Ambidexterity

Definition (Atiyah)

An n-dim^{l} TQFT is a symmetric monoidal functor $\left\{ \begin{array}{c} (n-1)\text{-mflds} \\ \text{bordisms} / \text{Diffeo.} \end{array} \right\} \xrightarrow{Z} \mathrm{Vect}_{\mathbb{C}}$.

In this talk, manifold means smooth, compact, oriented.

Example $n=2$.

$Z(S^1) = A \in \mathrm{Vect}_{\mathbb{C}}$ say. We equip A with a multiplication map be evaluating on pants.

$Z($  $) = (m : A \otimes A \longrightarrow A)$.

One can check m gives A the structure of a commutative $\mathbb{C}$-algebra. It is unital, via $Z($  $) : \mathbb{C} \xrightarrow{1} A$.

Also have $Z($  $) : A \xrightarrow{Tr} \mathbb{C}$

Exercise: The trace pairing

$A \otimes A \xrightarrow{m} A \xrightarrow{Tr} \mathbb{C}$

is non-degenerate. In particular A is finite dimensional.

Summary: Every 2d TQFT determines a commutative Frobenius algebra.

Folk theorem: The converse also holds: all commutative Frobenius algebras arise from 2d TQFTs.

Using this data, we can evaluate Z on closed 2-manifolds to compute a number.

Example:

$$Z(\ \ ) = Z(\ \ ) \circ Z(\ \ )$$

$$: \mathbb{C} \xrightarrow{1} A \xrightarrow{Tr} \mathbb{C}$$

so $Z(\ \ ) = Tr(1)$ in the Frobenius algebra.

Example.

$$Z(\ \ ) = Z(\ \ ) \circ Z(\ \ )$$

$$: \mathbb{C} \xrightarrow[\text{dual trace pairing}]{} A \otimes A \xrightarrow[\text{trace pairing}]{} \mathbb{C}$$

$A \cong A^\vee$ by trace pairing, so think

$$\mathbb{C} \xrightarrow{1} \mathrm{End}(A) \xrightarrow{Tr} \mathbb{C}$$

so $Z(\ \ ) = Tr(\mathrm{Id}_A) = \dim A$.

Dijkgraaf - Witten Theory

Fix a finite group G. For a topological space X, we can talk about G-bundles on it, i.e. $\tilde{X} \to X$ such that $G \circlearrowright \tilde{X}$ freely, & $\tilde{X}/G \cong X$. There's a classifying space for such data.

$$\left(\begin{matrix} G\text{-bundles on} \\ X \end{matrix} \right) \Big/ \text{iso} \longleftrightarrow \left(\text{maps } X \to BG \right) \Big/ \text{htpy}$$

BG is a $K(G,1)$, & is unique up to homotopy.

Fix a dimension n. We'll define an n-dim TQFT counting such G-bundles, roughly.

Let M^n be connected. We'll find

$$Z(M) = \frac{\#\text{ of homs } \pi_1 M \to G}{|G|}$$

$$= \#\text{ of } G\text{-bundles on } M,\text{ counted with mass}$$

i.e. $\displaystyle\sum_{\substack{G\text{-bundles} \\ \tilde{X} \to X}} \frac{1}{|\mathrm{Aut}(\tilde{X})|}$ (even if M is not connected).

These are the same, as

$$G\text{-bundles on } X/_{\text{iso}} = \text{homs } \pi_1 M \longrightarrow G/_{\text{conjugacy}}$$

so $$\sum_{\tilde{X}} \frac{1}{|\mathrm{Aut}(\tilde{X})|} = \sum_{\substack{\alpha : \pi_1 M \to G \\ \text{up to conjugacy}}} \frac{1}{|C_G(\alpha)|} = \sum_{\alpha : \pi_1 M \to G} \frac{1}{|G|}.$$

Now, let M be an $n-1$ manifold. Set

$Z(M) =$ locally constant functions on $\mathrm{Map}(M, BG)$

noting that the connected components of $\mathrm{Map}(M, BG)$ are indexed by G-bundles, & the components are homotopy equivalent to things like $B\,\mathrm{Aut}(\tilde{M})$

This is a $\mathbb{C}$-vector space of dim^n # of G-bundles up to isomorphism.

Given a bordism $B : M \longrightarrow N$, consider the diagram

$$\begin{array}{ccc} & \mathrm{Map}(B, BG) & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ \mathrm{Map}(M, BG) & & \mathrm{Map}(N, BG) \end{array}$$

& pull & push locally constant functions. Restrict & "integrate". Fibres of p_2 have finitely many connected components each the classifying space of a finite group. So we just sum with mass.

Concretely, if $Z(M) \ni a : \mathrm{Map}(M, BG) \longrightarrow \mathbb{C}$, then $Z(B)(a) \in Z(N)$, so compute

$$\underset{\substack{\uparrow \\ \mathrm{Map}(N, BG)}}{Z(B)(a)(x)} = \sum_{\substack{C \text{ component} \\ \text{of } p_2^{-1}(x)}} \frac{a(p_1(c))}{|\pi_1(c)|}$$

Example $M = N = \emptyset$, B^n closed

So $\mathrm{Map}(M, BG) = \mathrm{Map}(N, BG) = pt$,

& we compute the map

$$\mathbb{C} \xrightarrow{\times Z(B)} \mathbb{C}$$

where $Z(B)$ is as before.

This defines a TQFT.

Example $n=2$

$Z(S^1) = \{$ loc const functions $\mathrm{Map}(S^1, BG) \longrightarrow \mathbb{C}\}$.

What is a G-bundle on S^1? If we specify a trivialisation at a point, get G itself.

So

$$G\text{-bundles on } S^1/_{\text{iso}} = G/_{\text{conjugation}}.$$

Thus $\mathcal{O}(\mathrm{Map}(S^1, BG)) = $ (class functions on G).

$$= \mathrm{Rep}(G) \otimes_{\mathbb{Z}} \mathbb{C}.$$

What is $Z(\text{torus})$? We'll compute it in 2 ways.

Firstly, $Z(T) = \#\{\text{homs } \pi_1 T \to G\}/|G|$

$= \#\{\text{pairs of commuting elts of } G\}/|G|$.

Alternatively,

$$Z(T) = \dim(\mathcal{O}(\mathrm{Map}(S^1, BG)))$$
$$= \# \text{ of conjugacy classes in } G.$$

What parameters in this definition could we vary?

- The dimension n.
- The finite group G. Could we replace BG by other spaces? Count maps $M \to X$? Need only finitely many such maps.
- The base field $\mathbb{C}$. Replace by other rings. Something different will happen for a field whose characteristic divides $|G|$.

Remark (from last time)

For a Frobenius algebra A, one can think of Tr as an element of $A^\vee$. Having a Frobenius algebra is having a f.d. algebra A with an iso of A-modules $A \xrightarrow{\sim} A^\vee$. So these are equivalent to Gorenstein rings of Krull dim 0.

e.g. $\mathbb{C}[x_1, \dots, x_n] / (f_1, \dots, f_n)$ is a comm. Frob. algebra if f.d.

In particular, there are many examples.

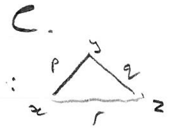
Let X be a topological space, & $\mathcal{C}$ a category.

Definition

A local system $\mathcal{L}$ on X with values in $\mathcal{C}$ is

a) For every $x \in X$, an object $\mathcal{L}_x \in \mathcal{C}$

b) For every path $p : [0,1] \to X$, an isomorphism $\mathcal{L}_p : \mathcal{L}_{p(0)} \xrightarrow{\sim} \mathcal{L}_{p(1)}$ in $\mathcal{C}$.

c) For every 2-simplex σ : 

in X, $\mathcal{L}_r = \mathcal{L}_q \circ \mathcal{L}_p$.

Equivalently, a local system on X is a functor $\mathcal{L} : \Pi_{\leq 1} X \to \mathcal{C}$.

$\Pi_{\leq 1}$ notation for fundamental groupoid

Concretely, if X is connected, & $x \in X$ is a base point, a local system on X is an object of $\mathcal{C}$ with an action of $\pi_1(X, x)$ (given by b, & c ensures it is an action).

Example $\mathcal{C} = \mathrm{Vect}_{\mathbb{C}}$.

Say we assign $\mathbb{C}$ to every point.
So we must assign an element of $\mathbb{C}^\times$ to every path, such that composition agrees with multiplication.

In other words, this is the data of a 1-cocycle on X with values in $\mathbb{C}^\times$. We made a choice, namely an identification $\mathcal{L}_x \cong \mathbb{C}$.

Upshot: Iso classes of rk 1 local systems on X are in bijection with $H^1(X; \mathbb{C}^\times)$.

Twisted Dijkgraaf-Witten

G a finite group, $\eta \in H^n(BG; \mathbb{C}^\times)$.
This defines an n-dim TQFT Z. At top level:

$$Z(M^n) = \frac{1}{|G|} \sum_{\alpha : \pi_1 M \to G} (\bar{\alpha}^* \eta)[M]$$

where $\alpha : \pi_1 M \to G$ determines $\bar{\alpha} : M \to BG$.

Now, let M be an $n-1$ manifold, & look at the classifying space for G-bundles $\mathrm{Map}(M, BG)$. Consider the diagram

$$\begin{tikzcd} M \times \mathrm{Map}(M, BG) \arrow[r, "ev"] \arrow[d] & BG \\ \mathrm{Map}(M, BG) & \end{tikzcd}$$

& produce the cohomology class

$$\int_M ev^* \eta \in H^1(\mathrm{Map}(M, BG); \mathbb{C}^\times)$$

This, by our discussion, is a rank 1 local system $\mathcal{L}_M$ on $\mathrm{Map}(M, BG)$.

__Definition__

$Z(M^{n-1})$ is the space of sections of $\mathcal{L}_M$.

In general, if $\mathcal{L}$ is a local system of $\mathbb{C}$-vector spaces on X, the space of sections is
$$H^0(X;\mathcal{L}) = \{(v_x \in \mathcal{L}_x)_x : (v_x) \text{ is holonomy invariant}\}$$

e.g. if $\mathcal{L}$ comes from a vector bundle with flat connection, this is the space of flat sections.

One can do this in any setting where $\mathcal{C}$ has limits. $H^0(X;\mathcal{L}) = \varprojlim \mathcal{L}$, under the diagram given by paths in X.

Now, say M & N are $(n-1)$-manifolds, & B is a bordism between them. Again, consider the diagram

$$\begin{tikzcd} & \mathrm{Map}(B, BG) \arrow[dl] \arrow[dr] & \\ \mathrm{Map}(M, BG) & & \mathrm{Map}(N, BG) \end{tikzcd}$$

$\mathcal{L}_M$ $\qquad\qquad$ $\mathcal{L}_N$

The local systems $\mathcal{L}_M$, $\mathcal{L}_N$ have common pullback $\mathcal{L}_B$, as the defining cocycles are cohomologous (via B).

Now, let $X = \mathrm{Map}(M, BG)$. X has finitely many connected components with finite homotopy:
$$X \cong \coprod_{\substack{\text{iso classes} \\ \text{of } G \text{ bundles} \\ P \to M}} B\,\mathrm{Aut}(P).$$

Let $X_0 \subseteq X$ be a connected component. Say $X_0 = BH$, for H a finite group. Restricting $\mathcal{L}_M|_{X_0}$, get a representation V of $H = \pi_1(X_0)$.

In these terms,

$$H^0(\mathcal{L}_M|_{x_0}) = V^H = \{v \in V : hv = v \ \ \forall h \in H\}$$

Dually, one constructs

$$H_0(\mathcal{L}_M|_{x_0}) = V_H = V/_{\{(hv-v) : v \in V, h \in H\}}$$

Note: there is a canonical isomorphism $V_H \longrightarrow V^H$ given by the norm map. There's a map

$$V \xrightarrow{v \mapsto \sum_{h \in H} hv} V$$

factoring through V_H, & V^H, so inducing

$$V_H \xrightarrow{Nm} V^H$$

If M is an abelian group, & $H \leq M$ we can do this construction. It is an isomorphism provided you can divide by $|H|$. The inverse is just $x \longmapsto x/_{|H|}$. In particular, this applies for $\mathbb{C}$-vector spaces.

So in this setting $H^0 = H_0$, & we have pull-back and push-forward maps, & can define the TQFT.

Question:

When can we say that the homology & cohomology of a local system are isomorphic? For now, we'll just mean degree 0.

$\mathcal{C}$ will be an arbitrary category with small limits & colimits.

For any space X, the collection of $\mathcal{C}$-local systems on X forms a category, denoted $\mathcal{C}^X$.

Given a cts map $f: X \longrightarrow Y$, one can pull back local systems on Y, i.e. we have a functor
$$f^*: \mathcal{C}^Y \longrightarrow \mathcal{C}^X$$
$$\mathcal{L} \longmapsto f^*\mathcal{L}$$
with $(f^*\mathcal{L})_x = \mathcal{L}_{f(x)}$.

As $\mathcal{C}^X$ has limits & colimits, f^* has adjoints on both sides: $f_! \dashv f^* \dashv f_*$. Category theoretically one takes left & right Kan extensions.

More explicitly, if f is a fibration say,
$$(f_! \mathcal{L})_y = H_0(f^{-1}\{y\} ; \mathcal{L}|_{f^{-1}\{y\}})$$
$$(f_* \mathcal{L})_y = H^0(f^{-1}\{y\} ; \mathcal{L}|_{f^{-1}\{y\}})$$

<u>Example</u>:

If $Y = pt$, $\mathcal{C}^Y \simeq \mathcal{C}$, & if $\pi: X \to pt$,
$\pi_* \mathcal{L} = H^0(X; \mathcal{L})$, $\pi_! \mathcal{L} = H_0(X; \mathcal{L})$.
So f_*, $f_!$ give relative versions of (co)homology.

<u>Question</u>:

When can we say $f_! \cong f_*$?

We'll try to construct an isomorphism (we'll go over it in detail next time). $f_! \xrightarrow{\sim} f_*$

WLOG, f is a fibration, by taking a replacement. Consider the pullback

$$\begin{tikzcd} X \times_Y X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

$X \xhookrightarrow{\Delta} X \times_Y X$ diagonally, & $\pi_i \circ \Delta = \mathrm{Id}$.
Start by supposing $\Delta_! \cong \Delta_*$. We'll construct an analogue of the norm map.

__Observe__ $\mathrm{Hom}(f_!, f_*) \cong \mathrm{Hom}(\mathrm{id}, f^* f_*)$ by definition of $f_!$. (Here id means $\mathrm{id}_{\mathcal{C}^X}$).

Use the diagram

$$\begin{tikzcd}
X \arrow[rr, "\mathrm{id}"] \arrow[dr, "\Delta"] \arrow[dd, "\mathrm{id}"'] & & X \arrow[dd, "f"] \\
& X \times_Y X \arrow[ur, "\pi_2"] \arrow[dl, "\pi_1"] & \\
X \arrow[rr, "f"] & & Y
\end{tikzcd}$$

$$\begin{aligned}
\mathrm{id}_{\mathcal{C}^X} = \mathrm{id}_* \mathrm{id}^* &= \pi_{2*} \Delta_* \Delta^* \pi_1^* \\
&\cong \pi_{2*} \Delta_! \Delta^* \pi_1^* && \text{by assumption} \\
&\longrightarrow \pi_{2*} \pi_1^* && \text{(Counit)} \\
&\cong f^* f_* && \text{From the cartesian square.}
\end{aligned}$$

This is our construction. Call the map Nm.

__Definition__
Say f is __ambidextrous__ if
1) $\Delta : X \longrightarrow X \times_Y X$ is ambidextrous
2) $\mathrm{Nm} : f_! \longrightarrow f_*$ is an isomorphism.

Obviously, this is circular. We mean the collection of ambidextrous maps is the smallest collection of maps containing htpy equivalences, & closed under this property. (__coinduction__).

Lurie 3

We'll explain later what $\mathrm{Nm}: f_! \longrightarrow f_*$ means.

Recall our coinductive definition.

Definition

We define a class of maps $f: X \longrightarrow Y$ called <u>ambidextrous</u> maps, and for each ambidextrous map a <u>norm</u> map $\mathrm{Nm}_f: f_! \xrightarrow{\sim} f_*$, with

1) If f is a homotopy equivalence, then f is ambidextrous, with Nm_f the obvious map ($f_!$ & f_* are both inverses to f^*).

2) If $f: X \longrightarrow Y$ is a fibration, & $\Delta: X \longrightarrow X \times_Y X$ is ambidextrous, define $\mathrm{Nm}_f: f_! \longrightarrow f_*$ using $\alpha = \mathrm{Nm}_\Delta$ in composite

$$f_! \simeq f_! \, id_* \, id^* \simeq f_! \, \pi_{2*} \Delta_* \Delta^* \pi_1^* \xrightarrow{\alpha} f_! \, \pi_{2*} \Delta_! \Delta^* \pi_1^* \longrightarrow f_! \, \pi_{2*} \pi_1^* \simeq f_! f^* f_* \longrightarrow f_*$$

For almost all purposes, it suffices to take $Y = *$.

Definition

We will say a space X is <u>ambidextrous</u> if the map $X \longrightarrow *$ is ambidextrous. In that case, for any local system $\mathcal{L}$ on X, we get an isomorphism

$$H_0(X; \mathcal{L}) \xrightarrow{\sim} H^0(X; \mathcal{L}).$$

Say $f: X \longrightarrow *$, $\alpha: f_* \longrightarrow f_!$. What does α give you? Suppose C and D are objects of $\mathcal{C}$, and suppose we have a map $X \xrightarrow{\rho} \mathrm{Hom}_{\mathcal{C}}(C, D)$.

ρ locally constant

Take f^*C, f^*D, constant local systems on X. Have a possibly interesting map $\rho: f^*C \longrightarrow f^*D$, hence composite

$$C \longrightarrow f_* f^* C \xrightarrow{f_* \rho} f_* f^* D \xrightarrow{\alpha} f_! f^* D \longrightarrow D$$

Call this map $\int_X \rho \, d\alpha$. It is a <u>single</u> map coming from a <u>family</u> of maps parameterised by X

Lets describe Nm in more concrete terms. Let $f : X \longrightarrow pt$, $\Delta : X \longrightarrow X \times X$ diagonal. We're given a natural transformation $\alpha : \Delta_* \to \Delta_!$

Say $\mathcal{L}$ is a local system on X. Remember
$f_! \mathcal{L}$ means $\varinjlim_{x \in X} \mathcal{L}_x$, and
$f_* \mathcal{L}$ means $\varprojlim_{y \in X} \mathcal{L}_y$.

So giving a map $f_! \mathcal{L} \to f_* \mathcal{L}$ is giving a map $\mathcal{L}_x \to \mathcal{L}_y$ for every x & $y \in X$. So think of a <u>matrix</u> $(Nm_f(\mathcal{L})_{xy})$, and

$$Nm_f(\mathcal{L})_{xy} = \int_{\substack{\text{paths } p \\ \text{from } x \text{ to } y}} \mathcal{L}_p \, d\alpha$$

<u>Examples:</u>

0) $X = \emptyset$. So our diagram becomes

$$\begin{tikzcd}
\emptyset \arrow[dr, "id"] \arrow[drr, bend left] \arrow[ddr] & & \\
& \emptyset \arrow[r] \arrow[d] & \emptyset \arrow[d, "f"] \\
& \emptyset \arrow[r, "f_!"] & *
\end{tikzcd}$$

In particular, $\emptyset \to \emptyset$ is an isomorphism.

Now $f_!, f_* : \underset{\{*\}}{\overset{\cong}{\mathcal{C}^\emptyset}} \longrightarrow \underset{\mathcal{C}}{\overset{\cong}{\mathcal{C}^*}}$

So write $f_!$ & f_* for the objects they map to in $\mathcal{C}$. $f_!$ is the initial object
f_* is the terminal object,
& there's a unique map $f_! \longrightarrow f_*$.

Upshot: $\emptyset$ is ambidextrous $\Leftrightarrow$ $\mathcal{C}$ is pointed, i.e. $\mathcal{C}$ has a zero object: initial and final.

The integration rule in this case says: given $\rho : \emptyset \longrightarrow \mathrm{Hom}(C, D)$, get
$$\int_\emptyset \rho = 0 \in \mathrm{Hom}(C, D)$$

The map is the composite $C \longrightarrow 0 \longrightarrow D$.

1) Given this, we can deal with the case of X discrete. Then all fibres of $\Delta : X \to X \times X$ are empty or contractible, so Δ is ambidextrous if $\mathcal{C}$ is pointed.

$\mathcal{C}^X$ is now just maps $X \to \mathcal{C}$. So
$$(f_! \mathcal{L}) = \coprod_{x \in X} \mathcal{L}_x, \qquad (f_* \mathcal{L}) = \prod_{y \in X} \mathcal{L}_y$$

The map $Nm_f : f_! \mathcal{L} \to f_* \mathcal{L}$ is now the identity matrix: the map $\coprod \longrightarrow \prod$.

X is ambidextrous if this is an isomorphism. e.g. this is true if $\mathcal{C} = Ab$ if X is finite, but not if X is infinite.

Again, if this holds we get an integration procedure. Suppose finite sets are ambidextrous for $\mathcal{C}$. Then $\mathcal{C}$ is semi-additive, i.e. $\mathrm{Hom}_{\mathcal{C}}(C, D)$ is a commutative monoid, & integration is addition of morphisms
$$C \longrightarrow \prod_{x \in X} C \longrightarrow \prod_{x \in X} D \longrightarrow D$$

equiv to
$E(G\times G)/_G$
$\downarrow$
$E(G\times G)/_{G\times G}$

2) Now, if $\mathcal{C}$ has this property, then we now know $\Delta: X \longrightarrow X\times X$ is ambidextrous if equivalent to a finite covering space, e.g $X = BG$ for G a finite group.

$\mathcal{C}^{BG}$ is the category of objects of $\mathcal{C}$ with a G-action. Denote such objects V.

$f_! V = V_G$, $f_* V = V^G$ in this setting. We're constructing $Nm: V_G \longrightarrow V^G$, which we can see is the same as the classical norm map.

What if it is an isomorphism? (e.g if $\mathcal{C} = Vect_{\mathbb{C}}$). Then we can construct new norm maps.

3) Suppose BG is ambidextrous for $\mathcal{C}$, & G is abelian. Then take $X = K(G,2)$. Let $f: X \longrightarrow *$, & consider the diagonal $X \xrightarrow{\Delta} X\times X$. This has homotopy fibre BG, so that means f is ambidextrous.

If $\mathcal{L}$ is a local system on $X = K(G,2)$, then we get $Nm_f: H_0(\mathcal{L}) \longrightarrow H^0(\mathcal{L})$.
<u>However</u> $\mathcal{L}$ cannot be interesting. X is simply connected, so $\mathcal{L}$ must be trivial, & $H_0(\mathcal{L})$, $H^0(\mathcal{L})$ are just evaluation at a point.

The map however is a little interesting: it is multiplication by $1/|G|$.

To get something more interesting, let $\mathcal{C}$ now be an $(\infty, 1)$-category. The definition of local systems now includes a homotopy between objects for every two-simplices, & analogous data for higher simplices.

Examples:

$\mathcal{C}$ = Topological spaces

$\mathcal{C}$ = Chain complexes of R-modules, R a ring.

$\mathcal{C}$ = Spectra

Let k be a field, & take $\mathcal{C}$ to be chain complexes of k-vector spaces. If X is a space & $\mathcal{L}$ is the constant local system with value k, then note $f_! \mathcal{L} = C_*(X; k)$, $f_* \mathcal{L} = C^*(X, k)$.

Here
- $\emptyset$ is ambidextrous
- Finite sets are ambidextrous
- BG for G finite. Is BG ambidextrous? If and only if char $k \nmid |G|$.

If we're in the good case, then we can go on. Take $K(G, 2)$. Actually all local systems are still trivial in this case. Nm is still multiplication by $\frac{1}{|G|}$, so $K(G, 2)$ is ambidextrous. Same for higher $K(G, n)$.

We'd like to access the case where BG was not ambidextrous: that's where the interesting local systems live.

$\mathcal{C}$ is going to be an $(\infty, 1)$-category with limits & colimits. Last time we introduced the notion of an ambidextrous space X for $\mathcal{C}$. If $p: X \to *$, we have an isomorphism

$$Nm_X : p_! \xrightarrow{\sim} p_* .$$

via a matrix of maps

$$\int_{\substack{\text{Paths } \gamma \\ \text{from } x \text{ to } y}} \mathcal{L}_\gamma \, dNm^{-1}_{\text{Path}(x,y)} : \mathcal{L}_x \longrightarrow \mathcal{L}_y$$

In this lecture, we'll make some interesting choices for $\mathcal{C}$.

Stable Homotopy Theory

Recall: A cohomology theory is a sequence of functors

$$E^n : \left\{ \begin{matrix} \text{Pairs of spaces} \\ Y \subseteq X \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{Abelian} \\ \text{groups} \end{matrix} \right\}$$

e.g $\quad (Y \subseteq X) \longmapsto H^n(X, Y; A)$

for an abelian group A. These should satisfy some familiar axioms. We could also restrict to the case where $Y = \emptyset$ for simplicity.

Brown representability:

For any cohomology theory E, there are spaces $Z(n)$, such that

$$E^n(X) = (\text{htpy classes of maps } X \to Z(n)).$$

Example:

If $E^n(X) = H^n(X; A)$, then $Z(n) = K(A, n)$, Eilenberg-Maclane spaces.

These spaces $Z(n)$ are related to one another. If X is a pointed space, $E^n_{red}(\Sigma X) = E^{n-1}_{red}(X)$.

This implies that there are homotopy equivalences $Z(n-1) \sim \Omega Z(n)$. So we have a sequence of spaces, each of which is a delooping of the previous one, unique up to homotopy. This data describes a <u>spectrum</u>.

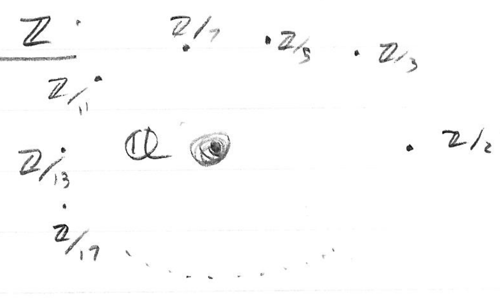
There is an $(\infty, 1)$ category Sp of spectra, and isomorphism classes of objects correspond to cohomology theories.

Ab sits inside Sp as a full subcategory, via sending a group to the corresponding Eilenberg-Maclane spectrum.

Last time, we got some mileage out of $\mathcal{C} = Ab$. In Sp also, there is a zero object, & finite products & direct sums agree. So $\emptyset$ & finite sets are Sp-ambidextrous. In Ab, BG for finite G was only <u>sometimes</u> ambidextrous. The same is true in Sp.

If instead of taking Ab, we took $Vect_{\mathbb{Q}}$ inside it, BG was <u>always</u> ambidextrous. We'll try a similar method: finding certain subcategories in Sp where BG is more often ambidextrous.

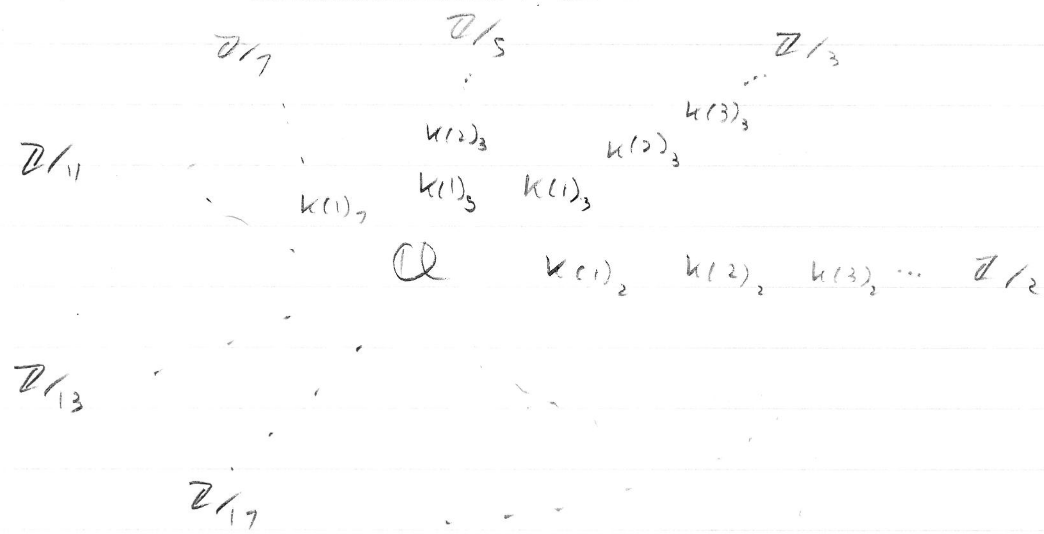
<u>Picture of Spec $\mathbb{Z}$</u>:



labels of points by residue field.

There are various ways we can simplify $\mathbb{Z}$, e.g. reducing mod p, or completing at a prime. These are organised by the picture. "geometry of the category of abelian groups"

<u>Corresponding picture for Spectra:</u>



If k is a field, we have some special properties: the <u>künneth formula</u>

$$H_*(X \times Y; k) \cong H_*(X; k) \underset{H_*(pt; k)}{\otimes} H_*(Y; k)$$

This is less simple if you replace k by $\mathbb{Z}$, or a more complicated group (künneth spectral sequence).

<u>Definition:</u>

A spectrum E is a "<u>field</u>" if there is a künneth formula for E-cohomology.

There aren't many spectra with this property. Cohomology with coefficients in a field is an example. If we restrict to prime fields, there's a complete classification, drawn in the picture above. $\mathbb{Q}$, $\mathbb{Z}/p$, & $K(n)_p$. The latter are called <u>Morava k-theories</u>.

Fix a prime p. For any natural number n, there is an interesting spectrum $K(n)$, with

$$K(n)^i(*) = \begin{cases} \mathbb{Z}/p & \text{if } (2p^n - 2) \mid i \\ 0 & \text{otherwise} \end{cases}$$

There's a sense in which $K(0)$ is rational cohomology, & $K(n)$ is mod p cohomology theory.

Analogy:

Abelian groups	Spectra
prime ideals in $\mathbb{Z}$	Morava K-theories
$\otimes$	$\wedge$
reduction mod p = $\otimes \mathbb{Z}/p$	$- \wedge K(n)$

Definition

A spectrum X is <u>$K(n)$-acyclic</u> if $X \wedge K(n) \simeq 0$.

Definition

Write $Sp^{K(n)}$ for the ∞-category of <u>$K(n)$-local spectra</u>, i.e. $Sp / K(n)\text{-acyclic spectra}$.

$Sp^{K(n)} \subseteq Sp$ is an orthogonal to $K(n)$-acyclic spectra. We can define it to be the spectra such that maps from $K(n)$-acyclic spectra are zero.

Note: For fixed p, $K(1)$ agrees with K/pK, where K is complex K-theory. (or a sum of copies of such a thing).

<u>Definition</u>

A topological space X is <u>π-finite</u> if $\pi_0 X$ is finite, & for each connected component Y, each $\pi_i(Y, x)$ is finite, & $\pi_i(Y, x) = 0$ for all but finitely many i.

These are the only reasonable candidates for ambidexterity.

<u>Theorem</u> (Hopkins, L)

Let $\mathcal{C}$ be the $(\infty, 1)$-category of $K(n)$-local spectra. Then every π-finite space is ambidextrous.

When $n=0$, this is the ambidexterity we saw last time. If $n=\infty$, this would be false.

Let X be π-finite, let $p: X \longrightarrow *$, & consider $p_* : \mathcal{C}^X \longrightarrow \mathcal{C}$

<u>Corollary</u>:

p_* commutes with colimits
& $p_!$ commutes with limits.

<u>Corollary</u>:

Take E to be a $K(n)$-local E_∞ ring spectrum. Then one can construct a version of Dijkgraaf-Witten theory with coefficients in E.

If G is a finite p-group we can get invariants for manifolds. <u>e.g</u>, plugging in T^2 yields

$$Z(T^2) = \# \text{ of conjugacy classes of elements}$$

in the complex theory. If we instead take E-coefficients, we produce integers

$Z(T^2) = \#$ of conjugacy classes of group homs
$\mathbb{Z}^{n+1} \longrightarrow G$

<u>Remarks on Proof</u>

One reduces to the interesting case
$$X = K(\mathbb{Z}/p, m),$$
& inducts on m. Say, for any local system $\mathcal{L}$, you have norm $Nm: p_! \mathcal{L} \to p_* \mathcal{L}$.

The hardest case is the trivial local system with value $K(n)$. The map goes.
$$Nm: K(n)_*(X) \to K(n)^*(X).$$

Ravenel & Wilson computed these groups. $K(n)$-cohomology looks like functions on the p-torsion of a certain formal group. Homology is the dual, & we're finding a certain pairing. replace $K(n)$ by Morava E-theory, which is torsion-free, & understand the trace pairing.

If $X = BG$, G a finite p-group, & we take $K_p^\wedge$: completed complex K-theory.

Then $K_p^\wedge(BG) = Rep(G) \otimes_{\mathbb{Z}} \mathbb{Z}_p$

The phenomenon is saying that this ring is self-dual. This duality is familiar: take intertwiners between representations.