String Topology and Hochschild homology

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Today we will relate the string operation to Hochschild homology. Namely we will prove:

Theorem: (Cohen-Jones) There is an isomorphism of algebras

$$\mathbb{H}_*(M) \xrightarrow{\simeq} HH_*(C^*M) \qquad \text{provided } M \text{ is 1-conn.}$$

We will do this by constructing a particular cosimplicial model for LM^{-TM}, that carries a natural product indued from Hochschild cohomology. We will then show that this product is indeed a model for the string product.

I. Hochschild homology

Chris has already mentioned a reason Hochschild homology comes up when studying 2-dimensional TFTs. Here we review the basics of Hochschild homology. Let A be an associative algebra, and M a A-bimodule. Define the chain complex

$$CH_*(A;M): \qquad \cdots \longrightarrow M\otimes A^{\otimes 3} \xrightarrow{b} M\otimes A^{\otimes 2} \xrightarrow{b} M\otimes A$$

where

$$b_n := \sum_{i=0}^{n} (-1)^i d_i \quad , \quad d_i(m\otimes a_1\otimes\cdots\otimes a_n) = \begin{cases} ma_1\otimes\cdots\otimes a_n \ , & i=0 \\ m\otimes a_1\otimes\cdots\otimes a_i a_{i+1}\otimes\cdots\otimes a_n \ , & 0<i<n \\ a_n m\otimes a_1\otimes\cdots\otimes a_{n-1} \ , & i=n \end{cases} .$$

And set $HH_*(A;M) = H(CH_*(A;M)) \longleftarrow$ Hochschild homology of A w/ coeff's in M.

A more invariant way of defining Hochschild is $\boxed{HH_*(A;M) = M \overset{\mathbb{L}}{\underset{A\otimes A^{op}}{\otimes}} A}$.

Similarly define cohomology $HH^*(A;M) := \mathbb{R}\mathrm{Hom}_{A\otimes A^{op}}(A,M)$.

Set $HH_*(A;A) \equiv HH_*(A)$

Ex: $HH_0(A;M) = M/\{am-ma\} \rightsquigarrow HH_0(A) = A/[A,A]$. Also, $HH^0(A) = Z(A)$, the center.

"Hochschild cohomology is a derived center".

"differential forms of an algebra"

Ex: If A is a smooth k-algebra, then $HH_*(A;M) \cong \Omega^*_A \otimes_A M$

II. Jones' identification of $H_*(LM)$

In this section we sketch the existence of an isomorphism

$$H_*(LX) \longrightarrow HH^*(C^*(X); C_*(X)) \quad \text{as } \underset{\sim}{\text{modules}}.$$

Let $\mathrm{Map}(S^1_*, X)$ be the following cosimplicial set. The k-simplices are k-tuples

$$(x_0, \dots, x_k) \in X^{k+1} \simeq \mathrm{Map}(S^1_k, X)$$

and the coface / codeg. maps are

$$\delta_i(x_0, \dots, x_n) = \begin{cases} (x_0, \dots, x_{i-1}, x_i, x_i, x_{i+1}, \dots, x_n), & 0 \le i < n \\ (x_0, \dots, x_n, x_0), & i = n \end{cases}$$

$$\sigma_i(x_0, \dots, x_{n+1}) = (x_0, \dots, \widehat{x_{i+1}}, \dots, x_n), \quad 0 \le i \le n.$$

It is easy to see $\mathrm{Tot}(\mathrm{Map}(S^1_*, X)) \cong LX$. Define maps

$$f_k : |\Delta^k| \times LX \longrightarrow X^{k+1}, \quad (y_1, \dots, y_k; \gamma) \longmapsto (\gamma(0), \gamma(y_1), \dots, \gamma(y_k)) \quad ???$$

Take adjoints $\bar{f}_k : LX \to \mathrm{Map}(|\Delta^k|, X^{k+1})$, and consider the big map

$$f : LX \xrightarrow{\Pi f_k} \prod_k \mathrm{Map}(|\Delta^k|, X^{k+1})$$

Thm: (Jones) f is a homeo onto its image. Moreover $\mathrm{Im}(f) \cong \mathrm{Tot}(\mathrm{Map}(S^1_*, X)) = LX$.

$\mathrm{Tot}(\mathrm{Map}(S^1_*, X)) = \{$maps commuting w/ structure maps$\}$

We can also consider the induced chain maps $f_k^* : C^*(X)^{\otimes k+1} \longrightarrow C^{*-k}(LX)$.

Thm (Jones): The homomorphisms f_k^* assemble to a map $CH_*(C^*(X)) \longrightarrow C^*(LX)$, which is an equiv. when X is 1-conn.

Now, dualize to get equiv. $H_*(LX) \xrightarrow{\simeq} HH^*(C^*(X); C_*(X))$

III : Relation to String topology

We now get to proving our main theorem. We first define a convenient cosimplicial model for LM. Consider

$$\begin{tikzcd} {|\Delta^k| \times LM} \arrow[r, "f_k"] \arrow[d, "ev_0"'] & M^{k+1} \arrow[d, "p_1"] \\ M \arrow[r, equal] & M \end{tikzcd}$$

$-TM$ $\quad$ $-TM$

We get a map $(f_k)_* : e^*(-TM) \longrightarrow p_1^*(-TM)$, and applying $Th(-)$:

$$f_k : (\Delta_k)_+ \wedge LM^{-TM} \longrightarrow M^{-TM} \wedge (M^k)_+ .$$

Where we recall the S-duality

$$Map(S^0, M_+) \simeq M^{-TM}$$

adjoints $\rightsquigarrow$

$$f : LM^{-TM} \xrightarrow{\prod f_k} \prod_k Map((\Delta^k)_+, M^{-TM} \wedge (M^k)_+).$$

Note this is the same as taking the $Th(-)$ of the map

$$LM \longrightarrow \prod_k Map(\Delta^k, M^{k+1}) \quad \text{above.}$$

$\mathbb{L}M_* := M^{-TM} \wedge (M^k)_+$ is the cosimplicial spectrum we want. See Cohen for the structure maps.

Insert Structure Maps.

By taking $Th(-)$ of Jones' result we see f is a homeo onto its image which is identified with $Tot(\mathbb{L}M)$. So indeed $\mathbb{L}M$ is a model. Applying S-duality again we get an equivalence

$$C^*(M^{-TM}) \simeq C_{n-*}(M_+)$$

So we get maps

$$f_k^* : C_{n-*}(M_+) \otimes C^*(M)^k \xrightarrow{\simeq} C^*(M^{-TM} \wedge M^k_+) \longrightarrow C^{*-k}(LM^{-TM})$$

Dualizing :

$$\begin{aligned}(f_k)_* : C_{*-k}(LM^{-TM}) \longrightarrow Hom(C^*(M)^{\otimes k} \otimes C_{-*}(M) ; \mathbb{Z}) &\simeq Hom(C^*(M)^{\otimes k}; C^*(M)) \\ &\simeq CH^k(C^*(M); C^*(M))\end{aligned}$$

Again, applying Jones' result to the $Th(-)$ of above we get a map

$$f_* : C_*(LM^{-TM}) \longrightarrow CH^*(C^*(M); C^*(M))$$

which is an equiv. when M is 1-connected.

Let's review the product structure on CH^*. Given $\varphi, \psi \in CH^*(A;A)$ in deg k, r resp., then

$$\varphi \cup \psi \in CH^{k+r}(A;A) \text{ is defined by } (\varphi \cup \psi)(a_1 \otimes \cdots \otimes a_k \otimes \cdots \otimes a_{k+r}) = \underbrace{\varphi(a_1 \otimes \cdots \otimes a_k)\,\psi(a_{k+1} \otimes \cdots \otimes a_{k+r})}_{}$$

product in A.

Taking adjoints and recalling the identification $\Delta^* = \cup$ via S-duality, this product is realized by:

$$\tilde{\mu}_{k,r} : (M^{-TM} \wedge M^k_+) \wedge (M^{-TM} \wedge M^r_+) \longrightarrow M^{-TM} \wedge M^{k+r}_+$$
$$(u; x_1, \dots, x_k) \wedge (v; y_1, \dots, y_r) \longmapsto (\Delta^*(u,v); x_1, \dots, x_k, y_1, \dots, y_r)$$

These maps define maps $(\mathbb{L}M)_k \wedge (\mathbb{L}M)_r \to (\mathbb{L}M)_{k+r}$, and further define a map

$$\tilde{\mu} : Tot(\mathbb{L}M) \wedge Tot(\mathbb{L}M) \longrightarrow Tot(\mathbb{L}M).$$

<u>Note</u>: This map is A_∞. Let's collect what we know

<u>Thm</u>: Using $Tot(\mathbb{L}M) \simeq LM^{-TM}$, the map $\tilde{\mu}$ gives LM^{-TM} the structure of an A_∞-ring spectrum.

Moreover, this product commutes with $\cup$:

$$\begin{tikzcd} C_*(LM^{-TM}) \otimes C_*(LM^{-TM}) \arrow[r, "\tilde{\mu}"] \arrow[d, "\simeq"] \arrow[dr, phantom, "\circlearrowright"] & C_*(LM^{-TM}) \arrow[d, "\simeq"] \\ CH^*(C^*M) \otimes CH^*(C^*M) \arrow[r, "\cup"] & CH^*(C^*M) \end{tikzcd}$$

So we really need to prove that

$$\begin{tikzcd} LM^{-TM} \wedge LM^{-TM} \arrow[r, "\mu"] \arrow[d, "\simeq"] \arrow[dr, phantom, "\circlearrowright"] & LM^{-TM} \arrow[d, "\simeq"] \\ Tot(\mathbb{L}M) \wedge Tot(\mathbb{L}M) \arrow[r, "\tilde{\mu}"] & Tot(\mathbb{L}M) \end{tikzcd}$$

String product

And we'd be

<u>Done</u>

Pf: Suffices to show

$$\begin{tikzcd}
\Delta^{k+r}_+ \wedge LM^{-TM} \wedge LM^{-TM} \arrow[r, "1\wedge\mu"] \arrow[d, "w_{k,r}\wedge 1"'] & \Delta^{k+r}_+ \wedge LM^{-TM} \arrow[dd, "f_{k+r}"] \\
(\Delta^k \times \Delta^r)_+ \wedge LM^{-TM} \wedge LM^{-TM} \arrow[d, "f_k \wedge f_r"'] & \\
M^{-TM} \wedge M^k_+ \wedge M^{-TM} \wedge M^r_+ \arrow[r, "\tilde{\mu}_{k,r}"] & M^{-TM} \wedge M^{k+r}_+
\end{tikzcd}$$

commutes $\forall\ k,r$.

Here, $w_{k,r} : \Delta^{k+r} \longrightarrow \Delta^k \times \Delta^r$, $(t_1,\dots,t_{k+r}) \longmapsto (t_1,\dots,t_k) \times (t_{k+1},\dots,t_{k+r})$. Consider the P.B.

$$\begin{tikzcd}
\Delta^k \times \Delta^r \times LM \times_M LM \arrow[r, hook] \arrow[d, "f_{k,r}"'] & \Delta^k \times \Delta^r \times LM \times LM \arrow[d, "f_k \times f_r"] \\
M^k \times M^r \times M \arrow[r, "\Delta"] & M^k \times M^r \times M \times M
\end{tikzcd}$$

By naturality of Pontryagin-Thom we get a square:

$$\begin{tikzcd}
(\Delta^k \times \Delta^r)_+ \wedge LM^{-TM} \wedge LM^{-TM} \arrow[r, "1\wedge\tau"] \arrow[d, "f_k \wedge f_r"] & (\Delta^k \times \Delta^r)_+ \wedge (LM \times_M LM)^{-TM} \arrow[d, "f_{k,r}"] \\
M^{-TM} \wedge M^k_+ \wedge M^{-TM} \wedge M^r_+ \arrow[r, "\tau'"] & M^{-TM} \wedge M^k_+ \wedge M^r_+
\end{tikzcd} \qquad (1)$$

Now consider the map $\gamma : LM \times_M LM \longrightarrow LM$, loop composition. We have a diagram:

$$\begin{tikzcd}
\Delta^{k+r} \times LM \times_M LM \arrow[r, "1\times\gamma"] \arrow[d, "w_{k,r}\times 1"'] & \Delta^{k+r} \times LM \arrow[dd, "f_{k+r}"] \\
\Delta^k \times \Delta^r \times LM \times_M LM \arrow[d, "f_{k,r}"'] & \\
M^k \times M^r \times M \arrow[r] & M^{k+r+1}
\end{tikzcd}$$

Applying $Th(-)$ we get:

$$\begin{tikzcd}
\Delta^{k+r}_+ \wedge (LM \times_M LM)^{-TM} \arrow[r, "1\wedge\gamma"] \arrow[d, "w_{k,r}\wedge 1"'] & \Delta^{k+r}_+ \wedge LM^{-TM} \arrow[dd, "f_{k+r}"] \\
(\Delta^k \times \Delta^r)_+ \wedge (LM \times_M LM)^{-TM} \arrow[d, "f_{k,r}"'] & \\
M^{-TM} \wedge M^k_+ \wedge M^r_+ \arrow[r, "="] & M^{-TM} \wedge M^{k+r}_+
\end{tikzcd}$$

Conglomerating:

$$\begin{tikzcd}
\Delta^{k+r}_+ \wedge LM^{-TM} \wedge LM^{-TM} \arrow[r, "1\wedge\tau"] \arrow[d, "w_{k,r}\wedge 1"'] \arrow[rr, bend left, "1\wedge\mu"] \arrow[dr, phantom, "3"] & \Delta^{k+r}_+ \wedge (LM \times_M LM)^{-TM} \arrow[r, "1\wedge\gamma"] \arrow[d, "w_{k,r}\wedge 1"'] \arrow[ddr, phantom, "2"] & \Delta^{k+r}_+ \wedge LM^{-TM} \arrow[dd, "f_{k+r}"] \\
(\Delta^k \times \Delta^r)_+ \wedge LM^{-TM} \wedge LM^{-TM} \arrow[r, "1\wedge\tau"] \arrow[d, "f_k\wedge f_r"] \arrow[dr, phantom, "1"] & (\Delta^k \times \Delta^r)_+ \wedge (LM \times_M LM)^{-TM} \arrow[d, "f_{k,r}"] & \\
M^{-TM} \wedge M^k_+ \wedge M^{-TM} \wedge M^r_+ \arrow[r, "\tau'"] & M^{-TM} \wedge M^k_+ \wedge M^r_+ \arrow[r, "="] & M^{-TM} \wedge M^{k+r}_+
\end{tikzcd}$$

3 is the only new square, but it obvs. commutes.

$\square$