Holomorphic Chern-Simons Theory and the (Super) Twistor Correspondence

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1 Motivation from Gauge Theory

In this seminar, we've spent a fair amount of time discussing three-dimensional quantum field theories. A *topological* 3d quantum field theory assigns invariants to smooth compact 3-manifolds that can be computed by cutting and pasting along 2-manifolds. Better still, if the theory is extended to codimension 2 one can cut and paste along 2-manifolds with boundary, as Sean explained. The particular example we've focused on is *Chern-Simons Theory*, which was discussed by Nilay and Pyongwon. Chern-Simons theory is an example of a *gauge theory*: it depends on the choice of a compact gauge group G, and the action functional is invariant under a natural action of the group

$$\mathcal{G} = C^\infty(M, G)$$

of *gauge transformations* on the fields. In addition, it has the special property of being *manifestly* topological, i.e. the action functional only depends on the underlying smooth oriented manifold, and not on any additional data ("background fields").

I'm more interested in studying four-dimensional gauge theories. These are important both for physics – the standard model is an example of a 4d gauge theory – and for a wealth of mathematical applications. Of course, just like we used Chern-Simons theory to study the topology of 3-manifolds, one can use 4d gauge theories to study the topology of 4-manifolds: Donaldson and Seiberg-Witten invariants are examples of the successful application of this principle. The applications I'm most interested in are a little different. Instead of assigning numerical invariants to closed 4-manifolds, one can think of a 4d gauge theory as assigning categorical invariants to closed 2-manifolds, or more precisely to pairs (Σ, G) where Σ is a closed 2-manifold and G is a choice of gauge group. These invariants often coincide with invariants from the field of *geometric representation theory*, which is to say that they can be studied using the algebraic geometry of stacks built from Σ and G.

Examples 1.1. 1. The principal example of a 4d gauge theory, from which many other examples can be built, is *Yang-Mills theory*. As a classical theory this theory is defined as follows.

Definition 1.2. *Yang-Mills theory* with gauge group G on a smooth oriented Riemannian 4-manifold X is the classical field theory whose fields are given by connections A on principal G-bundles, and whose Lagrangian density is

$$\mathcal{L}(A) = \mathrm{Tr}(F_A \wedge *F_A)$$

where Tr is the map $\Omega^4(X; \mathfrak{g}_P \otimes \mathfrak{g}_P) \to \Omega^4(X)$ associated to a choice of invariant pairing on $\mathfrak{g}$. The equations of motion in Yang-Mills theory say that $d^* F_A = 0$.

2. A connection A is called *self-dual* if $F_A = *F_A$, and *anti-self-dual* if $F_A = - * F_A$. One observes, since F_A is closed, if A is self-dual or anti-self-dual then it automatically satisfies the equations of motion of Yang-Mills theory. One can modify the classical field theory above to *self-dual Yang-Mills theory*, which has only the self-dual solutions. One considers fields of form (A, B), where B is a self-dual $\mathfrak{g}_P$-valued 2-form, and the Lagrangian density

$$\mathcal{L}_{\text{self-dual}}(A, B) = \mathrm{Tr}(F_A \wedge *F_A + F_A^+ \wedge *B)$$

where now F_A^+ is the self-dual part of F_A. The equations of motion of this theory say that $F_A = *F_A$ and $F_A^+ = B$.

3. Yang-Mills theory on $\mathbb{R}^4$ admits *supersymmetric extensions*. These are theories with a $\mathbb{Z}/2$-graded space of fields which is acted on by a supersymmetry algebra – a $\mathbb{Z}/2$ graded algebra extending the Poincaré algebra – preserving the space of solutions to the equations of motion. There are different versions of this theory depending on which supersymmetric extension of the Poincaré algebra one chooses. The odd parts of supersymmetry algebras are always spinorial representations of $\mathfrak{so}(4)$, so one speaks of $N = 1, 2$ and 4 super Yang-Mills theory, where one takes 1,2 or 4 copies of the spin representation (we skip 3, because it turns out that there's no extension of Yang-Mills theory which is acted on by the $N = 3$ supersymmetry algebra but not the $N = 4$ supersymmetry algebra.) Super Yang-Mills theories also can be restricted to their self-dual parts.

4. There's a procedure called *twisting*, which takes a supersymmetric classical field theory and returns a theory which is invariant under some of the symmetries in the Poincaré group. I won't describe this procedure in this talk, but this procudure allows you to construct topological field theories from supersymmetric field theories. From the point of view of mathematics this provides a strong motivation for considering supersymmetry.

   Remark 1.3. After twisting, the difference between Yang-Mills theory and its self-dual part frequently vanishes.

My aim today is to describe a relationship between these four-dimensional gauge theories and a version of Chern-Simons theory which makes sense in 3 *complex* dimensions, i.e. 6 real dimensions.

Definition 1.4. *Holomorphic Chern-Simons theory* on a complex manifold of complex dimension 3 equipped with a Calabi-Yau structure Ω [1] is the theory whose fields are $(0,1)$-connections on G-bundles (with an action of the usual gauge transformations), and whose Lagrangian density is

$$\mathcal{L}(A) = \Omega \wedge \mathrm{Tr}(A \wedge \overline{\partial} A + \frac{2}{3} A \wedge [A, A]).$$

The solutions to the equations of motion are *holomorphic* principal G-bundles.

This theory also admits a supersymmetric extension: given a complex *supermanifold* whose even part has complex dimension 3, with a super Calabi-Yau structure, the above theory still makes sense.

Remark 1.5. A different way of defining a classical field theory, which is maybe more satisfying to a mathematician, is to define it in terms of its phase space. One can define a classical field theory to be a *sheaf of derived stacks* whose global sections admit a (-1)-shifted symplectic structure, in the sense of [PTVV13]. One can recover holomorphic Chern-Simons theory on X from this point of view by assigning to an open set $U \subseteq X$ the moduli space $\mathrm{Bun}_G(U)$. The Calabi-Yau structure on X makes $\mathrm{Bun}_G(X)$ into a (-1)-shifted symplectic stack by the AKSZ construction.

Remark 1.6. For further information about holomorphic Chern-Simons theory, twistor geometry and its applications to gauge theory and gravity, I recommend the comprehensive accounts of Ward and Wells [WW91], and Mason, Skinner and Woodhouse [MW96].

2 The Penrose-Ward Transform

The relationship between holomorphic Chern-Simons theory and Yang-Mills theory proceeds via the idea of *compactification* of a classical field theory. This idea is very simple, it's just a version of pushforward.

Definition 2.1. Given a smooth map $p \colon X \to Y$ (in examples, this will usually be a fiber bundle), and a classical field theory $\mathcal{M}$ on X, the *compactification* along p is the classical field theory on Y whose fields and Lagrangian density on $U \subseteq Y$ are those of $\mathcal{M}$ on $p^{-1}(U) \subseteq X$.

[1] that is, a nowhere vanishing $(3,0)$-form.

The idea, introduced by Ward [War77] following a program of Roger Penrose, is that self-dual Yang-Mills theory on a Riemannian 4-manifold X arises by compactifying holomorphic Chern-Simons theory on a suitable complex 3-fold $\mathrm{Tw}(X) \to X$, called the *twistor space* of X. This idea is sometimes called the *Penrose-Ward correspondence*. In order to realise this proposal, we'll need to impose a curvature condition on X, which will need a little background from differential geometry.

Recall that the *curvature* of a connection on a vector bundle E over a manifold X is an $\mathrm{End}(E)$-valued 2-form. If X is a Riemannian manifold then there is a canonical connection on the tangent bundle (the Levi-Civita connection), and therefore a canonical $\mathrm{End}(T_X)$-valued 2-form called the *Riemann tensor* of X. Using the metric we can identify the Riemann tensor with a section of $T_X^{*\otimes 4}$. The bundle $T_X^{*\otimes 4}$ receives an action of the principal $\mathrm{SO}(4)$-bundle of oriented frames, and under this action it decomposes into a number of irreducible summands. In particular the Riemann tensor decomposes into several summands, living in these irreducible summands.

Recall that $\mathrm{SO}(4) \cong (\mathrm{SU}(2)\times\mathrm{SU}(2))/(\mathbb{Z}/2)$, where $\mathbb{Z}/2$ is embedded diagonally. Irreducible complex representations of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ are given by pairs (i, j) where i and j are half-integers ≥ 0, corresponding the the representation $V_1^{\otimes 2i} \otimes V_2^{\otimes 2j}$, where V_1, V_2 are the fundamental representations of the two copies of $\mathrm{SU}(2)$. The diagonal $\mathbb{Z}/2$ acts trivially whenever $i + j$ is an integer; in these cases the complex representation also admits a real form.

Proposition 2.2. The *Weyl tensor* $W(X)$ is the summand of the Riemann tensor corresponding to the 10-dimensional representation $(2, 0) + (0, 2)$. It decomposes into irreducible summands $W_+(X)$ and $W_-(X)$.

Remark 2.3. The other irreducible summands of the Riemann tensor live in the representations $(0, 0)$ (1-dimensional) and $(1, 1)$ (9-dimensional), and are built from the Ricci scalar and the Ricci tensor respectively, along with the metric.

Atiyah, Hitchin and Singer provided a mathematical realization of Ward's proposal, provided X satisfies a suitable flatness condition.

Definition 2.4. A Riemannian 4-manifold X is called *self-dual* if the negative Weyl tensor $W_-(X)$ vanishes.

Theorem 2.5 (Atiyah-Hitchin-Singer [AHS78])**.** If X is a self-dual oriented Riemannian 4-manifold, there exists a complex manifold $\mathrm{Tw}(X)$ and a smooth map $p\colon \mathrm{Tw}(X) \to X$ so that holomorphic Chern-Simons theory on $\mathrm{Tw}(X)$ compactifies to self-dual Yang-Mills theory on X.

Remarks 2.6.

1. For this to be literally true, we need to assume G is simply connected, otherwise we need to consider only the part of holomorphic Chern-Simons theory where the bundles are trivialisable on the fibers of p.
2. When I say "holomorphic Chern-Simons compactifies to self-dual Yang-Mills" here, I mean that there's a bijection on the level of coarse moduli spaces. That is, there's a canonical bijection between flat G-bundles on $\mathrm{Tw}(X)$ and self-dual G-bundles on X. One can actually prove something stronger: not only is there a bijection of classical solutions, but given two solutions there's an equivalence of "perturbative field theories": there's an L_∞ algebra classifying deformations of a holomorphic bundle, and of a self-dual bundle, modulo gauge, and these algebras are canonically quasi-isomorphic (see Boels, Mason and Skinner [BMS07], who also prove the supersymmetric version).
3. One would like to prove something stronger still: that these could be glued together to an actual equivalence of derived moduli stacks. The obstacle to doing this is that there isn't currently any way of describing the moduli space of self-dual bundles as a derived stack. Indeed, there's no reason to expect this moduli space to admit an algebraic, or even analytic structure, so doing so would likely need some currently undeveloped formalism of derived smooth stacks.
4. One can define *algebraic* Chern-Simons instead of holomorphic Chern-Simons, as the theory whose phase space is the moduli space of algebraic G-bundles on an algebraic 3-fold. These, however, are only relevant for the Penrose-Ward correspondence in a few examples: for compact X only S^4 or $\mathbb{CP}^3$. Hitchin proved [Hit81] that if X is compact, $\mathrm{Tw}(X)$ is only Kähler in these two examples. Since it's also proper, it can only admit an algebraic structure in these two cases. If X is not required to be compact then a few other possibilities arise: open subvarieties of these two options, plus certain open subvarieties in Fano 3-folds.

2.1 Constructing Twistor Spaces

I'll explain how to construct $\mathrm{Tw}(X)$ along with its complex structure, then say a few words about the proof of theorem 2.5. As a smooth 6-manifold, $p\colon \mathrm{Tw}(X) \to X$ is straightforward to define.

Definition 2.7. The twistor space $\mathrm{Tw}(X)$ is the total space of the $\mathbb{CP}^1$ bundle $\mathbb{P}(S_+)$ over X. Here S_+ is the positive Weyl spinor bundle: the rank 2 complex vector bundle associated to the frame bundle by the representation $(1/2, 0)$ of $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. This bundle might fail to be globally defined if X is not spin, but its projectivisation is always defined.

We can define an almost-complex structure on $\mathrm{Tw}(X)$ using the Levi-Civita connection on T_X. The pullback of this connection to twistor space allows us to canonically split the tangent bundle to $\mathrm{Tw}(X)$ as $T_{\mathrm{Tw}(X)} = p^*T_X \oplus T_{\mathrm{vert}}$; the almost-complex structure will preserve this decomposition.

- The fibers of the second summand are isomorphic to the tangent space in the fiber direction, so the second summand inherits a complex structure from the complex structure on the fibers (which, we recall, are copies of $\mathbb{CP}^1$).
- The fibers of the first summand are isomorphic to tangent spaces at point $x \in X$. If we choose a spinor $s \in (S_+)_x$, the Clifford multiplication map $\rho\colon (S_+)_x \otimes T_xX \to (S_-)_x$ induces an isomorphism of real vector spaces $T_xX \to (S_-)_x$, and therefore a complex structure on T_xX. This isomorphism is invariant under rescaling, so given a point in the fiber $p^{-1}(x) \cong \mathbb{P}(S_+)_x$ we obtain a complex structure on the first summand of the tangent space.

It remains to check that this almost-complex structure is integrable if (and, as it turns out, only if) the Weyl tensor $W_-(X)$ vanishes. I'll outline Atiyah-Hitchin-Singer's approach to proving this ([AHS78, Proposition 3.1 and 4.1]). Firstly, integrability is a local condition, so it suffices to check that the almost-complex structure, defined as above, on the 8-dimensional real vector space $\mathbb{R}^4 \times S_+$ is integrable (locally we can define S_+, not just its projectivisation).

We first check that the almost-complex structure, viewed as a sub-bundle of $T_{\mathbb{C}}(\mathbb{R}^4 \times S_+)$, can be identified as the kernel of a first-order differential operator. More precisely, given a differential operator D on a vector space E, one defines a sub-bundle $V(D)$ of the complexified tangent bundle $T_{\mathbb{C}}E$ whose fiber over $x \in E$ is generated by tangent vectors v such that $\partial_v f = 0$ for all $f \in \ker(D)$. Our almost complex-structure arises in this way for the operator

$$C^\infty(\mathbb{R}^4; S_+) \xrightarrow{A} \Omega^1(\mathbb{R}^4; S_+) \xrightarrow{\overline{\rho}} \ker(\rho)$$

where the second map is orthogonal projection onto the kernel of the Clifford multiplication map $\rho\colon \Omega^1(\mathbb{R}^4; S_+) \to C^\infty(\mathbb{R}^4; S_-)$, using the metric. The idea is that this operator, the *twistor operator* $\overline{D}$ is the $\overline{\partial}$ operator for the complex structure, i.e. solutions to the twistor equation $\overline{D}f = 0$ are local holomorphic functions on twistor space.

Atiyah, Hitchin and Singer now prove a general proposition about the integrability of such sub-bundles.

Proposition 2.8. Let $D\colon C^\infty(E) \to C^\infty(F)$ be a first-order differential operator of the form σA, where A is a connection on T_E, and $\sigma\colon \Omega^1(E) \to C^\infty(F)$ (the symbol of D). The sub-bundle $V(D)$ of $T_{\mathbb{C}}E$ is integrable if and only if the following two conditions hold.

1. The exterior derivative d_A sends $\ker(\sigma)$ to $\ker(\sigma) \wedge \Omega^1(E)$.
2. The curvature F_A of the connection lies in $\ker(\sigma) \wedge \Omega^1(E)$.

In the case of the twistor space, condition 1 is automatically satisfied, so it suffices to check that the curvature lies in $\ker(\sigma) \wedge \Omega^1(E)$. Here the symbol σ is the projection $\overline{\rho}$, so F_A is in the kernel of σ if and only if it's orthogonal to the kernel of ρ. Atiyah, Hitchin and Singer check that, with respect to the decomposition of the space of 2-forms as an $\mathrm{SO}(4)$-representation, the component corresponding to $\ker(\rho)$ is exactly the $(2, 0)$-component, so the curvature condition is exactly the vanishing of the negative Weyl tensor, as required.

Examples 2.9. 1. The twistor space of S^4 is $\mathbb{CP}^3$. The twistor map has a nice description, by identifying S^4 with $\mathbb{HP}^1$: there's a map $\mathbb{CP}^3 \to \mathbb{HP}^1$ given by identifying $\mathbb{C}^4$ with $\mathbb{H}^2$, and sending a complex line L in $\mathbb{C}^4$ to $L \otimes_{\mathbb{C}} \mathbb{H}$. This map is a fibration, and its fiber over a quaternionic line is the space of complex lines inside it, which is isomorphic to $\mathbb{CP}^1$.

2. The twistor space of $\mathbb{R}^4$ is obtained by removing a complex line "at infinity" from $\mathbb{CP}^3$. We note that even though $\mathbb{R}^4$ can be given a complex structure, the twistor map $\mathbb{CP}^3 \setminus \mathbb{CP}^1$ is not holomorphic for any choice.

3. The twistor space of $\mathbb{CP}^2$ is the variety $\mathrm{Fl}(\mathbb{C}^3)$ of complete flags in $\mathbb{C}^3$. The twistor map here is the obvious forgetful map from flags to lines, whose fiber is given by the space of planes over a given line, isomorphic to $\mathbb{CP}^2$.

4. In fact twistor spaces can be characterised. It turns out that a complex 3-fold arises as the twistor space for some self-dual Riemannian 4-manifold if and only if it admits a holomorphic foliation by complex projective lines, and has an antiholomorphic involution which restricts to the antipodal map on each line.

2.2 The Penrose-Ward Correspondence

Having defined the twistor space, let's talk briefly about why the Penrose-Ward correspondence is true. First recall a fact that Pyongwon proved in his lectures.

Proposition 2.10. If X is a complex manifold, and $P \to X$ is a smooth principal G-bundle with connection A such that the curvature of A is a $\mathfrak{g}_P$-valued $(1,1)$-form, then P has a unique holomorphic structure with $\overline{\partial}$-operator given by A.

We use this result to show that if (P, A) is a G-bundle with self-dual connection on a self-dual 4-manifold X, then (p^*P, p^*A) is a holomorphic G-bundle on $\mathrm{Tw}(X)$. It suffices to check that if A is self-dual p^*A has curvature of type $(1,1)$. Indeed, when we pull back a self-dual 2-form on X we obtain an anti-self-dual horizontal 2-form on $\mathrm{Tw}(X)$ with respect to the hermitian metric. It's anti-self-dual rather than self-dual because the complex structure we defined on horizontal forms has the opposite orientation to that of X. One then verifies in local coordinates that anti-self-dual 2-forms on a Hermitian surface are all, in particular, $(1,1)$-forms.

To complete the proof it's necessary to verify that all holomorphic structures arise in this way. Atiyah, Hitchin and Singer do this by observing that a holomorphic connection on twistor space $\mathrm{Tw}(X)$ is always covariant constant along the fibers of p, so comes from a connection on X, then noting that $(1,1)$-forms on $\mathrm{Tw}(X)$ that are pulled back from 2-forms on X are necessarily pulled back from self-dual 2-forms.

3 Supersymmetric Version

We conclude this talk with a bried discussion of the supersymmetric version of the Penrose-Ward correspondence. For simplicity we'll restrict to the case where $X = \mathbb{R}^4$. It's possible to consider other 4-manifolds, but for any specified amount of supersymmetry there are geometric restrictions on exactly when it makes sense to talk about supersymmetric Yang-Mills theory. The main theorem is due to Boels, Mason and Skinner [BMS07], but I learned about it from a paper of Costello [Cos11].

Definition 3.1. *Super projective space* $\mathbb{CP}^{m|n}$ is the moduli space of one-dimensional subspaces in the $\mathbb{Z}/2$-graded vector space $\mathbb{C}^{m+1|n}$, where the superscript $m+1|n$ denotes a space with $m+1$ even dimensions and n odd dimensions. Concretely, it's the total space of the vector bundle $\mathcal{O}(1) \otimes \mathbb{C}^n$, placed in odd degree.

One can check that the bundle of holomorphic m-forms on $\mathbb{CP}^{m|n}$ is isomorphic to $\mathcal{O}(n-m-1)$. In particular, if $n = m+1$ then this bundle is trivializable, and $\mathbb{CP}^{m|n}$ is Calabi-Yau.

Theorem 3.2. The $N = 4$ self-dual super Yang-Mills theory on $\mathbb{R}^4$ arises by compactification of holomorphic Chern-Simons theory on $\mathbb{CP}^{3|4} \setminus \mathbb{CP}^1$: the complement of a line in "super twistor space".

This is actually a theorem in the stronger sense I mentioned in remark 2.6 part 2. Checking it is quite a neat calculation which I discuss in my paper with Philsang Yoo [EY15, section 3.3], following an argument presented by Ward and Wells [WW91]. One can see the appearance of all the fields that appear in physicists' descriptions of this super Yang-Mills theory: a self-dual gauge field, 4 spinor fields and 6 adjoint valued scalar fields.

Remark 3.3. There are versions of this theorem that recover other super Yang-Mills theories, including those with a choice of matter multiplet.

References

[AHS78] Michael Atiyah, Nigel Hitchin, and Isadore Singer. Self-duality in four-dimensional Riemannian geometry. *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, 362(1711):425–461, 1978.

[BMS07] Rutger Boels, Lionel Mason, and David Skinner. Supersymmetric gauge theories in twistor space. *Journal of High Energy Physics*, 2007(02):014, 2007.

[Cos11] Kevin Costello. Notes on supersymmetric and holomorphic field theories in dimensions 2 and 4. *arXiv preprint arXiv:1111.4234*, 2011.

[EY15] Chris Elliott and Philsang Yoo. Geometric Langlands twists of $N = 4$ gauge theory from derived algebraic geometry. *arXiv preprint arXiv:1507.03048*, 2015.

[Hit81] Nigel Hitchin. Kählerian twistor spaces. In *Proc. London Math. Soc.*, volume 43, pages 133–150, 1981.

[MW96] Lionel J Mason and Nicholas Michael John Woodhouse. *Integrability, self-duality, and twistor theory.* Number 15. Oxford University Press, 1996.

[PTVV13] Tony Pantev, Bertrand Toën, Michel Vaquié, and Gabriele Vezzosi. Shifted symplectic structures. *Publications mathématiques de l'IHÉS*, 117(1):271–328, 2013.

[War77] Richard Ward. On self-dual gauge fields. *Physics Letters A*, 61(2):81–82, 1977.

[WW91] Richard Samuel Ward and Raymond O'Neil Wells. *Twistor geometry and field theory.* Cambridge University Press, 1991.

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