Potential papers, with brief summaries

Key:

- Papers marked ★ are the key papers that we definitely want to include.
- Papers marked $\star$ are important papers that we would also like to include if we have time.
- Papers not marked don't fit quite so well into the theme, or might be too long or technical, but could be covered if someone feels particularly strongly about including them.

We're open to other suggestions, let me know if there's another paper you might like to hear about, and I can add it to the list.

Kirillov, Unitary representations of nilpotent Lie groups, 1962 ★

This paper introduces the *orbit method* (for nilpotent groups). That is, for a nilpotent Lie group G with Lie algebra $\mathfrak{g}$, Kirillov constructs a canonical bijection between irreducible unitary representations of G and orbits for the coadjoint action of G on $\mathfrak{g}^*$. He goes on to discuss the representation theoretic meaning of the geometry of the coadjoint orbits (for instance, how representations are induced from subgroups). [Kir62] 52 pages.

Bernstein-Gelfand-Gelfand, Differential operators on the base affine space and a study of $\mathfrak{g}$-modules, 1975 $\star$

This paper has two independent parts. The *base affine space* in the title refers to the space G/N where N is a maximal unipotent subgroup. The first part discusses relationships between differential operators on G/N and functions on G. More specifically they conjecture that there exists a nice map $\mathcal{O}(G) \to D(G/N)$, compatible with the left and right G actions and the pullback $\mathcal{O}(G/N) \to \mathcal{O}(G)$. The second part introduces the BGG-resolution, explicit resolutions of highest weight modules $L(\lambda)$ associated to positive weights of a semisimple Lie group, where the terms in the resolution are constructed from Verma modules. [BGG75] 44 pages.

Kostant, On Whittaker vectors and representation theory, 1978

Whittaker modules give a procedure for constructing irreducible representations of a Lie algebra distinct from the construction by highest weight modules. These are cyclic modules generated by *Whittaker vectors*, which are simultaneous eigenvectors for the nilpotent radical with eigenvalue given by a character. Kostant also studies representations V, for instance their decomposition into irreducible representations, using their spaces of Whittaker vectors $\mathrm{Wh}(V)$. [Kos78] 84 pages.

Beilinson-Bernstein, Localisations de $\mathfrak{g}$-modules, 1981 ★

The localisation referred to in the title means localisation of $U(\mathfrak{g})$ at a central character χ (that is, a character χ induces a morphism $Z(\mathfrak{g}) \to D_\chi(G/B)$: the localisation is the quotient of $U(\mathfrak{g})$ by the kernel of this map). The main theorem establishes an equivalence between modules over this localisation and χ-twisted D-modules on G/B. They discuss the relationship with Harish-Chandra modules. [BB81] 4 pages.

Springer, Quelques applications de la cohomologie d'intersection, 1981 ⋆

Taking the Poincaré polynomial of IC sheaves of closures of Bruhat strata in a flag variety G/B yields elements of a Hecke algebra. These are basically Kazhdan-Lusztig polynomials, and products of these polynomials is inherited from convolution of IC sheaves. He uses the decomposition theorem of Beilinson-Bernstein-Deligne-Gabber to give a geometric proof of the Kazhdan-Lusztig conjecture. [Spr82] 25 pages.

Beilinson-Bernstein-Deligne, Faisceaux pervers, 1982

This is more of a comprehensive monograph on the theory of perverse sheaves than a paper, but it's not clear to me at the moment what a more suitable reference might be. [BBD82] 164 pages.

Borho-Macpherson, Partial resolutions of nilpotent varieties, 1983 ★

A version of the Springer correspondence is constructed, giving a resolution of the singularities of the nilpotent cone of a Lie algebra $\mathcal{N} \subseteq \mathfrak{g}$ isomorphic to $T^*(G/B)$. Further, they generalise to *partial* resolutions associated to more general parabolic subgroups of G. They also study generalisations of *Steinberg varieties*, which are the fibres of these resolutions. .... The proof uses the decomposition theorem for semismall maps, and the first part of the paper introduces the necessary background. [BM83] 52 pages.

Beauville-Laszlo, Conformal blocks and generalized theta functions, 1993 ★

This is the first paper on the list that's at chromatic level one, i.e. fundamentally about loop groups rather than complex semisimple groups. They describe the affine Grassmannian and the moduli space of G-bundles on a curve for $G = SL_n$ as ind-schemes, and the determinant line bundle, and they show that the cohomology of the moduli space with coefficients in a power of this line bundle is isomorphic to a space of conformal blocks (a space associated to a certain representation of the loop group). This cohomology can therefore be computed by a Verlinde formula. [BL94] 43 pages.

Beilinson-Bernstein, A Proof of Jantzen Conjectures, 1993 ★

This paper extends the localisation theorem of [BB81] so as to match up certain natural filtrations on objects: *Jantzen* filtrations on Verma modules, as objects in the localisation of the category of $U(\mathfrak{g})$-modules, match up with weight filtrations on perverse sheaves. Hodge theory then can be used to prove properties of the Jantzen filtration, for instance relating filtered pieces of Verma modules at different weights. [BB93] 50 pages.

Beilinson-Ginzburg-Soergal, Koszul Duality Patterns in Representation Theory, 1996 ⋆

The "Koszul duality patterns" referred to in the title are Koszul dualities between algebras constructed from highest weight modules for a semisimple Lie algebra: Ext-algebras of certain sums of highest weight modules. It can be viewed as a categorification of certain categorical results about Kazhdan-Lusztig polynomials, which can be identified as the Poincaré polynomials of such Ext-algebras. [BGS96] 55 pages.

Laumon, Transformation de Fourier généralisée, 1996 ⋆

The classical Fourier transform gives an equivalence between the bounded derived categories of quasicoherent sheaves on an abelian group scheme A and its dual, $A' = \mathrm{Pic}(A)$, realised as an integral transform. Laumon gives a *twisted* version of this, where on one side we replace quasicoherent sheaves by D_A-modules, and on the other we replace A' by $A^\natural$: the moduli space of line bundles on A with *flat connection*. This was also done independently by Rothstein in the same year [Rot96]. [Lau96] 47 pages.

Mirkovic-Vilonen, Geometric Langlands duality and representations of algebraic groups over commutative rings, 2004 ★

The main theorem of the paper is the *geometric Satake isomorphism*: for a reductive group G with Langlands dual G^L, they give an equivalence of tensor categories between perverse sheaves on the stack $G(\mathcal{O})\backslash G(K)/G(\mathcal{O})$ (with the convolution tensor structure, which they describe) and representations of G^L (with the ordinary tensor product). The proof uses Tannakian methods: they show that the former category is equivalent to the category of representations of *some* group scheme by formal category-theoretic methods, then show that its root datum is dual to that of G. [MV07] 44 pages.

Frenkel-Gaitsgory, Localization of $\hat{\mathfrak{g}}$-modules on the affine Grassmannian, 2005 ★

We are interested in an affine analogue of the Beilinson-Bernstein equivalence of [BB81], where the flag variety G/B is replaced by the *enhanced affine flag variety* $G(K)/I^0$ where I^0 is the unipotent radical of an Iwahori, the character χ is replaced by a level κ, the twisted D-modules are twisted by a power of the determinant line bundle $\mathcal{L}^{\otimes\kappa}$, and the localised enveloping algebra of $\mathfrak{g}$ is replaced by the Kac-Moody algebra $\hat{\mathfrak{g}}_\kappa$ at level κ. In this paper, Frenkel and Gaitsgory work in particular in the case of the *critical level*, and conjecture an equivalence between Hecke eigensheaves on the affine Grassmannian at this level and regular representations of the Kac-Moody algebra. Although they don't prove this conjecture, they prove a version where one further imposes invariance with respect to an action of I^0 [FG05] 34 pages.

Bezrukavnikov, Noncommutative Counterparts of the Springer Resolution, 2006

This paper was derived from an ICM talk, so contains a summary of ideas from other papers by Bezrukavnikov and others. He describes a construction of a 'noncommutative' Springer resolution, which is a certain torsion-free sheaf on a space related to the Springer resolution $\widetilde{\mathcal{N}}$ of the nilpotent cone so that modules for this sheaf are equivalent to quasicoherent sheaves on $\widetilde{\mathcal{N}}$. The equivalence sends the standard t-structure on the category of modules to a new "exotic" t-structure on $D(\widetilde{\mathcal{N}})$. Bezrukavnikov shows that this t-structure is related to ones that arise naturally through the geometric Langlands program. [Bez06] 23 pages.

Bezrukavnikov-Finkelberg, Equivariant Satake category and Kostant-Whittaker reduction, 2008

An extension is given of the geometric Satake equivalence of [MV07] to an equivalence of derived categories, where the Hecke category naturally sits at the heart of a full bounded derived category of D-modules, and the category of representations of the dual group sits at the heart of a category of perfect G^L-equivariant dg-modules for $\mathrm{Sym}(\mathfrak{g}^L[-2])$. This almost immediately proves the geometric Langlands conjecture on $\mathbb{P}^1$ [Laf09]. Broadly speaking, an extension of the functor to derived categories is uniquely determined by compatibility with a *Kostant functor* (a version of the functor Wh of [Kos78]). It is defined on objects using the classical geometric Satake equivalence, since objects in the heart generate the derived category, and is fully faithful by construction. Showing it is fully faithful is a calculation of equivariant cohomology of objects: the induced map on these cohomology groups is shown to be an isomorphism using purity of certain Hodge structures. [BF08] 31 pages.

Arinkin-Gaitsgory, Singular support of coherent sheaves, and the geometric Langlands conjecture, 2012

This paper defines a notion of singular support for ind-coherent sheaves, explores its properties and uses it to formulate a plausible geometric Langlands conjecture for which they check several compatibility conditions. For an ind-coherent sheaf $\mathcal{F}$ on a quasi-smooth scheme Z, its *singular support* is defined to be its support as a module for the even Hochschild cohomology of $\mathcal{O}_Z$, which naturally lives inside a shifted cotangent bundle of Z. For Z a moduli space of G-local systems, (H^0 of) the shifted cotangent bundle has fibre $\mathfrak{g}^*$ (which are identified with 'Arthur parameters"), and one defines a *global nilpotent cone* by taking the Nilpotent cone fibrewise. The version of geometric Langlands Arinkin and Gaitsgory gives considers on the Galois side the category of ind coherent sheaves on $\mathrm{Loc}_G(X)$ with singular support contained in this global nilpotent cone. They check that this conjecture is compatible with derived geometric Satake equivalence of [BF08], and with an Eisenstein series functor. [AG12] 153 pages.

Donagi-Pantev, Langlands duality for Hitchin systems, 2012 $\star$

For a reductive group G and its Langlands dual group G^L, one can define a moduli space of Higgs bundles, which admits the structure of an integrable system (in particular, a ramified torus fibration over an affine "Hitchin base" space). This paper proves an isomorphism of integrable systems between these moduli spaces for G and G^L. That is, from a canonical isomorphism of the Hitchin bases they produce a canonical isomorphism of torus bundles over the complements of the ramification loci, extending the isomorphism on the base. This extends earlier work for specific examples of G, such as $G = GL_n$. [DP12] 75 pages.

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