The string topology product

Brian Williams

October 20, 2013

Today I'm going to define the product which gives rise to the structure called string topology. It is a product defined on the homology of the free loop space of a manifold. We will discuss some algebraic structures this product exhibits.

Fix a smooth, closed, orientable n-manifold M. The main object of study is the free loop space of M, define by

$$LM := \mathrm{Map}(S^1, M)$$

where we take piecewise smooth maps. Poincaré dual to the cup product on M is the so-called intersection product

$$H_p(M) \times H_q(M) \to H_{p+q-n}(M)$$

which can be interpreted geometrically as follows. A p-chain can be represented by a p-dimensional submanifold P. Similarly a q-chain is represented by Q. We can perturb P, Q so as to make their intersection transverse. Then a standard theorem in differential topology says that $P \cap Q$ is $p+q-n$ dimension submanifold of M, hence determines a chain. Passing to homology this all is well defined and reproduces the above product.

1 The product

There are a couple of ways to define the aforementioned product. Originally, it was defined by Chas and Sullivan as a type of intersection product. Cohen and Jones came up with a definition that passes through the so-called Pontryagin-Thom collapse map. Furthermore, they extended this to a product on a related spectrum and upon taking homology realizes the original string product.

1.1 Via the intersection product

Here we discuss the original definitoin of the string product a la Chas-Sullivan. We consider $C_*(LM)$. Each loop has a marked point, namely the image of $0 \in S^1$. Consider $\alpha \in C_p(LM)$ and $\beta \in C_q(LM)$. The set of marked points of these chains can be viewed as dimension p, q submanifolds of M, i.e p, q-chains on M. We can intersect these to get a $p+q-n$ chain γ on M. Along this chain, the marked points of α, β coincide, so we can consder forming a loop by traversing α then β at each point of γ. This defined a $p+q-n$-chain of LM. On homology we denote this product by

$$\circ : H_p(LM) \times H_q(LM) \to H_{p+q-n}(LM).$$

Before talking about any algebraic structures, let's go over an equivalent construction of $\circ$.

1.2 Via Thom collapse

Let's recall some general notions about Thom spaces/spectra. Let $i : N \hookrightarrow M$ be a k-dimensional submanifold. Take a tubular neighborhood around N which we can identify with the total space of the normal bundle to the embedding denoted ν_N. Define the map

$$\tau : M \to \nu_i \cup \{\infty\} \ , \ \tau|_{\nu_i} = \mathrm{id}_{\nu_i} \ , \ \tau|_{M \setminus \nu_i} = \infty.$$

Notice that $\nu_i \cup \infty$ is nothing but the the Thom space $\mathrm{Th}(\nu_N)$. So we are really producing a map

$$\tau : M \to \mathrm{Th}(\nu_i).$$

Applying homology we get

$$i_! : H_p(N) \xrightarrow{\tau} H_p(\mathrm{Th}(\nu_i)) \xrightarrow[\simeq]{\text{Thom}} H_{q-n+k}(N)$$

where we have postcomposed with the Thom isomorphism. Such a "wrong-way" map is called an Umkher map.

Example 1.1. Lets consider the embedding

$$\Delta : M \hookrightarrow M \times M$$

so that $\nu_\Delta \simeq TM$. The induced map

$$\Delta_! : H_q(M \times M) \to H_{q-d}(M)$$

is just the intersection product discussed above.

We can talk about some stringy stuff again. Consider the pull-back space

$$X = LM \times_M LM.$$

This space comes equipped with an obvious map $\gamma : X \to LM$ which can be viewed as extending the usual product on ΩM. X can also be viewed as the mapping space of figure eights into M. It also fits into the pull-back square

$$\begin{array}{ccc} X & \xrightarrow{\tilde{\Delta}} & LM \times LM \\ \downarrow{\scriptstyle \mathrm{ev}} & & \downarrow{\scriptstyle \mathrm{ev}\times\mathrm{ev}} \\ M & \xrightarrow{\Delta} & M \times M. \end{array} \tag{1}$$

Now LM is certainly not an infinite dimensional manifold. Nevertheless, one can see that the map $\mathrm{ev} : LM \to M$ is a (locally trivial) fibre bundle. Since the square is a pull-back, we can therefore view $X \to LM \times LM$ as a codimension n embedding. Namely, we can take tubular neighborhoods, and it turns out that

$$\nu_{\tilde{\Delta}} \simeq \mathrm{ev}^* \nu_\Delta \simeq \mathrm{ev}^* TM.$$

So we get a Thom collapse map as above

$$\tau_{\tilde{\Delta}} : LM \times LM \to X^{\mathrm{ev}^* TM}.$$

Combining all this we get

Theorem 1.1. *There is a product*

$$\circ : H_*(LM) \times H_*(LM) \longrightarrow H_*(LM \times LM) \xrightarrow{\tilde{\Delta}_!} H_{*-d}(X) \xrightarrow{\gamma_*} M.$$

that coincides with the Chas-Sullivan product mentioned above.

Remark. Nothing was special about ordinary homology here. This works just as well for any multiplicative generalized cohomology theory h_* so long as M is appropriately oriented.

We now explain how to realize this product as coming from the ring structure on an associated ring spectrum. For this we will need a "twisted" version of the Thom collapse map. Let ζ be a vector bundle over M. The embedding $N \hookrightarrow M$ extends to an embedding of total spaces of bundles via pull-back:

$$\begin{array}{ccc} i^*\zeta & \xrightarrow{\tilde{i}} & \zeta \\ \downarrow & & \downarrow \\ N & \xrightarrow{i} & M. \end{array}$$

Then $\nu_{\tilde{i}} \simeq i^* \oplus \nu_i$ so that the Thom collapse map has the form

$$\begin{array}{ccc} \xi \cup \{\infty\} & \xrightarrow{\tau_\zeta} & \nu(i^*\zeta) \cup \{\infty\} \\ \| & & \| \\ \mathrm{Th}(\zeta) & \longrightarrow & \mathrm{Th}(i^*\zeta \oplus \nu_i). \end{array}$$

This construction actually works for any virtual bundle over M. Suppose $\zeta = -E$ where E is some rank k bundle over M. We form the Thom spectrum $\mathrm{Th}(\zeta)$ over M as follows. Choose an integer k' such that $E \hookrightarrow M \times \mathbb{R}^{k+k'}$ and let $E^\perp$ be the k'-dimensional orthogonal complement taken in $\mathbb{R}^{k+k'}$. Define the spectrum as

$$\mathrm{Th}(\zeta) = \mathrm{Th}(-E) = \Sigma^{-(k+k')}\mathrm{Th}(E^\perp).$$

In this setting the Thom isomorphism takes the form

$$H_*(\mathrm{Th}(-E)) \simeq H_{*+k}(M).$$

Example 1.2. Take $\zeta = -TM \times -TM$ and the diagonal embedding $M \hookrightarrow M \times M$. Then the induced map

$$\begin{array}{ccc} (M \times M)^{-TM\times -TM} & \xrightarrow{\tau} & M^{TM \oplus \Delta^*(-TM\times -TM)} \\ \downarrow \simeq & & \downarrow \simeq \\ \mathrm{Th}(-TM) \wedge \mathrm{Th}(-TM) & \longrightarrow & \mathrm{Th}(-TM). \end{array}$$

Atiyah showed that $\mathrm{Th}(-TM)$ is actually the Spanier-Whitehead dual of M with a disjoint basepoint added. Moroever, one can check that the above product just gives Spanier-Whitehead dual of $\Delta : M \to M \times M$.

Again consider diagram (1). What we do now is pull back the virtual bundle $-TM \times -TM$ to $LM \times LM$, and twist by this. The relevant collapse map is

$$(LM \times LM)^{(\mathrm{ev}\times\mathrm{ev})^*(-TM\times -TM)} \longrightarrow X^{\mathrm{ev}^*TM \oplus \mathrm{ev}^*(\Delta^*(-TM\times -TM))}.$$

But, $\Delta^*(-TM \times -TM) = -2TM$, so we have a map

$$LM^{-TM} \wedge LM^{-TM} \longrightarrow X^{-TM}$$

where we drop pulling back by ev^* for notational convenience. The map $\gamma : X \to LM$ extends to a map of Thom spectra

$$X^{-TM} \to LM^{-TM}$$

and post-composing with this we get the product

$$LM^{-TM} \wedge LM^{-TM} \to LM^{-TM}.$$

Taking homology and applying appropriate Thom isomoprhisms this reproduces the above product.

What all of this says is that $LM^{-TM} = \mathrm{Th}(-TM)$ is a *ring spectrum*. Morally, and rigourously proven by Chas-Sullivan, the ordinary string product plays nicely with intersection products and the standard loop product on ΩM_+. This manifests itself at the spectrum level as the existence of ring maps

$$LM^{-TM} \to M^{-TM}$$

and

$$\Sigma^\infty(\Omega M_+) \to LM^{-TM}.$$

The first map is simpy induced by evaluation. The second map is induced by the fibration:

$$\begin{array}{ccc} \Omega M_+ & \longrightarrow & LM \\ \downarrow & & \downarrow \\ \{*\} & \longrightarrow & M \end{array}$$

and comes from an appropriate Thom collapse map. Namely pulling back the tangent bundle across the fibration.

1.3 Algebraic structure

It is clear that $\circ$ provides the structure of an associative, commutative, graded algebra on

$$\mathbb{H}_*(M) := H_{*-d}(LM) = H_*(LM)[d].$$

There is slightly more structure here which we mention now.

The original structure that Chas-Sullivan produce is a so-called Batalin-Vilkovisky (BV-) algebra on $\mathbb{H}_*(M)$. This algebraic structure crops up everywhere in mathematical physics and is related to the framed little 2-discs operad. Namely it is a result of Getzler that there is a bijective correspondence between BV algebras and algebras over the framed little 2-discs operad E_2. Actually, its an algebra over the homology of the framed little 2-discs operad, but by formality this is the same.

We can realize this at the spectra level as well. It is a result of Salvatore and Gruher that given a fiberwise monoid E over M that carries a fiberwise action of E_n, then E^{-TM} has the structure of a E_n-ring spectrum. But I we have just mentioned that on homology level, string topology carries an E_2-structure. Cohen-Jones remedy this by constructing an explicity action of the cactus operad on LM^{-TM}, which is homotopy equivalent to the framed discs operad. Moreover, one can show that this induces the correct E_2 structure on homology.