

# Notes on “Partial Resolutions of Nilpotent Varieties” by Borho and Macpherson

Chris Elliott

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## 1 Setup

Let  $G$  be a complex reductive Lie group with Lie algebra  $\mathfrak{g}$ . The paper [BM83] relates two kinds of object:

- *Parabolic subgroups*  $P \subseteq G$ . Recall that a subgroup  $P$  is parabolic if it contains a Borel subgroup, or equivalently if the quotient space  $G/P$  is a projective variety.
- *Partial resolutions*  $\pi: \tilde{\mathcal{N}} \rightarrow \mathcal{N}$ , where  $N \subseteq \mathfrak{g}$  is the *nilpotent cone*: the set of elements  $X \in \mathfrak{g}$  such that the endomorphism  $[X, -]$  is nilpotent (which implies that  $\rho(X)$  is nilpotent in all representations  $\rho$  of  $\mathfrak{g}$ ). A *partial resolution* is a surjective smooth map which restricts to an isomorphism on the preimage of the smooth locus in  $\mathcal{N}$ .

From a parabolic subgroup  $P$  we'll construct a partial resolution  $\pi_P: \tilde{\mathcal{N}}_P \rightarrow \mathcal{N}$ , prove some nice properties of  $\pi_P$ , then study the geometry of its singularities and its fibres. If  $P = B$  is a Borel then this recovers the *Springer resolution* of  $\mathcal{N}$ , which is a bona fide resolution of singularities with total space isomorphic to  $T^*(G/B)$ .

Unless I say otherwise, all proofs are those given in [BM83].

## 2 The Partial Resolutions

We fix a parabolic subgroup  $P$  containing a Borel  $B$  once and for all with Lie algebra  $\mathfrak{p}$ . The quotient space  $G/P$  can be viewed as the space of parabolic subgroups  $P' \subseteq G$  conjugate to  $P$ , or equivalently parabolic subalgebras  $\mathfrak{p}' \subseteq \mathfrak{g}$  conjugate to  $\mathfrak{p}$ .

**Definition 2.1.** The partial resolution  $\tilde{\mathcal{N}}_P$  is the space of pairs

$$\{(X, \mathfrak{p}') : \mathfrak{p}' \in G/P \text{ and } X \in \mathfrak{p}' \cap \mathcal{N}\} \subseteq \mathcal{N} \times G/P$$

This maps to  $\mathcal{N}$  by the natural projection. Call this map  $\pi_P$ .

We can likewise define a resolution  $\pi_B: \tilde{\mathcal{N}} \rightarrow \mathcal{N}$  associated to the Borel subgroup  $B$ : the so-called *Springer resolution*. This map factors through the partial resolution  $\pi_P$ . We should check that  $\pi_B$  really is a resolution of singularities. To do so, we first study the geometry of the total space  $\tilde{\mathcal{N}}$ .

Let  $\pi_2$  be the projection map  $\tilde{\mathcal{N}} \rightarrow G/B$ . The fibre of  $\pi_2$  over a Borel subalgebra  $\mathfrak{b}'$  is the set of nilpotent elements of  $\mathfrak{b}'$ . We can understand this by decomposing  $\mathfrak{b}' \cong \mathfrak{h} \oplus \mathfrak{n}$  where  $\mathfrak{h}$  is the Cartan subalgebra and  $\mathfrak{n} = [\mathfrak{b}', \mathfrak{b}']$  is the nilradical. The nilpotent elements are those elements in  $\{0\} \oplus \mathfrak{n} \subseteq \mathfrak{b}'$ , so the projection  $\pi_2$  is a vector bundle.

**Proposition 2.2.** The Springer resolution  $\tilde{\mathcal{N}}$  is isomorphic to  $T^*(G/B)$  as a vector bundle over the flag variety.

*Proof.* [CG97] First, note that our description above identifies  $\tilde{\mathcal{N}}$  as the vector bundle

$$G/B \times_{\mathfrak{b}} \mathfrak{n} \cong G \times_B \mathfrak{n},$$

where on the left-hand side  $\mathfrak{b}$  is the tautological  $\mathfrak{b}$ -bundle over  $G/B$ , and on the right-hand side  $B$  acts on  $\mathfrak{n}$  by the adjoint action. We can identify  $\mathfrak{n}$  as the annihilator of  $\mathfrak{b}$  in  $\mathfrak{g}$  with respect to the Killing form  $B(X, Y) = \text{Tr}(\text{ad}(X)\text{ad}(Y))$ . Thus we've identified  $\tilde{\mathcal{N}}$  with  $G \times_B \mathfrak{b}^\perp$ , where  $\mathfrak{b}^\perp \subseteq \mathfrak{g}^*$  is the annihilator of  $\mathfrak{b}$ , acted on by  $B$  by the coadjoint action.

To see that this recovers the cotangent bundle of the flag variety, we consider the dual vector bundle  $G \times_B (\mathfrak{g}/\mathfrak{b})$ . This can be viewed as the quotient of the trivial  $\mathfrak{g}$  bundle on  $G/B$  by the tautological  $\mathfrak{b}$ -bundle. However, the tautological  $\mathfrak{b}$ -bundle is *also* the kernel of the bundle map

$$G/B \times \mathfrak{g} \rightarrow T(G/B)$$

given by the infinitesimal  $G$ -action. Thus  $G \times_B (\mathfrak{g}/\mathfrak{b}) \cong T(G/B)$  or dually  $G \times_B \mathfrak{b}^\perp \cong T^*(G/B)$  as required.  $\square$

Now, this proves that  $\tilde{\mathcal{N}}$  is smooth. To see it forms a resolution, we just observe that  $\pi_B$  is an isomorphism away from the singularities of  $\mathcal{N}$ . We'll need a description of this smooth locus: it will consist of the *regular* elements of  $\mathcal{N}$ .

**Definition 2.3.** An element  $X \in \mathfrak{g}$  is called *regular* if the characteristic polynomial of  $\text{ad}(X)$  has degree  $\text{rk}(\mathfrak{g})$  (recall that the *rank* of  $\mathfrak{g}$  is the maximal degree of such a characteristic polynomial, or equivalently the dimension of any Cartan). The regular elements are dense in  $\mathfrak{g}$ .

**Remark 2.4.** For  $G = GL(n)$ , the regular elements of  $\mathfrak{g}$  are just those matrices all of whose eigenvalues are distinct.

**Theorem 2.5.** For  $X \in \mathcal{N}$ , the Springer fibre  $\pi_B^{-1}(X)$  has dimension

$$d_X = \frac{1}{2}(\dim(\mathfrak{g}) - \dim(O_X))$$

where  $O_X$  is the orbit of  $X$  under the adjoint action. In particular, the Springer fibre is zero-dimensional if and only if  $X$  is regular. In this case it consists of a single point.

This is a theorem of Steinberg proven in [Ste76]. Much of the proof holds for more general parabolic subgroups than just Borels. In particular we notice that the Springer resolution  $\pi_B$ , hence all the  $\pi_P$ , are *semismall maps* with respect to the stratification of  $\mathcal{N}$  by adjoint orbits. In general a map  $\pi: Y \rightarrow X$  is *semismall* with respect to a stratification of  $X$  if for every stratum  $S \subseteq X$ , and for every point  $x \in S$ , one has

$$2 \dim(\pi^{-1}(x)) \leq \dim(X) - \dim(S).$$

This will become important later on when we apply the decomposition theorem. A stratum is called *relevant* if equality holds in the above inequality; for the Springer resolution all strata are relevant.

### 3 Intersection Cohomology and the Decomposition Theorem

The main tool we're going to use to study the topology of  $\mathcal{N}$  and its resolutions is the theory of *intersection cohomology*, or more generally of *perverse sheaves*. These are sheaves of vector spaces well-suited for studying the topology of singular spaces; they admit nice notions of push-forward and pull-back along functions between spaces, and a natural notion of *duality* related to things like Poincaré and Alexander duality for smooth manifolds. Good expository references for the theory of intersection cohomology include [Rie03] and [dCM09].

To start, we should be more precise about what kind of spaces we’re talking about. The idea is to consider spaces that admit *stratifications*, i.e. decompositions into a disjoint union of smaller spaces, where the strata look like smooth complex manifolds. For instance, singular complex varieties should have this form. Not just any stratification will do – one wants to avoid pathological examples – so we restrict to stratifications satisfying the *Whitney conditions*

**Definition 3.1.** A topological space  $X$  is called a *stratified pseudomanifold* if it can be decomposed as a finite disjoint union  $X = S_0 \sqcup \cdots \sqcup S_n$  where

- The stratum  $S_i$  is a smooth  $2n - 2i$ -manifold.
- The stratum  $S_0$  is dense in  $X$ .
- For every  $x \in S_i$ , there is an neighbourhood  $U$  such that the intersection of  $U$  with the strata makes  $U$  into a stratified pseudomanifold, with strata  $U \cap S_j \cong \mathbb{R}^{2n-2i} \times \text{cone}^o(\Lambda_i)$ , where  $\Lambda \sqcup \Lambda_i$  is a stratified pseudo-manifold of dimension  $2i$  called the *link* of  $x$ , and where  $\text{cone}^o$  denotes the open cone.

The latter is the appropriate local triviality condition for stratified spaces. The stratification is called a *Whitney stratification* if given two strata  $S_i$  and  $S_j$ , and a sequence  $(x_k) \in S_i$  converging to  $y \in S_j$ , if the tangent spaces  $T_{x_k} S_i$  converges to a tangent space  $T \subseteq T_y X$ , then this space  $T$  contains the tangent space  $T_y S_j$ .

From now on, we’ll use “stratified space” to mean “Whitney stratified pseudomanifold”. Whitney proved that every complex variety admits a Whitney stratification, so this includes many fundamental examples. Relevant to our purposes, the nilpotent cone  $\mathcal{N}$  admits a Whitney stratification by its adjoint orbits with dense stratum given by the regular semisimple elements (this is proven in [CG97]).

Now, let’s explain what kind of sheaves we’ll be dealing with on these stratified spaces.

**Definition 3.2.** A *local system* on a space  $X$  is a locally constant sheaf of finite-dimensional (complex) vector spaces. A sheaf  $\mathcal{F}$  of vector spaces on  $X$  is called a *constructible sheaf* if  $X$  admits a stratification  $\sqcup S_i$  so that  $\mathcal{F}|_{S_i}$  is a local system for every  $i$ . We’ll deal with the *bounded derived category*  $D_c^b(X)$  of constructible sheaves, so objects are finite cochain complexes of constructible sheaves.

For a map  $f: X \rightarrow Y$  of spaces there are natural (derived) pushforward and pullback functors  $f_*: D_c^b(X) \rightarrow D_c^b(Y)$  and  $f^*: D_c^b(Y) \rightarrow D_c^b(X)$ . I won’t go into detail about how to define these, but they satisfy the properties one would expect from more familiar versions of sheaf theory (for instance  $f^*$  is left adjoint to  $f_*$ ). One also has exceptional pullbacks and pushforwards  $f^!, f_!$ , Verdier dual to the functors above.

The *intersection complexes* we’ll study will be objects in this category which arise from a particularly nice topological origin. I’ll give two constructions. Let  $X$  be a stratified space of real dimension  $2n$ .

1. We can define a sheaf  $IC(L_X X)$  purely sheaf-theoretically. Start with a local system  $\mathcal{C}_0 = L_X$  on the open stratum  $S_0$ , placed in degree  $-n$ . For each  $i$ , inductively define

$$\mathcal{C}_i = \tau_{\leq i-n}(j_i)_* \mathcal{C}_{i-1}$$

where  $j_i$  is the inclusion of  $\mathcal{C}_{i-1}$  into  $\mathcal{C}_i$ , and where  $\tau_{\leq k}$  are the truncation functors:

$$\tau_{\leq k} \mathcal{F} = (\cdots \rightarrow \mathcal{F}_{k-2} \rightarrow \mathcal{F}_{k-1} \rightarrow \ker(d_k) \rightarrow 0).$$

The complex  $IC(L_X)$  is the result of this procedure:  $IC(L_X) = \mathcal{C}_n$ .<sup>1</sup>

2. We can also construct  $IC(\mathbb{C}_X)$  in a more explicit, topological way, in terms of singular cochains. For simplicity, we’ll work with singular *chains*: one can dualise to produce a complex agreeing with the one produced above. We say a  $p$ -chain  $C$  is *allowable* if, for every stratum  $S_i$ , one has

$$\begin{aligned} \dim(|C| \cap S_i) &\leq p - i \\ \dim(|\partial C| \cap S_i) &\leq p - i - 1 \end{aligned}$$

where  $|C|, |\partial C|$  denote the supports of the chain and its boundary. The *intersection complex*  $IC_*(X)$  is the subcomplex of the complex of singular chains (with  $\mathbb{C}$ -coefficients) that are allowable.

<sup>1</sup>Here I’m defining intersection cohomology in the “middle perversity”, which is the version that gives something Poincaré self-dual. One gets other versions by using harsher or less harsh truncations.

For a proof that these two constructions give isomorphic complexes, see [Rie03] 4.9.

**Remark 3.3.** To really make sense of the latter definition we should choose a piecewise linear structure on  $X$ , and only work with piecewise linear chains. The coincidence with the sheaf-theoretic construction proves that this construction is independent of the choice of piecewise linear structure.

Given any locally closed stratified subspace  $i: Y \hookrightarrow X$ , we'll denote the sheaf  $i_*IC(L_Y) \in D_c^b(X)$  by  $IC_Y(L_Y)$ . These intersection cohomology sheaves of subspaces form the building blocks for a special (abelian) subcategory of  $D^b(X)$ , the category of *perverse sheaves*. These objects have another description. There is always a canonical natural transformation of functors  $i_! \rightarrow i_*$ . The image of the map  $i_!\mathcal{F} \rightarrow i_*\mathcal{F}$  for  $\mathcal{F} \in D_c^b(Y)$  is called the *minimal* or *IC* extension of  $\mathcal{F}$ , and denoted  $i_{!*}\mathcal{F}$ . If  $\mathcal{F}$  is a local system, this agrees with the *IC* sheaf  $IC_Y(L_Y)$  defined above. For more details, see [HT08] section 8.2.

### 3.1 The Decomposition Theorem

The *decomposition* theorem tells us, if we have a map  $\pi: Y \rightarrow X$  between spaces, how to decompose the pushforward  $\pi_*(\mathbb{C}_Y)$  into nice irreducible objects of the above form: objects built from IC sheaves of strata. The statement is a little fiddly in general, but the theorem has a very nice form for *semismall* maps. Recall the definition.

**Definition 3.4.** A map  $\pi: Y \rightarrow X$  is called *semismall* with respect to a stratification of  $X$  if for every stratum  $S \subseteq X$ , and for every point  $x \in S$ , one has

$$2 \dim(\pi^{-1}(x)) \leq \dim(X) - \dim(S).$$

The map  $\pi$  is called *small* if equality only holds for the dense stratum. In general, a stratum is called *relevant* if equality holds in the inequality above.

**Theorem 3.5** (Decomposition theorem for semismall maps). Let  $\pi: Y \rightarrow X$  be a semismall map with respect to a stratification  $X = \sqcup S_i$  of  $X$ . Then there is an isomorphism

$$\pi_*(\mathbb{C}_Y) \cong \bigoplus_{S_i \text{ relevant}} IC_{\overline{S}_i}(L_{S_i})$$

where  $L_{S_i}$  is the local system on  $S_i$  induced from the monodromy action on the components of the fibres of  $\pi$  over  $S_i$ .

We can give a further decomposition according to irreducible representations of the fundamental groups  $\pi_1(S_i)$ . Specifically, for each such irreducible representation  $\chi$  we get a local system  $L_\chi$  on  $S_i$ , and we can decompose  $L_{S_i}$  according to isotypic components:

$$L_{S_i} \cong \bigoplus_{\chi \in \text{Irrep}(\pi_1(S_i))} \chi \otimes V_i^\chi.$$

This then gives a total decomposition

$$\pi_*(\mathbb{C}_Y) \cong \bigoplus_{S_i, \chi} IC_{\overline{S}_i}(L_\chi) \otimes V_i^\chi$$

where the sum is over  $S_i$  relevant strata and  $\chi$  irreducible representations of  $\pi_1(S_i)$ . In what follows we'll apply this to the nilpotent cone  $\mathcal{N}$  of a Lie algebra, stratified by orbits for the adjoint action.

## 4 Applications to the Topology of $\mathcal{N}$ and its Springer fibres

We'll conclude by returning to the nilpotent cone, and describing some applications of the decomposition theorem to understanding its topology. Firstly, Borho and Macpherson prove the following:

**Theorem 4.1.** The nilpotent cone  $\mathcal{N}$ , and the partial Springer resolutions  $\tilde{\mathcal{N}}_P$ , are *rational smooth*.

**Definition 4.2.** An  $n$ -dimensional complex variety  $X$  is *rationally smooth* if for every  $x \in X$  we have

$$H_*(X, X \setminus \{x\}; \mathbb{C}) \cong \mathbb{C}[-2n].$$

This admits an equivalent description in terms of intersection cohomology:

**Lemma 4.3.** A stratified space  $X$  is rationally smooth if and only if there is a quasi-isomorphism  $IC(\mathbb{C}_X) \cong p_*\mathbb{C}_X$ , where  $p$  is the map from  $X$  to a point.

*Proof sketch.* That rational smoothness implies this condition on  $IC(\mathbb{C}_X)$  is a Verdier duality calculation. The relative homology  $H_*(X, X \setminus \{x\}; \mathbb{C})$  is a definition of the stalk of the Borel Moore homology at  $x$ , which is to say the Verdier dualising sheaf of  $X$ , which we'll denote  $\omega_X$ . Rational smoothness is equivalent to the isomorphism of the shifted dualising sheaf  $\omega_X[-2n]$  with the constant sheaf  $\mathbb{C}_X$ . Pushing forward:

$$\begin{aligned} p_*\mathbb{C}_X &\cong p_*\omega_X[-2n] \\ &\cong (p!\mathbb{C}_X[-2n])^\vee \end{aligned}$$

using Verdier duality. The map from the right to the left always factors through the IC sheaf, so since it is an isomorphism, the map to the IC sheaf must be an isomorphism also.

The converse is a little harder. We proceed by simultaneous induction on the dimension of  $X$  and the codimension of a stratum. The claim is clear for the dense stratum: points in such strata have smooth neighbourhoods. The Borel-Moore homology (or stalk of the dualising sheaf) at a point  $x$  can be computed as the homology of the link  $\Lambda_x$  of the stratum containing  $x$ , which is rationally smooth by the induction hypothesis. One then computes

$$H_i(\Lambda_x) \cong IH^{2c-i-1}(\Lambda) \cong IH_x^{2c-i-1}(X)$$

for  $c \leq i < 2c$ , where  $c$  is the codimension of our stratum. This establishes our claim in this range of dimensions, and Poincaré duality for the link (using the inductive hypothesis) completes the proof.  $\square$

Thus to prove the theorem we need to compute the intersection complex of  $\mathcal{N}$  and its partial resolutions. Let's discuss the nilpotent cone  $\mathcal{N}$  only; one doesn't need to do much more work to extend the calculation I'll explain to partial resolutions also. We can use the decomposition theorem for this calculation. To compute the stalk of the IC complex at a point  $x$ , we observe

$$\begin{aligned} H^*((G/B)_x; \mathbb{C}) &\cong ((\pi_B)_*\mathbb{C}_{\tilde{\mathcal{N}}})_x \\ &\cong \bigoplus_{i, \chi} (IC_{\overline{S}_i}(L_\chi)_x \otimes V_i^\chi) \end{aligned}$$

in the notation of the statement of the decomposition theorem above, where  $(G/B)_x$  is the Springer fibre over  $x$  and  $p$  is the map from  $\mathcal{N}$  to a point. We have to investigate the isotypic component corresponding to the dense stratum and the trivial representation.

There is a Weyl group action on the complex  $(\pi_B)_*\mathbb{C}_{\tilde{\mathcal{N}}}$ , due to Lusztig. To define this, we generalise the Springer resolution to the so-called *Grothendieck resolution* of  $\mathfrak{g}$ :

$$\xi: \tilde{\mathfrak{g}} = \{(X, \mathfrak{b}') \in \mathfrak{g} \times G/B: X \in \mathfrak{b}'\} \rightarrow \mathfrak{g}.$$

When one restricts to regular semisimple elements, one finds that this map is a  $W$ -fold covering map, i.e.  $W$  acts on  $\tilde{\mathfrak{g}}_{rs}$  by deck transformations. Thus  $W$  acts on the sheaf  $\xi_*(\mathbb{C}_{\tilde{\mathfrak{g}}_{rs}})$ . Now, the map  $\xi$  is *small* – semismall with the dense stratum the only relevant one – so the decomposition theorem tells us

$$\xi_*(\mathbb{C}_{\tilde{\mathfrak{g}}}) \cong IC(\xi_*(\mathbb{C}_{\tilde{\mathfrak{g}}_{rs}}))$$

and upon pulling back along the inclusion  $i: \mathcal{N} \rightarrow \mathfrak{g}$  (i.e. restricting to nilpotent elements), the Weyl group action on the right gives a Weyl group action on  $i^*\xi_*(\mathbb{C}_{\tilde{\mathfrak{g}}}) \cong (\pi_B)_*\mathbb{C}_{\tilde{\mathcal{N}}}$  as required. Now we can use the following crucial theorem, proven in [BM81].

**Theorem 4.4** (Springer Correspondence). The terms in the decomposition for  $((\pi_B)_*\mathbb{C}_{\tilde{\mathcal{N}}})_x$  are in bijection to the irreducible characters of the Weyl group, and the decomposition agrees with the decomposition as a Weyl group representation.

Thus, we need to compute the piece corresponding to the trivial character, i.e. we must show that  $H^*((G/B)_x; \mathbb{C})^W = \mathbb{C}$ .

To do so, let's introduce a last piece of notation. We can factor a parabolic subgroup  $P$  as a product of its unipotent radical  $U$  and a semisimple group  $L \cong P/U$ . This group is called a *Levi subgroup*, and plays the same role inside a parabolic subgroup as a Cartan plays inside a Borel. In particular we can take the Weyl group  $W(L)$  of  $L$ , which sits inside the Weyl group  $W$  of  $G$ . For  $x \in \mathcal{N}$ , denote by  $(G/P)_x$  the fibre over  $x$  in the partial resolution  $\pi_P: \tilde{\mathcal{N}}_P \rightarrow \mathcal{N}$ .

**Proposition 4.5.** For the partial Springer fibre  $(G/P)_x$ , its cohomology arises as  $W(L)$  invariants in the cohomology of the full Springer fibre:

$$H^*((G/P)_x; \mathbb{C}) \cong H^*((G/B)_x; \mathbb{C})^{W(L)}$$

where the  $W(L)$ -action is the restriction of the Weyl group action defined above, using rational smoothness for  $\mathcal{N}$  to identify this with the ordinary cohomology.

*Proof.* The proposition follows from a sheaf-theoretic statement by taking stalks over a nilpotent element  $x \in \mathcal{N} \subseteq \mathfrak{g}$ . We'll prove that  $(\xi_P)_*\mathbb{C}_{\tilde{\mathfrak{g}}_P} \cong (\xi_*\mathbb{C}_{\tilde{\mathfrak{g}}})^{W(L)}$  for the Weyl group action described above. One checks that, on the regular semisimple locus, the partial Grothendieck resolution  $\tilde{\mathfrak{g}}_P \rightarrow \mathfrak{g}$  is obtained from the full Grothendieck resolution by taking the subcover associated to the quotient  $W/W(L)$ . Therefore we have

$$(\xi_P)_*(\mathbb{C}_{\tilde{\mathfrak{g}}_{rs,P}}) \cong \xi_*(\mathbb{C}_{\tilde{\mathfrak{g}}_{rs}})^{W(L)}$$

which gives the statement we want for *all* of  $\mathfrak{g}$  rather than just the dense stratum by applying the decomposition theorem to the desired statement, and using that the maps  $\xi$  and  $\xi_P$  are small, so only the dense stratum contributes.  $\square$

This allows us to complete the proof of 4.1, by plugging in  $P = G$  above. Proving the full theorem requires the result for all  $P$  and a little more care with the Weyl group actions discussed above.

## References

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