# Motives and Motivic L-functions

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# 1 Introduction

This report aims to be an exposition of the theory of $L$-functions from the motivic point of view. The classical theory of pure motives provides a category consisting of 'universal cohomology theories' for smooth projective varieties defined over – for instance – number fields. Attached to every motive we can define a function which is holomorphic on a subdomain of $\mathbb{C}$ which at least conjecturally satisfies similar properties to the Riemann zeta function: for instance meromorphic continuation and functional equation. The properties of this function are expected to encode deep arithmetic properties of the underlying variety. In particular, we will discuss a conjecture of Deligne and its more general forms due to Beilinson and others, that relates certain "special" values of the $L$-function – values at integer points – to a subtle arithmetic invariant called the *regulator*, vastly generalising the analytic class number formula. Finally I aim to describe potentially fruitful ways in which these exciting conjectures might be rephrased or reinterpreted, at least to make things clearer to me.

There are several sources from which I have drawn ideas and material. In particular, the book [1] is a very good exposition of the theory of pure motives. For the material on $L$-functions, and in particular the Deligne and Beilinson conjectures, I referred to the surveys [9], [11] and [4] heavily. In particular, my approach to the Deligne conjecture follows the survey [11] of Schneider closely. Finally, when I motivate the definition of the motivic $L$-function I use the ideas and motivation in the introduction to the note [5] of Kim heavily.

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# 2 Pure Motives

## 2.1 The Weil Conjectures

The following extremely well-known conjectures of André Weil on the number of points on varieties over finite fields will serve as the starting point for all of the theory that will follow:

**Conjecture 2.1.** Let $X$ be a smooth projective variety over $\mathbb{F}_q$ of dimension $n$. We define the zeta function of $X$ by

$$\zeta(X,s) = \exp\left(\sum_{m=0}^{\infty} \frac{N_m}{m}(q^{-s})^m\right)$$

where $N_m$ denotes the number of $\mathbb{F}_{q^m}$ points of $X$. Write $t = q^{-s}$. Then

1. The zeta function can be written as a rational function of $t$. Furthermore

$$\zeta(X,s) = \frac{P_1(t)P_3(t)\cdots P_{2d-1}(t)}{P_0(t)P_2(t)\cdots P_{2d}(t)}$$

where $P_i(t) \in \mathbb{Z}(t)$,

2. It satisfies a *functional equation* of the form
$$\zeta(X,s) = \pm q^{(n/2-s)\chi}\zeta(X,n-s)$$
where $\chi = \chi(X)$ is the Euler characteristic.

3. (The Riemann hypothesis) We can factor $P_i(t)$ as
$$P_i(t) = \prod_{j=0}^{n_j}(q - \alpha_{ij}t),$$
and $|\alpha_{ij}| = q^{i/2}$. Thus the zeroes of $P_i$ lie on the *critical line* $\Re s = \frac{i}{2}$.

The zeta function is a generating function for the number of points of $X$ over finite extensions $\mathbb{F}_{q^n}$ of $\mathbb{F}_q$. We'll see how these zeta functions fit into a more general picture later on. This particular viewpoint is the main focus of this report, with the Weil conjectures as one motivating aspect, justifying how studying these meromorphic functions might tell about concrete geometric facts like the number of points over various finite fields (i.e. solving congruences).

The case where $X$ is an elliptic curve was proved by Weil himself (he also proved the conjecture for all curves and abelian varieties), and is particularly easy. The reason is that we have a nice linearisation of the curve to use – the *Tate module* $T_\ell(E)$ for a prime $\ell \neq p$ – which allows us to reduce to an argument of linear algebra. This method is essentially *cohomological*. In general, associated to a variety $X$ we can produce a number of cohomology theories, which one should think of as *linearisations* of $X$, that remember as much of its structure as they can. One might imagine a simple proof of the Weil conjectures in their full generality proceeding as follows: produce a sufficiently nice cohomology theory for smooth projective varieties, so that one has in particular a Lefschetz trace formula, then mimic Weil's proof for the elliptic curve.

In fact, once the theory of étale cohomology was invented and sufficiently well developed, this was sufficient for Deligne to prove the Weil conjectures in full generality, though not quite with the beautiful proof invisioned by Grothendieck: the Riemann hypothesis proved particularly tricky, and needed more subtle methods.

## 2.2 Cohomology Theories

Although cohomology groups have a similar use in algebraic geometry to their use in topology: as linearisations of geometric objects, allowing one to attack geometric problems by methods of linear algebra, the situation is quite difficult for the following reason. In the topological setting there is essentially only one possible choice of a cohomology theory, at least if one imposes certain reasonable restrictions on what functors are allowed. This is the content of the Eilenberg-Steenrod theorem. In the algebro-geometric setting however, things are nowhere near this nice.

Indeed, we have a sensible notion of a 'reasonable' cohomology theory: what is called a Weil cohomology theory. One can define these over various different base fields. The problems arise in part due to the non-existence of a Weil cohomology theory over the rational numbers. This makes it quite difficult to compare, for instance, cohomology theories with $\mathbb{C}$ coefficients to those with $\mathbb{Q}_\ell$ coefficients.

Slightly more formally, though omitting many details,

**Definition 2.2.** A *Weil cohomology theory* is a contravariant functor $H^*$ from the category of smooth projective varieties over a field $k$, to graded vector spaces over a field $K$ where $K$ has characteristic zero, satisfying a number of conditions:

1. $\dim H^2(\mathbb{P}^1) = 1$

2. $H^*$ preserves the monoidal structure, where $\otimes$ on the category of smooth projective varieties is given by the usual product. This automatically makes $H^*(X)$ into a commutative algebra, because the diagonal embedding of varieties automatically gives a variety $X$ the structure of a coalgebra. We get a *Künneth formula* $H^*(X \times Y) \cong H^*(X) \otimes H^*(Y)$ for free.

3. We have *Poincaré duality*: loosely, ignoring technical issues of weights, this is a perfect pairing
$$H^i(X) \times H^{2d-i}(X) \to K,$$
where $d = \dim X$.

4. There is a *cycle class map*, which assigns a cohomology class of degree $i$ to an algebraic cycle of codimension $i$. Loosely, this is a homomorphism of rings for each $X$
$$\mathrm{cl}_X : \mathrm{CH}^*(X) \to H^{2*}(X)$$
where $\mathrm{CH}^*(X)$ is the Chow ring of algebraic cycles modulo rational equivalence, with the intersection product. These homomorphisms should be compatible with the induced morphisms coming from morphisms $f : X \to Y$

There are a number of ways to prove that there is no Weil cohomology theory when $K = \mathbb{Q}$ that underlies the other Weil cohomology theories. One could for instance give the explicit counterexample of a supersingular elliptic curve. This is described in Milne's expository article [8]. The most important examples of Weil cohomology theories for our purposes are:

- If $K = \mathbb{C}$, the *de Rham cohomology* $H^*_{dR}(X(\mathbb{C}))$, where $X(\mathbb{C})$ is given the analytic topology.
- If $K = \mathbb{Q}_\ell$, the *$\ell$-adic topology* $H^*_{\text{ét}}(X, \mathbb{Q}_\ell)$. This is well behaved whenever $\operatorname{char} k \neq \ell$, where $k$ is the field of definition of $X$.
- If $\operatorname{char} k = 0$, given an embedding $\nu : k \hookrightarrow \mathbb{C}$, the *singular cohomology* $H^*((X \times_{k,\nu} \mathbb{C})(\mathbb{C}), K)$ for any subfield $K$ of $\mathbb{C}$. Again we give the complex points the analytic topology.

Comparison theorems between these rings and others are not easy to come by. For a truly unified approach we would need some kind of *universal* cohomology theory. Since there is no genuine universal Weil cohomology theory, to produce such a thing we will need a larger target abelian category. This is exactly the role the category of pure motives plays.

## 2.3 The construction

As before, assume $X$ is smooth and projective over $K$ any field. We will enlarge the category of smooth projective varieties over $K$, first by producing extra morphisms from the intersection theory of subvarieties of $X$ (correspondences), then splitting idempotents, then inverting the Lefschetz motive $h^2(\mathbb{P}^1)$ to ensure Poincaré duality works. We will only sketch the construction here. For full details, see the book [1] by Yves André.

So, first of all, let $\mathcal{V}\mathrm{ar}(K)$ denote the category of smooth projective varieties over $K$. For $X$ in this category, let $C^*(X)$ denote the group of algebraic cycles in $X$, graded by codimension. There are a number of sources where these notions of intersection theory are explained, but I found [7] and [12] section III to be particularly helpful. This comes equipped with an *intersection product* and there are a number of equivalence relations we can impose on cycles to make this into a genuine multiplication on the group, compatible with the grading. Choosing rational equivalence we produce the Chow ring, but this will turn out to be too coarse for our needs. For now we will denote by $C^*_\sim(X)$ the ring of cycles produced by choosing *some* adequate equivalence relation $\sim$.

We enlarge the hom sets of $\mathcal{V}\mathrm{ar}(K)$ by putting $\mathrm{Hom}_{Corr}(X, Y) = C^*_\sim(X \times Y)$. Note that this includes all former morphisms, as their graphs describe an algebraic cycle. One defines composition by pushing forward to the triple product $X \times Y \times Z$, applying the intersection product, then pulling back to $X \times Z$, and this produces a category $\mathrm{Corr}(K)$ which contains all the data of algebraic cycles that we might need for the cycle class map to exist. It is not abelian however. To solve this we can take the *idempotent completion*, which involves formally splitting idempotents, or equivalently ensuring that they have kernels and cokernels in the category. This produces an abelian category provided $\sim$ is coarse enough. Conjecturally, numerical equivalence should suffice, but this depends on the so-called Grothendieck *standard conjectures* on algebraic cycles, which we will not discuss here.

This category, the category of *effective motives*, admits a natural faithful functor from the category of smooth projective varieties. We denote the image of a variety $X$ by $h(X)$. Objects in the category of effective motives have the general form $(X, e)$, where $e\colon X \to X$ is an idempotent in $\mathrm{Corr}(K)$ : a *projector*. Again conditional on the standard motives, every projector should split into a sum $e = e_0 + \cdots + e_{2d}$, making the category of effective motives *semisimple*. We denote the summands by $(X, e_i) = h^i(X)$. Furthermore, when we take the final step to produce the category of pure motives, every Weil cohomology functor $H^*$ should factor through $h^*$ in a way that preserves the grading. These maps to Weil cohomology theories are called *realisations*.

Finally, as mentioned above, we formally adjoin an inverse to the *Lefschetz motive* $\mathbb{L} = h^2(\mathbb{P}^1)$, thus producing the category of *pure motives*. This is, conditional on the standard conjectures at least, the universal cohomology theory we desired. The underlying conjectures are described in the article [7] of Kleiman.

# 3 $L$-functions

An *$L$-function*, or sometimes *zeta function*, is – loosely – a holomorphic function defined on some domain in $\mathbb{C}$ that encodes certain properties of a geometric or number theoretic object, for instance a scheme of finite type over a number field $K$, or a motive. The story that follows will motivate the study of these objects.

## 3.1 The Riemann zeta Function

The simplest, and historically earliest, example of an $L$-function is the *Riemann zeta function*

$$\zeta(s) = \sum_{n \in \mathbb{N}} \frac{1}{n^s}$$

introduced by Euler, but studied in more depth by Riemann in the 1850s. The first clue to the deep connections between the zeta function and the behaviour and distributions of the primes is the Euler product expression

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

In particular, Euler famously noted that the fact that the zeta function has a pole at $s = 1$ immediately implies the infinitude of the primes. One should think of the Riemann zeta function as encoding arithmetic behaviour of the scheme $\operatorname{Spec} \mathbb{Z}$, the ring of integers of the zero-dimensional variety $\mathbb{Q}$.

The Riemann zeta function, a priori defined on $\{\Re s > 1\}$ admits a meromorphic continuation to the whole complex plane, with a simple pole at $s = 1$ only. It also admits a certain symmetry called a *functional equation*

$$\xi(s) = \xi(1 - s)$$

where the *completed* zeta function $\xi(s)$ is given by

$$\xi(s) = \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s)$$

for $\Gamma$ the usual gamma function. As we will see later, the additional 'gamma' factor should be thought of as an extra Euler factor corresponding to the infinite place of $\mathbb{Q}$. More general $L$-functions for motives will also have gamma factors coming from the Hodge structure on their Betti (singular cohomology) realisations: powers of the gamma function depending on the Hodge numbers. We'll see examples of this later on.

Certain special values of the Riemann zeta function can be computed readily, whereas others are less tractable. The residue at the pole at $s = 1$ is easily computed to be 1. At negative integers and even positive integers there is a classical formula for the values of $\zeta$ involving the Bernoulli numbers. The odd positive integer values are much less well understood.

Thus we have outlined the general problems one might want to solve for a general $L$-function: one expresses it as an Euler product, with factors coming from the arithmetic behaviour of some object. Then one tries to prove theorems on meromorphic continuation and functional equation, and tries to study the locations of its poles and zeroes, and values at integer points. The significance of the study of the locations of zeroes of the Riemann zeta function needs no introduction, so this is a sign that we're on the right track.

## 3.2 Dedekind zeta functions

The simplest generalisation we might try is to pass from a function with Euler factors coming from each prime of $\mathbb{Z}$ to a function with Euler factors coming from each prime ideal of $\mathcal{O}_K$: the ring of integers of some number field $K$. We might then hope to prove similar results to those discussed above, to understand the arithmetic of the number field $K$. With a few moments thought we guess what we should write down:

$$\zeta(K,s) = \sum_{I \lhd \mathcal{O}_K} \frac{1}{\mathrm{Nm}(I)^s} = \prod_{p \lhd \mathcal{O}_K \text{ prime}} \frac{1}{1-\mathrm{Nm}(p)^{-s}},$$

where $\mathrm{Nm}(I) = |\mathcal{O}_K/I|$ is the (absolute) norm of the ideal $I$. Thus $\zeta(\mathbb{Q},s) = \zeta(s)$. This is the *Dedekind zeta function* of the number field $K$.

The crucial theorems – generalising Riemann's results – on these Dedekind zeta functions were proved by Hecke. Namely, $\zeta(K,s)$ always has a meromorphic continuation to $\mathbb{C}$ with a simple pole only at $s=1$, and satisfies a functional equation

$$\xi(K,s) = \xi(K,1-s),$$

where $\xi(K,s)$ is the result of completing $\zeta(K,s)$ with one gamma factor for each infinite place:

$$\begin{aligned} \Gamma_{\mathbb{R}}(s) &= \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}) \\ \Gamma_{\mathbb{C}}(s) &= 2(2\pi)^{-s}\Gamma(s) \\ \text{so } \xi(K,s) &= \sqrt{|\Delta_K|}^s\,\Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{C}}(s)^{r_2}\zeta(K,s) \end{aligned}$$

where $\Delta_K$ is the discriminant of $K$, and $r_1$ and $r_2$ are the number of real and complex places of $K$ respectively.

It is in this situation that we first begin to see just how much information the special values can encode. Hecke proved the following theorem:

**Theorem 3.1** (Analytic Class Number Formula)**.** The residue of the pole of $\zeta(K,s)$ at $s=1$ is given by

$$\mathrm{Res}_{s=1}\,\zeta(K,s) = \frac{2^{r_1}(2\pi)^{r_2} h_K \,\mathrm{Reg}_K}{w_K\sqrt{|\Delta_K|}}$$

where $h_K$ is the class number of $K$, $w_K$ is the number of roots of unity in $K$, and $\mathrm{Reg}_K$ is the classical *regulator*.

Recall the regulator encodes in a sense how dense the roots of unity are in the number field $K$. By Dirichlet's unit theorem the units span a rank $t = r_1 + r_2 - 1$ lattice in $\mathbb{R}^{[K:\mathbb{Q}]}$. More precisely the lattice is given by the $\mathbb{Z}$-span

$$\Lambda = \langle\{(\log|\sigma_1 u_i|,\ldots,\log|\sigma_{r_1}u_i|, 2\log|\sigma_{r_1+1}u_i|,\ldots,2\log|\sigma_{r_1+r_2}u_i|) : i = 1,\ldots,t\}\rangle.$$

The regulator is then the volume of a fundamental domain for this lattice. It plays the same role for $\mathcal{O}_K^\times$ that the discriminant plays for $\mathcal{O}_K$, but is often difficult to calculate in practice. The theorem gives us an easy way of computing at least the product $h_K\,\mathrm{Reg}_K$.

These Dedekind zeta functions will give us our first concrete example of a *motivic decomposition* of $L$-functions. The idea behind this is that if a motive can be decomposed as

$$M = M_1 \oplus \cdots \oplus M_k$$

then there should be a corresponding decomposition of their associated $L$-functions

$$L(M,s) = L(M_1,s) \times \cdots \times L(M_k,s)$$

(we will discuss more general $L$-functions of motives later on). Before we do this, we will introduce $L$-functions of more general varieties, and explain how to recover these Dedekind zeta functions as the zero-dimensional case.

### 3.3 $L$-functions of varieties

First of all, let $X$ be a scheme of finite type over $\mathbb{Z}$. We will explain how to pass from these objects to those of finite type over $\mathbb{Q}$, i.e. varieties over number fields. The right analogue of the primes of $\mathcal{O}_K$ is the *closed points* of $X$. Since $X$ is of finite type over $\mathbb{Z}$ any closed point $x$ has finite residue field $k(x)$ of size $N(x)$. This generalises the *norms* of ideals we had above. Thus we define

$$\zeta(X,s) = \prod_{x \in X_0} \frac{1}{1 - N(x)^{-s}}$$

where $X_0$ is the set of closed points of $X$.

What if now $X$ is a (smooth, projective) scheme of finite type over $\mathbb{Q}$? One thing we might try is taking an *integral model* of $X$: that is, a proper flat scheme $\mathcal{X}$ of finite type over $\mathbb{Z}$ whose generic fibre is $X$. Essentially, we think of $X$ as cut out by equations with coefficients in $\mathbb{Q}$, and *clear denominators* to get equations with coefficients in $\mathbb{Z}$ instead. The problem is that there is no canonical way of doing this, and we might well get different results in different ways (e.g. we could clear the denominators, and multiply all the coefficients by a constant).

However, suppose we work away from a finite set of primes $S$. That is, we find a model $\mathcal{X}_S$ for $X$ over the localisation $\mathbb{Z}[1/S]$ which is *smooth*. We can choose such an $S$ to be minimal uniquely (the set of primes where $X$ has 'bad reduction'), and if we do this the resulting *incomplete zeta function*

$$\zeta_S(X,s) = \zeta(\mathcal{X}_S, s) = \prod_{x \in (\mathcal{X}_S)_0} \frac{1}{1 - N(x)^{-s}}$$

is independent of the choice of model. The problem now is, what should we do to 'fill in' these missing factors? We can see that the missing factors really do correspond to the missing primes (those in $S$) because

$$\zeta(\mathcal{X}, s) = \prod_p \zeta(\mathcal{X} \otimes \mathbb{F}_p, s)$$

and hence

$$\zeta(\mathcal{X}_S, s) = \prod_{p \notin S} \zeta(\mathcal{X} \otimes \mathbb{F}_p, s).$$

To see how to do this naturally, we will work through a simple example: the elliptic curve:

**Example 3.2.** (following [5]) Let $E$ be an elliptic curve over $\mathbb{Q}$. Suppose it has bad reduction at $S$, so we can produce a smooth integral model $\mathcal{E}_S$ away from $S$ with well-defined zeta function $\zeta_S(E,s)$. We can now proceed by explicitly calculating the Euler factor at each prime $p \notin S$, and writing it in a way that is independent of the choice of model $\mathcal{E}_S$. This will make it clear what missing factors to insert, and will generalise to the setting of all smooth projective varieties.

So, Let $p$ be a good prime ($p \notin S$). Consider the base change

$$\overline{\mathcal{E}}_p = \mathcal{E} \otimes \overline{\mathbb{F}}_p$$

to the algebraic closure of $\mathbb{F}_p$. Then the zeta function of $\mathcal{E}_p$ has a classical description as we described in section 2.1 when we discussed the Weil conjectures: counting points over $\mathbb{F}_{p^n}$. Using the Lefschetz trace formula, we can explicitly write this factor at $p$ as

$$\zeta(\mathcal{E}_p, s) = \frac{\det(1 - \mathrm{Frob}_p\, p^{-s} | H^1_{\text{ét}}(\overline{\mathcal{E}}_p, \mathbb{Q}_\ell))}{\det(1 - \mathrm{Frob}_p\, p^{-s} | H^0_{\text{ét}}(\overline{\mathcal{E}}_p, \mathbb{Q}_\ell))\det(1 - \mathrm{Frob}_p\, p^{-s} | H^2_{\text{ét}}(\overline{\mathcal{E}}_p, \mathbb{Q}_\ell))}.$$

The notation here means we consider the given operator acting on the various étale cohomology groups. I won't explicitly describe how Frobenius acts on these groups here.

We'd like to write this in a way that is manifestly independent of the choice of model $\mathcal{E}$. But one can do this by recalling the notion of the *inertia group at p*. For a finite Galois extension $K/\mathbb{Q}$, one defines the decomposition group at $p$ by fixing a prime $\mathbf{p}/p$, and considering the subgroup $D_p \leq \mathrm{Gal}(K/\mathbb{Q})$ fixing $\mathbf{p}$. This is independent of the choice of $\mathbf{p}$, and there is a natural map

$$D_p \to \mathrm{Gal}((\mathcal{O}_K/\mathbf{p})/(\mathbb{Z}/p)) \cong \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p).$$

The inertia group $I_p$ is the kernel of this homomorphism. For the infinite extension $\overline{\mathbb{Q}}/\mathbb{Q}$ this is a little more subtle, but we deal with it by fixing an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$ and putting $D_p = \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$. So there is a natural map

$$D_p \to \mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p),$$

and we define the inertia group $I_p$ to be its kernel.

Now, one can prove the following crucial fact:

**Theorem 3.3.** The $\ell$-adic cohomology of $\overline{\mathcal{E}}_p$ is isomorphic to the cohomology of the base change $\overline{E}$ of $E$ to $\overline{\mathbb{Q}}$ fixed by inertia $I_p$. In symbols:

$$H^i_{\text{ét}}(\overline{\mathcal{E}}_p, \mathbb{Q}_\ell) \cong H^i_{\text{ét}}(\overline{E}, \mathbb{Q}_\ell)^{I_p}.$$

Given this, we have achieved our goal of writing the zeta factor of $E$ at $p$ in a manifestly model independent way, namely as

$$\zeta(E_p, s) = \frac{\det(1 - \mathrm{Frob}_p\, p^{-s} | H^1_{\text{ét}}(\overline{E}, \mathbb{Q}_\ell)^{I_p})}{\det(1 - \mathrm{Frob}_p\, p^{-s} | H^0_{\text{ét}}(\overline{E}, \mathbb{Q}_\ell)^{I_p}) \det(1 - \mathrm{Frob}_p\, p^{-s} | H^2_{\text{ét}}(\overline{E}, \mathbb{Q}_\ell)^{I_p})}.$$

So it is now clear what factors to insert at the bad primes $p \in S$: precisely Euler factors of the above form.

Note also that this gives an example of a motivic decomposition as described above. Namely, we have the motivic decomposition

$$h(E) = h^0(E) \oplus h^1(E) \oplus h^2(E),$$

and when we write out our L function, we can write it as a product of Euler factors of form

$$\zeta(E, s) = \prod_p L_p(h^0(E), s) \times L_p(h^1(E), s)^{-1} \times L_p(h^2(E), s)$$

with each local factor $L_p(h^i(E), s)$ given by a determinant of the characteristic polynomial of Frobenius acting on the $i^{\text{th}}$ $\ell$-adic cohomology group.

**Example 3.4.** If $X$ is zero-dimensional, one can fairly easily unpack the definitions to see that the full subcategory of motives generated by zero-dimensional objects is equivalent to the category of *Galois representations* (as described in section 3 of [10]). Explicitly, the motive $h(\operatorname{Spec} K)$ maps to the regular representation of $\mathrm{Gal}(K/\mathbb{Q})$, and its irreducible summands are precisely the irreducible representations of this group. This gives another example of a motivic decomposition when we take $L$-functions. Taking the $L$-function of a Galois representation recovers the classical notion of an *Artin $L$-function*: the Euler factor for a representation $\rho\colon G \to GL(V)$ at a prime $p$ of $K$ is given by

$$L_p(K, \rho, s) = \det(1 - \rho(\mathrm{Frob}_p)\, \mathrm{Nm}(p)^{-s} | V^{I_p})^{-1}.$$

In particular, plugging in the regular representation we recover the Euler factors for the Dedekind zeta function. Thus the Dedekind zeta function can be written as a product of Artin $L$-functions for the irreducible representations of $\mathrm{Gal}(K/\mathbb{Q})$. This can be done particularly explicitly when the extension is cyclotomic: in this case all irreducible representations are one-dimensional, given by so called *Dirichlet characters*. The $L$-function of a Dirichlet character is a classically understood object, so one can understand the Dedekind zeta function in terms of these classical objects.

## 3.4 Motivic $L$-functions

With this in mind, we can give the definition of the $L$-function associated to any pure motive $M$ through its $\ell$-adic realisations. We work in slightly more generality than above: over an arbitrary number field $K$ rather than just $\mathbb{Q}$. The definition should directly generalise the above, where the motive in question is $h^i(X)$. Say $M$ has $\ell$-adic realisations

$$M_\ell = H^i_{\text{ét}}(X, \mathbb{Q}_\ell)$$

for some fixed prime $\ell$. This vector space comes equipped with a continuous $\ell$-adic representation of the Galois group $\mathrm{Gal}(\overline{K}/K)$, and hence at each non-archimedean place $v$ of $K$ a well-defined action of Frobenius on the subrepresentation fixed by the inertia subgroup $M_\ell^{I_v}$ (the definition of inertia groups over $K$ is a straightforward generalisation of the definition we gave over $\mathbb{Q}$). Then, inspired by our above discussion in 3.2 we define the *local L-factor* at a place $v$ not dividing $\ell$ to be

$$L_v(M,s) = \det(1 - \mathrm{Frob}_v(\mathrm{Nm}(v))^{-s} | M_\ell^{I_v})^{-1}$$

the determinant of the characteristic polynomial of Frobenius. At least conjecturally this is indepedent of the choice of prime $\ell$ , so we can define the *L-function of M* to be the product

$$L(M,s) = \prod_v L_v(M,s)$$

where the product is over the non-archimedean places.

There is a way of dealing with the infinite places also, corresponding to the gamma factors we inserted when we defined Dedekind zeta functions in 3.2. The data for these comes from the *Betti* realisations of $M$, which come equipped with natural pure Hodge structures, and will we given by certain combinations of zeta functions analogously to the case of Dedekind zeta functions. We will see the details shortly. Let $L_\infty(M,s)$ denote the product of these infinite factors, and write

$$\Lambda(M,s) = L(M,s)L_\infty(M,s).$$

These motivic $L$-functions are conjectured to have the following properties. These are analogous to properties possessed by the more classical $L$ and zeta functions introduced earlier on, but are largely only known for certain special cases. Throughout, $i$ will denote the *weight* of the pure motive $M$, which will be assumed to be irreducible.

1. In the region $\Re s > 1 + \frac{i}{2}$, the Euler product expansion for $L(M,s)$ converges, so the $L$-function is well-defined and holomorphic in this region.
2. $L(M,s)$ admits a meromorphic continuation to the entire complex plane, with only possibly one pole at $s = 1 + \frac{i}{2}$. $L(M,s)$ never has a zero at that point, even if it is holomorphic.
3. With the infinite factors, $\Lambda(M,s)$ satisfies a functional equation:
$$\Lambda(M,s) = \Lambda(M, 1+i-s).$$

Let's now be more specific about the nature of the infinite factors. Our motive $M$ has a *Betti realisation* coming from each embedding $\nu\colon K \to \mathbb{C}$. For a motive $h^i(X)$ this is just the singular cohomology

$$M_\nu = H^i((X \times_K \mathbb{C})(\mathbb{C}), \mathbb{Q})$$

where the fibre product is with respect to the given embedding $\nu$. This is a pure Hodge structure of weight $i$:

$$M_\nu \otimes \mathbb{C} = \bigoplus_{p+q=i} H^{p,q}.$$

An important structure on this realisation is the involution given by complex conjugation of the $\mathbb{C}$ points $(X \times_K \mathbb{C})(\mathbb{C})$, which we will denote by $F_\infty$ for 'infinite Frobenius' (which is the standard terminology, even if it is a little pretentious. Just calling it 'complex conjugation' seems good enough to me). This flips the Hodge diamond over:

$$F_\infty(H^{p,q}) = H^{q,p}.$$

Write $h^{p,q}$ for $\dim H^{p,q}$, and write $h^{p\pm}$ for the dimensions of the two eigenspaces of $F_\infty$ acting on $H^{p,p}$. Then we define the infinite factor to be the following ugly gamma factor

$$L_\infty(M,s) = \prod_\nu \begin{cases} \left(\prod_{p+q=i,\ p<q} \Gamma_\mathbb{C}(s-p)^{h^{p,q}}\right) & \text{if } i \text{ is odd} \\ \left(\prod_{p+q=i,\ p<q} \Gamma_\mathbb{C}(s-p)^{h^{p,q}}\right) \Gamma_\mathbb{R}(s-\frac{i}{2})^{h^{\frac{i}{2}+}}\ \Gamma_\mathbb{R}(s+1-\frac{i}{2})^{h^{\frac{i}{2}-}} & \text{if } i \text{ is even} \end{cases}$$

where the outer product is over all the Betti realisations corresponding to different embeddings. This is rather intimidating. Let's at least check that it matches our Gamma factor for the Dedekind zeta function. Here $i = 0$. At a real place we just get $\Gamma_{\mathbb{R}}(s)$ as we should. At a complex place we get

$$\begin{aligned}\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1) &= \pi^{s-1/2}\Gamma(s/2)\Gamma(s/2+1/2)\\ &= 2^{1-s}\pi^{-s}\Gamma(s) = \Gamma_{\mathbb{C}}(s)\end{aligned}$$

as required.

I don't have a better explanation for this than "it's what should make the functional equation work, based on known cases". Deninger has proposed a unified approach, describing the infinite factors in terms of characteristic polynomials on certain 'Archimidean' cohomology groups, in [3]. This is, as far as I know, the most concrete understanding we have of this phenomenon. But why are the Hodge numbers involved? I'm sure people know a good answer to this question, but at the moment I don't know it. This is something I'd like to understand.

There is one final type of realisation that we'll have to refer to later: the *de Rham realisation*. The motive $h^i(X)$ has de Rham realisation

$$M_{dR} = H^i_{dR}(X) = H^i(X, \Omega^\bullet_X)$$

which comes equipped with a decreasing filtration

$$F^k M_{dR} = H^i(X, \Omega^{\geq k}_X)$$

where $\Omega^{\geq k}_X$ denotes the truncation.

# 4 Deligne's Conjecture

We have already seen that many of the expected properties of the category of pure motives rely on the Grothendieck standard conjectures, which are still unknown, but the conjectural theory, and the theory of $L$-functions, leads us in the direction of other interesting and deep open problems. We will try to describe one important aspect of the theory where very little is known. An important topic that I will not really discuss is the extension of the category of pure motives to include motives of varieties which are not necessarily smooth projective: the theory of *mixed motives*, whose name we will justify by a connection with mixed Hodge structures. There is currently no satisfactory construction of this category known, though various people, in particular Voevodsky and Levine, have proposed constructions for its derived category. One can produce 'motivic cohomology' groups as the Hom spaces in this derived category, i.e. as Ext groups between mixed motives. This is only one of many equivalent ways of defining motivic cohomology.

The area we will describe is the quest to understand *special values* of motivic $L$-functions, i.e. the values of motivic $L$-functions at certain integer points. There are a number of ambitious conjectures for how to relate these special values to arithmetic properties of the motive, generalising – for instance – the analytic class number formula for the Dedekind zeta function of a number field.

## 4.1 Special Values of $L$-functions

Let $M$ be a pure motive. We already defined the motivic $L$-function $L(M, s)$ of $M$, by a completed Euler product where the data for the factors came from realisations of the motive. First, let's observe where the poles of $L$ lie. Then we can investigate the values of $L$ at integer points, and, with some more work, the residues at its poles. We will see that conjecturally these values are given by so-called 'higher regulator' maps, relating cohomology groups associated to $M$.

The $L$-function $L(M, s)$ should only have one simple pole at $1 + \frac{i}{2}$. What about the archimedean factors? These are given by expressions involving gamma functions, which have poles exactly at the negative integers, so it shouldn't be too hard to work this out:

**Proposition 4.1.** $L_\infty(M,s)$ has poles only at the integers $m < \frac{i}{2}$. The orders of the poles are given by an expression involving the Betti realisations of $M$. Explicitly, the order of the pole at $m$ is

$$dim_{\mathbb{C}} M_B^{(-1)^{i-m}} - \dim_{\mathbb{C}} F^{i+1-m} M_{dR}$$

where $M_B$ is any choice of Betti realisation, and $M_B^{(-1)^k}$ is the $(-1)^k$-eigenspace of the involution $F_\infty$.

This is, unsurprisingly, just an expression involving the Hodge numbers of $M_B$. Using the functional equation, we can investigate the order of poles and zeroes of $L(M,s)$ at integer points. The functional equation says that

$$\frac{L(M,s)}{L(M,i+1-s)} = \frac{L_\infty(M,i+1-s)}{L_\infty(M,s)}$$

and hence the order of the zero at $m < \frac{i}{2}$ is given by

$$\begin{aligned} \operatorname*{ord}_{s=m} L(M,s) &= \operatorname*{ord}_{s=i+1-m} L(M,s) + \operatorname*{ord}_{s=i+1-m} L_\infty(M,s) - \operatorname*{ord}_{s=m} L_\infty(M,s) \\ &= -\operatorname*{ord}_{s=m} L_\infty(M,s) \\ &= \dim_{\mathbb{C}} M_B^{(-1)^{i-m}} - \dim_{\mathbb{C}} F^{i+1-m} M_{dR} \end{aligned}$$

where ord counts the multiplicity of zeroes positively and poles negatively. If we wanted, we could describe the order at $\frac{i}{2}$ in a similar way. We will interpret this difference of dimensions in a different way, as the dimension of a single cohomology group that takes into account the structure of the infinite Frobenius involution.

**Definition 4.2.** For a smooth projective variety $X$ over $K$, we define the *Deligne cohomology groups* as follows: first let $\mathbb{R}(p)$ be the twist $(2\pi i)^p\,\mathbb{R}$. It fits into a chain complex $\mathbb{R}(p)_{\mathcal{D}}$ as

$$\mathbb{R}(p) \longrightarrow \mathcal{O}_{X(\mathbb{C})} \longrightarrow \Omega_X^1 \longrightarrow \cdots \longrightarrow \Omega_X^{p-1} \longrightarrow 0$$

where the first arrow is the natural inclusion. We have a short exact sequence of complexes

$$0 \longrightarrow \Omega_X^{<p}[-1] \longrightarrow \mathbb{R}(p)_{\mathcal{D}} \longrightarrow \mathbb{R}(p) \longrightarrow 0\ .$$

Then the Deligne cohomology of $X$ of weight $p$ is defined to be the cohomology of this complex:

$$H^i_{\mathcal{D}}(X,\mathbb{R}(p)) = H^i(X_{an},\mathbb{R}(p)_{\mathcal{D}})$$

where we consider $X$ as a complex manifold in the natural way.

**Remark 4.3.** Perhaps more naturally, one can think of the Deligne cohomology groups as $\mathrm{Ext}^1$ groups in a category of mixed Hodge structures over $\mathbb{R}$:

$$H^i_{\mathcal{D}}(X,\mathbb{R}(p)) \cong \mathrm{Ext}^1(\mathbb{R}(0), H_B^{i-1}(X_{an})(p))$$

where $H_B^\bullet$ denotes the Betti cohomology. See [4] for more details.

For simplicity, we will now assume $K=\mathbb{Q}$. Our goal is to formulate Deligne's conjecture predicting special values of $L$-functions, at least in $\mathbb{R}^\times/\mathbb{Q}^\times$, using a 'generalised regulator'.

Taking the long exact sequence on cohomology of the short exact sequence of complexes described above, we get

$$\cdots \longrightarrow H^i(X(\mathbb{C}),\mathbb{R}(p)) \xrightarrow{\theta_p} H^i_{dR}(X(\mathbb{C}))/F^p \longrightarrow H^{i+1}_{\mathcal{D}}(X,\mathbb{R}(p)) \longrightarrow \cdots\ .$$

If $i$ is less than $2p$, then the map $\theta_p$ is actually injective: anything in the kernel has to actually land in $F^p \cap \overline{F^p}$ because complex conjugation on the right hand side is just multiplication by $(-1)^p$ on the left hand side. But

$$F^p \cap \overline{F^p} = \left(\bigoplus_{q\geq p} H^{q,r}\right) \cap \left(\bigoplus_{r\geq p} H^{q,r}\right) = \bigoplus_{q,r>p} H^{q,r} = 0.$$

Thus our long exact sequence splits into short exact sequences

$$0 \longrightarrow H^i(X(\mathbb{C}), \mathbb{R}(p)) \longrightarrow H^i_{dR}(X(\mathbb{C}))/F^p \longrightarrow H^{i+1}_{\mathcal{D}}(X, \mathbb{R}(p)) \longrightarrow 0\ ,$$

or equivalently

$$0 \longrightarrow F^p H^i_{dR}(X(\mathbb{C})) \longrightarrow H^i(X(\mathbb{C}), \mathbb{R}(p-1)) \longrightarrow H^{i+1}_{\mathcal{D}}(X, \mathbb{R}(p)) \longrightarrow 0\ .$$

Now, taking invariants with respect to $F_\infty$, we get

$$0 \longrightarrow F^p H^i_{dR}(X(\mathbb{R})) \longrightarrow H^i(X(\mathbb{C}), \mathbb{R}(p-1))^{(-1)^p} \longrightarrow H^{i+1}_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(p)) \longrightarrow 0$$

which gives us an interpretation of the order of zeroes at critical values in terms of our Deligne cohomology groups. Precisely we have, for $m < \frac{i}{2}$,

$$\begin{aligned} \operatorname*{ord}_{s=m} L(h^i(X), s) &= \dim_{\mathbb{R}} H^i(X(\mathbb{C}), \mathbb{R}(i-m))^{(-1)^{i+1-m}} - \dim_{\mathbb{R}} F^{i+1-m} H^i_{dR}(X(\mathbb{C})) \\ &= \dim_{\mathbb{R}} H^{i+1}_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(i+1-m)) \end{aligned}$$

So $L(h^i(X), m)$ has a non-zero value at an integer $m < \frac{i}{2}$ if and only if the Deligne cohomology group $H^{i+1}_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(i+1-m))$ vanishes. Such an integer $m$ is called *critical*, and it is these special values we will describe. As this Deligne cohomology group vanishes, our short exact sequence turns into an isomorphism

$$F^{i+1-m} H^i_{dR}(X_{\mathbb{R}}) \xrightarrow{\sim} H^i(X(\mathbb{C}), \mathbb{R}(i-m))^{(-1)^{i-m}}.$$

Both sides carry natural underlying $\mathbb{Q}$ vector space structures, so choosing a $\mathbb{Q}$-basis we have a well-defined determinant $c_M(m) \in \mathbb{R}^\times/\mathbb{Q}^\times$. This number is called the *regulator* or *period* of the twisted motive $M(m)$. That this generalises the classical regulator is not an easy fact, and traditionally involves a different equivalent construction of the above isomorphism as a map between $K$-groups. Now we can state Deligne's conjecture on special values:

**Conjecture 4.4** (Deligne)**.** Let $m < \frac{i}{2}$ be a critical value of the $L$-function $L(M, s)$. Then the special value at $m$ is given by this regulator:

$$L(M, m) \equiv c_M(m) \mod \mathbb{Q}^\times.$$

There is an even more wide-reaching generalisation of this conjecture due to Beilinson [2], which covers the behaviour of the non-critical values as well. To define this we would need to set up an isomorphism between a Deligne cohomology group, and a so-called 'motivic' cohomology group with natural $\mathbb{Q}$-structures on both sides to define a regulator. There are a number of equivalent ways of doing this, which we will not discuss here. There are full accounts in the surveys [4], [9] or [11]. One can also conjecturally say something about the rational parts of the special values as we did in the analytic class number formula. This is the content of the *Bloch-Kato conjecture* on Tamagawa numbers. For details see the survey [6] of Kings.

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