Twisting Supersymmetric Field Theories

Chris Elliott

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The idea of *twisting* supersymmetric field theories is a useful idea for two reasons:

1. It allows us to take a very complicated and unwieldy field theory (like super Yang-Mills) and produce a much simpler theory, both simpler in the sense of having a more understandable Lagrangian description, but also often simpler in terms of invariance under the action of certain symmetries. For instance one can often produce topological or conformal field theories from supersymmetric theories that are only Euclidean.
2. It allows us to produce theories that can manifestly be defined on general Riemannian manifolds from supersymmetric theories that only make sense on very special manifolds: $\mathbb{R}^n$, or manifolds admitting covariant constant spinors.

In this talk I'll explain what it means to twist a supersymmetric theory, and explain how these two simplifications can arise. In addition to the original papers of Witten [4], [5] a good reference on cohomological field theories and topological theories is a minicourse by Marcel Vonk [3]. A more modern perspective on twisting is described by Costello [1], [2].

1 Supersymmetric Field Theories

Let's recall the notion of *supersymmetry* from the point of view of classical Lagrangian field theories on a flat spacetime. We'll use the following definition of a classical field theory

Definition 1.1. A *classical Lagrangian field theory* on $\mathbb{R}^n$ is a pair $(\Phi, \mathcal{L})$, where Φ is a coherent sheaf of super vector spaces on $\mathbb{R}^n$, and $\mathcal{L}$ is a map of sheaves into the sheaf of densities

$$\mathcal{L}\colon \Phi \to \mathrm{Dens}(\mathbb{R}^n).$$

Requiring that this is a map of sheaves ensures *locality*, i.e. that the Lagrangian density cannot depend on interaction between values of the fields which are separated in spacetime.

Remark 1.2. Usually we equip $\mathbb{R}^n$ with a pseudo-Riemannian metric, and allow the Lagrangian density $\mathcal{L}$ to depend on this metric. To make this dependence explicit we will often write $\mathbb{R}^{p,q}$ instead of $\mathbb{R}^n$, to indicate $\mathbb{R}^n$ equipped with the standard diagonal metric of signature (p,q), and consider field theories with this metric fixed.

The action of the additive group $\mathbb{R}^n$ on itself by translation induces an action of $\mathbb{R}^n$ on the super vector space $\Phi(\mathbb{R}^n)$ of global fields by pullback (assuming throughout that the sheaf of fields is equivariant for the action of the isometry group of spacetime). The derivative of this action at the identity describes an action of the abelian Lie algebra $\mathbb{R}^n$ by *infinitesimal symmetries*: that is, translation manifestly preserves the action.

Definition 1.3. An *infinitesimal symmetry* of the Lagrangian field theory $(\Phi, \mathcal{L})$ is a vector field $\xi \in \mathrm{Vect}(\Phi(\mathbb{R}^n) \times \mathbb{R}^n)$, that is *local*, and *preserves the action*: that is,

$$\mathrm{Lie}_\xi \mathcal{L} = d\alpha_\xi$$

for some (twisted) $n-1$-form α_ξ on Φ. This means the infinitesimal variation of the Lagrangian density along ξ integrates to zero by Stokes' theorem, so the integral of the Lagrangian density (the action) is preserved.

The idea of *supersymmetry* is that under certain circumstances it is possible to extend the translation action to also include *fermionic* symmetries of the super vector space of fields. If this happens, the Lagrangian theory is called supersymmetric. More precisely:

Definition 1.4. Let S be a non-trivial finite-dimensional real spinorial representation of $\mathrm{Spin}(p,q)$, and let $\Gamma\colon S \otimes S \to \mathbb{R}^{p+q}$ be a spin-equivariant symmetric bilinear pairing (i.e. a map of $\mathrm{Spin}(p,q)$-representations). The *super translation algebra* associated to this data is the super Lie algebra with underlying vector space

$$T = \mathbb{R}^{p+q} \oplus \Pi S,$$

whose only non-trivial bracket comes from the pairing Γ.

A field theory $(\Phi, \mathcal{L})$ on $\mathbb{R}^{p,q}$ is said to be *supersymmetric* if the translation action by $\mathbb{R}^{p+q}$ extends to the action of a super translation algebra T by infinitesimal symmetries.

If $p - q \neq 0 \mod 8$, then there is a unique irreducible spinorial representation S_0. If $S \cong S_0^N$ the theory is said to have N supersymmetries. If $p - q \equiv 0 \mod 8$ then there are two irreducible spinorial representations S_1 and S_2. If $S \cong S_1^{N_1} \oplus S_2^{N_2}$ then the theory is said to have (N_1, N_2) supersymmetries.

1.1 R-Symmetries

So far, we've only discussed the action of translation on a field theory, naturally induced from the action of $\mathbb{R}^n$ on itself. However, $\mathbb{R}^{p,q}$ has more isometries than just translation, we have the infinitesimal action of a full *Poincaré algebra* of the form $\mathfrak{so}(p,q) \rtimes \mathbb{R}^{p+q}$ preserving the metric and orientation of spacetime. This action describes infinitesimal symmetries of any Lagrangian field theory on $\mathbb{R}^{p,q}$ in an analogous way to the action of translation.

Introducing a fermionic piece S acted on by $\mathfrak{so}(p,q)$ as before (with this action defining the additional brackets) leads to a *super Poincaré algebra*, which exponentiates to a *super Poincaré group*. The spin group acts on the odd symmetries. We might ask, are these the only ways to modify the supersymmetries? In general no, and the additional automorphisms are called *R-symmetries*.

Definition 1.5. The *R-symmetry group* associated to the super translation group T is the group

$$G_R = \{\alpha \in \mathrm{Out}(T)\colon \alpha(v) = v \text{ for all } v \in \mathbb{R}^{p+q}\},$$

the group of outer automorphisms fixing the bosonic part.

When we include the left multiplication action also, there is an action of the product group $\mathrm{Spin}(p,q) \times G_R$ on the odd part of the super translation algebra. Understanding the action of this group will be important for understanding whether a theory is topological or not, and in order to *force* theories to be topological we'll modify this action using the R-symmetries.

2 Topological Field Theories

In many important examples of field theories the Lagrangian density $\mathcal{L}$ manifestly has no dependence on the metric on spacetime (for instance, Chern-Simons theory has this property). Such theories are particularly easy to work with, and usually admit obvious extensions to theories defined on more general manifolds, where they may have applications computing topological invariants and so on. However, even if a theory isn't explicitly metric-independent it is sometimes possible to modify the observables in a theory so as to produce a topological result. In this section I'll explain a general procedure due to Witten [4] [5] for identifying such theories. This will be the foundation for the idea of twisting.

First, recall that *Noether's theorem* tells us that for each symmetry there is associated a *conserved current*: a current j whose divergence is zero. In particular, we can apply this to the translation symmetries of a field theory on $\mathbb{R}^n$, to produce conserved currents for each element of $\mathbb{R}^n$. One usually chooses a basis x^α for $\mathbb{R}^n$ and denotes the associated conserved current $T_{\alpha\beta}\, dx^\beta$. The corresponding 2-tensor is called the *energy-momentum tensor* associated to the field theory. If the theory is topological then the translation symmetries act trivially on the fields, and therefore, the energy-momentum tensor vanishes. We'll construct a theory where this happens by starting with the next best thing. Suppose we have a fermionic symmetry Q such that:

1. $Q^2 = 0$
2. The energy-momentum tensor $T_{\alpha\beta}$ is Q-exact.
3. The vacuum state is Q-invariant (more on this later).

Remarks 2.1.

- Since Q is fermionic, $Q^2 = \frac{1}{2}[Q, Q]$ makes sense.
- We can ensure that condition 2 holds by requiring the Lagrangian itself to be Q-exact: asking for another map $\Lambda\colon \Phi \to \mathrm{Dens}(X)$ of sheaves such that $\mathcal{L}(\phi) = \Lambda(Q\phi)$. If this is the case then perturbations of the action under infinitesimal translation symmetries are themselves given by Q-exact observables, which vanish in the cohomology.

Now, the idea is to *pass to the cohomology* with respect to Q, i.e. to consider Q-closed observables modulo Q-exact observables, so that the energy-momentum tensor – being Q-exact – vanishes. What does this mean?

Intuitively, observables are just *functions* on the space of fields $\Phi(\mathbb{R}^n)$. The symmetry Q acts on observables by pulling back along the action on functions, so it's clear what it means to be Q-closed or Q-exact. Now, for the Q-cohomology of observables to be a sensible thing to consider we need computable quantities in the quantum field theory – path integrals –to be independent on the choice of representative in a cohomology class. That is, we need the expectation value of Q-exact observables to vanish.

Here's a sketch proof of this fact using the Hamiltonian formalism (following Vonk [3]). The exact observable $Q\mathcal{O}$ corresponds to an operator on the Hilbert space of the theory $[S, \mathcal{O}]$ (abusing notation to identify the functional $\mathcal{O}$ with the corresponding operator). Here S is the canonical quantisation of the conserved current of Q. The expectation value of this observable is computed to be

$$\langle 0|S\mathcal{O}|0\rangle \pm \langle 0|\mathcal{O}S|0\rangle$$

in the usual physics notation. Now we use the invariance of the vacuum: condition 3 above. Invariance of the vacuum under Q is equivalent to the condition $S|0\rangle = 0$, so this expectation value must vanish. The result is that these observables behave as observables in a quantum field theory.

While intuitively it suffices to work with any heuristic definition of observables, one can make this more mathematically explicit in the language of factorisation algebras. In this language, the *observables* in the quantum field theory are modelled by an elliptic moduli problem, which in particular is a super cochain complex (perhaps with, for instance, a dgla or L_∞-structure). Intuitively, we should just *add* Q to the differential in the moduli problem. Of course, for this to make sense, we really need Q to be degree one for the grading in the moduli problem (ghost number): a priori it just acts on fields, i.e. is degree zero in the ghost number grading, but fermionic. So we must fix this somehow if we want to get anywhere with this description. For concreteness, let's describe this for twisted supersymmetric theories.

3 Topological Twisting

Given our discussion on cohomological field theories above, we are led to wonder whether we could cook up a fermionic symmetry Q that could define a topological field theory from the supersymmetries in a supersymmetric theory. We'll see that while the energy momentum tensor is not exact for any supersymmetry, we can often *make it* exact by modifying the action of the isometry group of spacetime on the fields.

The procedure begins as follows:

1. Find an odd element Q of the super translation algebra such that $[Q, Q] = 0$.
2. Choose a homomorphism $\rho\colon \mathrm{Spin}(p, q) \to G_R$ – hence an embedding of $\mathrm{Spin}(p, q)$ into $\mathrm{Spin}(p, q) \times G_R$ by $g \mapsto (g, \rho(g))$ – such that Q is invariant for the new $\mathrm{Spin}(p, q)$-action.

Since Q is $\mathrm{Spin}(p, q)$-invariant (the fermionic symmetries of the system form a $\mathrm{Spin}(p, q)$ representation: and we're asking for Q to lie in a trivial irreducible summand), the energy-momentum tensor is Q-exact. Indeed, if this is the case the image of the map $[Q, -]$ from odd symmetries to even symmetries is itself an irreducible $\mathrm{Spin}(p, q)$-representation. We just have to check that *some* $[Q, Q']$ is a translation because the translations $\mathbb{R}^{p+q}$ describe the defining representation of $\mathrm{Spin}(p, q)$, which is irreducible. If every translation is of the form $[Q, Q']$ then the conserved currents for these translations are also Q-exact.

So applying the recipe in the previous section produces a topological field theory. As mentioned in the previous section, if we just want to add Q to the BV-BRST differential on the complex of observables we'll need to fix its degree. We can do this explicitly in this supersymmetric setting. To do so, choose a copy of $\mathbb{C}^\times \hookrightarrow G_R$. This $\mathbb{C}^\times$ acts on Q, and we'd like it to act with weight 1 (given the data we've already given ourselves we can always find such an embedding). We'll use this $\mathbb{C}^\times$ to change the degrees in the BV-BRST complex so that Q is bosonic of degree 1 as required. If (Obs, d) is the complex of observables in our Lagrangian field theory, we'll consider the new complex

$$\mathrm{Obs}_Q = (\mathrm{Obs}((t)), d + tQ)^{U(1)},$$

where t is a degree 1 complex fermionic parameter, so tQ is degree 1 bosonic as required, and where $\mathbb{C}^\times$ acts on t with weight -1.

Remark 3.1. We could perform this procedure even if Q was not $\mathrm{Spin}(p, q)$-invariant. In that case we generally wouldn't get a topological theory, but we might still get a twisted theory that was easier to manage in some ways than the original theory. For instance, for $N = 1$ super Yang-Mills there is no topological twist, but there is a *conformal* twist.

References

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