Quantisation of Cotangent Theories

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1 Recognising Cotangent Theories

In Kevin's talk last week we learned about the idea of a *cotangent theory*: a classical field theory that can be described as a sheaf of dg-schemes of the form $T^*[-1]M$, where M is a smooth scheme. I'll explain how many of the classical field theories we've discussed in this seminar – theories that arise as twists of supersymmetric theories – fit into this framework. In particular we'll see in concrete examples how to relate cotangent theories in the language Kevin used and theories described by an action functional.

Many of the action functionals we've seen so far have a particularly nice form: we have fields $A_1, \ldots, A_n$; $B_1, \ldots, B_n$, differential operators $D_1, \ldots D_n$, and action given by a certain pairing

$$\sum_{i=1}^{n} \langle B_i, D_i A_i \rangle.$$

We'll see that this form is characteristic of cotangent theories. Indeed, let's demonstrate this by computing the moduli space of classical solutions in some examples.

Example 1.1 (Twisted $N = 1$ Theory)**.** The first example is one we've discussed several times recently. In the $\frac{1}{2}$-twisted $N = 1$ Super Yang-Mills theory on a complex surface X (for a fixed principal G-bundle P) we have fields

$$A \in \Omega^{0,*}(X; \mathfrak{g}_P)[1]$$
$$B \in \Omega^{2,*}(X; \mathfrak{g}_P)$$

and action

$$S(A, B) = \int B \wedge \overline{\partial}(A)$$

which we recognise as an example of the general form described above, with differential operator $\overline{\partial}$ and the integration pairing (we might further split the action into two terms corresponding to $A \in \Omega^{0,0}$ and $\Omega^{0,1}$, with non-degenerate pairings for each term). We compute the equations of motion to be

$$\overline{\partial} A = 0, \overline{\partial} B = 0.$$

so A is now an element of the Dolbeault *complex*, and B is an element of the dual complex $(\Omega^{2,*}, \overline{\partial})$. We might interpret A as a point in the derived moduli space of holomorphic structures on P, and B as an element of the cotangent complex to the moduli space at this point, where the action describes the symplectic pairing.

This example is characteristic: the action describes a (-1)-symplectic pairing on the classical moduli, and the description of fields as A and B describe a global system of Darboux coordinates on this moduli space, identifying it as $T^*[-1]N$, where N is the space of solutions to the equations of motion for A.

I've worked with a fixed P here. One could extend to include all topological types of bundles, and obtain as the phase space the shifted cotangent bundle to the whole moduli space of holomorphic G-bundles.

Example 1.2 (Twisted $N = 4$ Theories)**.** The other examples Kevin discussed last week were $N = 4$ theories. We can cut to the chase and describe the cotangent theory for the B $\frac{1}{2}$-twisted theory (The Kapustin-Witten $\frac{1}{2}$-twist) directly. The moduli space of 'A' fields solving the equations of motion are now

$$A \in \Omega^{*,*}(X; \mathfrak{g}_P)[1],$$

with differential $\overline{\partial}$. We interpret this as the moduli of holomorphic structures and Higgs fields on P. The cotangent directions also look like $\Omega^{*,*}(X; \mathfrak{g}_P)$, but shifted down by 1, with the (-1)-symplectic pairing coming from the integration pairing.

Another $N = 4$ twisted theory is the full B-twist. We'll find that the phase space in this theory is the shifted cotangent bundle to the moduli space of flat connections on P. That is, as Kevin explained, we find that the differential on the base of the cotangent bundle is deformed from $\overline{\partial}$ to $d = \overline{\partial} + \partial$. Notice that the resulting complex now has finite virtual dimension.

1.1 The Phase Space

Now, let $Y \subseteq X$ be a codimension 1 submanifold. The *phase space* of a classical system on the submanifold Y is defined to be the space of germs of solutions to the equations of motion on Y. Again, let's discuss this in an example where we can see what's going on explicitly.

Example 1.3 (Twisted $N = 1$ Theory)**.** Let's consider the simplest example above, on a complex surface of the form $\Sigma \times \mathbb{C}$ (though we could take the second factor to be a cylinder or torus instead if we preferred). Let γ be a circle in $\mathbb{C}$ around the origin, and let A be an annulus around the origin containing γ. We'll investigate the phase space on the codimension 1 submanifold $\Sigma \times \gamma$. Consider the space of classical solutions on $\Sigma \times A$. As we saw above, we should find classical solutions of the form

$$(\Omega^{0,*}(\Sigma \times A; \mathfrak{g}_P)[1], \overline{\partial}) \times (\Omega^{2,*}(\Sigma \times A; \mathfrak{g}_P), \overline{\partial})$$

We're really interested in the limit where A becomes arbitrarily thin. To see that this limit is well-behaved, we use the fact that A is flat, so the canonical bundle is trivial. Thus the covector field (B above) is a $\overline{\partial}$-closed element of

$$\Omega^*(X \times A; \mathfrak{g}_P \otimes K_\Sigma \otimes_{\mathbb{C}} \mathbb{C}((t)))$$

for all annuli A, which we can think of as a *Higgs field* for the loop group. Thus we see that the phase space on $\Sigma \times \gamma$ should be isomorphic to

$$T^* L \operatorname{Bun}_G(\Sigma) \cong T^* \operatorname{Bun}_{LG}(\Sigma),$$

the shifted cotangent bundle of the loop space (at least heuristically).

2 Cotangent Quantisation

Cotangent theories are particularly natural from the point of view of quantisation. One can produce the Hilbert space of the quantisation of a classical cotangent theory by following directly the *canonical quantisation* recipe from quantum mechanics, or from the quantisation of free scalar field theories. For the moment we'll focus on the case where M is *finite-dimensional*, where everything works very smoothly. There are additional subtleties in the infinite-dimensional case which we'll discuss at the end.

So consider a finite-dimensional cotangent theory on X. That is, the phase space on a codimension 1 submanifold $Y \subseteq X$ should be a finite-dimensional cotangent space. Let's focus on the case where $X = \Sigma_1 \times \Sigma_2$ is a product of two Riemann surfaces, and let our codimension 1 submanifold be $\Sigma_1 \times \gamma$ for a closed curve γ in Σ_2. Call the phase space on this submanifold $M = T^*N$. The Hilbert space associated to this submanifold will be the space $\mathcal{H} = L^2(N)$. To explain why, we should say something about quantisation of operators.

2.1 Canonical Quantisation of Observables

The most classical example of cotangent quantisation is *canonical quantisation* of functions on a vector space. We'll review this now. Consider the example where $N \cong \mathbb{R}^n$, so $M \cong \mathbb{R}^{2n}$. Classical polynomial observables are functions on the phase space, so elements of the polynomial algebra $\mathbb{R}[p_1, \dots, p_n, q_1, \dots q_n]$. Here I have chosen *canonical coordinates* on M: that is, Darboux coordinates for the cotangent bundle: the q_i are coordinates in the base and the p_i are coordinates in the cotangent fibre. This algebra comes equipped with a *Poisson bracket*, which on the coordinates is given by

$$\{p_i, p_j\} = \{q_i, q_j\} = 0, \quad \{p_i, q_j\} = \delta_{ij},$$

which is antisymmetric and is extended to the whole algebra as a derivation: $\{fg, h\} = f\{g, h\} + g\{f, h\}$. To follow the story from above more closely we might take the p_i to be in degree -1, but this grading won't be important for what follows.

Quantisation is supposed to promote these observables f to self-adjoint operators $\mathcal{O}_f$ on the Hilbert space $\mathcal{H}$ in such a way that

$$[\mathcal{O}_f, \mathcal{O}_g] = i\hbar\{f, g\}.$$

For the operators $\mathcal{O}_{p_i}, \mathcal{O}_{q_j}$ quantising the coordinate operators on the phase space, the resulting commutation relations we require are called the *canonical commutation relations*. We see that these operators, along with $\mathcal{O}_1$, quantising the observables on the phase space must form a representation of the *Heisenberg algebra* generated by self-adjoint operators, or equivalently a unitary representation of the *Heisenberg group*.

Now, we can cook up such a representation in a standard way, by setting $\mathcal{O}_{q_i}$ to be a multiplication operator by the coordinate q_i, and $\mathcal{O}_{p_i}$ to be the differential operator $-i\hbar\frac{\partial}{\partial q_i}$ acting on $L^2(\mathbb{R}^n)$. This is called the *Schrödinger representation* of the Heisenberg group, and is guaranteed to be *unique* up to unitary equivalence by the Stone-von Neumann Theorem.

Let's extend this to more general cotangent bundles, now that we satisfactorily understand the local story. The recipe is the following:

1. To each classical observable f we associate a Hamiltonian vector field v_f.
2. We can build an operator on $L^2(M)$ by the formula
$$\mathcal{O}_f^{\mathrm{pre}}(\psi) = -i\hbar\nabla_{v_f}\psi + f \cdot \psi.$$
3. We produce an operator $\mathcal{O}_f$ on $L^2(N)$ by restricting $\mathcal{O}_f^{\mathrm{pre}}$ to the zero section of the cotangent bundle.

We can readily check that on $T^*\mathbb{R}^n$, or locally, this recipe recovers the Schrödinger representation described above, and that globally the quantised polynomial observables satisfy suitable commutator relations.

Example 2.1. A situation where the phase space is finite-dimensional, as we saw above, is the full B-twist of $N = 4$ super Yang-Mills. We'll find the phase space on a manifold of form $\Sigma \times \gamma$ will be

$$T^* \operatorname{Loc}_G(\Sigma \times \gamma)$$

which is a finite-dimensional smooth (derived) manifold. So the Hilbert space there is

$$\mathcal{H} = L^2(\operatorname{Loc}_G(\Sigma \times \gamma)).$$

We can think of this as a (completed) version of the Hochschild (co)homology of the ring of functions on $\operatorname{Loc}_G(\Sigma)$ (the homology and cohomology should be isomorphic).

What about familiar observables in this theory, like Wilson operators? As functions only on the base of the cotangent bundle, these should become multiplication operators under quantisation. The interesting Wilson operators from the point of view of this phase space are those corresponding to loops of the form $\{\sigma\} \times \gamma$, for $\sigma \in \Sigma$. The Wilson operator $W = W_R(\{\sigma\} \times \gamma)$ quantises to become

$$\mathcal{O}_W(\psi) = W \cdot \psi.$$

We'll look at this in more detail next week, when we try to tie together Wilson and 't Hooft operators with operators on the categories in geometric Langlands.

2.2 Infinite-dimensional Phase Spaces

Now, broadly speaking this method will still work even for infinite-dimensional cotangent theories. But we'll have to be more careful, because a new subtlety can arise. In the discussion above we never explicitly referred to the *dynamics* of the classical field theory. The phase space assigned to a codimension 1 submanifold retains a piece of data that we haven't used: the *Hamiltonian*: a Hamiltonian vector field describing infinitesimal time evolution. Under quantisation, this should be promoted to an operator on the Hilbert space.

A problem that might arise in the infinite-dimensional setting has to do with the spectrum of this operator: for the quantised system to be meaningful we require that the Hamiltonian operator has bounded-below spectrum. In finite dimensions this is not an onerous requirement to impose.

Let's consider an infinite-dimensional example: the twisted $N = (0, 2)$ supersymmetric sigma model with target $\mathbb{C}$, working in a formal neighbourhood of the constant map 0. We find the phase space is $T^*[-1]\mathbb{C}((t))$: the cotangent bundle to the space of formal loops in $\mathbb{C}$. In this language, the Hamiltonian is given by the vector field $-t\frac{d}{dt}$ on $\mathbb{C}((t))$, which quantises to the *operator* $-t\frac{d}{dt}$ on $\widehat{\mathrm{Sym}}(\mathbb{C}((t)))$. This has arbitrarily negative eigenvalues, since $-t\frac{d}{dt}$ has eigenfunctions t^a for all $a \in \mathbb{Z}$ on Sym^1.

To fix this, we need to choose our canonical coordinates in a more careful way than just taking the zero section of the cotangent bundle: we choose some *other* Lagrangian submanifold of the phase space on which the Hamiltonian has better behaved spectrum. A suitable choice – suggested by the state-field-correspondence – is to take those functions on the phase space that extend from formal loops to formal discs and consider the cotangent bundle of *that* as our Lagrangian, namely

$$L = T^*[-1]\mathbb{C}[[t]],$$

on which the Hamiltonian has only non-negative eigenvalues as required.