The Affine Grassmannian

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1 Introduction

The *affine Grassmannian* is an important object that comes up when one studies moduli spaces of the form $\mathrm{Bun}_G(X)$, where X is an algebraic curve and G is an algebraic group. There is a sense in which it describes the *local* geometry of such moduli spaces. I'll describe the affine Grassmannian as a moduli space, and construct it concretely for some concrete groups. References, including the construction of perverse sheaves on affine Grassmannians, include the papers [4], [1], the book [5] and set of seminar notes [3] by Gaitsgory.

2 Informal Description

Let's first try to understand what the affine Grassmannian *should* be. We work over a field k, which we'll later specialise to the complex numbers, but for now it could be anything. Let G be a (connected reductive) algebraic group over k. The affine Grassmannian describes

$$\frac{\left\{\text{trivialisable } G\text{-bundles on the formal punctured disc}\right\}}{\left\{\text{trivialisable } G\text{-bundles on the formal disc}\right\}}.$$

This object comes up naturally when one is studying G-bundles on a curve. For instance, if $k = \mathbb{C}$, any G-bundle over a Riemann surface becomes trivialisable after puncturing the curve finitely many times. The bundle can then be described by taking the punctured curve and a small disc around each puncture as a trivialising cover, and specifying a transition function on each punctured disc. However these small discs can be made arbitrarily small, so only germs of such transition functions are required. These are precisely trivialisable G-bundles on the formal punctured disc, and the affine Grassmannian describes such transition functions modulo change of trivialisation on the formal disc (i.e. trivialisable bundles that extend across the puncture).

Let $\mathcal{K} = k((t))$, and let $\mathcal{O} = k[[t]]$. The *formal disc* and *formal punctured disc* over k are the affine schemes $\mathbb{D} = \operatorname{Spec} \mathcal{O}$ and $\mathbb{D}^\times = \operatorname{Spec} \mathcal{K}$ respectively. How do we describe the moduli of G-bundles over these schemes? I'll give an informal description first, then later say something more precise. Informally, the set of *trivialisable* G-bundles on an affine scheme $\operatorname{Spec} A$ (where A is a k-algebra) is the same as the set of trivialisations of a trivial bundle, i.e. the set of G-valued functions

$$\mathrm{Maps}(\operatorname{Spec} A, G) = G(A),$$

the set of A-points of G. The set of trivialisable G-bundles on $\mathbb{D}^\times$ is precisely the set of G-bundles that extend across the puncture, so this tells us that our affine Grassmannian should be modelled on the quotient

$$\mathcal{G}r_G = G(\mathcal{K})/G(\mathcal{O}).$$

Another way of describing this is as the group of *formal loops* in G modulo the subgroup of *formal arcs*, i.e. formal loops that one can "fill in" to formal discs.

3 Some Algebraic Geometry

Since this talk is about the *geometry* of the affine Grassmannian, we'll certainly need to describe some structure on it beyond that of a set. This will require a bit of care, because the affine Grassmannian is *not* a scheme, but something more general.

In order to construct $\mathcal{G}r_G$ as an algebro-geometric object (one arising from a moduli problem), we'll describe the functor that it's supposed to represent:

$$\mathcal{G}r_G \colon (\text{Commutative } k\text{-algebras}) \to (\text{Sets})$$

sending an algebra A to the set of pairs (E, γ), where E is an A-family of G-bundles over $\mathbb{D}$, and γ is an isomorphism from $E|_{\mathbb{D}^\times}$ to the trivial A family of G-bundles over $\mathbb{D}^\times$. Notice that if $A = \operatorname{Spec} k$ that this is just a G-bundle on $\mathbb{D}$ and a trivialisation away from the origin. We'll see below that this is the same data as we described before (as a set).

To make sense of this, I'd should say what exactly an A-family of G-bundles is, but to keep things reasonably simple, I'll just describe an A-family of *vector* bundles over $\mathbb{D}$ and $\mathbb{D}^\times$. Recall the notion of a vector bundle from commutative algebra:

Definition 3.1. The *rank* of a finitely generated projective A-module M at a point $p \in \operatorname{Spec} A$ is the rank of the free A_p module M_p. We say M is rank n if $\operatorname{rk}_M(p) = n$ for all points p.

Definition 3.2.
- An A-family of rank n vector bundles on $\mathbb{D}$ is a rank n finitely generated projective module over the ring $A[[t]]$.
- Similarly, an A-family of rank n vector bundles on $\mathbb{D}^\times$ is a rank n finitely generated projective module over the ring $A((t))$.

One can modify this definition directly to work for groups other than GL_n, but more generally and indirectly one can define G-bundles in the following way:

Definition 3.3. An A-family of G-bundles on $\mathbb{D}$ is an exact tensor functor $\operatorname{Rep}(G) \to \operatorname{Vect}(\mathbb{D})$, where $\operatorname{Vect}_A(\mathbb{D})$ is the tensor category of A-families of vector bundles (of any rank) as above. Similarly for $\mathbb{D}^\times$.

That is, we define the associated vector bundle in every representation of G.

Now we must investigate what kind of object might represent the affine Grassmannian functor. The first thing to note is that it is not represented by a scheme. Indeed, consider the example of $G = \mathbb{G}_a$. Even $\mathbb{G}_a(\mathcal{K}) = k((t))$ cannot be represented by a scheme: it sits inside the scheme $k[[t, t^{-1}]] \cong \prod_{i=1}^{\infty} \mathbb{A}^1$, but the condition that only finitely many negative coefficients are non-zero cannot be expressed by polynomials.

However, it is very close to being representable by a scheme. Specifically, we have the following:

Proposition 3.4. There are a sequence of finite-type projective schemes $\mathcal{G}r_G^i$ for $i \in \mathbb{N}$, and closed immersions $\mathcal{G}r_G^i \hookrightarrow \mathcal{G}r_G^{i+1}$ such that $Gr_G = \varinjlim \mathcal{G}r_G^i$, i.e.

$$\mathcal{G}r_G(A) \cong \varinjlim \operatorname{Hom}(\operatorname{Spec} A, \mathcal{G}r_G^i).$$

A functor which is isomorphic to a direct limit of schemes of this form is called a (strict) *ind-scheme*, and is almost as nice to work with as a scheme (some details on the theory of ind-schemes can be found in [3] and the appendix to [2]). In particular we can define categories of (e.g. constructible) sheaves on an ind-scheme, where every object is in fact supported on a finite subscheme. Furthermore we can do things like investigate the topology of the $\mathbb{C}$-points of an ind-scheme over $\mathbb{C}$. We'll do this in some examples later.

3.1 Construction

So let's describe the schemes $\mathcal{G}r^i_G$. In doing so we'll start to make a connection with the description of the k-points of $\mathcal{G}r_G$ as $G(\mathcal{K})/G(\mathcal{O})$ from the previous section. We'll concentrate on the example $G = GL_n$.

Let $E^0(A)$ denote the trivial A-family of rank n vector bundles on $\mathbb{D}$, i.e. the module $(A[[t]])^n$. Define $\mathcal{G}r^i_n(A)$ to be the set of all finitely generated projective $A[[t]]$-submodules $E(A) \subseteq E^0(A) \otimes_{A[[t]]} A((t))$ such that

$$t^i E^0(A) \subseteq E(A) \subseteq t^{-i} E^0(A).$$

If we take the direct limit $i \to \infty$ we recover the affine Grassmannian functor for GL_n as described above. Indeed, any $E(A)$ in $\mathcal{G}r^i_n(A)$ for some i is a rank n vector bundle on $\mathbb{D}$, and the restriction $E(A) \otimes_{A[[t]]} A((t))$ to $\mathbb{D}^\times$ comes with an isomorphism to the trivial bundle $E^0(A) \otimes_{A[[t]]} A((t))$ via

$$t^i E^0(A) \otimes_{A[[t]]} A((t)) \hookrightarrow E(A) \otimes_{A[[t]]} A((t)) \hookrightarrow t^{-i} E^0(A) \otimes_{A[[t]]} A((t)).$$

All points in $\mathcal{G}r_n(A)$ arise in this way for some i: given a finitely generated projective module we can view it as a submodule of $(A((t)))^n$ by embedding $(A[[t]])^{mn} \hookrightarrow (A((t)))^n$ as a submodule for each m. If the submodule fails to meet any of these finiteness conditions then it cannot be isomorphic to the trivial bundle on $\mathbb{D}^\times$.

We'll use this description of vector bundles later. Such $A[[t]]$-submodules of $(A((t)))^n$ are sometimes called *lattices*, and $E^0(A)$ is referred to as the *standard lattice*. This approach is described in [5].

Remark 3.5. If we restrict ourselves to lattices such that the determinant lattice $\bigwedge^n E(A) = (A[[t]])^n$ is the standard lattice $\mathcal{O} \subseteq \mathcal{K}$, then we find a description of the A-points of the affine Grassmannian for SL_n. If we quotient the space of lattices by the diagonal action of $k[t]$ on $(A((t)))^n$ then we find a description of the A-points of the affine Grassmannian for PGL_n.

Now, in order to see that the functors $\mathcal{G}r^i_n$ are representable by (projective finite-type) schemes, we'll describe an isomorphic functor. Namely we define

$$Z^i_n(A) = \{\text{projective finite-generated quotients of the } A[t]/t^{2i}\text{-module } t^{-i}E^0(A)/t^iE^0(A)\},$$

which is isomorphic to $\mathcal{G}r^i_n$ by the map which sends $E(A)$ to

$$t^{-i}E^0(A)/E(A),$$

with inverse the map that sends a quotient module N to the vector bundle which is the pre-image of N under

$$t^{-i}E^0(A) \to t^{-i}E^0(A)/t^iE^0(A).$$

One has to check that these maps are well-defined, i.e. that their images are finitely-generated projective (this check is done in [3], but I'll omit it for brevity).

Finally, note that Z^i_n is a closed subscheme of a usual Grassmannian of subspaces in the finite-dimensional k-vector space $t^{-i}\mathcal{O}^n/t^i\mathcal{O}^n$, therefore representable by a finite-type projective scheme. So $\mathcal{G}r^i_n$ also has this property as required.

Remark 3.6. If we look at the k-points of $\mathcal{G}r_n$, we can interpret the filtration $\mathcal{G}r^i_n$ as a filtration on the group $GL_n(\mathcal{K})/GL_n(\mathcal{O})$. The group $GL_n(\mathcal{K})$ acts on the vector space $\mathcal{K}^n$ in the usual way. The subgroup $GL_n(\mathcal{O})$ fixes the standard lattice $\mathcal{O}^n \subseteq \mathcal{K}^n$, so a coset in $GL_n(\mathcal{K})/GL_n(\mathcal{O})$ is characterised by the image of the standard lattice. The subset $\mathcal{G}r^i_n$ is then precisely the set of cosets $\alpha GL_n(\mathcal{O})$ such that

$$t^i\mathcal{O}^n \subseteq \alpha\mathcal{O}^n \subseteq t^{-i}\mathcal{O}^n.$$

4 Examples

We now have the machinery necessary to compute the geometric structure of some examples of affine Grassmannians for some specific groups. From now on we'll focus on the special case $k = \mathbb{C}$.

4.1 Stratification of $\mathcal{G}r_2(\mathbb{C})$

We'll use the lattice description from the previous section to understand the complex geometry of the $\mathbb{C}$-points of the affine Grassmannian for some small groups. We'll try to find representative elements in each filtered piece in order to stratify the affine Grassmannian by affine subvarieties.

So for the moment let $G = GL_2$. Following remark 3.6 above, we identify $\mathbb{C}$-points in the affine Grassmannian with lattices of the form $\alpha\mathcal{O}^2$, where $\alpha \in GL_2(\mathcal{K})$. Some natural choices for α are diagonal matrices of the form

$$\alpha_{a,b} = \begin{pmatrix} t^a & 0 \\ 0 & t^b \end{pmatrix}$$

for $a, b \in \mathbb{Z}$. The lattice $\alpha_{a,b}\mathcal{O}^2$ lies inside the filtered piece $\mathcal{G}r_2^d(\mathbb{C})$ where $d = \max(|a|, |b|)$. We think of it as follows: we have a natural basis for $\mathcal{K}^2$ as a $\mathbb{C}$-vector space consisting of elements $t^i x_1$ and $t^j x_2$ with $i, j \in \mathbb{Z}$, where x_1 and x_2 are a $\mathcal{K}$-basis for $\mathcal{K}^2$. Then $\alpha_{a,b}\mathcal{O}^2$ is the subspace

$$\mathrm{span}_{\mathbb{C}}\{t^i x_1, t^j x_2 \colon i \geq a \text{ and } j \geq b\}$$

Now, this suggests a more refined decomposition of $\mathcal{G}r_2(\mathbb{C})$. Specifically, we decompose the affine Grassmannian as a disjoint union

$$\mathcal{G}r_2(\mathbb{C}) = \coprod_{b \leq a \in \mathbb{Z}} S_{a,b}$$

where $S_{a,b}$ is the collection of lattices $\mathcal{L}$ in $\mathcal{K}^2$ such that $t^a\mathcal{O}^2 \subseteq \mathcal{L} \subseteq t^b\mathcal{O}^2$, and such that this is the *tightest* possible such restriction on $\mathcal{L}$. That is, $t^{a-1}\mathcal{O}^2 \nsubseteq \mathcal{L}$ and $\mathcal{L} \nsubseteq t^{b+1}\mathcal{O}^2$. So each $\mathcal{G}r_2^i(\mathbb{C})$ decomposes as a finite union of such strata:

$$\mathcal{G}r_2^i(\mathbb{C}) = \coprod_{-i \leq b \leq a \leq i} S_{a,b}.$$

We'll refer to this as a *stratification* of the space $\mathcal{G}r_2(\mathbb{C})$. In each stratum we've already found a representative element, namely the lattice $\alpha_{a,b}\mathcal{O}^2 \in \S_{a,b}$. An important thing to understand will be the gluing properties of the strata, so first let's compute which strata lie in the boundaries of other strata.

Proposition 4.1. The closure of the stratum $S_{a,b}$ is the union of all $S_{a-i,b+i}$ for $0 \leq i \leq \frac{1}{2}|a-b|$.

Proof. Choose a lattice $\alpha\mathcal{O}^2$ in $S_{a-i,b+i}$ as in the hypothesis. We'll construct a sequence of lattices $\alpha_n\mathcal{O}^2$ in $S_{a,b}$ converging to $\alpha\mathcal{O}^2$. Let x_1, x_2 denote the standard $\mathcal{O}$-basis for $\mathcal{O}^2$. The group $GL_2(\mathcal{O})$ acts on $\mathcal{G}r_2(\mathbb{C})$ on the left transitively on the strata, so we can find β so that $\beta\alpha\mathcal{O}^2$ is the lattice $\alpha_{a-i,b+i}\mathcal{O}^2$ spanned by $t^{b+i}x_1$ and $t^{a-i}x_2$. It suffices to find lattices converging to this standard element, then multiplying them on the left by β^{-1}.

Now, this lattice is certainly in the closure of $S_{a,b}$: a sequence of lattices converging to it is given by the lattices spanned by $t^{b+i}x_1$ and $\frac{1}{n}t^b x_1 + t^{a-i}x_2$: this lattice is certainly contained in $t^b\mathcal{O}^2$, and a little arithmetic shows it also contains $t^a\mathcal{O}^2$.

So we've shown that the closure $\overline{S_{a,b}}$ is contained in the required union as a dense subspace, so it remains to show that the union is closed. We do this via the determinant map

$$\wedge^2 \colon \mathcal{G}r_2(\mathbb{C}) \to \mathcal{G}r_1(\mathbb{C}).$$

The space $\mathcal{G}r_1(\mathbb{C})$ is just $\mathbb{Z}$, with points given by rank 1 lattices $t^i\mathcal{O} \subseteq \mathcal{K}$. The pre-image of such a lattice $t^i\mathcal{O}$ is the set of lattices in $S_{a,b}$ such that $a + b = i$ (as before we need only compute this for $\alpha_{a,b}\mathcal{O}^2$, where it is clear). These sets are precisely the sets that we are trying to show are closed. $\square$

This argument has also shown us that the connected components in $\mathcal{G}r_2(\mathbb{C})$ are precisely the unions

$$\bigcup_{i \geq 0} S_{a+i,a-i} \text{ and } \bigcup_{i \geq 0} S_{a+1+i,a-i}$$

for $a \in \mathbb{Z}$. The affine Grassmannian for SL_2 consists of those lattices whose determinant is the standard lattice, which is precisely the component $\bigcup S_{i,-i}$.

4.2 Geometry of the strata in $\mathcal{G}r_{PGL_2}(\mathbb{C})$

Now, let's pass to the group PGL_2 instead of GL_2. In that case, things simplify in an important way: we now identify lattices that are rescalings of one another, so in particular the strata $S_{a,b}$ and $S_{a+n,b+n}$ are identified for every $n \in \mathbb{Z}$, but multiplying the lattices by the diagonal matrix

$$\begin{pmatrix} t^n & 0 \\ 0 & t^n \end{pmatrix}.$$

We may as well rescale all out lattices so that they contain $\mathcal{O}^2$, but not $t^{-1}\mathcal{O}^2$, and label the resulting strata $S_{0,-b} = S_b$ for $b \geq 0$. We'll compute the geometry of the strata starting from some small examples.

$\mathbf{b = 0}$: The stratum S_0 contains the single lattice $\mathcal{O}^2$ itself only: $S_0 = \mathrm{pt}$.

$\mathbf{b = 1}$: The stratum S_1 consists of lattices $\mathcal{L}$ such that $\mathcal{O}^2 \subseteq \mathcal{L} \subseteq t^{-1}\mathcal{O}^2$, *apart* from the lattice $\mathcal{O}^2$ itself, *nor* the lattice $t^{-1}\mathcal{O}^2$, which is equal to $\mathcal{O}^2$ in the affine Grassmannian for PGL_2. So if x_1, x_2 is the standard $\mathcal{O}$-basis for $\mathcal{O}^2$, the lattices $\mathcal{L}$ must be generated by $\mathcal{O}^2$ and a single non-zero element in $\langle t^{-1}x_1, t^{-1}x_2 \rangle$. There are a $\mathbb{P}^1$ worth of such lattices, so $S_1 \cong \mathbb{P}^1$.

$\mathbf{b = 2}$: In order to understand the points of S_2, first choose a lattice $\mathcal{L}_0 \in S_1$. We'll study the lattices $\mathcal{L} \in S_2$ such that $\mathcal{L}_0 \subseteq \mathcal{L}$. Without loss of generality we choose $\mathcal{L}_0$ to be the $\mathcal{O}$-module spanned by $t^{-1}x_1$ and x_2. Lattices in S_2 containing $\mathcal{L}_0$ are spanned as an $\mathcal{O}$-module by $\mathcal{O}^2$ itself, $t^{-1}x_1$ and some non-zero $at^{-2}x_1 + bt^{-1}x_2$, where $a \neq 0$ (note that there can be no $t^{-2}x_2$ coefficient, or $\mathcal{L}$ would contain $t^{-1}\mathcal{O}^2$). Since we can rescale a to 1, we see that the fibre in S_2 over $\mathcal{L}_0$ is a copy of $\mathbb{A}^1$. If we allowed a to tend to zero, the lattice would converge to $\mathcal{O}^2$, so S_0 lies in the closure of each fibre. Since every point in S_2 contains a *unique* lattice in S_1 we have proven that S_2 is the total space of a line bundle over $S_1 \cong \mathbb{P}^1$. A little more work allows us to check that the line bundle in question is $\mathcal{O}(-1)$.

This stratum is not closed: its closure is a singular complex surface with an additional point at infinity adjoined. In other words, it's the result of blowing down the zero-section in the Hirzebruch surface Σ_1.

For general strata, we can use a similar method: every lattice in S_b contains a unique sublattice in S_{b-1}. The fibre in S_b over such a sublattice is the affine line $\mathbb{A}^1$. Thus, by induction, S_b is b-dimensional, consisting of the total space of a $b-1$-times iterated line bundle over $\mathbb{P}^1$. What's more, it admits a decomposition into $\mathbb{A}^b \cup \mathbb{A}^{b-1}$ from the decomposition $\mathbb{P}^1 = \mathbb{A}^1 \cup \mathbb{A}^0$ (for $b > 0$). Thus we have proven the following:

Proposition 4.2. The affine Grassmannian for PGL_2 has two connected components: $Gr|_{\mathrm{even}} = \bigcup S_{\mathrm{even}}$ and $\mathcal{G}r|_{\mathrm{odd}} = \bigcup S_{\mathrm{odd}}$. Each of these has cohomology

$$H^i(\mathcal{G}r|_{\mathrm{even}}) = H^i(\mathcal{G}r|_{\mathrm{odd}}) = \begin{cases} \mathbb{Z} & i \text{ even} \\ 0 & i \text{ odd} \end{cases}.$$

Proof. The statement about the connected components follows from the argument about closures of strata: 4.1 above. The cohomology calculation follows because each component is built as a union of cells in even real dimensions, one in each dimension. □

Remark 4.3. A similar calculation works for GL_2 and SL_2 also: one finds only one connected component for SL_2, and one connected component for each integer in the case of GL_2, and the cohomology of each component is the same as in the PGL_2 example.

References

[1] Arnaud Beauville and Yves Laszlo. Conformal blocks and generalized theta functions. *Com. Math. Phys*, 164(2):385–419, 1994.

[2] Dennis Gaitsgory. Construction of central elements in the affine Hecke algebra via nearby cycles. *Inv. Math.*, 144:253–280, 2000.

[3] Dennis Gaitsgory. Seminar notes, affine Grassmannian and the loop group. Unpublished notes available online at `http://www.math.harvard.edu/~gaitsgde/grad_2009/SeminarNotes/Oct13%28AffGr%29.pdf`, 2009.

[4] Victor Ginzburg. Perverse sheaves on a loop group and Langlands duality, preprint alg-geom/9511007, 1995.

[5] Ulrich Görtz. Affine Springer fibres and affine Deligne-Lusztig varieties. In Alexander Schmitt, editor, *Affine Flag Manifolds and Principal Bundles*, Trends in Mathematics. Springer, 2010.