathrm{Bord}_1^{\mathrm{or}} \to \mathrm{Bord}_2^{\mathrm{nc}}$ we produce a 1d TQFT $Z_1\colon\ \mathrm{Bord}_1^{\mathrm{or}} \to \mathcal{C}$ which necessarily factors through $\mathcal{C}_1$: the $(\infty, 1)$-category obtained by discarding all non invertible 2-morphisms in $\mathcal{C}$. Applying 5.3 to this, we produce an *open-closed* theory fitting into the commutative diagram

$$\begin{array}{ccccc} \mathcal{OC} & \xrightarrow{\partial} & \mathrm{Bord}_1^{\mathrm{or}} & \xrightarrow{\iota} & \mathrm{Bord}_2^{\mathrm{nc}} \\ \downarrow{\scriptstyle \overline{Z}} & & \downarrow{\scriptstyle Z_1} & & \downarrow{\scriptstyle Z} \\ \overline{\mathcal{C}} & \longrightarrow & \mathcal{C}_1 & \longrightarrow & \mathcal{C} \end{array}.$$

Not all open-closed theories should be expected to arise in this way: the functor $\overline{Z}$ will satisfy a technical condition related to the cocartesian fibration structure of the maps $\partial\colon \mathcal{OC} \to \mathrm{Bord}_1^{\mathrm{or}}$ and $\overline{\mathcal{C}} \to \mathcal{C}_1$. Still, we might expect open-closed theories satisfying such a suitable condition to correspond to Calabi-Yau objects, as in theorem 4.5.

At this point we come back around to Calabi-Yau objects, via the main theorem of [Cos07]. Let $\mathcal{C}$ be a symmetric monoidal $(\infty, 1)$-category (Costello uses the category of chain complexes over a field of characteristic zero).

Theorem 5.6. [Theorem A1 of [Cos07]] There is a natural equivalence between *open* TQFTs with target $\mathcal{C}$ and Calabi-Yau objects in the Morita category internal to $\mathcal{C}$.

[2]Being a cocartesian fibration is a technical condition which I won't dwell on. See Lurie's definition 3.3.3 and 3.3.4 in [Lur09b], and section 2.4.2 of [Lur09a].

As usual, it's instructive to understand how to recover the Calabi-Yau algebra from an open TQFT. We do so using the left Kan extension discussed above, which associated to an open TQFT Z an open-closed TQFT Z' that we can then evaluate on S^1. The pair of pants makes $Z'(S^1)$ into an algebra, so an object in the Morita category. This object is naturally self-dual, and the cap $Z'(S^1) \to k$ provides a Calabi-Yau structure.

Now, we can explain why theorem 4.5 should be true, using the theory of open and open-closed TQFTs (and, again, following the exposition in [Lur09b]). Start with a target $(\infty, 2)$-category $\mathcal{C}$ and associated $(\infty, 1)$-category $\mathcal{C}_1$. Choose a Calabi-Yau object in $\mathcal{C}$. The cobordism hypothesis *in dimension 1* associates a one-dimensional oriented field theory to A, call it $Z_1\colon \mathrm{Bord}_1^{\mathrm{or}} \to \mathcal{C}_1$. Then precomposition with $\partial\colon \mathcal{OC} \to \mathrm{Bord}_1^{\mathrm{or}}$ yields an open-closed theory.

In order to produce an extended (non-compact) 2d theory, we again use 5.3. If we can lift this theory along the cocartesian fibration $\overline{\mathcal{C}} \to \mathcal{C}_1$ associated to the embedding $\mathcal{C}_1 \to \mathcal{C}$ then the equivalence of that theorem gives us the desired extended 2d theory. We'll try to construct this lift using Costello's theorem 5.6. The open-closed theory Z_1 restricts to an open theory, and thus by 5.6 corresponds to a Calabi-Yau algebra in the Morita category $\mathrm{Mor}_{\mathcal{C}_1}$. We know exactly what this algebra is: it's the object assigned by Z_1 to the circle, namely the endomorphism algebra $\mathrm{End}(A) = A \otimes A^\vee$ with Calabi-Yau structure coming from the evaluation map: the induced map $HH_*(\mathrm{End}(A)) \to HH_*(k) = k$.

Using Costello's theorem again, lifting to a theory with target $\overline{\mathcal{C}}$ corresponds to lifting $\mathrm{End}(A)$ to a Calabi-Yau object in this category. That is, we must put a Calabi-Yau structure on the object $\mathrm{coev}\colon k \to A \otimes A^\vee$ in $\overline{\mathcal{C}}$ (using the Grothendieck construction). To do so, we'd like an ($SO(2)$-invariant) morphism in $\overline{\mathcal{C}}$ from $\mathrm{coev}\colon k \to A \otimes A^\vee$ to $\mathrm{id}\colon k \to k$ yielding an induced map on Hochschild homology. Post-composition with the evaluation map gives such a morphism provided we have an $SO(2)$-invariant *2-morphism* from $\mathrm{ev} \circ \mathrm{coev}$ to id_k, which is precisely a Calabi-Yau structure *on A itself*.

This shows one direction of the theorem: to a Calabi-Yau object we have associated a non-compact extended 2d TQFT. All that remains for the other direction is to show that there is an essentially unique way of extending this open theory to an open-closed theory. Lurie says that this follows from a "relative" version of Costello's theorem, but I haven't tried to work out the details.

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