Chris Elliott – Research Statement

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Introduction

My research addresses the mathematical structure of quantum field theory and its applications to geometric representation theory using tools from derived algebraic geometry and homotopical algebra. I'm particularly interested in developing rigorous mathematical models for constructions that appear in physics – for instance the notion of "twisting" for supersymmetric field theory – and using those models to turn properties that have been observed in the physics literature into novel mathematical structures.

The notion of a topological twist of a supersymmetric field theory has already been used to motivate and explain a number of wide-reaching research programs in mathematics, for instance the theory of mirror symmetry from twists of 2d $\mathcal{N} = (2, 2)$ sigma models, the theory of symplectic duality and 3d mirror symmetry from twists of 3d $\mathcal{N} = 4$ theories, the theory of Donaldson and Seiberg-Witten invariants from 4d $\mathcal{N} = 2$ gauge theories and Kapustin and Witten's approach to the geometric Langlands correspondence from 4d $\mathcal{N} = 4$ gauge theories. One aim of my research is to develop systematic mathematical techniques for the study of such supersymmetric twists that includes these examples – in many cases promoting an analogy to a more precise statement – and allows for the development of new examples. In this research statement I'll summarise five themes of my research including ongoing projects and proposals for future work. I'll begin with a short introduction to each topic: more detailed statements can be found in the sections below.

1. The idea of twisting, to a first approximation, says that if you're given a quantum field theory with an odd symmetry Q satisfying $Q^2 = 0$ you can obtain a simpler quantum field theory by restricting to the Q-invariant part. For example one piece of data one can associate to a quantum field theory is a vector space $\mathrm{Obs}(U)$ of observables on any open set U; in the twisted theory these observables are replaced by the cohomology of $\mathrm{Obs}(U)$ with respect to Q. The expectation coming from physics is that for special choices of Q called "topological" symmetries the twisted quantum field theory is itself topological, meaning that measurements made on an open set U only depend on the topology of U and not on any extra geometric structure like a metric. This idea is a unifying theme for my research: I'm interested in making the idea of twisting precise and studying the mathematical structures coming from twisted field theories, both when Q is topological but also more generally.

   In joint work with Safronov I investigated the physical expectation above: in what sense is the twist by a topological supercharge actually topological? One model for the observables in a topological field theory in dimension n is given by a "little n-disk algebra", meaning we have a single algebra A of local observables along with a multiplication $A^{\otimes k} \to A$ for every way of embedding k n-dimensional balls into a big n-ball. We don't get this structure automatically after twisting, but there's only a single condition to verify.

   Theorem ([ES18])**.** The algebra of local operators in a topologically twisted quantum field theory in dimension n has the structure of a little n-disk algebra whenever the canonical map extending observables defined on a ball of radius r to a ball of radius $R > r$ is a quasi-isomorphism.

   The condition in the theorem is not automatic, but we check that it is satisfied under many physically reasonable circumstances, for instance for a special type of field theory called "superconformal". The main source of field theories with an odd symmetry we can twist by is the theory of *supersymmetry*: supersymmetric field theories have an action of a $\mathbb{Z}/2\mathbb{Z}$-graded extension of the group of isometries of $\mathbb{R}^n$ which in particular typically includes many odd elements Q such that $Q^2 = 0$. We went on to classify all possible twists for supersymmetric field theories in dimensions up to 10. There are quite a few interesting twists which have not, as far as we are aware, been studied in the mathematics literature. I discuss our proposed research in this direction in Section 1.

2. My original motivation for studying these ideas came from Kapustin and Witten's work [KW07] on the geometric Langlands program. Kapustin and Witten described a pair of topological twists of a supersymmetric gauge theory in dimension 4 and argued that to each twist and any Riemann surface C one can attach a category of "boundary conditions" along C. Specifically these categories are categories of sheaves on certain representation-theoretic moduli spaces that depend on the curve C and the gauge group G. The claim is that the categories thus obtained are the same as the categories that appear in the geometric Langlands program. The physical expectation is that these two categories should be equivalent, where on the one side the group G occurs and on the other its Langlands dual group $G^\vee$ occurs. This is exactly what geometric Langlands predicts.

   In joint work with Yoo [EY18, EY17] I gave a formalism for twisting that allowed us to calculate the categories in Kapustin and Witten's twists, verifying the expectation that the geometric Langlands categories occur. This required some ingredients that did not appear in Kapustin and Witten's original work to see the correct algebraic structures on the moduli spaces, and to see a crucial "singular support condition" for the geometric Langlands correspondence discovered by Arinkin and Gaitsgory [AG15]. In future work we propose to extend this calculation to a much larger family of twisted gauge theories. There is a physical expectation that this larger family is related by a vast web of "Gaiotto" dualities [Gai12], so we expect this calculation to lead to a large family of new conjectural equivalences generalizing the geometric Langlands correspondence. See Section 2 for more details.

3. With Pestun I'm currently pursuing another novel analogue of the geometric Langlands correspondence, this time related to twists of supersymmetric gauge theories in five dimensions. This work begins with an algebro-geometric result, where we construct a hyperkähler structure (in particular a $\mathbb{CP}^1$-indexed family of holomorphic symplectic structures) on a moduli space of "multiplicative Higgs bundles" on a curve C. Points in this moduli space are principal G-bundles on C equipped with a bundle automorphism $\phi \colon P \to P$ with a fixed set of singularities. So far we've been studying the "rational" case where $C = \mathbb{C}$ with a boundary condition at infinity, but we hope to extend our analysis to the "trigonometric" case where $C = \mathbb{C}^\times$ with boundary conditions at 0 and infinity. In Section 3 I'll explain our results and how they lead to a multiplicative generalization (meaning functions valued in the Lie algebra $\mathfrak{g}$ are replaced by functions valued in the Lie group G) of the geometric Langlands conjecture.

4. Not all quantum field theories and not all twists are topological. For example, it's natural to study field theories in dimension $2n$ where the observables on an open set U depend on a complex structure on U. We call such theories *holomorphic*. Holomorphic theories can occur via twisting by certain "holomorphic" symmetries Q; being holomorphic is a weaker condition on Q than being topological and so holomorphic twists are more plentiful than topological twists.

   In joint work with Brian Williams I'm investigating an example of holomorphic twisting coming from the theory of gravity. There is an intricate physical theory of field theories extending the classical theory of gravity but also including supersymmetry, these theories are called theories of *supergravity*. Because these theories are supersymmetric we can study their twists. Starting from a supergravity theory in four real dimensions we give a description of its holomorphic twist, and explain how this twisted theory can very easily be extended to include matter fields: something that is much trickier in the full supersymmetric setting. In Section 4 I explain this project and its potential applications.

5. Finally I'll describe an application to the theory of quantum groups. The Yangian $Y(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is an infinite-dimensional Hopf algebra that quantizes the algebra of functions on the group $G_1[z]$ of G-valued polynomials with constant term 1 (which can be naturally equipped with a Poisson bracket). Costello [Cos13] explained a relation between the Yangian and a twist of a gauge theory in dimension 4: the local observables in this twisted theory are not equal to the Yangian: instead they generate the Yangian by applying the process of *Koszul duality*.

   I propose to extend this example to describe not only the Yangian itself, but also the *twisted Yangian* $Y(\mathfrak{g}, \mathfrak{g}^\theta) \subseteq Y(\mathfrak{g})$ associated to an involution θ of $\mathfrak{g}$. This is a subalgebra which is a coideal with respect to the comultiplication. By generalizing recent ideas from the work of Weelinck we will be able to produce the twisted Yangian from a twisted gauge theory on the quotient of $\mathbb{R}^4$ by an involution fixing a line. The local observables on a ball away from this fixed line will be Koszul dual to the Yangian, and the local observables on a ball meeting the fixed line will be Koszul dual to the twisted Yangian. This analysis should lead to a description of the "universal K-matrix": a datum describing the coideal structure coming from the physical structure of the twisted quantum field theory. For more details on this proposal see Section 5.

Project 1 Supersymmetric Twisting

In joint work with Safronov [ES18] I gave a mathematical description of the concept of *twisting* in quantum field theory. The idea (both of twisting in general and of our work specifically) is the following. Suppose you're given a field theory over $\mathbb{R}^n$ with an action of the group of translations – we use factorization algebras as in the work [CG17, CG18] of Costello and Gwilliam as a model for perturbative field theory. Typically the data of the local observables on an open set $U \subseteq \mathbb{R}^n$ depends on the geometry of U. However, if we're given the data of a homotopy trivialization of the translation action then we prove that the local observables actually only depend on the homotopy type of U. In that sense, the field theory is *topological*. More precisely we prove the following.

Theorem ([ES18, Corollary 2.28])**.** Let Obs be a factorization algebra on $\mathbb{R}^n$ with a homotopically trivialized action of the group of translations, and suppose the factorization map $\mathrm{Obs}(B_r(0)) \to \mathrm{Obs}(B_R(0))$ associated to the inclusion of concentric disks of radii $r < R$ is a quasi-isomorphism. Then $\mathrm{Obs}(B_1(0))$ can be canonically given the structure of an $\mathbb{E}_n$-algebra.

Note that this theorem does not appeal to Lurie's result from [Lur17] comparing $\mathbb{E}_n$ algebras and locally constant factorization algebras. In fact it is constructive, it builds an action of a specific model for the $\mathbb{E}_n$ operad.

Topological twisting is a way of constructing theories with homotopically trivial translation action. Suppose now that our field theory is acted upon by a supertranslation algebra: a super Lie algebra $\mathcal{T}$ whose even part is the translation algebra $\mathbb{R}^n$. Suppose Q is an odd element of $\mathcal{T}$ satisfying $[Q, Q] = 0$. One can form the twist with respect to Q: a new field theory roughly speaking given by taking the homotopy invariants with respect to the group $\Pi\mathbb{C}Q$ generated by Q (see [ES18, Section 3.2] for a careful definition). The Q-twisted theory has the property that all translations in the image of $[Q, -]$ act homotopically trivially. If the map $[Q, -]$ is surjective we say Q is a *topological supercharge*. More precisely we prove the following.

Theorem ([ES18, Proposition 3.12, Theorem 3.37])**.** Now let Obs be a supersymmetric factorization algebra on $\mathbb{R}^n$ and let Obs^Q be the twist by a topological supercharge Q. If the condition from the previous theorem is satisfied then $\mathrm{Obs}^Q(B_1(0))$ has the structure of an $\mathbb{E}_n$-algebra. In many examples (such as superconformal field theories) the condition is satisfied automatically, and we can additionally extend the $\mathbb{E}_n$-algebra to an $\mathrm{SO}(n) \ltimes \mathbb{E}_n$-algebra, or a *framed* $\mathbb{E}_n$-algebra.

This analysis is fairly complete for perturbative field theories – indeed, we went on to classify the possible twists of supersymmetric theories in all dimensions up to ten – but it would be very interesting to extend this analysis from algebras of observables to more sophisticated objects: categories or stacks. For example, in [EY18] I developed, with Yoo, a formalism for topological twists of the moduli stacks of solutions to the equations of motion in certain classical field theories. In supersymmetric gauge theories we can model the classical moduli spaces as formal derived thickenings of stacks $\mathrm{Bun}_G(X)$ of algebraic principal G-bundles on algebraic varieties (one could consider the same notion where $\mathrm{Bun}_G(X)$ was replaced by any base stack fixed by Q). These can equivalently be viewed as Lie algebroids over $\mathrm{Bun}_G(X)$, and we can then form the twist as a Lie algebroid.

Proposal 1. Using the classification from [ES18] we can compute all the field theories arising by twisting and compactifying supersymmetric Yang-Mills theories in dimensions up to ten. This calculation will proceed as follows. Begin by calculating the twist of $\mathcal{N} = k$ super Yang-Mills theory with gauge group G on $\mathbb{R}^n$ as an L_∞-algebra. The compactification of this twisted theory on a smooth variety X^m can be modelled as a factorization algebra valued in Lie algebroids on the stack $\mathrm{Bun}_G(X)$. This can then be further twisted using a supersymmetry algebra in dimension $n - m$. If this latter twist was topological and the assumption of rescaling invariance is satisfied then the local observables of the compactified theory define an $\mathbb{E}_{n-m}$-algebra. However we can also investigate other structures associated to these twisted compactified theories such as categories of line operators or boundary conditions.

Note that many interesting twisted theories that occur are A-type twists, meaning the moduli stacks that occur are de Rham stacks, and in particular have contractible tangent complex. We should always describe these examples in families, as deformations of a holomorphically twisted theory. While the Lie algebra data at the fully twisted point is contractible it arises as a fiber of a family of Lie algebras over $\mathbb{A}^1$ whose fiber at 0 is non-contractible.

Proposal 2. In the classification of topological twists we came across an interesting example that doesn't appear to have been studied in the mathematics literature. In 8 dimensional $\mathcal{N} = 1$ there is a topological supercharge with stabilizer Spin(7). Alternatively, reducing down to 7 dimensions there is a topological supercharge with stabilizer G_2. According to the theory of factorization homology – which allows one to build topological quantum field theories defined on G-structured n-dimensional manifolds from a G-equivariant $\mathbb{E}_n$-algebra – the twists of super Yang-Mills theories in these dimensions will be gauge theories defined on manifolds with special holonomy: Spin(7) in dimension 8 or G_2 in dimension 7. What are these twisted theories exactly? For instance, the 8-dimensional example yields a Spin(7)-equivariant $\mathbb{E}_8$-algebra: what is it? What are the $\mathbb{E}_n$-algebras in lower dimensions arising from compactification? We speculate that these theories are related to categorifications of Donaldson-Thomas theory.

Project 2 Geometric Langlands and Gaiotto Duality

My motivation for studying twists of supersymmetric field theory originally came from the work of Kapustin and Witten [KW07]. They described an approach to the geometric Langlands conjecture using twists of $\mathcal{N} = 4$ super Yang-Mills theory in dimension 4. To summarise their approach, Kapustin and Witten described a family of topological twists of $\mathcal{N} = 4$ super Yang-Mills with gauge group G parameterized by points Ψ in $\mathbb{CP}^1$. They then compactify these twisted theories on a Riemann surface C and argue that the theories thus obtained look like the A-model with target $T^* \mathrm{Bun}_G(C)$ when $\Psi = 0$ and the B-model with target $\mathrm{Loc}_G(C)$ when $\Psi = \infty$. Here $\mathrm{Bun}_G(C)$ is the moduli stack of holomorphic G-bundles on C and $\mathrm{Loc}_G(C)$ is the moduli space of G-local systems on C. Finally they argue that S-duality swaps the group G with its Langlands dual $G^\vee$, and acts antipodally on the $\mathbb{CP}^1$ family of twists. The geometric Langlands equivalence then arises by comparing the categories of boundary conditions in the two dual theories at $\Psi = 0$ and ∞. There are two main mathematical issues that Kapustin and Witten didn't address.

1. The moduli stack $\mathrm{Loc}_G(C)$ of G-local systems on C and the moduli stack $\mathrm{Flat}_G(C)$ of flat G-bundles on C are analytically equivalent but algebraically distinct. The geometric Langlands conjecture is usually formulated with the latter algebraic structure (though see [BZN16] for a discussion of the former case). Therefore to compare this approach to work in mathematics we need a field-theoretic origin for the algebraic structure on the moduli space.
2. The "best-hope" geometric Langlands correspondence comparing the categories D-modules on $\mathrm{Bun}_G(C)$ and coherent sheaves on $\mathrm{Flat}_{G^\vee}(C)$ needs correcting, as shown in the work of Arinkin and Gaitsgory [AG15]. One must consider not all coherent sheaves on $\mathrm{Flat}_{G^\vee}(C)$, but only those satisfying a singular support condition. This condition does not appear in the work of Kapustin and Witten and its sequels.

In my work with Yoo [EY18, EY17] we gave a construction of the Kapustin-Witten twists resolving both of these issues.

1. As described in the previous section, in [EY18] we introduced a non-perturbative description for topological twists of gauge theories. In the specific example of $\mathcal{N} = 4$ supersymmetric gauge theory we obtain a description of the classical field theory and its twists with a canonical algebraic structure using the Penrose-Ward correspondence, which says that 4d supersymmetric gauge theories arise as dimensional reductions of holomorphic Chern-Simons theory on twistor space – a complex 3-fold. This Chern-Simons theory has a canonical algebraic structure therefore so does its dimensional reduction. Using these ideas we proved the following.

   Theorem ([EY18, Corollary 4.16, Proposition 4.26])**.** The Kapustin-Witten A- and B-twists of $\mathcal{N} = 4$ super Yang-Mills assign to an algebraic curve C the following derived moduli spaces of germs of solutions to the equations of motion.

   $$\mathrm{EOM}_A(C) = (\mathrm{Higgs}_G(C))_{\mathrm{dR}}$$
   $$\mathrm{EOM}_B(C) = T^*[1]\mathrm{Flat}_G(C).$$

   In particular the moduli space on the B side admits the expected algebraic structure for geometric Langlands.

   One obtains the categories from geometric Langlands by the ansatz of categorical geometric quantization. Each moduli space admits a canonical Lagrangian subspace, and the geometric Langlands categories arise as the categories of ind-coherent sheaves on those Lagrangians.

2. In [EY17] we described the physical meaning of Arinkin and Gaitsgory's singular support conditions. We describe a very general concept: a category of boundary conditions in a topological field theory is acted upon by the algebra of local observables, and therefore one can discuss the support of a boundary condition in the moduli space of vacua: the spectrum of the algebra of local observables. We proved the following.

   Theorem ([EY17, Theorem 3.24])**.** The space of vacua in the Kapustin-Witten B-twist is isomorphic to $\mathfrak{h}^*/W$ [1]. The full subcategory of boundary conditions on a curve C set-theoretically supported at the vacuum $0 \in \mathfrak{h}^*/W$ is equivalent to the Arinkin-Gaitsgory category $\mathrm{IndCoh}_{\mathcal{N}_G}(\mathrm{Flat}_G(C))$ of sheaves with nilpotent singular support.

   In fact we conjecture somewhat more. It makes sense to consider the full subcategory of objects supported at any point in $\mathfrak{h}^*/W$. We discussed these categories in the example of the B-twist of $\mathcal{N} = 4$ super Yang-Mills theory and conjectured that the category of boundary conditions supported at $v \in \mathfrak{h}^*/W$ is equivalent to $\mathrm{IndCoh}_{\mathcal{N}_L}(\mathrm{Flat}_L(C))$ where L is the stabilizer of the point v. We refer to this as the "gauge-symmetry breaking" conjecture.

   The gauge-symmetry breaking conjecture leads to an interesting new conjectural factorization structure on $\mathrm{IndCoh}(\mathrm{Flat}_G(C))$. Restricting attention to the case where $G = \mathrm{GL}_n$ so that $\mathfrak{h}^*/W$ can be identified with the configuration space of n points in $\mathbb{C}$ we make the following conjecture.

   Conjecture. There is a factorization algebra over $\mathbb{C}$ where the stalk over a point $x \in \mathrm{Ran}(\mathbb{C})$ is equivalent to the algebra of Hochschild cochains

$$\bigoplus_{n \geq 1,\ \tilde{x} \in \mathrm{Sym}^n(\mathbb{C})} \mathrm{HC}^\bullet(\mathrm{IndCoh}_{\mathcal{N}_{L_{\tilde{x}}}}(\mathrm{Flat}_{L_{\tilde{x}}}(C)))$$

   where $\tilde{x} \in \mathrm{Sym}^n(\mathbb{C})$ is a lift of $x \in \mathrm{Ran}(\mathbb{C})$ and $L_{\tilde{x}}$ is its stabilizer in GL_n.

   This conjecture has a physical interpretation in terms of the motion of D3-branes, and we conjecture that it is related to the factorization structure on cohomological Hall algebras as in [KS11b] viewed analogously as the Hilbert space on a stack of D0-branes.

This research leads to several natural questions, as well as an intriguing potential generalization to Gaiotto's duality for general theories of class S, as I'll now explain.

Proposal 3. We propose to develop the necessary formalism to prove the analogous result to the theorem on vacuum conditions on the A-side. That is, beginning with the category of ind-coherent D-modules as discussed in [Gun17] (see also the category of renormalized D-modules of [AG15]) to prove that the full subcategory of D-modules set-theoretically supported at $0 \in \mathfrak{h}^*/W$ is equivalent to the usual geometric Langlands category $\text{D-mod}(\mathrm{Bun}_G(C))$. Concretely this would involve studying the action of a subring of the Hochschild cohomology $\mathrm{HH}^\bullet(\text{D-mod}_{\mathrm{coh}}(\mathrm{Bun}_G(C)))$ freely generated by $\mathfrak{h}/W$.

Proposal 4. Gaiotto [Gai12] introduced a wide-reaching generalization of the S-duality that is conjectured to implement a generalization of geometric Langlands duality for the so called theories of "Class S". These are $\mathcal{N} = 2$ supersymmetric field theories obtained by reducing the six-dimensional $(2, 0)$ superconformal field theory on a curve Σ: one obtains the usual 4d $\mathcal{N} = 4$ super Yang-Mills theories when Σ is an elliptic curve.

Theories in dimension 4 with $\mathcal{N} = 2$ supersymmetry admit a holomorphic-topological twist which we call the Kapustin twist after [Kap06]. Take a theory of class S associated to a curve Σ and a self-dual group G, form the Kapustin twist and compactify it on a curve C to obtain a 2d topological field theory which we'll denote by $Z_{\Sigma,G}(C)$. This topological field theory should admit a category $\mathcal{B}_{\Sigma,G}(C)$ of branes. Gaiotto's duality then tells us, in particular, the following.

Conjecture. The collection of categories $\mathcal{B}_{\Sigma,G}(C)$ for each Σ assembles to form a sheaf of categories over the moduli space $\mathcal{M}_{g,n}$ of curves. This sheaf extends to the boundary of the moduli space, i.e. extends to a sheaf of categories over the compactification $\overline{\mathcal{M}_{g,n}}$.

[1] Up to a degree shift by 2: see the paper for a discussion of the role of the degree shift.

The 4d theory at the boundary of the moduli space is obtained by RG-flowing to a particular low energy limit for the family of theories in the interior. These limiting theories do not necessarily have Lagrangian descriptions, however we can seek out examples where the twisted reduced theories nevertheless have amenable descriptions.

There are examples in which we can explicitly calculate the categories $\mathcal{B}_{\Sigma,G}(C)$ and therefore investigate this conjecture. To begin with we can compute many examples in the interior of the moduli space of curves by calculating the Kapustin twist of $\mathcal{N} = 2$ super Yang-Mills with an arbitrary matter hypermultiplet. Taking this as a starting point we can proceed to the following examples.

- The examples where $g = 0, n = 4$ are given by $\mathcal{N} = 2$ super Yang-Mills theory with four fundamental matter hypermultiplets. The dependence on a point in the moduli space $\mathcal{M}_{0,4}$ is given by a coupling parameter. In the case where $G = \mathrm{SU}(2)$ there is a *triality* isomorphism, here realized as an action of the group S_3 on the category occuring in the Kapustin twist. Which, as a first example, we can identify.
- A large family of dualities generalizing Argyres-Seiberg theory were described by Gaiotto and then greatly expanded by Chacaltana and Distler [CD10]. These dualities occur whenever a nodal curve at the boundary of $\overline{\mathcal{M}_{g,n}}$ can be obtained by gluing two curves at a point. Typical examples relate an $\mathrm{SU}(n)$ supersymmetric gauge theory with appropriate matter to an $\mathrm{SU}(2)$ gauge theory coupled to a non-Lagrangian superconformal matter theory – for instance the theories denoted by $\mathcal{T}_n$. While these theories are hard to describe in general, in the special case where our curve C has genus 1 we have a Lagrangian description of the 2d compactified theory as a sigma model valued in a Hitchin system [GMN13] – alternatively we could have reversed the order of our compactification along C and along Σ (although one needs to keep careful track of the twists one uses). By identifying the categories $\mathcal{B}_{\Sigma,G}(C)$ in each case we can hope to obtain interesting conjectural equivalences as consequences of Gaiotto duality.
- Finally we might investigate the ansatz that the Kapustin twists, even of the theories occuring at non-Lagrangian points with gauge group G, have Lagrangian descriptions of the form $\mathrm{Coh}(\mathrm{Sect}_G(C, V_\Sigma))$ where $\mathrm{Sect}_G(C, V_\Sigma)$ is the moduli stack of G-bundles along with sections of an associated bundle in some representation V_Σ, possibly twisted by the canonical bundle of C. We hope to check this ansatz and identify V_Σ in some simple examples.

Project 3 Multiplicative Langlands Duality

In current work with Pestun I am pursuing another connection between the geometric Langlands program and supersymmetric gauge theory. The version of the geometric Langlands conjecture here is a "multiplicative" version of the conjecture that is, as far as we are aware, a new idea. Our work involves the following moduli space (versions of which have been studied previously by Hurtubise-Markman [HM02], Bouthier [Bou14] and Frenkel-Ngô [FN11]).

Definition. The moduli space of *multiplicative G-Higgs bundles* on a curve C consists of pairs (P, ϕ) where P is a principal G-bundle on C and ϕ is a meromorphic automorphism of P. We fix the locations of the poles of ϕ at a divisor $D = \{z_1, \ldots, z_k\}$. We can also fix the local behaviour near the poles – controlled by a dominant coweight $\omega^\vee_{z_i}$ of G at each puncture. Denote these moduli spaces by $\mathrm{mHiggs}_G(C, D)$ (without fixed local behaviour) and $\mathrm{mHiggs}_G(C, D, \omega^\vee)$ (with fixed local behaviour where $\omega^\vee$ denotes a k-tuple of dominant coweights) respectively.

We focus our attention on the following rational/trigonometric/elliptic trichotomy:

- (Rational) $C = \mathbb{CP}^1$ and we fix a framing at the point ∞.
- (Trigonometric) $C = \mathbb{CP}^1$ and we fix a B_+ reduction at ∞ and a B_--reduction at 0 so that the respective induced T-reductions coincide.
- (Elliptic) $C = E$ is an elliptic curve (with no additional decorations).

In these cases the moduli space – like the ordinary Hitchin system – has the structure of an algebraic integrable system which we can naturally describe using the theory of shifted Poisson and coisotropic structures [CPT+17, MS18]. In particular it has an algebraic symplectic structure. If one doesn't fix dominant coweights at the punctures the full infinite-type moduli space has a Poisson structure and the moduli spaces with fixed coweights are symplectic leaves.

A theorem of Charbonneau and Hurtubise [CH10] (for GL_n) and Smith [Smi15] (for general G) tell us that the moduli space of multiplicative G-Higgs bundles (or rather its polystable locus) is analytically isomorphic to the moduli space of G-monopoles on $C \times S^1$. In the rational case this moduli space of periodic monopoles can be realized as a hyperkähler quotient. In particular it is holomorphic symplectic. In this case Pestun and I prove the following.

Theorem ([EP18] (In Progress))**.** In the rational case, the isomorphism identifying the moduli space of periodic monopoles and the moduli space of multiplicative Higgs bundles is compatible with the holomorphic symplectic structures on both sides. The holomorphic symplectic structure on the multiplicative Higgs moduli space can be identified with the pullback of the Poisson Lie structure under the map $\mathrm{mHiggs}_G^{\mathrm{fr}}(\mathbb{CP}^1, D) \to G_1[[z^{-1}]]$ given by restriction to a formal neighbourhood of the framed point ∞. For $G = \mathrm{GL}_n$ the symplectic leaves coincide with the symplectic leaves classified by Shapiro [Sha16].

Our equivalence promotes the holomorphic symplectic structure on $\mathrm{mHiggs}_G^{\mathrm{fr}}(\mathbb{CP}^1, D, \omega^\vee)$ to a hyperkähler structure. We can identify the holomorphic symplectic space obtained by rotating to a point q in the twistor sphere with the moduli space of *q-connections*: principal G-bundles P equipped with a meromorphic isomorphism $P \to q^*P$ from P to its translate.

This work is motivated in part by the work of Nekrasov and Pestun [NP12], which implies that moduli spaces of multiplicative Higgs bundles arise as the Seiberg-Witten integrable system associated to ADE quiver gauge theories. In particular they should admit natural hyperkähler structures. Nekrasov and Pestun also conjectured that the deformation quantization of the algebras of functions on the moduli spaces (without fixing data at the singularities) should be closely related to the Yangian – our result shows that there is an algebra map from the Yangian $Y(\mathfrak{g})$ to this deformation quantization of our moduli space. If we fix data at the singularities the resulting deformation quantization therefore has the structure of a $Y(\mathfrak{g})$-module. Our approach follows the work of Hurtubise and Markman [HM02] who studied the elliptic analogue of our symplectic structure.

Proposal 5. We hope to extend this work in two ways. Firstly we plan on studying the trigonometric analogue of this result, where we expect there to exist a natural holomorphic symplectic structure related to symplectic leaves in the loop group with its trigonometric R-matrix.

We also conjecture that it's possible to construct symplectomorphisms from the rational multiplicative Higgs moduli space – after rotating away from 0 in the twistor sphere – to a version of the trigonometric Zastava space studied by Finkelberg, Kuznetsov and Rybnikov [FKRD18] via a Nahm transform; in other words by comparing monopoles with suitable boundary conditions on $\mathbb{C} \times S^1$ with monopoles on $\mathbb{C}^\times \times \mathbb{R}$. Likewise in the trigonometric case it should be possible to compare the multiplicative Higgs moduli space to the space of monopoles on $E \times \mathbb{R}$ with scattering-type boundary conditions at $\pm\infty$.

Part of our motivation for this work is to give a multiplicative analogue of the geometric Langlands conjecture. To a first approximation this conjecture should take the following form.

Conjecture (Multiplicative Geometric Langlands)**.** Let G be a Langlands self-dual group. In the rational, trigonometric and elliptic examples there is an equivalence of categories

$$A_{q^{-1}}\text{-branes}(\mathrm{mHiggs}_G(C, D, \omega^\vee)) \cong \mathrm{Coh}(q\text{-Conn}_G(C, D, \omega^\vee))$$

where $q\text{-Conn}_G$ is the moduli space of q-connections where q is an automorphism of C. If G is obtained from a self-dual group $\widetilde{G}$ via an automorphism ψ of the Dynkin diagram, there is an equivalence of categories

$$A_{q^{-1}}\text{-branes}(\mathrm{mHiggs}_{\widetilde{G}}(C, D, \omega^\vee)^\psi) \cong \mathrm{Coh}(q\text{-Conn}_G(C, D, \omega^\vee))$$

where on the left ψ acts on $q\text{-Conn}_{\widetilde{G}}(C, D, \omega^\vee)$ via its action on the Dynkin diagram of $\widetilde{G}$, but also by rotating the circle (where we identify $\mathrm{mHiggs}_{\widetilde{G}}(C)$ with $\mathrm{Bun}_G(C \times S^1_{\mathrm{Betti}})$).

We can also pose a symmetrical 2-parameter form of the conjecture analogous to the quantum Langlands correspondence. In order to make these conjectures precise we need a concrete mathematical model for the category of $A_{q^{-1}}$-branes. We'll discuss in a moment the abelian case where we have such a model, but first let's discuss the limit $q \to \mathrm{id}$, in which we obtain a classical limit for this multiplicative Langlands conjecture.

Conjecture. Let $G, \widetilde{G}$ and ψ be as above. There is an equivalence of categories

$$\mathrm{Coh}(\mathrm{mHiggs}_G(C, D, \omega^\vee) \cong \mathrm{Coh}(\mathrm{mHiggs}_{\widetilde{G}}(C, D, \omega^\vee)^\psi).$$

From the physical point of view these conjectures are motivated by S-duality for 5d $\mathcal{N} = 2$ super Yang-Mills theory. Just as the usual Langlands duality arises by dimensionally reducing a 6d $(2, 0)$ superconformal field theory on a torus, we can study the compactification of the 6d theory on a torus then take the radius of only one of the circles to zero, keeping the other circle finite. The result is a 5d $\mathcal{N} = 2$ gauge theory compactified on a circle, and we can study the (partially) topological twists of such 5d theories. The categories of A- and B-branes above arise as the categories of boundary conditions in such twists on 3-manifolds of the form $C \times S^1$, with the insertion of monopole operators at a finite set of points.

Proposal 6. Make this reasoning precise using the techniques of Section 2.

Proposal 7. For a first example we can consider the abelian case of the conjecture on the elliptic curve. Write $\mathrm{Diff}_q(X)$ for the category of q-difference modules on X, i.e. modules for the sheaf of q-difference operators. Our conjecture suggests we should look for an equivalence of categories

$$\mathrm{Diff}_{q_1}(q_2\, \mathrm{Conn}_{\mathrm{GL}(1)}(E) \cong \mathrm{Diff}_{q_2^{-1}}(q_1^{-1}\, \mathrm{Conn}_{GL(1)}(E)$$

given by a twisted Fourier-Mukai transform analogoue to that of Polishchuk and Rothstein [PR01]

Project 4 Holomorphic Theories and Holomorphic Supergravity

In a joint paper with Williams and Yoo [EWY18] we investigated how the theory of asymptotic freedom and the notion of the β-function fit into the factorization algebra formalism for perturbative quantum field theory. In a future project we propose to use these ideas to study RG flow for holomorphic field theories on $\mathbb{C}^n$, and in particular to work towards a conceptual understanding of higher analogues of the c-theorem of Zamolodchikov. We are also currently investigating a specific holomorphic theory in two complex dimensions arising as a twist of $\mathcal{N} = 1$ supergravity.

In our earlier paper we obtained the following result.

Theorem ([EWY18])**.** There is a quantum observable – the *β-functional* – associated to any translation-invariant field theory, describing the first order deformations of the effective interaction. This observable is closed for the BV differential. The contribution at k-loops is individually closed if the ℓ-loop contributions vanish for $\ell < k$. In particular if the theory is classically scale-invariant then its cohomology class – the 1-loop β-function – is well-defined. This class is a homotopy invariant in the space of all quantum field theories and independent of the choice of 1-loop quantization of a classical field theory. It can be computed as a 1-loop counterterm or as a 1-loop scale anomaly. Altogether this allows us to verify in the formalism of factorization algebras the physical computation of the β-function of Yang-Mills theory.

Now, let's consider holomorphical factorization algebras on $\mathbb{C}^n$, meaning quantum field theories with an action of the Lie algebra of holomorphic vector fields. There is a classification of the possible anomalies associated to the holomorphic translation action in terms of the Lie algebra cohomology of the algebra W_n of Lie algebras on the formal n-disk: see [Wil18, Section 4.5]. One can describe these classes explicitly as local observables in the factorization envelope of the algebra of holomorphic vector fields. In joint work with Williams we hope to use this in order to understand the c-theorem and glimmers of its higher analogues for higher dimensional holomorphic field theories.

Proposal 8. Every conformal anomaly cocycle pulls back to a class in the space of anomalies for holomorphic vector fields. We propose to identify the a- and b-classes [DS93] as concrete observables in the factorization envelope of the algebra of holomorphic vector fields. In particular we would like to investigate the geometric interpretation of the a-class in complex dimension 2 occuring in the a-theorem of Komargodski and Schwimmer [KS11a].

In joint work with Williams I am also investigating the holomorphic twist of $\mathcal{N} = 1$ supergravity in dimension 4. The theory of twisted supergravity has recently been studied in the mathematics literature by Costello and Li [CL16].

In the physics literature some relevant aspects of twisted supergravity have been discussed by Baulieu, Bellon and Reys [BBR13] who calculated the supersymmetry action on $\mathcal{N}=1$ supergravity as a superalgebra extending the group $\mathrm{U}(2) \ltimes \mathbb{C}^2$.

We are currently working on the following project.

Proposal 9. There is an equivalence between the holomorphic twist of $\mathcal{N}=1$ supergravity on a complex symplectic surface X – which we model in the first order formalism – and the cotangent theory of holomorphic symplectic vector fields on X. By an equivalence, we mean a quasi-isomorphism between the sheaves of dg Lie algebras that model the classical fields in the BV formalism. The latter theory has, as its classical BV complex, the dg Lie algebra of fields

$$(A, B) \in \Omega^{0,\bullet}(X)[1] \oplus \Omega^{2,\bullet}(X)$$

with Lie bracket given by the holomorphic Poisson bracket on $\Omega^{0,\bullet}(X)$.

Furthermore, if we couple the supergravity theory to a chiral matter multiplet valued in a vector space V and calculate the holomorphic twist, we obtain the theory of holomorphic symplectic vector fields coupled to a free $\beta\gamma$ system valued in V, where the field A acts on γ via the holomorphic Poisson bracket. In particular the complicated supergravity coupling becomes something very simple. We will demonstrate this by proving that the twisted coupling is the unique one that is compatible with the cotangent structure and the holomorphic covariant structure – this uniqueness is an obstruction theory calculation.

We intend to apply this in order to study the holomorphic twists of G_2-compactifications of 11-dimensional supergravity. In the 11d $\mathcal{N}=1$ supersymmetry algebra there is a twist with 9d image stabilized by the group $G_2 \times \mathrm{SU}(4)$. We can therefore describe the corresponding twist of maximal supergravity on a G_2-manifold as the coupled $\mathcal{N}=1$ 4d supergravity theory above with matter determined by the cohomology of the G_2-manifold. Alternatively we could start with 10d type IIB supergravity. The corresponding twist of this theory is conjectured to be equivalent to BCOV theory in [CL15]. Our proposal is compatible with this conjecture under compatification along a Calabi-Yau threefold.

After completing this analysis I propose to extend it in order to investigate twists of the AdS/CFT correspondence.

Proposal 10. Having computed the holomorphic twist of $\mathcal{N}=1$ supergravity in four dimensions coupled to matter, we can extend the calculation to analyze the holomorphic twists of $\mathcal{N}=2$ supergravity. We can then describe holomorphic 3d boundary theories associated to this holomorphic twist, whose observables we can model by factorization algebras on $\mathbb{R} \times \mathbb{C}$ which are topological in the real direction and holomorphic in the complex direction, or equivalently by $\mathbb{E}_1$-algebras valued in vertex algebras.

The AdS4/CFT3 correspondence leads us to speculate that there is such a boundary theory Koszul dual (see the proposal in [Cos17]) to the algebra of quantum observables in a twisted gauge theory, namely a twist of $\mathcal{N}=2$ superconformal Chern-Simons theory in dimension 3 in the large N limit (see e.g. [BKKS08]).

Project 5 Twisted Yangians

I'll finally discuss a project relating twisted Yangians – coideal subalgebras for the Yangian – to defects in 4d supersymmetric field theory. This generalizes work of Costello [Cos13] (and its sequels with Witten and Yamazaki [CWY17, CWY18]) and uses ideas recently introduced by Weelinck [Wee18].

The Yangian arises from a 4d partially holomorphic Chern-Simons theory – for instance arising as a deformation of holomorphically twisted 4d $\mathcal{N}=1$ super Yang-Mills theory. The local quantum observables in this theory form an $\mathbb{E}_2$-algebra valued in holomorphic factorization algebras. The Yangian is equivalent to the $\mathbb{E}_1$ Koszul dual of this algebra of local observables.

Proposal 11. Consider the quotient of $\mathbb{R}^2 \times \mathbb{C}$ with respect to the $\mathbb{Z}/2\mathbb{Z}$-action $(x, y, z) \mapsto (x, -y, -z)$ so that the $\mathbb{Z}/2\mathbb{Z}$-action fixes a line. Consider the partially holomorphic Chern-Simons theory with gauge group G on $\mathbb{R}^2 \times \mathbb{C}$. We can descend this theory to the quotient stack where $\mathbb{Z}/2\mathbb{Z}$ acts not only on spacetime but also on the gauge group by an involution $\theta \colon G \to G$ (alternatively we can think of this as a 4d theory with a line defect). Consider the categories of line operators along lines $\mathbb{R}_x \times \{y_0\} \times \{z_0\}$. There is a monoidal category $\mathcal{L}$ of line operators in the bulk (away from the fixed line) and an $\mathcal{L}$-module $\mathcal{L}_0$ of line operators along the defect. We have a pair of natural functors

$$\begin{aligned}(\mathcal{L}, \mathcal{L}_0) &\to (\mathrm{HC}^\bullet(\mathcal{L})\text{-mod}, \mathrm{HC}^\bullet(\mathcal{L}_0)\text{-mod}) \\ &\cong (\mathrm{Obs}\text{-mod}, \mathrm{Obs}_0\text{-mod})\end{aligned}$$

to the categories of modules for the local operators in the bulk and at the defect. Here $\mathrm{HC}^\bullet(\mathcal{C})$ denote the algebra of Hochschild cochains of $\mathcal{C}$. According to Costello's theorem, Obs is Koszul dual to the Yangian algebra $Y(\mathfrak{g})$. I conjecture that Obs_0 is Koszul dual to the twisted Yangian $Y(\mathfrak{g}, \mathfrak{g}^\theta)$ – a coideal subalgebra in the Yangian associated to the involution. Just as Costello's description characterizes the universal R-matrix as determined by the quantization of the factorization structure (the OPE) in the holomorphic direction this characterizes the universal K-matrix in a similar way in terms of the action of the local observables in the bulk on those near the defect.

This expectation comes both from calculating the classical limit and from the work of Weelinck. Weelinck proved that the data of a topological factorization category on the orbifold $\mathbb{R}^2/(\mathbb{Z}/2\mathbb{Z})$ is equivalent to a choice of quantum symmetric pair. Replacing one of the real directions by a complex direction I hope to find a similar relationship, where now the quantum symmetric pairs include spectral parameter. By Costello's theorem we already know that the algebra in the bulk is Koszul dual to the Yangian, in particular it has the same category of modules. At the defect we would therefore expect to obtain the category of modules for some coideal subalgebra, which is then fixed by observing that it has the correct classical limit.

This proposal is related to the project I described in Section 3. Conjecturally the four-dimensional partially holomorphic Chern-Simons theory is a boundary theory for the holomorphic-topological twist of 5d $\mathcal{N} = 2$ Yang-Mills theory which I hope will lead to a "holographic" explanation of the Koszul duality between their algebras of local observables.

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