

# Notes on “Conformal Blocks and Generalized Theta Functions” by Beauville and Laszlo

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In this talk I’ll review the geometric meaning of the affine Grassmannian  $G(\mathcal{K})/G(\mathcal{O})$  and its quotient  $G(\mathcal{O}) \backslash G(\mathcal{K})/G(\mathcal{O})$ , and compare them to spaces of *conformal blocks* arising in conformal field theory. There will be a bit of overlap with Aron’s talk last week, which I hope will be useful for people that aren’t already familiar with these sorts of constructions. The main reference is the paper [BL94] of Beauville and Laszlo, but I’ll also refer extensively to a later paper [Bea94] of Beauville. Another good reference with many details on spaces of conformal blocks and the Verlinde formula is the master’s thesis [Muk10] of Mukhopadhyay.

The first section of these notes has overlap with some longer notes I wrote on the geometry of affine Grassmannians, available at [http://www.math.northwestern.edu/~celliott/Affine\\_Grassmannian.pdf](http://www.math.northwestern.edu/~celliott/Affine_Grassmannian.pdf).

## 1 The Affine Grassmannian and Related Constructions

I’ll work over  $\mathbb{C}$  throughout (though all results should immediately generalise to any algebraically closed field of characteristic zero, and most of the definitions still make sense over much more general base rings). Write  $\mathcal{K}$  for the field  $\mathbb{C}((t))$  of Laurent series, and  $\mathcal{O}$  for the subring  $\mathbb{C}[[t]]$  of power series. The *formal disc* and *formal punctured disc* over  $\mathbb{C}$  are the affine schemes  $\mathbb{D} = \text{Spec } \mathcal{O}$  and  $\mathbb{D}^\times = \text{Spec } \mathcal{K}$  respectively. The *formal bubble* is my name for the fibre product of schemes  $\mathbb{B} = \mathbb{D} \times_{\mathbb{D}^\times} \mathbb{D}$ , so-called because I like to think of it as an infinitesimal version of  $\mathbb{P}^1$  but remembering that the transition function is given by  $z_1 = z_2$  rather than  $z_1 = z_2^{-1}$ .

The geometric objects we will study in this talk will be *moduli spaces of  $G$ -bundles* over such schemes, where  $G$  is a complex reductive group. One has to be a little careful, because we’ll be working with functors that are almost but not quite representable by schemes, that is, representable only by colimits of schemes, so called *ind-schemes* (in the category of complex algebraic spaces say, which admits appropriate colimits).

Let’s first give some informal definitions, then make them more precise by describing a functor-of-points.

- Definition 1.1.**
1. The *formal loop group*  $G(\mathcal{K})$  is the space of morphisms  $\mathbb{D}^\times \rightarrow G$ . Likewise, the *formal arc group*  $G(\mathcal{O})$  is the space of morphisms  $\mathbb{D} \rightarrow G$ .
  2. The *affine Grassmannian*  $\mathcal{G}r_G$  is the moduli space of  $G$ -bundles  $P$  on  $\mathbb{D}$  with trivialisations of the restriction  $P|_{\mathbb{D}^\times}$  to  $\mathbb{D}^\times$ .
  3. The moduli space  $\text{Bun}_G(\mathbb{B})$  is the moduli space of  $G$ -bundles on the formal bubble  $\mathbb{B}$ , or equivalently *pairs* of  $G$ -bundles  $P_1, P_2$  on  $\mathbb{D}$  with isomorphic trivialisations of their restrictions to  $\mathbb{D}^\times$ .

Interpreted correctly, we have isomorphisms  $\mathcal{G}r_G \cong G(\mathcal{K})/G(\mathcal{O})$  and  $\text{Bun}_G(\mathbb{B}) \cong G(\mathcal{O}) \backslash G(\mathcal{K})/G(\mathcal{O})$ , since  $G(\mathcal{K})$  describes maps  $\mathbb{D}^\times \rightarrow G$ , or trivialisated  $G$ -bundles on the punctured formal disc  $\mathbb{D}^\times$ , and  $G(\mathcal{O})$  likewise describes trivialisated  $G$ -bundles on  $\mathbb{D}$ .

To make this precise, we’ll describe the functors these objects represent, and at least hint at why they are representable by colimits of schemes (in the first case), or stacks (in the second case).

## 1.1 The Affine Grassmannian

In order to construct  $\mathcal{G}r_G$  as an algebro-geometric object (one arising from a moduli problem), we'll describe the functor that it's supposed to represent:

$$\mathcal{G}r_G: (\text{Commutative } \mathbb{C}\text{-algebras}) \rightarrow (\text{Sets})$$

sending an algebra  $A$  to the set of pairs  $(E, \gamma)$ , where  $E$  is an  $A$ -family of  $G$ -bundles over  $\mathbb{D}$ , and  $\gamma$  is an isomorphism from  $E|_{\mathbb{D}^\times}$  to the trivial  $A$  family of  $G$ -bundles over  $\mathbb{D}^\times$ . Notice that if  $A = \text{Spec } \mathbb{C}$  that this is just a  $G$ -bundle on  $\mathbb{D}$  and a trivialisation away from the origin. We'll see below that this is the same data as we described before (as a set).

To make sense of this, I'd should say what exactly an  $A$ -family of  $G$ -bundles is, but to keep things reasonably simple, I'll just describe an  $A$ -family of *vector* bundles over  $\mathbb{D}$  and  $\mathbb{D}^\times$ . Recall the notion of a vector bundle from commutative algebra:

**Definition 1.2.** The *rank* of a finitely generated projective  $A$ -module  $M$  at a point  $p \in \text{Spec } A$  is the rank of the free  $A_p$  module  $M_p$ . We say  $M$  is rank  $n$  if  $\text{rk}_M(p) = n$  for all points  $p$ .

**Definition 1.3.** • An  $A$ -family of rank  $n$  vector bundles on  $\mathbb{D}$  is a rank  $n$  finitely generated projective module over the ring  $A[[t]]$ .

- Similarly, an  $A$ -family of rank  $n$  vector bundles on  $\mathbb{D}^\times$  is a rank  $n$  finitely generated projective module over the ring  $A((t))$ .

One can modify this definition directly to work for groups other than  $GL_n$ , but more generally and indirectly one can define  $G$ -bundles in the following way:

**Definition 1.4.** An  $A$ -family of  $G$ -bundles on  $\mathbb{D}$  is an exact tensor functor  $\text{Rep}(G) \rightarrow \text{Vect}(\mathbb{D})$ , where  $\text{Vect}_A(\mathbb{D})$  is the tensor category of  $A$ -families of vector bundles (of any rank) as above. Similarly for  $\mathbb{D}^\times$ .

That is, we define the associated vector bundle in every representation of  $G$ .

Now we must investigate what kind of object might represent the affine Grassmannian functor. The first thing to note is that it is not represented by a scheme. Indeed, consider the example of  $G = \mathbb{G}_a$ . Even  $\mathbb{G}_a(\mathcal{K}) = \mathbb{C}((t))$  cannot be represented by a scheme: it sits inside the scheme  $\mathbb{C}[[t, t^{-1}]] \cong \prod_{i=1}^{\infty} \mathbb{A}^1$ , but the condition that only finitely many negative coefficients are non-zero cannot be expressed by polynomials.

However, it is very close to being representable by a scheme. Specifically, we have the following:

**Proposition 1.5.** There are a sequence of finite-type projective schemes  $\mathcal{G}r_G^i$  for  $i \in \mathbb{N}$ , and closed immersions  $\mathcal{G}r_G^i \hookrightarrow \mathcal{G}r_G^{i+1}$  such that  $\mathcal{G}r_G = \varinjlim \mathcal{G}r_G^i$ , i.e.

$$\mathcal{G}r_G(A) \cong \varinjlim \text{Hom}(\text{Spec } A, \mathcal{G}r_G^i).$$

A functor which is isomorphic to a direct limit of schemes of this form is called a (strict) *ind-scheme*, and is almost as nice to work with as a scheme (some details on the theory of ind-schemes can be found in [Gai09] and the appendix to [Gai00]). In particular we can define categories of (e.g. constructible) sheaves on an ind-scheme, where every object is in fact supported on a finite subscheme. Furthermore we can do things like investigate the topology of the  $\mathbb{C}$ -points of an ind-scheme over  $\mathbb{C}$ .

For the sake of time, I won't describe the construction of the  $\mathcal{G}r_G^i$ , but the *idea* – for  $G = GL_n$  – is to define  $\mathcal{G}r_G^i(A)$  to be the set of all finitely generated projective  $A[[t]]$ -submodules  $E(A) \subseteq E^0(A) \otimes_{A[[t]]} A((t))$  such that

$$t^i E^0(A) \subseteq E(A) \subseteq t^{-i} E^0(A).$$

This is sometimes called the *lattice description* of the affine Grassmannian: the modules  $E(A)$  are called *lattices*. For instance, the  $\mathbb{C}$  points of  $\mathcal{G}r_{GL_n}^i$  consist of the set of all  $\mathcal{O}$ -submodules  $M$  of  $\mathcal{K}^n$  such that  $t^i\mathcal{O}^n \subseteq M \subseteq t^{-i}\mathcal{O}^n$ . It remains to generalise this construction to more general groups  $G$ , to prove that the  $\mathcal{G}r_G^i$  are representable by schemes (indeed, by projective schemes of finite type) and to prove that the colimit of the  $\mathcal{G}r_G^i$  agrees with the description of  $\mathcal{G}r_G$  given above. We refer to [Gai09] for details.

## 1.2 Moduli stacks of $G$ -bundles

Now, let's describe  $\text{Bun}_G(\mathbb{B}) = G(\mathcal{O}) \backslash \mathcal{G}r_G = G(\mathcal{O}) \backslash G(\mathcal{K})/G(\mathcal{O})$ . The arc group  $G(\mathcal{O})$  is a genuine affine group scheme, not just an ind-group, so our moduli space is the quotient of an ind-scheme by a group. We might use abstract theory to describe the quotient as an algebraic stack, but instead we'll use a different, global approach which is special to  $SL_n$  and which will mesh nicely with the subsequent parts of the talk.

Let  $X$  be a Riemann surface, and let  $x \in X$  be a point. Let  $X^\times = X \setminus \{x\}$ . Let  $\mathcal{A}_x$  be the ring of algebraic functions on  $X^\times$ . The space  $\text{Bun}_G(\mathbb{B})$  is an example of the following sort of stack.

**Definition 1.6.** Let  $\mathcal{G}r_{G,X}$  be the space of  $G$ -bundles  $P$  on  $X$  equipped with a trivialisation of the restriction  $P|_{X^\times}$ . That is, the functor associating to a  $\mathbb{C}$  algebra  $A$  the set of principal  $G$ -bundles on  $X \times \text{Spec}(A)$  with trivialisation on  $X^\times \times \text{Spec}(A)$ . This is a *global* version of the affine Grassmannian, and is isomorphic to  $\mathcal{G}r_G$  [BL95], [Gai09].

Let  $\text{Bun}_G^x(X)$  be the quotient stack  $G(\mathcal{A}_x) \backslash \mathcal{G}r_{G,X}$ . That is, the functor associating to a  $\mathbb{C}$  algebra  $A$  the groupoid of principal  $G$ -bundles on  $X \times \text{Spec}(A)$  which are trivialisable on  $X^\times \times \text{Spec}(A)$ . We've forgotten the data of the trivialisation.

In the case where  $G = SL_n$ , this is isomorphic to the full moduli stack. That is  $\text{Bun}_G^x(X) \cong \text{Bun}_G(X)$ . This is proposition 3.4 in [BL94], and should be interpreted as the fact that all vector bundles on a non-compact complex algebraic curve are algebraically trivialisable. Points can be viewed as vector bundles with trivialisable *determinant*, where the determinant is the top exterior power of a line bundle. Although  $\mathbb{B}$  is not a Riemann surface, the stack  $\text{Bun}_G(\mathbb{B})$  is very similar to  $\text{Bun}_G(\mathbb{P}^1)$ . For instance, the categories of coherent or constructible sheaves on these two stacks are (non-canonically) equivalent [Laf09].

We can define a canonical line bundle on  $\text{Bun}_{SL_n}(X)$  called the *determinant* line bundle, whose fibre is not just the determinant of the corresponding line bundle, but rather the determinant of its *cohomology*.

**Definition 1.7.** To define the line bundle  $\det \rightarrow \text{Bun}_{SL_n}(X)$ , we must associate to each  $SL_n$ -bundle  $E$  on  $X \times \text{Spec}(A)$  a line bundle on  $\text{Spec}(A)$ , i.e. a rank one projective  $A$ -module. Define such a line bundle by

$$\det(\pi_*(E)_0) \otimes \det(\pi_*(E)_1)^\vee$$

where  $\pi: X \times \text{Spec}(A) \rightarrow \text{Spec}(A)$  is the projection, and noting that the pushforward  $\pi_*(E)$  is a complex concentrated in degrees 0 and 1.

We'll come back to the study of sections of this bundle and its tensor powers in section 3.

## 2 Conformal Blocks

Now we'll introduce the other main object of study in this talk, and give an idea of its origin in conformal field theory (though only briefly; I don't want to assume any familiarity with quantum field theory). As well as [BL94], I referred to a later descriptions of Beauville given in [Bea94]. One can more generally describe spaces of conformal blocks associated to a vertex algebra [FBZ01], or a chiral algebra [BD04]; we're just considering the space of conformal blocks associated to a Kac-Moody vertex algebra for  $SL_r$ .

Let  $G$  be a simple complex Lie group. We'll construct a *central extension*  $\widehat{G}$  of the loop group  $G(\mathcal{K})$  which will act on the bundles  $\pi^*(\det^{\otimes \ell})$  over  $\mathcal{G}r_G$ .

**Definition 2.1.** The *affine Kac-Moody algebra* associated to a simple Lie algebra  $\mathfrak{g}$  is the Lie algebra with underlying vector space  $\widehat{\mathfrak{g}} = (\mathfrak{g} \otimes \mathbb{C}((t))) \oplus c \cdot \mathbb{C}$ , where  $c$  is a central element and the bracket of other elements is given by

$$[X \otimes f, Y \otimes g]_c = [X, Y] \otimes fg + c\langle X, Y \rangle \text{res}(fdg)$$

where  $\langle -, - \rangle$  is the Killing form.

This algebra arises as the Lie algebra of a central extension of the *loop group*  $G(\mathcal{K})$  by  $\mathbb{G}_m$ : the *Kac Moody group*  $\widehat{G}$ . However, in what follows it should mostly suffice to consider only the Lie algebra.

**Remark 2.2.** Why does this algebra act on the bundle  $\pi^*(\det^{\otimes \ell})$  rather than the loop algebra itself? Beauville and Laszlo prove that the pullback line bundle admits a description closely related to the Kac-Moody algebra. Specifically, they prove (corollary 5.5 of [BL94]) that the Weil divisor corresponding to  $\pi^*(\det^{\otimes \ell})$  arises as the image of the divisor of zeroes and poles of a meromorphic function  $\tau$  on  $\widehat{G}$  under a natural projection map

$$\widehat{G} \rightarrow \mathcal{G}r_G.$$

This description yields a natural action of the Kac-Moody algebra  $\widehat{\mathfrak{g}}$ , projected from the infinitesimal action of  $\widehat{G}$  on functions on  $\widehat{G}$ .

We'll describe representations of  $\widehat{\mathfrak{g}}$  associated to integer *levels*  $\ell \geq 0$ . That is, the central charge  $c$  should act as multiplication by the level  $\ell$ . These are built from highest weight representations  $L_\lambda$  for  $\mathfrak{g}$ , where  $\lambda$  is a dominant weight such that  $\langle \lambda, \theta^\vee \rangle \leq \ell$ , where  $\theta$  is a highest root and  $\theta^\vee$  is the corresponding coroot. Such a representation is constructed as follows (following [Bea94]):

- Split  $\widehat{\mathfrak{g}}$  as  $\widehat{\mathfrak{g}}_- \oplus \mathfrak{g} \oplus c \cdot \mathbb{C} \oplus \widehat{\mathfrak{g}}_+$  where  $\widehat{\mathfrak{g}}_\pm$  are the spaces spanned by only the positive or negative powers of the parameter  $t$ . Consider the subalgebra  $\mathfrak{p} = \mathfrak{g} \oplus c \cdot \mathbb{C} \oplus \widehat{\mathfrak{g}}_+$ .
- Consider  $L_\lambda$  as a representation of  $\mathfrak{p}$  by letting  $\mathfrak{g}$  act as normal,  $\widehat{\mathfrak{g}}_+$  act trivially, and  $c$  act as multiplication by  $\ell$ .
- Induce to a representation  $\mathcal{V}_{\ell, \lambda} = L_\lambda \otimes_{U(\mathfrak{p})} \otimes U(\widehat{\mathfrak{g}})$ .
- Take the quotient  $\mathcal{H}_{\ell, \lambda}$  of  $\mathcal{V}_{\ell, \lambda}$  by the unique maximal  $\widehat{\mathfrak{g}}$  subrepresentation (the existence of such a thing uses the constraint on  $\lambda$  mentioned above).

The resulting representation is uniquely characterised by the action of  $c$  and the property that  $L_\lambda \otimes 1$  is precisely the subspace annihilated by  $\widehat{\mathfrak{g}}_+$ .

Now, let  $X$  be a Riemann surface, and let  $x \in X$  be a point (we'll eventually compare our constructions to the spaces considered in remark 1.6 with the same setup). Let  $\mathcal{A}_x$  be the ring of algebraic functions on  $X^\times = X \setminus \{x\}$ , and let  $\mathfrak{g}(\mathcal{A}_x) = \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{A}_x$ . There's a ring homomorphism  $\mathcal{A}_x \rightarrow \mathcal{K}$  by taking the Laurent expansion at  $x$  (that is, pulling back under the inclusion  $\mathbb{D}^\times \hookrightarrow X^\times$ , or taking the germ). This induces a Lie algebra homomorphism  $\mathfrak{g}(\mathcal{A}_x) \rightarrow \widehat{\mathfrak{g}}$ .

**Definition 2.3.** The space  $B_{\ell, \lambda}^\dagger$  of *covacua* for  $G$  of central charge  $\ell$  and weight  $\lambda$  is the space of coinvariants for this Lie algebra

$$B_{\ell, \lambda}^\dagger = (\mathcal{H}_{\ell, \lambda})_{\mathfrak{g}(\mathcal{A}_x)}.$$

Write  $B_{\ell, \lambda}$  for the dual vector space, the space of *vacua* or *conformal blocks*.

**Remark 2.4.** This definition naturally extends to a finite set of points  $x_i \in X$  and dominant weights  $\lambda_i$ ,  $i = 1, \dots, n$ . Varying these points (and keeping the weights fixed) preserves the space of conformal blocks up to isomorphism, so gives a vector bundle on the configuration space of  $n$  points in  $X$ , and varying the complex structure also gives a vector bundle on the moduli space  $\mathcal{M}_{g, n}$  of punctured curves.

The motivation for this definition comes from conformal field theory on the Riemann surface  $X$ . The local observables in such a theory are described by a vertex algebra (for instance, the affine Kac-Moody algebra described above induced a Kac-Moody vertex algebra by an enveloping algebra construction [Gwi12], which is the algebra of observables in a WZW model). The Hilbert space of the WZW model at level  $\ell$  is the sum

$$\bigoplus_{\lambda \in P_\ell} (\mathcal{H}_{\ell, \lambda} \oplus \overline{\mathcal{H}_{\ell, \lambda}})$$

where  $P_\ell$  is the set of dominant roots satisfying the constraint  $\langle \lambda, \theta^\vee \rangle \leq \ell$ . The  $\mathcal{H}_{\ell, \lambda}$  and  $\overline{\mathcal{H}_{\ell, \lambda}}$  are just two copies of the highest weight representation; the bar is just a label. To a state in the Hilbert space we associate a local observable on a disc by the state-operator correspondence, thus local operators admit a weight decomposition.

If we look at a set of points  $x_1, \dots, x_n$ , and the local observables of weights  $\lambda_1, \dots, \lambda_n$  in a small neighbourhood of this finite set, we find they satisfy the *chiral* and *antichiral Ward identities*. That is, they are invariant under a certain Lie algebra describing conformal transformations of the complex structure on  $X$ . Thus, in order to construct physical observables we might look at all functionals on a tensor product of the Hilbert space (or the weight  $\lambda$  piece) which are invariant under the Ward identities. The resulting space is precisely a copy of the space of conformal blocks plus its complex conjugate, because the algebra encoding the Ward identities is precisely the algebra  $\mathfrak{g}(\mathcal{A}_{x_1, \dots, x_n})$ . This description is given, for instance, in [Fre07].

### 3 Generalized Theta Functions

To conclude, we'll say what Beauville and Laszlo mean by "generalized theta functions", and describe their relationship with conformal blocks (the *Verlinde formula*). At this point, we once again restrict to  $G = SL_n$ . Let's recall the most classical notion of a theta function, as described in [Bea13].

**Definition 3.1.** Let  $T = V/\Gamma$  be a complex torus, and let  $(e_\gamma)_{\gamma \in \Gamma}$  be a set of holomorphic functions on  $V$  satisfying the cocycle condition

$$e_{\gamma+\delta}(z) = e_\gamma(z + \delta)e_\delta(z).$$

A *theta function* for this system is a holomorphic function on  $V$  such that

$$\theta(z + \gamma) = e_\gamma(z)\theta(z) \quad \text{for all } \gamma \in \Gamma, z \in V.$$

The complex tori we have in mind are the Picard varieties of Riemann surfaces. The systems  $e_\gamma$  precisely describe transition functions for line bundles on the torus, and a theta function for such a system is precisely a *section* of such a line bundle. That is, there is a bijection

$$\{\theta \text{ functions on } T \text{ for } L\} \leftrightarrow H^0(T; L).$$

Now, we'd like to generalise this by replacing a torus (i.e.  $\text{Bun}_{\mathbb{C}^m}(X)$ ) by a *non-abelian* analogue:  $\text{Bun}_{SL_n}(X)$ . The line bundles we'll consider will be powers of the determinant line bundle, as defined in 1.7. The resulting space of generalized theta functions (i.e. holomorphic sections of these line bundles) agrees with a space of conformal blocks, by the main theorem of [BL94].

**Theorem 3.2.** There is a canonical isomorphism between the space of sections of  $\det^{\otimes \ell}$  and the space of weight zero conformal blocks. That is

$$H^0(\text{Bun}_{SL_n}(X); \det^{\otimes \ell}) \cong B_{\ell, 0}.$$

**Remark 3.3.** We might ask for an interpretation of the spaces of conformal blocks of higher weight, or associated to more than one point in  $X$ . The case of additional points is described by a property called *propagation of vacua*. There is always a canonical isomorphism

$$B_{\ell, \lambda_1}(x_1) \otimes B_{\ell, \lambda_2}(x_2) \cong B_{\ell, \lambda}(x)$$

where  $\underline{x} = (x_1, x_2)$  and  $\underline{\lambda} = (\lambda_1, \lambda_2)$ . Propagation of vacua says that when  $x_2 = x_2$  is a single point, and  $\lambda_2 = 0$  then there is also a canonical isomorphism

$$B_{\ell, \underline{\lambda}_1}(\underline{x}_1) \cong B_{\ell, (\underline{\lambda}_1, 0)}(\underline{x}_1, x_2)$$

allowing us to freely introduce additional points with zero weight.

The introduction of non-zero weights corresponds to the introduction of parabolic structures – reduction of structure group to particular parabolic subgroups at specified points – into our moduli space [Pau96].

Let’s say at least a few words about the proof of theorem 3.2. The main input is a theorem of Kumar [Kum87] and Mathieu [Mat88] (independently) on the level of the affine Grassmannian.

**Theorem 3.4.** There is a canonical isomorphism of  $\widehat{\mathfrak{sl}}_n$ -representations

$$H^0(\mathcal{G}_{rSL_n}; \pi^* \det^{\otimes \ell}) \cong \mathcal{H}_{\ell, 0}^\vee.$$

The Kac-Moody action on the left hand side was described above in 2.2. The theorem is proved by checking the left hand side satisfies the properties uniquely characterising the right hand side, i.e. that the central element  $c$  acts with weight  $\ell$  and the subspace annihilated by  $\widehat{\mathfrak{g}}_+$  is isomorphic to the trivial representation  $L_0$ .

Passing from this theorem to 3.2 is mostly formal. Based purely on abstract properties of quotient stacks, there is an isomorphism

$$H^0(\text{Bun}_{SL_n}(X); \det^{\otimes \ell}) \cong H^0(\mathcal{G}_{rSL_n}; \pi^* \det^{\otimes \ell})^{\mathfrak{sl}_n(\mathcal{A}_x)}$$

and taking  $\mathfrak{sl}_n(\mathcal{A}_x)$ -invariants in the Kumar-Mathieu theorem give the desired result.

We conclude by mentioning a very famous result: the *Verlinde formula*. This is a formula for the *dimensions* of the spaces of conformal blocks, and therefore for the spaces of sections of powers of the determinant line bundle on moduli space. One proves the Verlinde formula using “fusion rules” (which allow for detailed analysis of the representation rings of Kac-Moody algebras at level  $\ell$ ), normalised by calculations of the dimensions for  $\mathbb{P}^1$ . We include the formula for completeness; details are available in [Bea94].

**Theorem 3.5.** Suppose  $\mathfrak{g}$  is a simple Lie algebra of type  $A, B, C, D$  or  $G$ . Then

$$\dim B_{\ell, \underline{\lambda}}(\underline{x}) = |T_\ell|^{g-1} \sum_{\mu \in P_\ell} \text{Tr}_{V_{\lambda_1} \otimes \dots \otimes V_{\lambda_n}} \left( e^{2\pi i \frac{\mu + \rho}{\ell + h^\vee}} \right) \prod_{\alpha > 0} \left| 2 \sin \left( \pi \frac{\langle \alpha, \mu + \rho \rangle}{\ell + h^\vee} \right) \right|^{2-2g}$$

where  $g$  is the genus of  $X$ ,  $P_\ell$  is the set of dominant roots satisfying the constraint  $\langle \lambda, \theta^\vee \rangle \leq \ell$ ,  $\rho$  is the half-sum of the roots, and  $h^\vee = \rho(\theta) + 1$ , and  $T_\ell$  is the set of elements  $t$  in the maximal torus such that  $\chi(t) = 1$  for all characters  $\chi$  in the lattice generated by  $\ell + h^\vee$  multiples of the long roots. The exponential map in the formula is the map associating a character to a root.

**Remark 3.6.** Teleman and Woodward [TW09] proved a more general version of the Verlinde formula, allowing the computation of the dimensions of any vector bundle on  $\text{Bun}_G(X)$ , not just powers of the determinant line bundle.

## References

- [BD04] Alexander Beilinson and Vladimir Drinfeld. *Chiral algebras*, volume 51. American Mathematical Society Providence, RI, 2004.
- [Bea94] Arnaud Beauville. Conformal blocks, fusion rules and the Verlinde formula. *arXiv preprint alg-geom/9405001*, 1994.
- [Bea13] Arnaud Beauville. Theta functions, old and new. In *Open Problems and Surveys of Contemporary Mathematics*, volume 6 of *Surveys of Modern Mathematics*, pages 99–131. International Press, 2013.

- [BL94] Arnaud Beauville and Yves Laszlo. Conformal blocks and generalized theta functions. *Communications in mathematical physics*, 164(2):385–419, 1994.
- [BL95] Arnaud Beauville and Yves Laszlo. Un lemme de descente. *Comptes Rendus de l'Academie des Sciences-Serie I-Mathematique*, 320(3):335–340, 1995.
- [FBZ01] Edward Frenkel and David Ben-Zvi. *Vertex algebras and algebraic curves*, volume 88. American Mathematical Society Providence, 2001.
- [Fre07] Edward Frenkel. Lectures on the langlands program and conformal field theory. In *Frontiers in number theory, physics, and geometry II*, pages 387–533. Springer, 2007.
- [Gai00] Dennis Gaitsgory. Construction of central elements in the affine Hecke algebra via nearby cycles. *Inv. Math.*, 144:253–280, 2000.
- [Gai09] Dennis Gaitsgory. Seminar notes, affine Grassmannian and the loop group. Unpublished notes available online at [http://www.math.harvard.edu/~gaitsgde/grad\\_2009/SeminarNotes/Oct13%28AffGr%29.pdf](http://www.math.harvard.edu/~gaitsgde/grad_2009/SeminarNotes/Oct13%28AffGr%29.pdf), 2009.
- [Gwi12] Owen Gwilliam. *Factorization Algebras and Free Field Theory*. PhD thesis, Northwestern University, 2012.
- [Kum87] Shrawan Kumar. Demazure character formula in arbitrary Kac-Moody setting. *Inventiones mathematicae*, 89(2):395–423, 1987.
- [Laf09] Vincent Lafforgue. Quelques calculs reliés à la correspondance de Langlands géométrique pour  $\mathbb{P}^1$  (version provisoire). Available online at <http://www.math.jussieu.fr/~vlafforg>, 2009.
- [Mat88] Olivier Mathieu. Formules de caracteres pour les algebres de Kac-Moody générales. *Astérisque*, 159, 1988.
- [Muk10] Swarnava Mukhopadhyay. Decomposition of conformal blocks. Master's thesis, University of North Carolina at Chapel Hill, 2010.
- [Pau96] Christian Pauly. Espaces de modules de fibrés paraboliques et blocs conformes. *Duke Mathematical Journal*, 84(1):217–236, 1996.
- [TW09] Constantin Teleman and Christopher Woodward. The index formula for the moduli of  $G$ -bundles on a curve. *Annals of mathematics*, 170:495–527, 2009.