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Gauge Theoretic Aspects of the Geometric Langlands Correspondence

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ABSTRACT

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In their revolutionary 2006 paper, Kapustin and Witten described a fascinating bridge between geometric representation theory and the quantum theory of supersymmetric gauge fields. They explained how, by performing a suitable topological twist, one can obtain categories of sheaves on moduli stacks of holomorphic and flat G-bundles as categories of boundary conditions in supersymmetric gauge theories, and why the physical phenomenon of S-duality should yield a conjectural equivalence of categories known as the *geometric Langlands correspondence*. In this thesis, I begin to make some of the structures introduced by Kapustin-Witten and other theoretical physicists mathematically rigorous, with the eventual aim of systematically using the huge amount of structure possessed by the panoply of supersymmetric gauge theories in the theoretical physics literature to draw new insights about geometric representation theory. The present work consists of two distinct approaches. Firstly I give a construction of a generalization of *abelian* gauge theories using the mathematical structure of a factorization algebra, and explain how S-duality for these theories can be described as a version of the Fourier transform. Then, I explain how to construct classical supersymmetric gauge theories using derived algebraic geometry, introduce an appropriate notion of *twisting* for such theories, and prove that the twists introduced by Kapustin and Witten yield the moduli stacks of interest for the geometric Langlands correspondence.

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CHAPTER 1

Introduction

1.1. Introduction to S-Duality

We will begin this thesis somewhat philosophically, with a discussion of what it should mean for a pair of quantum field theories to be "dual". Roughly speaking, when physicists speak about a duality, they mean a pair of prequantum field theories (by which I mean classical field theories equipped with some additional data, including a specific action functional with specified values for coupling constants, that determines a choice of quantization) along with an equivalence of their quantizations. For example, consider the following definition of duality, from a physical point of view.

Definition 1.1.1. *[**Tes16**] A pair of theories (Φ, τ, S_τ) and $(\Phi', \tau', S'_{\tau'})$ given by a space of fields and an action functional depending on some auxilliary parameters τ or τ' are* dual *if there exists a moduli space $\mathcal{M}$ of quantum field theories having boundary points z_0 and z_0', coordinates τ_0 and τ_0' on $\mathcal{M}$ near these boundary points, and maps $f\colon \mathrm{Obs} \to \mathcal{O}(\Phi)$ and $f'\colon \mathrm{Obs}' \to \mathcal{O}(\Phi')$ from the local observables near z_0 and z_0' such that there exist equivalences of asymptotic expansions*

$$\langle \mathcal{O} \rangle_z \cong \int D\phi e^{-S_{\tau_0}(\phi)} f(\mathcal{O})(\phi, \tau_0)$$
$$\textit{and } \langle \mathcal{O} \rangle_z \cong \int D\phi' e^{-S'_{\tau_0'}(\phi')} f'(\mathcal{O})(\phi', \tau_0')$$

near z_0 and z_0' respectively.

Making this definition precise relies on a number of things, not least an appropriate definition of a quantum field theory and an appropriate definition for the path integral expressions given on the right hand of the above equivalences. Nevertheless it captures the idea for what sort of thing a duality should be: a pair of apparently distinct prequantum field theories which nonetheless, when quantized, describe equivalent physics. Taking the non-rigorous parts for granted, definition 1.1.1 admits a neat simplification in the case where the moduli space $\mathcal{M}$ is a single point.

Example 1.1.2. A pair of prequantum theories (Φ, τ, S_τ) and $(\Phi', \tau', S'_{\tau'})$ are dual if there is a correspondence

$$\begin{tikzcd} & \mathrm{Obs} \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ \mathrm{Obs}_\Phi & & \mathrm{Obs}_{\Phi'} \end{tikzcd}$$

where Obs_Φ and $\mathrm{Obs}_{\Phi'}$ are *quantizations* of the two classical theories, and where $f(\mathcal{O})$ and $f'(\mathcal{O})$ have the same expectation value for each observable $\mathcal{O} \in \mathrm{Obs}$.

This thesis will attempt to capture, in a mathematically rigorous way, certain aspects of a particular duality called *S-duality*, which is deeply intertwined with topics of modern research interest in the field of geometric representation theory. In the rest of this section we'll describe the idea, and some of the history, of S-duality. Then we'll go on to talk about geometric representation theory, and more specifically the geometric Langlands program, and the connection – introduced by Kapustin and Witten – between these two disparate fields.

There are many surveys of S-duality (also called electric-magnetic duality, or Montonen-Olive duality) in the literature, for example [**Kap08**, **Vec97**, **Oli96**, **Har96**, **FO98**, **Kne99**]. I won't try to reproduce the full story in this introduction, but I'll try to at least explain what the classical idea is – what S-duality is and why one might conjecture its existence – as well as hinting at some of the theory's more modern developments.

1.1.1. Classical Electric-Magnetic Duality

It is obligatory to begin our story with the familiar form of Maxwell's equations in the vacuum [1]:

$$\nabla \cdot E = 0 \qquad \nabla \cdot B = 0$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \qquad \nabla \times B = \frac{\partial E}{\partial t}$$

where E and B are the electric and magnetic fields: time dependent vector fields on $\mathbb{R}^3$. Using the metric on $\mathbb{R}^3$ to identify E and B with a time dependent 1- and 2-form on $\mathbb{R}^3$ respectively, one can equivalently write these equations in terms of the electromagnetic field strength, the 2-form $F = E \wedge dt + B \in \Omega^2(\mathbb{R}^4)$ [2],

$$dF = 0 \quad d^*F = 0.$$

One observes that these equations admit a very natural symmetry, namely the symmetry $E \mapsto B, B \mapsto -E$, or equivalently the Hodge star operator $*\colon \Omega^2(\mathbb{R}^4) \to \Omega^2(\mathbb{R}^4)$. This symmetry is broken as soon as one introduces background charge and current distributions ρ and j: Gauss' law becomes $\nabla \cdot E = \rho$, and Ampère's law becomes $\nabla \times B = \frac{\partial E}{\partial t} + j$.

This broken symmetry can be repaired if one also introduces *magnetic* charge and current distributions μ, k coupled to the magnetic field, and ask for the symmetry to also send $\rho \mapsto \mu, \mu \mapsto -\rho, j \mapsto k, k \mapsto -j$. This means in particular that the integral of B over a sphere in space need no longer equal zero, so allows for the existence of *magnetic monopoles.*

It's natural to ask whether such a theory of gauge fields and particles carrying both electric and magnetic charges can admit a quantization. Furthermore, if it does, does there exist a duality of quantum field theories based on this classical symmetry? More specifically, in the terminology of definition 1.1.1, does there exist a duality where $f'(\mathcal{O}) = * \circ f(\mathcal{O})$ where $*$ is the pushforward

[1] Here, and throughout this introduction, we'll work in natural units where $c = \hbar = 1$.

[2] This interpretation is natural from the point of view of the Lorentz force law governing the trajectory of a charge particle through an electromagnetic field, which says that the force on a particle moving with velocity v is given by the vector field dual to the 1-form $\iota_v F$.

along this symmetry of classical theories? This question has been considered at least since the 1930's, when Dirac published a very influential paper [**Dir31**] demonstrating that if one tries to quantize the theory of a single charged particle with charge e in a classical electo-magnetic potential corresponding to a magnetic monopole of magnetic charge f (i.e. a solution to Maxwell's equations where the magnetic charge distribution μ is given by f times a delta function at the origin), then a quantization can only exist if the product ef of the charges lies in the lattice $2\pi\mathbb{Z}$.

Before we address the second part of the question, let's explain how to generalise this story to Yang-Mills theory for non-abelian gauge groups.

1.1.2. Montonen and Olive's Proposal

While Dirac's construction of a monopole state in classical electro-magnetism is fairly straightforward, a bit more ingenuity is required to demonstrate the existence of such states in non-abelian gauge theories. One attractive construction is originally due to 't Hooft [**tH74**] and Polyakov [**Pol74**] (independently), originally for the gauge group SU(2) (or SO(3)). I'll explain briefly how this solution arises, following the exposition in the surveys of di Vecchia and Figureoa-O'Farril [**Vec97, FO98**]. We consider a classical Yang-Mills-Higgs theory: that is, Yang-Mills theory with gauge group SU(2) coupled to an adjoint valued scalar field ϕ with potential

$$V(\phi) = \frac{\lambda}{4}\left(|\phi|^2 - \alpha^2\right)^2$$

where λ and α are non-zero constants (the "Mexican hat" potential). The Higgs mechanism breaks the gauge symmetry to a maximal torus $H \subseteq \mathrm{SU}(2)$. One observes that after symmetry breaking the SU(2) gauge boson breaks up into a massless $U(1)$ gauge boson – which we might call the photon – along with two bosons with charge 1 and -1 and mass αe, which we call *W-bosons*. Here e is the gauge coupling constant.

Polyakov and 't Hooft proposed a solution to the classical equations of motion of this theory using a certain ansatz for a spherically symmetric solution, which we'll write in index notation. Here $a = 1, 2, 3$ index a basis for $\mathfrak{su}(2)$, and $i, j = 1, 2, 3$ index a basis for a codimension 1 subspace of spacetime.

$$\begin{aligned}
\phi^a(x) &= \frac{x^a}{e|x|^2} H(\xi) \\
A_a^0(x) &= 0 \\
A_i^a(x) &= -\frac{x^j}{e|x|^2} \varepsilon_{aij}(1 - K(\xi)).
\end{aligned}$$

Here H and K are smooth real-valued functions of $\xi = \alpha e|x|$ satisfying certain natural boundary conditions constraining their behaviour in the limits $\xi \to 0$ and $\xi \to \infty$, ensuring that the total energy of a solution satisfying this ansatz is finite. If one plugs this ansatz into the equations of motion, one finds a pair of coupled second order ordinary differential equations:

$$\begin{aligned}
\xi^2 \frac{d^2 K}{d\xi^2} &= KH^2 + K(K^2 - 1) \\
\xi^2 \frac{d^2 H}{d\xi^2} &= 2K^2 H + \frac{\lambda}{e^2} H(H^2 - \xi^2).
\end{aligned}$$

While these equations generally do not admit closed form solutions, Jaffe and Taubes [**JT80**] proved that they do in fact admit solutions satisfying the appropriate boundary conditions. These solutions are usually called *'t Hooft-Polyakov monopoles*.

Example 1.1.3. One obtains a special example, which we can express in closed form, in the limit $\lambda \to 0$ (sometimes called the "Prasad-Sommerfield limit"). In this case we obtain a solution to the classical equations of motion using the 't Hooft-Polyakov ansatz of the form

$$H(\xi) = \xi \coth(\xi) - 1, \; K(\xi) = \xi \operatorname{cosech}(\xi).$$

This solution is sometimes called a *BPS monopole*, for reasons we'll explain shortly.

To see why the ‘t Hooft-Polyakov solution to the classical equations of motions describes a monopole, observe that we can extract the part of the gauge field corresponding to the photon as the dot product $A_a \phi^a$. The condition that A_a^0 is identically zero ensures that the electric field associated to this configuration vanishes. However, in the limit $\xi \to \infty$, the total electromagnetic field strength tends to

$$\begin{aligned} F_{ij} &= \frac{1}{\alpha} \phi^a \partial_j A_i^a \\ &\to -\frac{1}{e} \frac{x^a}{|x|} \partial_j \left(\frac{x^k}{|x|^2} \right) \varepsilon_{aik} \\ &= -\frac{x^k}{e|x|^3} \varepsilon_{ijk} \end{aligned}$$

using the condition that $K \to 0$ and $H/\xi \to 0$ as $\xi \to \infty$. This has the property that distinguishes a magnetic monopole: the electric field vanishes, and when one computes the flux of the magnetic field through any sphere around the origin, one obtains a positive number, which – as the radius goes to infinite – converges to $f = \frac{4\pi}{e}$. Note that this magnetic charge automatically satisfies the Dirac quantization condition, since $fe = 4\pi \in 2\pi\mathbb{Z}$.

Now, these monopole solutions are designed to have finite energy, or equivalently finite mass. This mass has a lower bound called the *Bogomol’nyi-Prasad-Sommerfield (BPS) bound*: one checks that

$$\begin{aligned} E &= \int_{\mathbb{R}^3} \frac{1}{2} \left(E_a^i E_a^i + B_i^a B_i^a + \partial_0 \phi^a \partial_0 \phi^a + \partial_i \phi^a \partial_i \phi^a \right) + V(\phi) dx \\ &\geq \int_{\mathbb{R}^3} \frac{1}{2} \left(B_i^a B_i^a + \partial_i \phi^a \partial_i \phi^a \right) dx \\ &\geq \int_{\mathbb{R}^3} \partial^i (B_i^a \phi_a) dx \end{aligned}$$

where B is the magnetic field strength. By Stokes’ theorem and using the expression for ϕ in our ansatz, this can be thought of as the surface integral $\alpha \int_{S^2_\infty} B \cdot ds$, where S^2_∞ is the “sphere at infinity” (more precisely, this integral is equal to the limit of the surface integrals over spheres of increasing radius). Thus the BPS bound says that $E \geq \alpha f = \frac{4\pi\alpha}{e}$ by the observation of the previous paragraph. It’s easy to see when the BPS bound is saturated: when $E = \partial_0 \phi = B_i^a - \partial_i \phi^a = 0$, and

$\lambda = 0$. In particular, this explains why the solution of example 1.1.3 was called a BPS monopole: its mass saturates the BPS bound, and is equal to αf.

Having explained a construction of monopoles for $G = \mathrm{SU}(2)$, let's explain how to generalise this to other gauge groups. One can analyse the spectrum of particles in Yang-Mills-Higgs theory for a general reductive G in much the same way as we did for SU(2). The Higgs mechanism still breaks the gauge symmetry to a maximal torus $H \subseteq G$, leaving $r = \mathrm{rank}(G)$ massless bosons, and $\dim(G) - r$ massive W-bosons. These W-bosons are charged for the H-gauge symmetry: there is a W-boson associated to each root, whose (electric) charges are given by the diagonal action of H on the root space. The mass of a W-boson is given by $\alpha e|q|$, where $q \in \mathfrak{h}^*$ is the electric charge of the boson.

Given a Lie algebra homomorphism $c \colon \mathfrak{su}(2) \hookrightarrow \mathfrak{g}$, we can construct an 't Hooft-Polyakov monopole solution to the Yang-Mills-Higgs theory with gauge group G. To do this, we simply take a monopole solution for the group SU(2), and take its image under the induced map from SU(2) gauge fields and adjoint-valued scalars to G-valued ones (although we might need to be modify this solution to ensure it still satisfies the appropriate boundary conditions, as done by Weinberg [**Wei80**]). Given such an embedding, the magnetic charge corresponds to an element of the Cartan subalgebra $\mathfrak{h}$, by taking the image of the SU(2)-magnetic charge under the restricted embedding $\mathfrak{u}(1) \hookrightarrow \mathfrak{h}$. The Dirac quantization condition ensures that this magnetic charge is not just any element of $\mathfrak{h}$, but an element of the *coroot lattice.*

For a general gauge group, the BPS bound says that the mass, or energy, is bounded by $E \geq \alpha|m|$, where m is the magnetic charge of a monopole state. So let's summarise what we've learned about W-bosons and BPS monopoles in the Yang-Mills-Higgs theory for a gauge group G in table 1.1.

Montonen and Olive observed that the masses and charges of the BPS monopoles match the masses and charges of the W-bosons in a *dual* theory: the Yang-Mills-Higgs theory but for the *Langlands dual group* $G^\vee$ – a Lie group so that the coroot lattice of $G^\vee$ is the root lattice of G, and vice

	W-Boson	*BPS Monopole*
Charge	root $q \in \mathfrak{h}^*$	coroot $m \in \mathfrak{h}$
Mass	$\alpha e\|q\|$	$\alpha f\|m\|$

Table 1.1. Masses and charges of electrically and magnetically charged states. Here $f = \frac{4\pi}{e}$.

versa – and with the electric and magnetic couplings e and f are interchanged. This group is characterized by the following theorem, a proof of which can be found in standard textbooks on algebraic groups, e.g. Springer [**Spr98**].

Definition 1.1.4. *A* root datum *is a quadruple $(X^*, \Delta, X_*, \Delta^\vee)$, where X^* is a finite rank lattice with dual lattice X_* and $\Delta \subseteq X^*$ and $\Delta^\vee \subseteq X_*$ are finite subsets, along with a bijection $\Delta \to \Delta^\vee$ that we denote by $\alpha \mapsto \alpha^\vee$, satisfying the following conditions.*

(1) *For each $\alpha \in \Delta$, $\langle \alpha, \alpha^\vee \rangle = 2$.*
(2) *For each $\alpha \in \Delta$, the map $\beta \mapsto \beta - \langle \beta, \alpha^\vee \rangle \alpha$ permutes the elements of Δ, and similarly the map $\beta^\vee \mapsto \beta^\vee - \langle \beta^\vee, \alpha \rangle \alpha^\vee$ permutes the elements of $\Delta^\vee$.*

If additionally Δ does not contain 2α for any $\alpha \in \Delta$, the root datum is called reduced.

Theorem 1.1.5. *We can associate to every reduced root datum $(X^*, \Delta, X_*, \Delta^\vee)$ a unique complex reductive algebraic group G so that X^* is the lattice of characters $G \to \mathrm{U}(1)$, containing the roots $\Delta \subseteq X^*$, and X_* is the lattice of cocharacters $\mathrm{U}(1) \to G$, containing the coroots $\Delta^\vee \subseteq X_*$.*

Definition 1.1.6. *The* Langlands dual *(also called the* GNO dual *or the* magnetic dual *in the physics literature) of a group G with root datum $(X^*, \Delta, X_*, \Delta^\vee)$ is the unique group $G^\vee$ with root datum $(X_*, \Delta^\vee, X^*, \Delta)$.*

Recall that each complex reductive algebraic group has a unique compact form, up to isomorphism, and every compact connected Lie group arises in this way, so we can just as well think of the

$G_{\mathbb{C}}$	G	$G^{\vee}$	$G^{\vee}_{\mathbb{C}}$
$\mathrm{GL}(n;\mathbb{C})$	$\mathrm{U}(n)$	$\mathrm{U}(n)$	$\mathrm{GL}(n;\mathbb{C})$
$\mathrm{SL}(n;\mathbb{C})$	$\mathrm{SU}(n)$	$\mathrm{PSU}(n)$	$\mathrm{PGL}(n;\mathbb{C})$
$\mathrm{SL}(mn;\mathbb{C})/C_m$	$\mathrm{SU}(mn)/C_m$	$\mathrm{SU}(mn)/C_n$	$\mathrm{SL}(mn;\mathbb{C})/C_n$
$\mathrm{SO}(2n+1;\mathbb{C})$	$\mathrm{SO}(2n+1)$	$\mathrm{Sp}(n)$	$\mathrm{Sp}(2n;\mathbb{C})$
$\mathrm{SO}(2n;\mathbb{C})$	$\mathrm{SO}(2n)$	$\mathrm{SO}(2n)$	$\mathrm{SO}(2n;\mathbb{C})$
$\mathrm{Spin}(2n+1;\mathbb{C})$	$\mathrm{Spin}(2n+1)$	$\mathrm{Sp}(n)/C_2$	$\mathrm{Sp}(2n;\mathbb{C})/C_2$
$\mathrm{Spin}(4n;\mathbb{C})$	$\mathrm{Spin}(4n)$	$\mathrm{Spin}(4n)/(C_2\times C_2)$	$\mathrm{Spin}(4n;\mathbb{C})/(C_2\times C_2)$
$\mathrm{Spin}(4n+2;\mathbb{C})$	$\mathrm{Spin}(4n+2)$	$\mathrm{Spin}(4n+2)/C_4$	$\mathrm{Spin}(4n+2;\mathbb{C})/C_4$
$\mathrm{Spin}(8n;\mathbb{C})/C_2^1$	$\mathrm{Spin}(8n)/C_2^1$	$\mathrm{Spin}(8n)/C_2^1$	$\mathrm{Spin}(8n)/C_2^1$
$\mathrm{Spin}(8n;\mathbb{C})/C_2^2$	$\mathrm{Spin}(8n)/C_2^2$	$\mathrm{Spin}(8n)/C_2^2$	$\mathrm{Spin}(8n;\mathbb{C})/C_2^2$
$\mathrm{Spin}(8n+4;\mathbb{C})/C_2^1$	$\mathrm{Spin}(8n+4)/C_2^1$	$\mathrm{Spin}(8n+4)/C_2^2$	$\mathrm{Spin}(8n+4;\mathbb{C})/C_2^2$
E_6	E_6	E_6/C_3	E_6/C_3
E_7	E_7	E_7/C_2	E_7/C_2
E_8	E_8	E_8	E_8
F_4	F_4	F_4	F_4
G_2	G_2	G_2	G_2

Table 1.2. Compact connected and complex reductive groups and their Langlands duals. Here C_n denotes the cyclic group of order n. The group $\mathrm{Spin}(4n)$ has center $C_2 \times C_2$, we write the two factors as C_2^1 and C_2^2. The group $\mathrm{SO}(4n)$ is the quotient by the diagonal subgroup. This table is adapted from the lecture notes of Figueroa-O'Farril [**FO98**, Table 6.4].

Langlands dual of a compact connected Lie group. In table 1.2 we present the Langlands duals of various compact and complex groups. We can see from the table, for example, the general phenomenon that Langlands duality exchanges the simply connected and adjoint forms associated to a simple Lie algebra.

We can now state Montonen and Olive's conjecture.

Conjecture 1.1.7 (Montonen-Olive Duality)**.** *There exists a duality between quantum field theories quantizing the classical Yang-Mills-Higgs theories with gauge groups G and $G^{\vee}$, in the BPS limit $\lambda \to 0$. This duality interchanges quantum states corresponding to W-bosons and BPS monopoles.*

1.1.3. S-Duality for Supersymmetric Gauge Theories

There are two potential obstructions to the existence of Montonen and Olive's duality: the fact that the monopole does not appear to have the correct spin (spin 1, to match the gauge boson of the dual theory), and the fact that the masses of the particles and the monopoles may be renormalized. In this section we'll explain how both of these problems are resolved when we consider not an ordinary Yang-Mills-Higgs theory, but a *supersymmetric* Yang-Mills theory.

The data of a quantization of the classical Yang-Mills-Higgs theory includes, in the perturbative path integral approach to quantization formalized by Wilson, a family of effective theories at each energy scale, compatible with the renormalization group flow. In particular, parameters like the mass of a particle receive *quantum corrections* – in any given effective theory one will generally have to modify the values of these parameters to preserve compatibility with the renormalization group. In order for Montonen-Olive duality to hold, the equality of the masses of electrically and magnetically charged states should hold now only in the classical theory, but also in the effective theories at different energy scales. There is, however, no reason why it should be possibile to arrange this.

It is possible to get around this problem by extending the classical theory with which we work. The idea that in a supersymmetric theory the quantum corrections to the mass formula might vanish – i.e. that the masses of W-bosons and monopole states might not flow under renormalization – was first discussed by d'Adda, Horsley and di Vecchia [**dHDV78**], who described BPS monopole solutions in $N = 2$ super Yang-Mills and showed that its mass does not admit quantum corrections at the one loop level. Witten and Olive [**OW78**] extended their result, and showed that the mass spectrum in these $N = 2$ theories does not admit quantum corrections at any level: that the classical formula holds exactly. As such, the problem discussed above does not arise.

We defer a detailed description of what supersymmetric Yang-Mills theories actually *are* to chapter 8, especially section 8.2 in the second part of this thesis. Roughly speaking, super Yang-Mills theories are extensions of Yang-Mills theories, where we introduce additional fields in order to promote the action of the Poincaré group to the action of a $\mathbb{Z}/2$-graded extension by a spinor representation, a "super Poincaré group". In $N = 2$ theories – those where the spinor representation has two irreducible summands – the fields include a $\mathfrak{g}$-valued scalar with a quartic potential, which plays the role of the Higgs field in the ordinary Yang-Mills-Higgs theory.

The problem remains that, even in $N = 2$ theories, the W-boson and BPS-monopole states appear to have different spins, i.e. they transform differently under the action of the super Poincaré group, which means that Montonen and Olive's conjecture still can't be valid (we refer to sections 2.3.3 and 2.3.4 of Figueroa-O'Farril [**FO98**] for details). Osborn [**Osb79**] demonstrated that this problem was also resolved if we pass from $N = 2$ to $N = 4$ super Yang-Mills theory. This motivates an improved form of Montonen and Olive's conjecture.

More modern, supersymmetric formulations of Montonen-Olive duality are more often called "S-dualities" (this term was used by Schwarz and Sen [**SS93**] in 1993 in the context of string theory, but it's possible that the name is older). 'S' here is short for "strong-weak": since S-duality inverts the gauge coupling constant, strongly coupled theories can have weakly coupled duals, allowing the calculation of correlation functions in the strongly-coupled theory using perturbative methods.

Conjecture 1.1.8 (S-duality)**.** *There exists a duality between quantum field theories quantizing the classical $N = 4$ super Yang-Mills theories with gauge groups G and $G^\vee$.*

1.1.4. Abelian Duality

In the case where the gauge group G is abelian, pure Yang-Mills theory is free, so even without supersymmetry there's no concern about quantum corrections to the mass formula, and one conjectures that there exists a duality between Yang-Mills theory with gauge group given by a torus H, and Yang-Mills theory with gauge group the dual torus $H^\vee$. This simple instance of electric-magnetic duality was called *abelian duality* by Witten in his 1995 paper on the subject [**Wit95a**], in which he addresses a relationship between the partition functions of abelian pure Yang-Mills theories on a compact 4-manifold. This relationship, and generalizations to higher degree, were also studied by Verlinde in [**Ver95**].

Analogues of this duality in lower dimensions have also been investigated in the physics literature, for instance the *T-duality* between sigma models with dual torus targets on a 2-manifold, and a duality between sigma models and abelian gauge theories on a 3-manifold (these are described in [**Wit99**]. Explicit calculations of duality for the partition functions in three dimensions have been performed by Prodanov and Sen [**PS00**] and by Broda and Duniec [**BD04**], and a detailed analysis of duality for more general observables was recently performed by Beasley [**Bea14b**, **Bea14a**]). These theories are the lowest-dimensional examples of a sequence of theories, whose fields model connections on higher torus bundles, models for which have been described by Freed [**Fre00**] using *ordinary differential cochains* to model the fields. The quantizations of these theories were further studied by Barbón [**Bar95**], who discussed abelian duality for the partition functions in higher degree theories, and Kelnhofer [**Kel09**], who explained how to compute the vacuum expectation values of gauge invariant observables in these theories using the language of ordinary differential cochains.

In part 1 of this thesis, we will prove a version of abelian duality for this family of theories. The statement we establish follows the paradigm of example 1.1.2, where we model quantum field theories as *factorization algebras*, using the techniques of Costello and Gwilliam [**Gwi12**, **CG15**].

Theorem 1.1.9. *Let X be a compact Riemannian n-manifold. There exists a correspondence of factorization algebras of the form*

$$\begin{tikzcd} & (\mathrm{Obs}_{\Omega^p})_0 \arrow[dl, "r"'] \arrow[dr, "r'"] & \\ \mathrm{Obs}^{\mathrm{q}}_{p,R} & & \mathrm{Obs}^{\mathrm{q}}_{n-p,\frac{1}{2R}} \end{tikzcd}$$

where $\mathrm{Obs}^{\mathrm{q}}_{p,R}$ *is the quantization of the factorization algebra of classical observables in a generalized Maxwell theory whose fields are connections on* $(p-2)$*-gerbes for the group* $\mathbb{R}/2\pi R\mathbb{Z}$*. Observables which are* incident*, i.e. images of the same observable in* $(\mathrm{Obs}_{\Omega^p})_0(U)$ *under the two maps, have the same expectation value.*

In chapter 6 we'll explain what exactly $(\mathrm{Obs}_{\Omega^p})_0$ is, and what the maps r and r' are, and argue that a large family of interesting observables (generalizations of Wilson and 't Hooft loop operators) lie in the images of r and r'.

1.1.5. Modern Topics

We'll conclude this section with a brief discussion of how the S-duality conjecture for $N = 4$ gauge theories has developed since it was first stated in the late 1970's. A great deal of the development of the subject relies on ideas from string theory; indeed S-duality is believed to arise as a low energy limit of a duality between string theories. This introduction will not attempt to explain any of these ideas (since I don't understand them myself); instead we'll focus on topics that can be understood purely from the point of view of field theory.

Firstly, the S-duality transformation can be promoted to a family of symmetries indexed by the modular group $\mathrm{SL}(2;\mathbb{Z})$. To do this, note that one can add a topological term of the form $\theta \int F_A \wedge F_A$ to the super Yang-Mills action functional without affecting the classical equations of motion. The quantization of these theories are now related by an additional, trivial symmetry, corresponding to the symmetry of the classical theory that shifts $\theta \mapsto \theta + 1$. Combining the ordinary Yang-Mills coupling constant e and the new parameter θ into a single element $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{e}$ in the complex upper half-plane, one finds that S-duality and this new trivial symmetry combine to generate a copy of $\mathrm{SL}(2;\mathbb{Z})$, where the two symmetries correspond to the matrices

$$t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2;\mathbb{Z}).$$

An important antecedent of the work of Kaputin and Witten (which we'll discuss below) is a 1994 paper of Vafa and Witten [**VW94**] which proposed evidence for S-duality, by arguing that it held for a certain *twist* of the full supersymmetric theory. The idea of twisting a classical field theory (which we discuss in detail in chapter 8) was introduced by Witten in an earlier paper [**Wit88a**]. Witten's idea was that, in a supersymmetric field theory, if one chooses a supersymmetry Q such that $[Q, Q] = 0$ and restricts attention to the Q-*cohomology* of the algebra of observables, all symmetries of the form $[Q, Q']$ will act trivially. If one chooses Q appropriately, *all* translations take this form, so observables in the Q-cohomology are translation invariant, and the resulting theory of such observables is physically comparatively simple, and mathematically often closely connected to the theory of well-studied topological invariants.

Vafa and Witten argued that, in a certain twist of $N = 4$ super Yang-Mills theory, the partition function coincides with generating functions for the Euler characteristic of the moduli space of instantons. They then investigate these Euler characteristics for various gauge groups, and various 4-manifolds, and prove that they are modular, i.e. $\mathrm{SL}(2;\mathbb{Z})$-invariant. This provides some evidence for the S-duality conjecture, but more importantly for the purposes of this thesis provided an early link between S-duality of gauge theories and topics in the geometry of moduli spaces.

An intriguing proposal for the origin of S-duality was made by Witten first in 1995 [**Wit95b**], and elaborated upon in several later papers [**Wit04a**, **Wit09**]. Witten suggested that there should exist a six-dimensional $(2, 0)$-superconformal quantum field theory, which includes a "self-dual higher gauge field" with semisimple simply-laced gauge group G – mathematicians have recently taken to referring to this theory as "Theory X" [3]. This quantum field theory does not generally arise as the quantization of a classical field theory in the usual sense: it does not admit an action functional. From a mathematical point of view this six-dimensional theory is highly speculative, and there isn't currently a mathematically satisfying description of almost any aspect of it. Nevertheless, if we assume its existence we can start to conjecture the existence of many interesting properties

[3] As in, for example, the proposal for the 2012 NSF grant DMS-1342948, entitled "In and Around Theory X".

of and dualities between quantum field theories of lower dimension. For example, one obtains an $N = 4$ super Yang-Mills theory by dimensionally reducing theory X along a torus; Witten proposed that S-duality for such theories should arise as follows. By dimensionally reducing along different tori, one obtains a family of four-dimensional quantum field theories parameterized by the upper half plane, thought of as the Teichmüller space $\mathcal{T}_{1,0}$. Inside the upper half-plane one can choose a fundamental domain for the group $\mathrm{SL}(2;\mathbb{Z})$, and each element of this group then yields a duality between points on the boundary, as in definition 1.1.1. In particular, the standard elements t and $u \in \mathrm{SL}(2;\mathbb{Z})$ correspond to the S-duality transformation, and a shift of the topological term respectively. One immediately observes that this perspective leads to a wide family of S-dualities corresponding to elements of general mapping class groups, by dimensionally reducing theory X along curves of higher genus. These dualities were first discussed by Gaiotto [**Gai12**]. They relate gauge theories with $N = 2$ supersymmetry called theories of *class S*, that generally fail to admit Lagrangian descriptions unless all the simple factors of the gauge group G have rank 1.

Finally, we should briefly mention recent work of Gaiotto and Witten, in which they describe an interpretation of S-duality as a 3-dimensional domain wall between 4-dimensional theories [**GW09b**, **GW09a**]. In [**GW09b**] Gaiotto and Witten construct certain boundary conditions in $N = 4$ super Yang-Mills theories that are invariant under exactly half of the supersymmetries (*half BPS boundary conditions*), characterized by certain data, including an $\mathfrak{sl}_2$-triple in the Lie algebra of the gauge group. In [**GW09a**], they go on to discuss S-duality for boundary conditions in $N = 4$ theories, using a theory with a codimension 1 defect invariant, again, under half of the supersymmetries, called a “Janus wall”, first introduced by Bak, Gutperle and Hirano [**BGH03**] using the AdS/CFT correspondence.

1.2. Introduction to the Geometric Langlands Program

Historically, the original motivation for the geometric Langlands conjecture comes from number theory: from trying to find the right analogue of the Langlands reciprocity conjecture in the realm of complex geometry. Because the objects of interest behave better in a geometric setting, one can prove stronger results in a cleaner way and hope to eventually transport some ideas from geometry to number theory. Ngô's proof [**Ngô10**] of the fundamental lemma using the geometry of the Hitchin system is an example of a striking success of this program (explained in an expository article of Nadler [**Nad12**]). In this section we will recall and attempt to motivate the heuristic categorical statement of the geometric Langlands conjecture introduced by Beilinson and Drinfeld. We will then go on to describe recent work of Arinkin and Gaitsgory [**AG12**] in formulating a more precise version of this conjecture.

I will not attempt to give a complete historical overview of the origin of the geometric Langlands program from number theory, nor will I attempt to give a comprehensive description of the state of the art of research into the categorical Langlands program. For a survey of the origins of the geometric Langlands program, along with some of its connections to quantum field theory, one cannot do better than the excellent surveys of Frenkel [**Fre07**, **Fre10**]. We refer the reader to Arinkin and Gaitsgory [**AG12**] for details of the modern formulation of the geometric Langlands conjecture, and Gaitsgory [**Gai13**] for a proof for the group GL(2), along with a description of a program by which experts hope to establish the full conjecture.

1.2.1. Motivation from Number Theory

There are many articles surveying the Langlands program in number theory, for example [**Tay04**, **Kna97**]. In this section I'll give a short summary, in order to motivate the geometric version of the conjecture. The Langlands program, viewed from a distance, describes a relationship between two kinds of representation object which can be constructed from a number field F (or, for that matter, many other kinds of field of number theoretic interest: there are many versions of the Langlands correspondence). One of these types of object has a purely algebraic flavour, while the other is defined and studied using techniques from functional analysis. Simultaneously, the correspondence relates representation theoretic objects defined using a reductive algebraic group G on the one side, and its Langlands dual ${}^L G$ on the other [4].

The algebraic objects that appear in the Langlands correspondence are, at least at the simplest level, representations of the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$ valued in the group ${}^L G$. Galois representations are ubiquitous in number theory (arising, for instance, from the étale cohomology of algebraic varieties over F), but their abstract structure is a priori difficult to access, making results that relate their structure to a different kind of object desirable.

The more analytic objects are slightly more difficult to motivate a priori, but are closely related to classically studied objects in number theory such as the ideal class group of a number field, and the ring of modular forms. These are objects called *automorphic representations*, which we won't define carefully, but will try to at least give the flavour of. We define the ring of *adèles* of a global field F to be the restricted product

$$\mathbb{A}_F = \prod_p{}' F_p$$

where the product is taken over all places p of F, and where F_p is the formal completion at p (for instance, if $F = \mathbb{Q}$ and p is a prime, these are the p-adic numbers). By the *restricted* product, we mean that the image of an adèle under the projection $\mathbb{A}_F \to F_p$ must land in the local ring $\mathcal{O}_p \subseteq F_p$ for all but finitely many p. Inside the adèles, we can define the ring of integers $\mathcal{O} \subseteq \mathbb{A}_F$ to be the product $\prod \mathcal{O}_p$. The *automorphic quotient* associated to the field F is the double quotient

$$G(F) \setminus G(\mathbb{A}_F)/G(\mathcal{O}).$$

As we'll argue below, this double quotient has a natural geometric interpretation as a version of a moduli space of G-bundles. An *automorphic representation* is, very roughly, an irreducible unitary subquotient of the $G(\mathbb{A}_F)$-module $L^2_\chi(G(F) \setminus G(\mathbb{A}_F))$ of L^2-functions twisted by a character χ of

[4] One should take care: since these groups need no longer be defined over the complex numbers, the definition of the Langlands dual in the previous section is not quite right. For the definition as written to hold, we need to assume that our group G is *split*, i.e. contains a split maximal torus, and even under this assumption one needs to form the semidirect product with the absolute Galois group of F. Because of this difference, and for compatibility with the number theory literature, for this section only we'll write ${}^L G$ for the Langlands dual group of G.

$Z(\mathbb{A}_F)$, where Z is the maximal F-split central torus in G, after equipping the quotient with an appropriate measure. These representations can be grouped together into "L-packets", consisting of representations that are isomorphic as $G(F_v)$-representations for almost all places v.

With this setup, we can state an (imprecise) version of the Langlands reciprocity conjecture.

Meta-Conjecture 1.2.1 (Langlands reciprocity). *Let F be a number field, and let G be a split reductive algebraic group over F. There is a bijection*

$$\left\{\ell\textit{-adic representations of } \mathrm{Gal}(\overline{F}/F) \textit{ valued in } {}^L G\right\}_{/\sim}$$

$$\updownarrow$$

$$\left\{L\textit{-packets of automorphic representations of } G(\mathbb{A}_F)\right\}$$

intertwining the natural symmetries on both sides.

By referring to this as a "meta-conjecture", I mean to indicate that this should not necessarily be taken at face value, but as a guiding principal for a great number of more precise conjectures (see for instance Clozel [**Clo90**] for a precise version of this conjecture for $G = \mathrm{GL}_n$, and Buzzard-Gee [**BG14**] for a precise version for a general group). At the very least, in order to make this precise we would have to define all of our terms, and to identify what exactly these "natural symmetries" on the two sides should be.

Remark 1.2.2. In a more careful formulation of the Langlands reciprocity conjecture, one might instead consider representations not of $\mathrm{Gal}(\overline{F}/F)$, but of $\mathrm{Gal}(\overline{F}/F) \times \mathrm{SL}(2;\mathbb{C})$. The homomorphism $\mathrm{SL}(2;\mathbb{C}) \to G^\vee$ is an additional piece of data called an "Arthur parameter", and the set of elements which only differ in this parameter is called an "Arthur packet".

The passage from the Langlands program in number theory to the geometric Langlands program uses an analogy sometimes called "Weil's Rosetta Stone" [**Wei79**]. The idea is the following. There are two kinds of global fields: number fields and global function fields, i.e. the function fields of algebraic curves Σ defined over a finite field $\mathbb{F}_q$. Weil's analogy relates algebraic properties of these fields to geometric properties of algebraic curves defined over the field $\mathbb{C}$ of complex numbers, using global function fields as a bridge.

- The places of a number field correspond to the closed points of an algebraic curve (note that there isn't an obvious geometric analogue for the collection of infinite places of a number field).
- The completion of a number field at a place v corresponds to the local field of an algebraic curve at a point p – the field of functions on a formal punctured neighbourhood of p. Its ring of integers corresponds to the local ring – the ring of functions on a formal (unpunctured) neighbourhood of p.
- The automorphic quotient $G(F) \setminus G(\mathbb{A}_F)/G(\mathbb{A}_F^f)$ of a number field corresponds to the moduli space $\mathrm{Bun}_G(\Sigma)$ of holomorphic G-bundles on Σ. To deduce this, first note that any holomorphic G bundle can be trivialized away from a finite set of points. In other words, one obtains a trivializing cover by taking small discs around a finite set $x_1, \ldots, x_k$ of points in Σ, together with the complement $\Sigma \setminus \{x_1, \ldots, x_k\}$. The set of G-bundles trivialized on this cover is in bijection with the set of double cosets

$$G(\Sigma \setminus \{x_1, \ldots, x_k\}) \setminus \prod_{i=1}^{k} G(D_{x_i}^{\times})/\prod_{i=1}^{k} G(D_{x_i})$$

by identifying the bundle by its transition functions, modulo different choices of trivialization on the trivializing cover. This can equivalently be written as

$$G(\Sigma \setminus \{x_1, \ldots, x_k\}) \setminus \left(\prod_{i=1}^{k} G(D_{x_i}^{\times}) \times \prod_{x \neq x_i \in \Sigma} G(D_x^{\times}) \right) / \prod_{x \in \Sigma} G(D_x).$$

Now, allowing the set $\{x_1, \ldots, x_k\}$ to vary through all finite subsets of Σ and replacing our discs by formal discs, we obtain a description of the set of all G-bundles on Σ as a set of double cosets. Namely

$$G(\Sigma \setminus \{x_1, \ldots, x_k\}) \setminus {\prod_{x \in \Sigma}}' G(\mathbb{D}_x^\times) / \prod_{x \in \Sigma} G(\mathbb{D}_x)$$

where the restricted product means that the G-valued function extends across the puncture for all but finitely many points. Since we can view $G(\mathbb{D}_x^\times)$ as the points of G valued in the local field at x, and $G(\mathbb{D}_x)$ as the points of G valued in the local ring, we observe that this set is exactly analogous to the set of points of the automorphic quotient.

- Galois extensions of the number field F correspond to Galois covers of the curve Σ.
- The absolute Galois group of F corresponds to the fundamental group of Σ.
- $G^\vee$-valued Galois representations of F correspond to $G^\vee$-valued representations of the fundamental group of Σ, or equivalently locally constant sheaves on Σ with a $G^\vee$-structure.

This gives us a geometric analogy for the main objects appearing in the Langlands conjecture. The remaining ingredient needed to pass from the Langlands reciprocity conjecture to the geometric Langlands conjecture is called the Grothendieck "function-sheaf" or "fonctions-fasciceaux" dictionary. This is an example of *categorification*, in which we replace the functions on the automorphic quotient by *constructible sheaves*, or *D-modules*. Over the complex numbers, the set-theoretic geometric Langlands correspondence takes the following form.

Conjecture 1.2.3. *There is a natural bijection*

$$\{\textit{irreducible } G^\vee\textit{-local systems on } \Sigma\}_{/\sim} \leftrightarrow \{\textit{Hecke eigensheaves on } \mathrm{Bun}_G(\Sigma)\}$$

sending $\mathcal{L}$ to an eigensheaf with eigenvalue $\mathcal{L}$.

Hecke eigensheaves here are D-modules that satisfy a particular equation, prescribing the action of local "Hecke" operators. We refer to Frenkel [**Fre07**] for an exposition of exactly what Hecke eigensheaves are – we'll shortly give another, stronger form of the conjecture which we'll state precisely. A version of conjecture 1.2.3 goes back to Drinfeld, who stated and proved it for the group $G = \mathrm{GL}_2$ [**Dri83**], after which Laumon formulated the conjecture GL_n [**Lau87**]. This conjecture for GL_n was proved by L. Lafforgue [**Laf02**], and later work of Frenkel, Gaitsgory and Vilonen [**FGV02**] combined with a result of Gaitsgory [**Gai04**] explicitly constructs a Hecke eigensheaf associated to an irreducible local system in unramified cases. More recently, V. Lafforgue formulated the conjecture for general reductive groups G, and gave a construction of a map from left to right [**Laf12**].

1.2.2. The Categorical Proposal

The above story motivates a geometric version of the Langlands conjecture, where now – thanks to the fonctions-fasciceaux correspondence – we can ask for the existence not just of a bijection of sets, but of an equivalence of categories. We'll begin by stating the most optimistic version of the geometric Langlands conjecture. We view this as a "meta-conjecture" rather than a true conjecture: an inspiring story which we hope will lead us to a true statement. Beilinson and Drinfeld proposed the following statement.

Meta-Conjecture 1.2.4 ("Drinfeld's best hope")**.** *There is a dg-equivalence*

$$\mathrm{D}(\mathrm{Bun}_G(\Sigma)) \simeq \mathrm{QC}(\mathrm{Loc}_{G^\vee}(\Sigma)),$$

between the dg-category of D-modules on $\mathrm{Bun}_G(\Sigma)$ *and the dg-category of quasi-coherent sheaves on* $\mathrm{Loc}_{G^\vee}(\Sigma)$*, intertwining the natural symmetries on both sides.*

Remark 1.2.5. In what follows we'll sometimes refer to the left- and right-hand sides of this equivalence as the *automorphic* and *spectral* sides of the correspondence respectively. After explaining the connection to S-duality and mirror symmetry in the next section, we will also refer to them as the A- and B-sides respectively.

We'll be more precise about what kind of "natural symmetries" we mean below. This statement extends earlier set-theoretic statements of the geometric Langlands correspondence, which propose a bijection between $G^\vee$-local systems $\mathcal{L}$ on Σ (i.e. skyscraper sheaves on the right hand side) and Hecke eigensheaves on $\mathrm{Bun}_G(\Sigma)$ with eigenvalue $\mathcal{L}$.

If G is abelian, this "best hope" statement is precisely correct, and was proven independently by Laumon [**Lau96**] and Rothstein [**Rot96**]. One can construct an equivalence between the abelian geometric Langlands categories by a modified Fourier-Mukai transform. If A is an abelian variety, let $A^\natural$ denote the moduli space of $\mathbb{G}_m$ flat bundles on A. Just like in the ordinary Fourier-Mukai transform, there's a Poincaré sheaf on $A \times A^\natural$ – a line bundle with flat connection – which restricts to $(\mathcal{L}, \nabla)$ on $A \times \{(\mathcal{L}, \nabla)\}$. Laumon and Rothstein proved that the integral transform associated with the Poincaré sheaf provides an equivalence of categories

$$\mathrm{QC}(A^\natural) \to \mathrm{D}(A),$$

which gives a version of the abelian geometric Langlands correspondence when one sets $A = \mathrm{Jac}(\Sigma)$, the Jacobian of Σ. This needs only small modifications to produce the best hope conjecture: on the A-side, $\mathrm{Bun}_{\mathbb{G}_m}(\Sigma) = \mathrm{Jac}(\Sigma) \times \mathbb{Z} \times B\mathbb{G}_m$, and on the B-side $\mathrm{Loc}_{\mathbb{G}_m}(\Sigma) = \mathrm{Jac}(\Sigma)^\natural \times (\mathrm{pt} \times_{\mathbb{A}^1} \mathrm{pt}) \times B\mathbb{G}_m$, so one obtains the abelian geometric Langlands correspondence by observing that $\mathrm{D}(\mathbb{Z}) \cong \mathrm{QC}(B\mathbb{G}_m)$ and $\mathrm{D}(B\mathbb{G}_m) \cong \mathrm{QC}(\mathrm{pt} \times_{\mathbb{A}^1} \mathrm{pt})$.

It turns out, however, that the abelian case is the *only* example where the best hope conjecture holds literally. In the rest of this section we'll discuss the example $\Sigma = \mathbb{P}^1$, or equivalently, the categories of *local operators* for the geometric Langlands correspondence. V. Lafforgue [**Laf09**] proved that, in the case where $\Sigma = \mathbb{P}^1$ and G is non-abelian, the category $\mathrm{QC}(\mathrm{Loc}_{G^\vee}(\mathbb{P}^1))$ is equivalent to a proper subcategory of $D(\mathrm{Bun}_G(\Sigma))$. "Large" objects in $D(\mathrm{Bun}_G(\mathbb{P}^1))$, including the D-module $D_{\mathrm{Bun}_G(\mathbb{P}^1)}$ fail to correspond to any quasi-coherent sheaf on $\mathrm{Loc}_{G^\vee}(\mathbb{P}^1)$. This result follows from a careful discussion of the local version of the conjecture, usually called the *geometric Satake correspondence*. In this discussion, we'll explain what the natural symmetries alluded to above are.

To start with, let's discuss the original version of the geometric Satake correspondence, due to Mirkovic and Vilonen [**MV07**].

Definition 1.2.6. *The* affine Grassmannian *of G is the ind-scheme $G(\mathcal{K})/G(\mathcal{O})$, where $\mathcal{K} = \mathbb{C}((t))$ and $\mathcal{O} = \mathbb{C}[[t]]$. A proof that this is represented by an ind-scheme can be found in lecture notes of Görtz [**Gör10**].*

Theorem 1.2.7 (Underived Geometric Satake [**MV07**])**.** *There is an equivalence of abelian categories*

$$\mathrm{D}^b_{G(\mathcal{O})}(\mathcal{G}\mathrm{r}_G)^\heartsuit \to \mathrm{Rep}(G^\vee)$$

where $\mathrm{D}^b_{G(\mathcal{O})}(\mathcal{G}\mathrm{r}_G)^\heartsuit$ [5] is the abelian category of bounded complexes of $G(\mathcal{O})$-equivariant D-modules on the affine Grassmannian, where $G(\mathcal{O})$ acts by left multiplication, and where $\mathrm{Rep}(G^\vee)$ is the category of finite-dimensional representations of $G^\vee$. The category $\mathrm{D}^b_{G(\mathcal{O})}(\mathcal{G}\mathrm{r}_G)^\heartsuit$ admits a monoidal structure by convolution making this equivalence monoidal.

Mirkovic and Vilonen proved this theorem by showing that the category on the left-hand side is *neutral Tannakian*, which – by a theorem of Deligne – means that it is equivalent to the category of representations of a group scheme. By analysing the structure of the category on the left, they show that this group scheme is actually a reductive algebraic group, then show that it has irreducible representations parameterized by dominant coweights of G. This allows them to prove that the unknown reductive group is actually isomorphic to the Langlands dual group $G^\vee$.

Remark 1.2.8. The name "geometric Satake" comes from an analogous theorem in number theory called the Satake isomorphism. This theorem says that if $\mathcal{K}$ is a non-archimedean local field, and $\mathcal{O}$ is its ring of integers, then the set of compactly supported functions on $G(\mathcal{K})$ which are invariant for the $G(\mathcal{O})$ action on both sides can be naturally made into a ring isomorphic to the representation ring of the Langlands dual group. This ring is called the *spherical Hecke algebra* of G.

[5] We're writing $\heartsuit$ here to indicate the heart of the natural t-structure.

It's natural to try to promote Mirkovic-Vilonen's theorem to an equivalence of derived, or dg-categories, but this turns out to be quite subtle. In particular, while it is possible to form the derived categories of the abelian categories in theorem 1.2.7 and thus obtain an equivalence of derived categories, the categories we obtain will *not* be the categories that naturally locally act on the geometric Langlands categories. The correct derived version of the geometric Satake correspondence was proven by Bezrukavnikov and Finkelberg [**BF08**], and is discussed at length by Arinkin and Gaitsgory [**AG12**, section 12]. Specifically, they prove that a certain category of sheaves on the "spectral Hecke stack"

$$\mathrm{Hecke}^{\mathrm{spec}}_{G^\vee} = BG^\vee \times_{\mathfrak{g}^\vee/G^\vee} BG^\vee$$

is equivalent to the dg-category $\mathrm{D}_{G(\mathcal{O})}(\mathcal{G}\mathrm{r}_G)$ as monoidal categories, equipped with the convolution monoidal structure. We'll state the theorem here, but some of the notation won't be introduced until the next section.

Theorem 1.2.9 (Derived Geometric Satake [**AG12**, 12.5.5])**.** *There is a canonical equivalence of monoidal dg-categories*

$$\mathrm{Sat}\colon \mathrm{D}_{G(\mathcal{O})}(\mathcal{G}\mathrm{r}_G) \to \mathrm{IndCoh}_{\mathcal{N}}(\mathrm{Hecke}^{\mathrm{spec}}_{G^\vee}).$$

To see why a statement of this form is natural, we'll explain why these categories naturally act on the geometric Langlands categories. We write $\mathbb{D}$ for $\operatorname{Spec}\mathcal{O}$, and $\mathbb{D}^\times$ for $\operatorname{Spec}\mathcal{K}$ – the *formal disk* and *formal punctured disk*. Denote by $\mathbb{B}$ the coproduct $\mathbb{D} \sqcup_{\mathbb{D}^\times} \mathbb{D}$ obtained by gluing two formal disks together along a formal punctured disk – the "formal bubble". The derived geometric Satake correspondence is a local version of the geometric Langlands correspondence because

$$\begin{aligned} \mathrm{Bun}_G(\mathbb{B}) &= G(\mathcal{O}) \setminus G(\mathcal{K})/G(\mathcal{O}) \\ \text{and } \mathrm{Loc}_{G^\vee}(\mathbb{B}) &\cong \mathrm{Loc}_{G^\vee}(\mathbb{D}) \times_{\mathrm{Loc}_{G^\vee}(\mathbb{D}^\times)} \mathrm{Loc}_{G^\vee}(\mathbb{D}) \\ &\cong BG^\vee \times_{\mathfrak{g}^\vee/G^\vee} BG^\vee \\ &= \mathrm{Hecke}^{\mathrm{spec}}_{G^\vee} \end{aligned}$$

so derived geometric Satake provides an equivalence between $\mathrm{D}(\mathrm{Bun}_G(\mathbb{B})$ and sheaves (of a sort to be elaborated on below) on $\mathrm{Loc}_{G^\vee}(\mathbb{B})$.

If we choose a $\mathbb{C}$-point x in Σ, gluing an extra formal neighbourhood near x allows us to form the diagram

$$\begin{tikzcd} & \mathbb{B} \arrow[d] & \\ & \Sigma \times_{\mathbb{D}_x^\times} \mathbb{D} & \\ \Sigma \arrow[ur] & & \Sigma \arrow[ul] \end{tikzcd}$$

of derived stacks, where the arrows are all given by the natural inclusions (there are two inclusions $\Sigma \hookrightarrow \Sigma \times_{\mathbb{D}_x^\times} \mathbb{D}$ corresponding to the two neighbourhoods of x). By taking algebraic G-bundles, or flat $G^\vee$-bundles, this structure gives us diagrams of the form

$$\begin{tikzcd} & \mathrm{Bun}_G(\mathbb{B}) & \\ & \mathrm{Bun}_G(\Sigma \times_{\mathbb{D}_x^\times} \mathbb{D}) \arrow[u] \arrow[dl] \arrow[dr] & \\ \mathrm{Bun}_G(\Sigma) & & \mathrm{Bun}_G(\Sigma) \end{tikzcd}$$

and

$$\begin{tikzcd}
 & \mathrm{Hecke}^{\mathrm{spec}}_{G^\vee} & \\
 & \mathrm{Loc}_{G^\vee}(\Sigma \times_{\mathbb{D}_x^\times} \mathbb{D}) \arrow[u] \arrow[dl] \arrow[dr] & \\
\mathrm{Loc}_{G^\vee}(\Sigma) & & \mathrm{Loc}_{G^\vee}(\Sigma).
\end{tikzcd}$$

Therefore, associated to any object in either $\mathrm{D}_{G(\mathcal{O})}(\mathcal{G}\mathrm{r}_G)$ or $\mathrm{IndCoh}_{\mathcal{N}}(\mathrm{Hecke}^{\mathrm{spec}}_{G^\vee})$, we obtain an endo-functor of $\mathrm{Bun}_G(\Sigma)$ or $\mathrm{Loc}_{G^\vee}(\Sigma)$ by the natural convolution. This makes the geometric Langlands categories into modules for the monoidal geometric Satake categories for each point $x \in \Sigma$: we call these *local operators* at the point x.

With this structure, we can now say what was meant by the natural symmetries in conjecture 1.2.4. The derived geometric Satake correspondence 1.2.9 makes both sides into modules for $\mathrm{D}_{G(\mathcal{O})}(\mathcal{G}\mathrm{r}_G)$, say. We require that the geometric Langlands equivalence is an equivalence of module categories for this structure.

Remark 1.2.10. V. Lafforgue [**Laf09**] deduced a version of the geometric Langlands correspondence for $\Sigma = \mathbb{P}^1$ from the geometric Satake correspondence. He showed that the geometric Langlands categories for $\mathbb{P}^1$ are torsors for the geometric Satake categories. That is, if we choose an object in the geometric Langlands category, the action of the geometric Satake category on this object defines an equivalence of dg-categories. Lafforgue's result was not quite of the form of theorem 1.2.9, but instead used an equivalence between $\mathrm{QC}(\mathrm{Loc}_{G^\vee}(\mathbb{B}))$ and a subcategory of $D(\mathrm{Bun}_G(\mathbb{B}))$ of objects satisfying a support condition.

1.2.3. Arinkin and Gaitsgory's Conjecture

In this section we'll explain a corrected form of the geometric Langlands conjecture due to Arinkin and Gaitsgory [**AG12**], and using the derived geometric Satake correspondence described above.

Arinkin and Gaitsgory's correction is motivated by compatibility with an additional piece of structure that the geometric Langlands correspondence should possess: compatibility with *geometric Eisenstein series* functors.

Let $P \subseteq G$ be a parabolic subgroup, and let $L = P/U$ be the associated Levi subgroup, where U is the unipotent radical of P. One can construct a natural functor between the categories on the automorphic side of the Langlands correspondence for the groups L and G.

Definition 1.2.11. *The* geometric Eisenstein series *functor* $\mathrm{Eis}^P \colon D(\mathrm{Bun}_L(\Sigma)) \to D(\mathrm{Bun}_G(\Sigma))$ *is the pull-push operator $p_! q^*$ of D-modules associated to the diagram*

$$\begin{tikzcd} & \mathrm{Bun}_P(\Sigma) \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathrm{Bun}_L(\Sigma) & & \mathrm{Bun}_G(\Sigma). \end{tikzcd}$$

This functor has a Langlands dual: there is a parabolic $P^\vee \subseteq G^\vee$ whose associated Levi subgroup is $L^\vee$ allowing us to make the following definition.

Definition 1.2.12. *The* geometric Eisenstein series *functor* $\mathrm{Eis}^{P^\vee}_{\mathrm{spec}} \colon \mathrm{QC}(\mathrm{Loc}_{L^\vee}(\Sigma)) \to \mathrm{QC}(\mathrm{Loc}_{G^\vee}(\Sigma))$ *is the pull-push operator $p_{\mathrm{spec},*} q^!_{\mathrm{spec}}$ of sheaves associated to the diagram*

$$\begin{tikzcd} & \mathrm{Loc}_{P^\vee}(\Sigma) \arrow[dl, "q_{\mathrm{spec}}"'] \arrow[dr, "p_{\mathrm{spec}}"] & \\ \mathrm{Loc}_{L^\vee}(\Sigma) & & \mathrm{Loc}_{G^\vee}(\Sigma). \end{tikzcd}$$

An expected property of the geometric Langlands correspondence, motivated by an example of *Langlands functoriality* in number theory, is that it intertwines these two geometric Eistenstein series functors. However, there's an immediate problem with this condition: the automorphic Eisenstein functor sends compact objects to compact objects (this was proven by Drinfeld and

Gaitsgory [**DG11**]), but the spectral functor does not. Therefore there cannot exist an equivalence as in the best hope conjecture 1.2.4 that intertwines the Eisenstein series functors.

The first step we will therefore take, following Arinkin and Gaitsgory, is to replace quasi-coherent sheaves by "ind-coherent sheaves". The category $\mathrm{IndCoh}(X)$ of ind-coherent sheaves is the ind-completion of the category of coherent sheaves on X (as discussed extensively by Gaitsgory, e.g. in [**Gai11a**]). The main advantage to this step is that the structure of IndCoh, for example the pull and push functors associated to morphisms of stacks, is better behaved than that of QC on singular stacks, such as $\mathrm{Loc}_G(\Sigma)$. For instance, for IndCoh but not QC there is a !-pullback functor right adjoint to the *-pushforward which sends compact objects to compact objects. The motivation for this modification is extensively discussed in Gaitsgory's article on the subject [**Gai11a**], as well as by Arinkin and Gaitsgory.

This modification fixes the problem with compact generation, but we know by analysis of the example of $\mathbb{P}^1$, as demonstrated by V. Lafforgue and discussed in the previous section, that the category $\mathrm{IndCoh}(\mathrm{Loc}_{G^\vee}(\Sigma))$ is too large to be equivalent to $\mathrm{D}(\mathrm{Bun}_G(\Sigma))$. Arinkin and Gaitsgory used the theory of *singular support* to fix this, by finding a *minimal* subcategory of $\mathrm{IndCoh}(\mathrm{Loc}_{G^\vee}(\Sigma))$ where the geometric Eisenstein series functor is well-behaved. This subcategory is generated by the images of $\mathrm{QC}(\mathrm{Loc}_{L^\vee}(\Sigma))$ under the geometric Eisenstein series functors, using a suitable (not the naïve) embedding of QC in IndCoh [**AG12**, Corollary 13.3.9].

Theorem 1.2.13 ([**AG12**, Corollary 13.3.9])**.** *The category* $\mathrm{QC}(\mathrm{Loc}_{G^\vee}(\Sigma))$ *and the images of* $\mathrm{QC}(\mathrm{Loc}_{L^\vee}(\Sigma))$ *under the geometric Eisenstein series functors for proper parabolic subgroups together generate the category of ind-coherent sheaves on* $\mathrm{Loc}_{G^\vee}(\Sigma)$ *with* nilpotent singular support*.*

The *singular support* of a sheaf in $\mathrm{IndCoh}(X)$ is a conical Zariski-closed subset of the degree 0 part of the shifted cotangent bundle $\mathrm{Sing}(X) = H^0(T^*[-1]X)$ (the *scheme of singularities* of X). Arinkin and Gaitsgory introduce the notion of support for an arbitrary triangulated category acted on by a dg-algebra, based on work of Benson, Iyengar and Krause [**BIK08**]. The *singular support* of an ind-coherent sheaf is its support as a module for the even Hochschild cohomology of X, which receives a canonical map from $\mathcal{O}(\mathrm{Sing}(X))$.

In the case where $X = \mathrm{Loc}_{G^\vee}(\Sigma)$, the scheme of singularities can be described explicitly. Its set of closed points looks like

$$\mathrm{Sing}(\mathrm{Loc}_{G^\vee}(\Sigma)) = \{(P, \nabla, \phi) \colon (P, \nabla) \in \mathrm{Loc}_{G^\vee}(\Sigma) \text{ and } \phi \text{ is a flat section of } \mathfrak{g}_P\}.$$

This is, the subcategory of sheaves with *nilpotent singular support* is the full subcategory of sheaves whose singular support lies in the *global nilpotent cone*, i.e. the conical subspace of $\mathrm{Sing}(\mathrm{Loc}_{G^\vee}(\Sigma))$ where ϕ is required to be nilpotent.

Remark 1.2.14. This subcategory can be (partially) motivated number theoretically, which Arinkin and Gaitsgory describe in the introduction to their paper. In the Langlands reciprocity conjecture, as we mentioned in remark 1.2.2, one may have to choose not just a Galois representation, but also a homomorphism $\mathrm{SL}(2;\mathbb{C}) \to G^\vee$. In particular, this homomorphism corresponds to a choice of nilpotent element in the Lie algebra $\mathfrak{g}^\vee$ (the image of the standard element $e \in \mathfrak{sl}(2;\mathbb{C})$). Thus, when one develops a geometric analogue of the spectral side of the Langlands correspondence one guesses that one needs not only to choose a flat $G^\vee$-bundle, but also a nilpotent section. The category of ind-coherent sheaves automatically includes this data, as the singular support lies in the stack parameterising such data, but the condition of nilpotence has to be imposed by hand.

1.3. The Approach of Kapustin and Witten

In this section I'll review some of the main ideas in Kapustin and Witten's work [**KW06**] drawing an explicit bridge between S-duality for $N = 4$ Yang-Mills theory and the geometric Langlands correspondence. We can split the approach into several steps.

(1) First, $N = 4$ theories admit a family of *topological twists* indexed by points in the Riemann sphere $\mathbb{CP}^1$. S-duality interchanges these twisted theories by acting by the antipodal map,

not on the sphere indexing twists, but on a sphere indexing a rational combination of the twisting parameter and the coupling constant. Kapustin and Witten call this combined parameter the *canonical parameter*, denoted by Ψ. Twisted theories at the special values $\Psi = 0$ and ∞ are independent of the coupling constant.

(2) S-duality between twisted theories corresponding to these special values reduces to a duality of the 2-dimensional topological quantum field theories obtained by compactifying on a Riemann surface Σ. These theories coincide with the A- and B-models with target the Hitchin system, equipped with two different complex structures.

(3) Thus, S-duality exchanges A-branes and B-branes on the Hitchin moduli space, in these two structures, and for Langlands dual gauge groups. The categories of A- and B-branes can be identified with $D(\mathrm{Bun}_G(\Sigma))$ and $\mathrm{QC}(\mathrm{Loc}_{G^\vee}(\Sigma))$ respectively, so S-duality provides an equivalence of these categories.

(4) After choosing a point x in Σ, there is an action of line operators through x in the two twisted theories on the categories of boundary conditions, and S-duality intertwines these two actions. These actions can be identified with the action of Hecke operators and tensoring operators respectively.

From a mathematical point of view, there are many parts of Kapustin and Witten's work which are unsatisfying (beyond the fact that non-perturbative quantum field theories like $N = 4$ super Yang-Mills aren't currently objects with a complete mathematical model). We'll discuss some of these issues below.

1.3.1. The Family of Kapustin-Witten Twists

Witten introduced the notion of a topological quantum field theory, and of a *topological twist* of a classical or quantum field theory in the context of $N = 2$ gauge theories [**Wit88a**], though the physical perspective on twisting was algebraically formalized by Eguchi and Yang [**EY90**]. A theory is called *topological* if its observable quantities depend only on spacetime as a smooth manifold and not, for instance, on a choice of metric. This might, but needn't necessarily, arise because the action functional is manifestly metric independent. Witten's idea was that, on flat spacetime, topological field theories can be constructed from general theories whenever there's a symmetry Q such that all translation symmetries are of the form $[Q, Q']$ [6], by passing to the Q-cohomology of the observables. In this twisted theory all translations necessarily act trivially, making the theory necessarily metric independent, and such theories can be extended to metric independent theories on curved spacetimes. We'll discuss what it means to twist a classical field theory on $\mathbb{R}^n$, in a general, abstract setting, in chapter 8.

The main source of symmetries Q to twist by is *supersymmetry*. By analysing supersymmetry algebras one can identify exactly which supersymmetries have the desired property – that all translations are Q-exact – and therefore characterise the possible twists of supersymmetric classical field theories. Kapustin and Witten identified one particular family of twists in the $N = 4$ supersymmetry algebra indexed by $\mathbb{CP}^1$. That is, they identified two linearly independent symmetries Q_A and Q_B such that any linear combination $\lambda Q_A + \mu Q_B$ has the desired property. The twisted theory only depends on Q up to scale, so we can think of these twists as varying in a $\mathbb{CP}^1$. Denote the choice of twist by a parameter t in the projective line.

It will be convenient to combine t and the coupling constant for our super Yang-Mills theory. If $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{e}$ is the complexified coupling constant, we define the *canonical parameter* Ψ by

$$\Psi = \frac{\tau + \overline{\tau}}{2} + \frac{\tau - \overline{\tau}}{2}\left(\frac{t - t^{-1}}{t + t^{-1}}\right).$$

Kapustin and Witten argue that the twisted theory does not depend on t and τ independently, but only on this complexified coupling constant. Further, they argue that S-duality interchanges twisted gauge theories with antipodal values of the parameter Ψ, i.e. it exchanges theories with dual gauge groups, and parameters Ψ and $-\frac{1}{\Psi}$ [7].

[6] Witten expressed this in slightly different language: he talked about the stress-energy tensor being Q-exact.

[7] Strictly speaking this is only correct when G is simply laced. For general groups S-duality is expected to exchange parameters Ψ and $-\frac{1}{n_{\mathfrak{g}}\Psi}$ where $n_{\mathfrak{g}}$ is the lacing number: the ratio of the lengths of a longest and shortest root.

From the point of view of geometric Langlands, we'll be interested particularly in the special values $\Psi = 0$ and ∞. We observe, from the definition of Ψ, that when $\Psi = \infty$ then $t = \pm i$, and τ is unconstrained. Likewise, if $\Psi = 0$ then if we additionally suppose that $\theta = 0$ then $t = \pm 1$ and e is unconstrained. So, if we don't introduce a topological term into our theory these special twisted theories are independent of the coupling constant e.

1.3.2. Compactification to Two Dimensions

Kapustin and Witten's second idea is to consider the compactification of these twisted theories along a compact Riemann surface Σ, and identify the resulting two-dimensional theories as topologically twisted supersymmetric sigma-models with a certain, interesting, target. The idea of *compactification* is straightforward: given a fibration $p\colon X \to Y$ and a classical field theory on X, we obtain a classical field theory on Y whose phase space on an open set $U \subseteq Y$ is the phase space of the original theory on $p^{-1}(U)$ (we discuss this at the beginning of chapter 9). While the original $N = 4$ theory was only defined on flat space, after twisting it makes sense on a more general 4-manifold. In particular, one can define the twisted theories on $\Sigma \times U$, and compactify along the projection $\Sigma \times U \to U$.

The 2d theories one obtains are very interesting from the point of view of geometric representation theory. Kapustin and Witten argue that the compactification of the $\mathbb{CP}^1$ of topologically twisted $N = 4$ theories along this map is a $\mathbb{CP}^1$ of topologically twisted $N = (2, 2)$ supersymmetric sigma models whose target is $\mathcal{M}_G(\Sigma)$: the Hitchin moduli space. The twisted theories depend on a choice of Kähler structure on this moduli space. This Kähler structure also varies in the $\mathbb{CP}^1$ of available structures (since the Hitchin space is hyperkähler) as Ψ varies.

At the special points $\Psi = 0$ and ∞, the structures one chooses identify $\mathcal{M}_G(\Sigma)$ with the moduli space $T^* \operatorname{Bun}_G(\Sigma)$ of Higgs bundles on Σ, and the the moduli space $\operatorname{Loc}_G(\Sigma)$ of principal G bundles with flat connection respectively. The twisted theories one obtains at these special points are the A- and B-models respectively, first defined by Witten in 1988 [**Wit88b**]. Therefore after twisting and compactification, S-duality relates the A-model with target $T^* \operatorname{Bun}_G(\Sigma)$, and the B-model with target $\operatorname{Loc}_{G^\vee}(\Sigma)$. These moduli spaces are supposed to be *T-dual*, so S-duality has compactified to T-duality, or mirror symmetry (Strominger, Yau and Zaslow explained the relationship between these two concepts [**SYZ96**]) for the Hitchin fibration. Viewing geometric Langlands as a consequence of (homological) mirror symmetry for the Hitchin fibration pre-dates the work of Kapustin and Witten: it was first proposed by Hausel and Thaddeus [**HT03**], and has been further pursued by Donagi and Pantev [**DP12**] who proved a *classical limit* of the geometric Langlands correspondence from this point of view.

1.3.3. Branes and Line Operators

It remains to explain how to recover the categories that appear in the geometric Langlands correspondence from these 2d quantum field theories. Kapustin and Witten argue that these categories occur as categories of *branes* in the two theories, i.e. A-branes for the stack $T^* \operatorname{Bun}_G(\Sigma)$, and B-branes for $\operatorname{Loc}_{G^\vee}(\Sigma)$.

The notion of a *brane* in a quantum field theory is, heuristically, an enhancement of the naïve idea of a boundary condition. In a classical field theory, for the moment viewed as a system of differential equations, one can consider the set of possible boundary conditions for solutions to the equations of motion on a manifold with boundary (for 2d theories one generally considers a half-plane). In addition one can imagine coupling the equations of motion to an auxilliary theory defined on the boundary (see for instance [**HKK$^+$03**, Chapter 19] for a discussion of what this means).

Given a pair of branes $\mathcal{B}_1, \mathcal{B}_2$ in an n-dimensional quantum field theory associated to an $(n-2)$-manifold M, one can consider the Hilbert space $\mathcal{H}_{\mathcal{B}_1,\mathcal{B}_2}$ of states on $M \times [0,1]$ where one imposes the boundary conditions and couples to the boundary theories associated to B_1 and B_2 on the two boundary components. What's more, gluing together states on two such manifolds yields a linear map $\mathcal{H}_{\mathcal{B}_1,\mathcal{B}_2} \otimes \mathcal{H}_{\mathcal{B}_2,\mathcal{B}_3} \to \mathcal{H}_{\mathcal{B}_1,\mathcal{B}_3}$. The idea (due, in the physics literature, to Douglas [**Dou01**]) is that these Hilbert spaces and structure maps will combine to form a *linear category* of branes, where $\mathcal{H}_{\mathcal{B}_1,\mathcal{B}_2}$ is the space of morphisms from $\mathcal{B}_1$ to $\mathcal{B}_2$.

An alternative – more mathematically rigorous – perspective is provided by the notion of extended topological quantum field theory, as developed most notably by Lurie [**Lur09b**], following ideas of Lawrence, [**Law93**], Freed and Quinn [**FQ93, Fre93**] and Baez and Dolan [**BD95**]. In an extended n-dimensional TQFT, one associates a category, thought of as the category of branes along a specified boundary type, to an $n-2$-manifold, so in particular in two dimensions there is a unique category associated to the point. According to the cobordism hypothesis (see [**Lur09b**]) the entire 2d TQFT can be reconstructed from the data of the category associated to the point.

We do not expect that the two-dimensional theories described above quite fit into this mathematical framework – they do not appear to be truly topological in the mathematical sense. One expects that they might be something more like "topological conformal field theories" in the sense of Eguchi and Yang [**EY90**], and indeed Costello has demonstrated how to construct TCFTs from A- and B-models [**Cos07**]. However, for targets like $\mathrm{Loc}_{G^\vee}(\Sigma)$, which is neither smooth nor compact, even this structure is likely to be too strict. I'll discuss some approaches to axiomatizing these quantum theories, especially on the B-side, in chapter 12.

So, accepting for now that we won't be rigorously defining the categories of branes in our theories, what should we expect the categories to be, from a physical point of view? On the B-side there's a standard answer: the derived category of coherent sheaves (on the target of the sigma model). There are basic B-branes corresponding to Dirichlet boundary conditions – imposing the condition that the *support* of a field should land in a certain (complex, closed) subspace of the boundary. By coupling to a gauge theory on the boundary these objects also involve a choice of vector bundle over this subspace. The category of coherent sheaves is therefore viewed as a natural category of fundamental B-branes, containing in particular the pushforwards of vector bundles along closed embeddings. Douglas [**Dou01**] gave a physical motivation for the appearance of the derived category, as proposed by Kontsevich as part of his homological mirror symmetry conjecture. One problem with this approach from the mathematical point of view is that it's not clear how to distinguish categories that differ in their "functional analysis", by which I mean that it's hard to motivate the different between perfect, quasi-coherent, coherent and ind-coherent complexes of sheaves: one must simply make a choice, which may *not* be compatible with duality.

On the A-side, there are also standard objects corresponding to Dirichlet boundary conditions associated to *Lagrangian* subspaces of the target, coupled to a certain kind of gauge theory. The usual physical argument says that the Hilbert spaces in the A-model associated to a pair of branes are given by Floer cohomology groups, and therefore the category of A-branes is a version of the Fukaya category. In Kapustin and Witten's paper, they argue that in fact, in the case of the theory dimensionally reduced from 4-dimensions, one can identify the category of A-branes with the category of *D-modules* on $\mathrm{Bun}_G(\Sigma)$, roughly by identifying this category with the category of modules for a "canonical coisotropic brane", then identifying this brane with the sheaf of differential operators. Of course, just as on the B-side there are issues of functional analysis: one must choose exactly what category of modules one considers.

We observe that the equivalence that S-duality implies agrees with the equivalence we're led to by homological mirror symmetry. While the category on the A-side of mirror symmetry is a version of the Fukaya category, a theorem of Nadler-Zaslow [**NZ09**] and Nadler [**Nad09**] provides an equivalence between a version of the Fukaya category of the cotangent bundle T^*X of a real analytic manifold and the category of D-modules on X. As such, either S-duality for twisted $N = 4$ gauge theories or mirror symmetry for twisted supersymmetric sigma models leads to the geometric Langlands conjecture, at least in its "best hope" form.

Kapustin and Witten also explained how one of the most important pieces of structure in the geometric Langlands conjecture arise from 4d gauge theory: the action of the local categories – of Hecke and tensoring operators – at a point $x \in \Sigma$ as we discussed in section 1.2.2 above. They argued that this action corresponds to the action of *line operators* along a line orthogonal to Σ through x. Specifically, they argue that the category $\mathrm{Rep}(G^\vee)$ is equivalent to the category of Wilson operators in the B-twisted theory, and that the spherical Hecke category is equivalent to the category of 't Hooft operators in the A-twisted theory. It's worth noting that this description is natural from the point of view of 4d supersymmetric gauge theory, not from the point of view of the dimensionally reduced 2d theory; this is one of the main advantages of thinking of geometric Langlands as originating from S-duality of 4d gauge theories, rather than T-duality of 2d supersymmetric sigma models into the Hitchin system.

1.3.4. Remaining Mathematical Questions

Although Kapustin and Witten's argument is deep and inspiring, there are problems that must be addressed in any mathematical approach to the study of the geometric Langlands program that draws from it.

(1) We've already hinted at one problem, when we talked about "functional analysis" issues with the definition of the category of B-branes above. Kapustin and Witten's approach suggests an equivalence along the lines of the "best hope" conjecture 1.2.4. However, as we saw, the best hope conjecture is false, and to make a plausible conjecture one needs to modify the categories of sheaves one uses, for example in the way proposed by Arinkin and Gaitsgory that we discussed in section 1.2.3. Correcting this involves making appropriate functional analytic choices in the construction of the categories of branes, but it appears that one must make *different* choices on the two sides of S-duality. We discuss some ideas and work in progress towards this aim in chapter 13.

(2) The whole story discussed by Kapustin and Witten fundamentally lives in the world of *analytic* geometry, for instance their A-branes are D-modules on the moduli space of analytic G-bundles on a Riemann surface. For this reason, they don't distinguish between the moduli spaces of flat connections and of local systems, which are not complex

algebraically isomorphic. On the other hand, the categories in the geometric Langlands correspondence do depend on the algebraic structure of an algebraic curve, in a way that the character variety, or more generally the moduli stack of G-local systems, does not. As such, we can't expect to deduce the geometric Langlands correspondence as it's usually stated from Kapustin and Witten's arguments without some additional, algebraic, input.

We address this issue in part 2 of this thesis, in which we construct the classical moduli spaces of the Kapustin-Witten twisted $N = 4$ theories as derived algebraic stacks. In particular, we recover the moduli spaces that appear in the geometric Langlands program from a classical field theoretic construction, complete with their algebraic structures. We see an interesting phenomenon occur: the $N = 4$ theories naturally splits into $2 + 2$ (real) directions, namely the directions we dimensionally reduce along and the remaining directions. The theory we construct involves a choice, as to whether to think of these directions as complex algebraic (*de Rham* directions), or as real and topological (*Betti* directions). From a physical point of view it is most natural to keep all the directions de Rham. To recover the usual geometric Langlands correspondence it seems most natural to make two of the directions Betti (though we don't have a good physical motivation for this). One might try making all four directions Betti: we expect that the resulting story will correspond to the "Betti Langlands" correspondence recently proposed by Ben-Zvi–Nadler [**BZN16**] and Ben-Zvi–Brochier–Jordan [**BZBJ15**].

1.3.5. Extensions and Generalizations

The ideas of Kapustin and Witten's paper have been extended and elaborated upon in many ways since their 2006 paper, by both of the authors along with many other researchers. In this section we'll discuss a few of those elaborations, especially those that are particularly relevant either for the geometric Langlands program or for the topics presented in this thesis.

One extremely important topic which, so far, we've barely addressed, is the idea of *ramification*. From the geometric point of view ramification means that we allow not just ordinary G-bundles on Σ, but also those with some kind of singularity at a discrete set of points. The simplest possibility is that these singularities are simple poles, so that – for instance – flat sections have polynomial growth near a pole (we say that the bundles have *parabolic structure* at a finite set of points). This type of ramification is called *tame*, and is analogous to the condition in number theory that the Langlands parameter $\mathrm{Gal}(\overline{F}/F) \to G^\vee$ factors through the quotient of the Galois group by the wild inertia.

Gukov and Witten [**GW06**] described an extension of the S-duality approach to geometric Langlands to explain a form of the geometric Langlands correspondence with tame ramification. They discuss S-duality in the presence of certain *surface operators* along surfaces N transverse to the Riemann surface Σ. Classically, these operators are 't Hooft type operators, where one imposes that the scalar field corresponding to the Higgs field has a simple pole along N, with a fixed residue. In the dimensionally reduced quantum theory, this has the result of replacing the Hitchin system in the target of the supersymmetric sigma model with a different moduli space, with parabolic structure at a finite set of points. Gukov and Witten argue that S-duality interchanges such theories with different residues; the theory they discuss is reminicent of Simpson's non-abelian Hodge theory on curves with parabolic structure [**Sim90**]. Gukov and Witten also discuss local operators in the tamely ramified theory. This theory has been studied (on the A-side) in geometric representation theory by Gaitsgory [**Gai01**], who described the action of unramified local operators on tamely ramified local operators.

Witten also has a proposal which would allow the analysis of duality for more general "wild" ramification (where we allow poles of order greater than one) [**Wit08**]. Witten's idea is to generalise the above by introducing surface operators along 2-manifolds N where now one prescribes the Higgs field ϕ to have a singularity along N which has not just a fixed residue, but a fixed z^{-k} term for

$1 \leq k \leq n$, required to land in a fixed Cartan subalgebra of the gauge Lie algebra. These higher order residues have an interpretation in terms of Stokes data.

In a different direction, but more directly relevant for this thesis, Kapustin [**Kap06**] described an $N = 2$ version of the Kapustin-Witten argument discussed in this section. He investigated a particular twist which is self-dual and *holomorphic-topological*, i.e. topological in two directions and holomorphic in the other two. We use twists of this form in an essential way in our approach to the A-twist in section 10.3. We also discuss some ideas regarding the extension of this approach to geometric Langlands to $N = 2$ theories, and connections to Gaiotto duality for theories of class S in chapter 14.

Finally, we should mention the 2010 work of Kapustin, Setter and Vyas [**KSV10**], in which they give descriptions of the full categories of line and surface operators in the B-twisted $N = 4$ theory, and in the abelian A-twisted theory (see also the thesis [**Set13**] of K. Setter).

1.4. Outline of this Thesis

The first part of the thesis is concerned with abelian duality. We will define factorization algebras of quantum observables in generalized Maxwell theories, and prove the existence of a correspondence between dual theories that preserve the expectation values of observables. In chapter 3 we begin by describing the general formalism we use to construct the factorization algebra of quantum observables, starting from a sheaf of fields and an action functional. This formalism (based on that of Batalin-Vilkovisky) was developed by Costello and Gwilliam [**CG15**] [**Gwi12**] as a formulation of quantization techniques common in the physics literature (described for instance in Witten's expository note [**Wit90**]) as homological algebra. While we only use the theory for free theories here (their methods can also be used to study the perturbative parts of interacting theories), we do need to allow for spaces of fields that are neither linear nor connected, so we go over the formalism with a certain amount of care.

Having set up the abstract theory we construct the main objects of study in chapter 4: the factorization algebras of observables in *generalized Maxwell theories*. These are free quantum field theories whose fields model connections on higher principal torus bundles, whose action functional generalises the Yang-Mills action. These theories are closely related to simpler free theories – where the fields are just p-forms and the action is just the L^2 norm – by mapping a connection to its curvature. Observables of interest (such as Wilson and 't Hooft operators in abelian Yang-Mills) factor through the curvature, so in a sense "come from" this simpler theory. As such we can prove results about the expectation values of observables purely in the world of curvatures.

The theory of expectation values arises naturally from the factorization algebra formalism. In chapter 5 we describe how to abstractly define expectation values of gauge invariant observables by viewing the observables as living in a cochain complex with canonically trivialisable cohomology. To compute expectation values we use classical physical techniques: *Feynman diagrams* and *regularization*. Since the theory is free these methods are very well-behaved, and encode results about convergent sequences of finite-dimensional Gaussian integrals.

To conclude the first part, in chapter 6 we introduce abelian duality for observables in our theories as a *Fourier dual*. This also admits a diagrammatic description, but we prove that duality preserves expectation values using a Plancherel's theorem at each regularized level. It is worth remarking that there are three different "levels" of factorization algebra necessary to make sense of Fourier duality for observables in the generalized Maxwell theory. The dual itself is defined for a theory where the fields consist of all p-forms, but at this level duality doesn't preserve expectation values. An observable in this theory restricts to an observable in a theory where the fields consist of only *closed* p-forms and at this level duality *does* preserve expectation values. However in order to define a dual we now need to *choose* an extension from an observable acting on closed p-forms to an observable acting on all p-forms. We can phrase this in terms of a correspondence of factorization algebras: observables are called *incident* if they are the images of the same observable under a pair of restriction maps (to the closed p-form theory and its dual). The third level is that of the generalized Maxwell theory we're really interested in. On an open set we can construct a map from observables in the closed p-form theory to observables in the generalized Maxwell theory, which is an isomorphism of local sections of the factorization algebra if the open set is contractible, for instance. This gives us a way of defining a dual of a local observable in the original generalized Maxwell theory.

The second part of the thesis deals with $N = 4$ super Yang-Mills theory specifically. We describe a construction, now classically, of twists of $N = 4$ theories that include Kapustin and Witten's $\mathbb{P}^1$ of topological twists, and prove that the moduli spaces of solutions to the classical equations of motion in these theories recover the moduli spaces that appear in geometric Langlands. We begin in chapter 8 by setting up the formalism for twists of supersymmetric field theories that we'll use in the rest of the document. We describe the $N = 4$ supersymmetry algebra in four dimensions and its square-zero supercharges: the *holomorphic* supercharges for which half of the translations are exact, and the *topological* supercharges for which all the translations are exact. In particular, we'll describe the A and B topological supercharges whose corresponding twists are discussed by Kapustin and Witten. The A supercharge is approximated by a $\mathbb{C}^\times$ family of *holomorphic-topological* supercharges for which three translation directions are exact. After performing a holomorphic twist all of these supercharges admit descriptions as vector fields on a superspace of form $\mathbb{C}^{2|3}$, which we'll describe, allowing us to generalize the twisted theories to classical field theories on curved manifolds. The background on supersymmetry algebras which we refer to is reviewed in appendix A.

We proceed by defining classical field theories, both locally and globally, in the language of derived algebraic geometry. We discuss what it means to *twist* a classical field theory by an action of the supergroup $\mathbb{C}^\times \ltimes \Pi\mathbb{C}$: examples of such twisting data arise naturally from square-zero supercharges in a supersymmetric field theory. Twists of non-perturbative field theories are defined as one-parameter deformations that are compatible with the perturbative twists described by Costello [**Cos11a**] when we restrict to the tangent complex. There are natural constructions of twists using results of Gaitsgory and Rozenblyum that identify derived stacks with formal maps from a base derived stack $\mathcal{X}$ with Lie algebroids on $\mathcal{X}$.

In chapter 9 we review the main constructions of $N = 4$ supersymmetric gauge theories. We begin by introducing the language of compactification and (informally) dimensional reduction for classical field theories. The first construction is sketched at a lower level of rigor: dimensional reduction from $N = 1$ super Yang-Mills theory on $\mathbb{R}^{10}$. More rigorous is the construction by compactification from holomorphic Chern-Simons theory on $N = 4$ twistor space, although there are still subtleties stemming from the non-holomorphicity of the relevant twistor map. We review some background from twistor theory, and then prove that the linearized BV complex in holomorphic Chern-Simons yields the linearized BV complex of $N = 4$ anti-self-dual super Yang-Mills theory under compactification.

The main results of the second part appear in chapter 10, where we compute the holomorphic, B- and A-twists of $N = 4$ super Yang-Mills theory as derived stacks, beginning from the twistor space perspective. We find the following

Theorem 1.4.1. *The moduli space of germs of solutions to the equations of motion in the B-twist of $N = 4$ super Yang-Mills near $\Sigma \times S^1$, where Σ is a compact curve, is equivalent to*

$$\mathrm{EOM}_B(\Sigma \times S^1) \cong T^*(\mathcal{L}\,\mathrm{Loc}_G(\Sigma))$$

as a 0-shifted symplectic derived stack, where $\mathcal{L}\,\mathrm{Loc}_G(\Sigma)$ is the derived loop space of $\mathrm{Loc}_G(\Sigma)$.

Theorem 1.4.2. *The moduli space of germs of solutions to the equations of motion in the A-twist of $N = 4$ super Yang-Mills near $\Sigma \times S^1$, where Σ is a compact curve, is equivalent to*

$$\mathrm{EOM}_A(\Sigma \times S^1) \cong T^*((\mathcal{L}\,\mathrm{Bun}_G(\Sigma))_{\mathrm{dR}})$$

as a 0-shifted symplectic derived stack, where X_{dR} is the de Rham prestack of X.

We conclude with a discussion of work in progress, and future directions related to both of these themes. In section 11 we discuss extensions of the theorem of part 1 to include supersymmetric abelian gauge theories, in particular in dimension 4. Duality for such theories should have the abelian version of the geometric Langlands correspondence, sometimes called “geometric class field theory” as a consequence, provided one can properly understand the interaction between duality and the twisting parameters. In section 12 we discuss approaches to constructing topological quantum field theories from the topological classical field theories discussed in part 2, and in particular how one should obtain the Hochschild homologies of the categories on the two sides of the geometric Langlands correspondence as the Hilbert spaces of these dual theories. In section 13 we discuss current work in progress explaining the appearance of the singular support conditions in Arinkin and Gaitsgory from a physical point of view, by restricting attention to boundary conditions compatible with a particular choice of vacuum condition. Finally in section 14 we discuss some ideas, currently very speculative, regarding how Gaiotto duality for theories of class S should yield new conjecture in geometric representation theory.

1.5. Conventions

It will be necessary, mainly for the second part of this thesis, to fix some conventions and notation regarding derived algebraic geometry. Throughout this paper we’ll work with $(\infty,1)$-categories, where between two objects one has a topological space – or a simplicial set – of morphisms. We won’t use any model-dependent arguments, but to be concrete one may consider the formulation in terms of quasi-categories, which is most extensively developed by Lurie [**Lur09a**]. Henceforth, we will usually just say category when we mean an $(\infty,1)$-category, use the word functor to mean a functor of $(\infty,1)$-categories, and a limit for a limit in $(\infty,1)$-categories, and so on, unless otherwise specified. As is usual in the subject, there are a lot of technicalities which must be stated in order to make subtle arguments, most of which we will omit when possible for simplicity.

Also throughout the paper (from chapter 4 on) we'll work over the complex number field $\mathbb{C}$, although most of the formal arguments would proceed under more relaxed hypotheses.

We won't offer an extensive exposition for the framework of derived algebraic geometry that we'll use. This is justified partially because our arguments are mainly formal, not using any deep result of algebro-geometric content, and also because there are a few great references, for instance due to Gaitsgory [**Gai11b**] [**Gai11c**] and Toën [**Toë05**] [**Toë14**]. For the reader's convenience, in appendix B we provide a summary of the aspects of formal derived algebraic geometry that we take advantage of throughout.

- By a (super) *cdga* R we'll always mean a (super) commutative differential graded algebra over $\mathbb{C}$. We denote the category of such by cdga. We also consider the functor (of ordinary categories) $(-)^\natural\colon \mathrm{cdga} \to \mathrm{cdga}$ by $R \mapsto R^\natural$, where $R^\natural$ is the underlying graded commutative algebra obtained after forgetting the differential. We use cohomological grading with respect to which we introduce the full subcategory $\mathrm{cdga}^{\leq 0} \subset \mathrm{cdga}$ of cdgas whose cohomology is concentrated in non-positive degrees. We denote the opposite category to $\mathrm{cdga}^{\leq 0}$ by dAff, the category of affine derived schemes, considering an object $R \in \mathrm{cdga}^{\leq 0}$ as the ring of functions on the space $\operatorname{Spec} R$. In particular, a classical affine scheme is an affine derived scheme.
- By a *derived scheme*, we mean a ringed space $(X, \mathcal{O}_X)$ where $\mathcal{O}_X$ is a sheaf valued in $\mathrm{cdga}^{\leq 0}$ such that $(X, H^0(\mathcal{O}_X))$ is a classical scheme and $H^i(\mathcal{O}_X)$ is a quasicoherent sheaf over the scheme $(X, H^0(\mathcal{O}_X))$. By definition, a scheme or an affine derived scheme forms a derived scheme in an obvious manner and a derived scheme yields a classical scheme as its classical truncation $X^{\mathrm{cl}} := (X, H^0(\mathcal{O}_X))$. Note that an affine derived scheme could have been defined to be a derived scheme whose classical truncation is an affine scheme. We call the category of derived schemes dSch.
- A *prestack* $\mathcal{X}$ is a functor

$$\mathcal{X}\colon \mathrm{cdga}^{\leq 0} \to \mathrm{sSet},$$

where sSet is the category of simplicial sets. A *derived stack* is a prestack satisfying a descent condition with respect to the étale topology and we denote the category of derived stacks by dSt. In particular, any simplicial set provides a constant derived stack, and any derived scheme defines a derived stack by its functor of points. That is, if X is a derived scheme we define the corresponding derived stack whose R-points are the simplicial set whose i-simplices are $\mathrm{Hom}_{\mathrm{dSch}}(\mathrm{Spec}(R \otimes \Omega^\bullet_{\mathrm{alg}}(\Delta^i)), X)$, where $\Omega^\bullet_{\mathrm{alg}}(\Delta^i)$ is the ring of algebraic de Rham forms on the standard i-simplex Δ^i. The reduced part $\mathcal{X}^{\mathrm{red}}$ of a prestack $\mathcal{X}$ is the functor $\mathrm{cRing}^{\mathrm{red}} \to \mathrm{sSet}$ from reduced commutative rings obtained by the restriction along the functor $\mathrm{cRing}^{\mathrm{red}} \to \mathrm{cdga}^{\leq 0}$.

- A derived stack is a *derived* 0*-Artin stack* if it is an affine derived scheme. A derived stack is a *derived* n*-Artin stack* if it is realized as a colimit over a smooth groupoid of derived $(n-1)$-Artin stacks. A derived stack is called a *derived Artin stack* if it is a derived n-Artin stack for some n. For arguments involving shifted symplectic structures we'll need to restrict attention to derived Artin stacks which are locally of finite presentation. This ensures that the cotangent complex is perfect, hence dualizable.
- For any two derived stacks $\mathcal{X}, \mathcal{Y}$, one can define the *mapping stack* $\underline{\mathrm{Map}}(\mathcal{X}, \mathcal{Y})\colon \mathrm{dAff}^{\mathrm{op}} \to \mathrm{sSet}$ by $U \mapsto \mathrm{Map}_{\mathrm{dSt}}(\mathcal{X} \times U, Y)$. As an example of a mapping stack, one defines the k-shifted tangent space $T[k]\mathcal{X}$ of $\mathcal{X}$ to be $T[k]\mathcal{X} := \underline{\mathrm{Map}}(\mathrm{Spec}\,\mathbb{C}[\varepsilon], \mathcal{X})$, where ε is a parameter of cohomological degree $-k$ with $\varepsilon^2 = 0$. As another example, we define the loop space $\mathcal{L}X := \underline{\mathrm{Map}}(S^1_B, X)$, where the *Betti circle* S^1_B is the simplicial set S^1 understood as a derived stack.
- For a derived stack $\mathcal{X}$, one defines its category $\mathrm{QC}(\mathcal{X})$ of *quasicoherent sheaves* as the limit

$$\mathrm{QC}(\mathcal{X}) := \lim_{U \in (\mathrm{dAff}_{/\mathcal{X}})^{\mathrm{op}}} \mathrm{QC}(U)$$

over the opposite category $(\mathrm{dAff}_{/\mathcal{X}})^{\mathrm{op}}$ of the category of affine derived schemes over $\mathcal{X}$, where $\mathrm{QC}(\mathrm{Spec}\,R)$ is defined to be the category R-mod of dg modules over R. Similarly, one defines the category $\mathrm{Perf}(\mathcal{X})$ of perfect complexes using finitely generated dg-modules, and the category $\mathrm{Coh}(\mathcal{X})$ of coherent sheaves using bounded complexes with coherent cohomology. Finally, one defines the category $\mathrm{IndCoh}(\mathcal{X})$ of ind-coherent sheaves on $\mathcal{X}$ as the ind-completion of the category $\mathrm{Coh}(\mathcal{X})$.

- Every derived Artin stack $\mathcal{X}$ admits a *cotangent complex* $\mathbb{L}_{\mathcal{X}} \in \mathrm{QC}(\mathcal{X})$ [**TV08**][2.2.3.3]. Since $\mathcal{X}$ is assumed to be of locally finite presentation, $\mathbb{L}_{\mathcal{X}}$ is a perfect complex and hence dualizable, allowing one to define the *tangent complex* $\mathbb{T}_{\mathcal{X}} := \mathbb{L}_{\mathcal{X}}^*$. We can recover this tangent complex from the previously defined notion of the tangent space $T[k]\mathcal{X}$ [**TV08**][1.4.1.9]. The shifted tangent complex $\mathbb{T}_{\mathcal{X}}[k]$ is obtained as the limit of the objects $T[k]\mathcal{X} \times_{\mathcal{X}} U$ over all $U \in \mathrm{dAff}_{/\mathcal{X}}$, each of which is affine and finitely generated over U so lies in $\mathrm{Perf}(U)$, and therefore the limit defines an object in $\mathrm{Perf}(\mathcal{X})$. One can then define the k-shifted cotangent stack as the relative spectrum $T^*[k]\mathcal{X} := \underline{\mathrm{Spec}}_{\mathcal{X}}(\mathrm{Sym}(\mathbb{T}_{\mathcal{X}}[-k]))$.
- For a prestack $\mathcal{X}$, we define its *de Rham prestack* $\mathcal{X}_{\mathrm{dR}}$ to be the functor $R \mapsto \mathcal{X}(R^{\mathrm{red}})$. For a map $\mathcal{X} \to \mathcal{Y}$ of prestacks, we introduce the *formal completion* $\mathcal{Y}^\wedge_{\mathcal{X}}$ of $\mathcal{Y}$ along $\mathcal{X}$ defined by $\mathcal{Y}^\wedge_{\mathcal{X}} := \mathcal{X}_{\mathrm{dR}} \times_{\mathcal{Y}_{\mathrm{dR}}} \mathcal{Y}$. Note that one recovers the usual notion when $\mathcal{X} \to \mathcal{Y}$ is a closed immersion of ordinary schemes, justifying the name. If $\mathcal{Y} = \mathrm{pt}$, then one obtains the de Rham prestack $\mathcal{X}_{\mathrm{dR}}$. If $\mathcal{Y} = T^*[k]\mathcal{X}$ is the k-shifted cotangent stack, then we set $T^*_{\mathrm{form}}[k]\mathcal{X} := (T^*[k]\mathcal{X})^\wedge_{\mathcal{X}}$ for the formal neighborhood of $\mathcal{X}$ inside $T^*[k]\mathcal{X}$.
- A *inf-scheme* [**GRb**] is a prestack $\mathcal{X}$ whose reduced part $\mathcal{X}^{\mathrm{red}}$ is a reduced scheme, and which *admits deformation theory* in the sense of [**GRa**] (in particular derived Artin stacks locally of finite presentation admit deformation theory). A morphism $\mathcal{X} \to \mathcal{Y}$ of prestacks is *inf-schematic* if the base change $\mathcal{X} \times_{\mathcal{Y}} \mathrm{Spec}\, R$ by any affine derived scheme is an inf-scheme. For instance, any map of prestacks $\mathcal{X} \to \mathcal{Y}$ induces an inf-schematic map $\mathcal{X} \to \mathcal{Y}^\wedge_{\mathcal{X}}$.

Part 1

Abelian Duality

CHAPTER 2

Introduction and Motivation

The aim of this part is to give a detailed account of the phenomenon of *S-duality* in a very simple situation, as a duality between families of *free* quantum field theories, in a way allowing explicit understanding and computation of the local structure of the duality.

In the context of these generalized Maxwell theories, abelian duality refers to the following phenomenon for generalized Maxwell theories on an n-manifold X:

Theorem 2.0.1. *To every gauge invariant observable $\mathcal{O}$ in a degree k generalized Maxwell theory with gauge group T, we can produce a gauge invariant* dual *observable $\widetilde{\mathcal{O}}$ in the theory of degree $n-k$ and with gauge group $T^\vee$ such that the vacuum expectation values agree:*

$$\langle \mathcal{O} \rangle_{R,T} = \langle \widetilde{\mathcal{O}} \rangle_{\frac{1}{2R},T^\vee}.$$

We do not require X to be compact, so abelian duality makes sense for *local* observables. As such we describe abelian duality as a relationship between a pair of *factorization algebras* modelling the local quantum observables in the quantum field theories. The duality is compatible with the structure maps in the factorization algebras, but does not extend to a morphism of factorization algebras because of an obstruction to defining a dual observable on non-contractible open sets, where observables may not be determined purely by the curvature of a field. Instead duality arises as a *correspondence* of factorization algebras.

Despite the theories being free, duality of observables is still a non-trivial phenomenon to investigate. The dual of a gauge-invariant observable can have a qualitatively different nature to the original observable. For instance, in abelian Yang-Mills we verify that the dual of an abelian

Wilson operator (a holonomy operator around a loop) is an 't Hooft operator (corresponding to imposing a singularity condition on the fields around a loop).

CHAPTER 3

The BV Formalism for Free Field Theories

3.1. The Idea of the BV Formalism

The *Batalin-Vilkovisky formalism* (hereafter referred to as the BV formalism) gives a description of the moduli space of solutions to the equations of motion in a classical field theory that is particularly amenable to quantization. When we quantise following the BV recipe, we will see – in the case of a free theory – that the Feynman path integral description of the expectation values of observables naturally falls out. This quantization procedure admits an extension to interacting theories: see [**Cos11b**] and [**CG15**] for details.

We start with a rough description and motivation of the classical BV formalism. In its simplest form, a classical field theory consists of a space Φ of *fields* (often the global sections of a sheaf over a manifold X which we call *spacetime*), and a map $S\colon \Phi \to \mathbb{R}$: the *action functional*. The physical states in this classical system are supposed to be those states which extremise the action, i.e. the *critical locus* of S: the locus in Φ where $dS = 0$. This can be written as an intersection, specifically as

$$\mathrm{Crit}(S) = \Gamma_{dS} \cap X$$

where Γ_{dS} is the graph of dS in the cotangent bundle T^*X, and X is the zero section. The classical BV formalism gives a model for functions on the *derived* critical locus of S: that is, more than just forming the pullback in spaces given by this intersection, one forms a derived pullback in a homotopy category of spaces, and considers its ring of functions.

We can describe the ring of functions on the derived critical locus explicitly as a derived tensor product by resolving $\mathcal{O}(\Gamma_{dS})$ as an $\mathcal{O}(T^*X)$-module. We choose the Koszul resolution. Explicitly this says:

$$\mathcal{O}(\Gamma_{dS}) \sim \left(\cdots \longrightarrow \mathcal{O}(T^*X) \otimes_{\mathcal{O}(X)} \bigwedge^2 T_X \longrightarrow \mathcal{O}(T^*X) \otimes_{\mathcal{O}(X)} T_X \longrightarrow \mathcal{O}(T^*X) \right)$$

where T_X denote the module of vector fields, and the differential is extended from the map $\mathcal{O}(T^*X) \otimes_{\mathcal{O}(X)} T_X \to \mathcal{O}(T^*X)$ sending $f \otimes v$ to $fv - f\iota_{dS}(v)$ as a derivation with respect to the wedge product Taking this complex and tensoring with $\mathcal{O}(X)$ we find the complex $\mathrm{PV}(X)$ of *polyvector fields* on X, i.e. exterior powers of the ring of vector fields placed in non-positive degrees, with the differential $-\iota_{dS}$ from vector fields to functions extended to a differential on the whole complex as a derivation for the wedge product. This model for functions on the derived critical locus is the BV model for the algebra of *classical observables* in the Lagrangian field theory.

Now, we motivate the *quantum* BV formalism by means of a toy example. Let Φ be a finite-dimensional vector space, and let S be a quadratic form on this vector space. In this toy example, quantum field theory (in Euclidean signature) boils down to the computation of the Gaussian integrals

$$\begin{aligned} \langle \mathcal{O} \rangle &= \frac{\int_\Phi \mathcal{O}(\phi) e^{-S(\phi)/\hbar} d\phi}{\int_\Phi e^{-S(\phi)/\hbar} d\phi} \\ &= \frac{1}{Z} \int_\Phi \mathcal{O}(\phi) e^{-S(\phi)/\hbar} d\phi \end{aligned}$$

for polynomial functions $\mathcal{O}$ on Φ (writing Z for the normalizing factor $\int e^{-S(\phi)/\hbar} d\phi$). Here $\hbar$ is a positive real number and $d\phi$ is a volume form on Φ. Equivalently, we can think of this as computing the cohomology class of a top degree element $\mathcal{O} d\phi$ in a twisted de Rham complex: the complex of polynomial differential forms $\Omega^*_{\text{poly}}(\Phi)$ with differential $d - \frac{1}{\hbar}(\wedge dS)$. Finally, contracting with the top form $d\phi$ gives an isomorphism of graded vector spaces $\mathrm{PV}(\Phi)[-\dim \Phi] \to \Omega^*(\Phi)$, which becomes an isomorphism of *complexes* when one gives the space of polyvector fields the differential $D - \frac{1}{\hbar}\iota_{dS}$, where D is the *BV operator* given by transferring the exterior derivative along the map $\iota_{d\phi}$. Concretely, let $x_1, \ldots x_n$ form a basis for Φ, and let $\partial_1, \ldots, \partial_n$ be the corresponding basis on

$T_0\Phi$. Say $d\phi = dx_1 \wedge \cdots dx_n$. Then

$$D = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \frac{\partial}{\partial(\partial_i)}.$$

If Φ is infinite-dimensional then we can no longer immediately make sense of the original Gaussian integral (though we can compute it as a suitable limit), nor of "top" degree forms in the twisted de Rham complex. But the complex $\mathrm{PV}(\Phi)$ in degrees ≤ 0 and the differential $D - \frac{1}{\hbar}\iota_{dS}$ still makes sense, and we can still compute the cohomology class of a degree zero element, thus defining its *expectation value* directly. What's more, we see that, considering instead the isomorphic complex with differential $\hbar D - \iota_{dS}$, in the "classical" limit as $\hbar \to 0$ we recover the BV description of the algebra of classical observables. So this explicitly gives a *quantization* of that algebra. This quantization is no longer a dg-algebra: the Leibniz identity receives a correction term proportional to $\hbar$ coming from the classical Poisson bracket.

The general BV formalism therefore gives a model for the classical and quantum observables in a free Lagrangian field theory (i.e. a theory with quadratic action) following this outline. The classical observables are constructed as an algebra of polyvector fields with the differential ι_{dS}, and a quantization is produced by deforming this differential with a BV operator analogous to the one above. One builds this operator by – approximately – identifying Darboux co-ordinates on the (shifted) cotangent bundle to the fields and defining an operator using a formula like the one given above. An easier way to describe this is to use the Poisson bracket on functions on the shifted cotangent bundle (the so-called *antibracket*), and to extend this to a BV operator by an inductive formula on the degree.

3.2. Derived Spaces from Cochain Complexes

When defining a classical field theory on a manifold X, it's not completely clear what kind of object one should use to define the "sheaf of fields" of a Lagrangian field theory on X. One can build a classical field theory starting from a cochain complex of vector spaces (this is the approach used by Costello and Gwilliam [**CG15**]), but this excludes the most natural treatment of many interesting non-linear examples, such as sigma models. It also doesn't allow for discrete data in the space of fields, for instance the choice of G-bundle for fields in Yang-Mills theory. Witten's work on abelian duality [**Wit95a**] shows that this discrete invariant is necessary for the existence of duality phenomena: one sees theta functions in the partition function of an abelian gauge theory only after summing over all topological sectors.

We'll use the following definition of a classical Lagrangian field theory which, while not the most general definition possible, allows for discrete and non-linear pieces in the fields suitable for the free theories we will consider.

Definition 3.2.1. *A* classical Lagrangian field theory *on a manifold X consists of a sheaf Φ of simplicial abelian algebraic groups on X (the* fields*), and a morphism of sheaves of simplicial schemes*

$$\mathcal{L} \colon \Phi \to \mathrm{Dens}$$

(the action functional*) where* Dens *denotes the sheaf of densities on X, thought of as a sheaf of abelian groups.*

If X is compact then we can integrate global densities. The resulting map $S \colon \Phi(X) \to \mathbb{R}$ is called the *action functional*. Similarly, we can integrate local *compactly supported* sections of Φ to define a local action functional. In the next chapter I'll explain how to produce the factorization space of classical observables from this data, but first I'll explain some ways in which one might produce such a simplicial abelian group from more naïve data.

Remark 3.2.2. Note that while definition 3.2.1 mixes the world of manifolds with the world of schemes, the two types of geometry play very different roles. For other applications we could relax the condition that X is a smooth manifold, and instead allow it to be any site. In contrast, the condition that the local sections $\Phi(U)$ on an open set U form a (simplicial) scheme will be essential.

Remark 3.2.3. It's worth noting that we could also define a theory with fermions by allowing Φ to be instead a sheaf of simplicial abelian super algebraic groups. We won't need this generality for the examples of this paper.

Suppose that instead of a sheaf of simplicial algebraic groups, Φ is a sheaf of simplicial abelian groups on the site

$$\mathrm{Open}(X) \times \mathcal{Z}\mathrm{ar}_{/\mathbb{R}}$$

where $\mathcal{Z}\mathrm{ar}_{/\mathbb{R}}$ denotes the big Zariski site of $\mathbb{R}$-algebras (equivalently, Φ is a sheaf on X taking values in a category of sheaves of simplicial abelian groups). One can build a sheaf $\widetilde{\Phi}$ of simplicial algebraic groups on X by the following procedure. Fix an open set $U \subseteq X$. Consider all maps of simplicial sheaves from a scheme $\mathcal{X}$ to $\Phi(U)$, and take the homotopy colimit of this diagram in the category of simplicial schemes. Call the result $\widetilde{\Phi}(U)$, and observe that varying U defines a sheaf of simplicial abelian algebraic groups on X as required. We could make the same construction over $\mathbb{C}$ instead of $\mathbb{R}$ (or for that matter over a more general field), and indeed many of the examples we consider will be complex-valued theories.

Example 3.2.4. Let V be a sheaf of cochain complexes of vector spaces on X concentrated in non-positive degrees. Applying the Dold-Kan correspondence, we can think of this as a sheaf of simplicial vector spaces $\mathrm{DK}(V)$. This induces a sheaf of simplicial vector spaces on the product site $\mathrm{Open}(X) \times \mathcal{Z}\mathrm{ar}_{/\mathbb{R}}$ by setting

$$\Phi(U, R) = \mathrm{DK}(V(U) \otimes_{\mathbb{R}} R).$$

We observe that, for fixed U, this sheaf is already representable by the dg-scheme $\mathrm{Spec}(\mathrm{Sym}(V^\vee))$, and it's unnecessary to take a further homotopy colimit. That is to say the homotopy colimit over all dg-schemes mapping to Φ is trivial.

Example 3.2.5 (Yang-Mills Theory)**.** Let G be a compact connected Lie group. Define a sheaf Φ of simplicial groups on $\mathrm{Open}(X) \times \mathcal{Z}\mathrm{ar}_{/\mathbb{R}}$ modelling the stack of connections on principal G-bundles:

set

$$\Phi(U,R) = \bigoplus_P \left(\Omega^0(U;\mathfrak{g}_P) \otimes_{\mathbb{R}} R[1] \to \Omega^1(U;\mathfrak{g}_P) \otimes_{\mathbb{R}} R\right)$$

where the sum is over principal G bundles $P \to X$ up to isomorphism. This gives, under Dold-Kan, a simplicial sheaf for each U, thus a simplicial scheme upon forming a homotopy colimit. This simplicial scheme then gives a model for a moduli stack of connections on principal G-bundles, and allows one to define classical Yang-Mills theory. We'll generalise this construction in the case where G is abelian.

Example 3.2.6 (Higher Maxwell Theory)**.** The main objects of study in this paper are a family of theories generalizing Yang-Mills theory with gauge group $U(1)$, following the description of generalized Maxwell theory through ordinary differential cochains described in [**Fre00**]. The fields in this theory should describe "circle $(p-1)$-bundles with connection", for p some positive integer (the reason we use $p-1$ will become clear later: the "curvature" of a field in such a theory will be a p-form). The starting point for the description of these theories is the *smooth Deligne complex*: a sheaf of cochain complexes of abelian groups on a manifold X given by

$$\begin{aligned} \mathbb{Z}(p)_{\mathcal{D}}(U) \;&=\; \mathbb{Z}[p] \hookrightarrow \Omega^0(U)[p-1] \xrightarrow{d} \Omega^1(U)[p-2] \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{p-1}(U) \;. \\ &\cong\; C^\infty(U,\mathbb{R}/\mathbb{Z})[p-1] \xrightarrow{-id\log} \Omega^1(U)[p-2] \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{p-1}(U) \end{aligned}$$

or its complexified version

$$\mathbb{Z}(p)_{\mathcal{D},\mathbb{C}}(U) \;=\; C^\infty(U,\mathbb{C}^\times)[p-1] \xrightarrow{-id\log} \Omega^1(U;\mathbb{C})[p-2] \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{p-1}(U;\mathbb{C}) \;.$$

We can extend this complex to a sheaf of cochain complexes of abelian groups (or, by Dold-Kan, of simplicial abelian groups) whose $\mathbb{C}$-points are the complexified Deligne complex. Indeed, for an $\mathbb{C}$-algebra R, define a cochain complex

$$C^\infty(U;R^\times)[p-1] \longrightarrow \left(\Omega^1(U;\mathbb{C}) \otimes_{\mathbb{C}} R\right)[p-2] \longrightarrow \cdots \longrightarrow \Omega^{p-1}(U;\mathbb{C}) \otimes_{\mathbb{C}} R$$

where the first map is given by $-id\log$, and the latter maps simply by the de Rham differential in the first variable. The colimit procedure described above produces a sheaf of simplicial algebraic groups on U.

We might think of these complexified fields as higher principal $\mathbb{C}^\times$ bundles with connection, or as higher complex line bundles with connection.

3.3. The Action Functional and the Classical Factorization Space

Once we have the sheaf of fields Φ we can apply the classical BV procedure to build a model for the derived critical locus of the action that is amenable to quantization. The first step is to describe the shifted cotangent bundle $T^*[-1]\Phi$ as a derived stack, or – to avoid requiring too much formalism from derived algebraic geometry – describing the algebra of functions $\mathcal{O}(T^*[-1]\Phi)$ as a cochain complex. Fixing an open set $U \subseteq X$, the local fields $\Phi = \Phi(U)$ form a simplicial abelian algebraic group, so in particular the (shifted) cotangent bundle should be trivialisable. That is, we define

$$T^*[-1]\Phi \cong T_0^*\Phi[-1] \times \Phi$$

$$\text{so } \mathcal{O}(T^*[-1]\Phi) \cong \mathcal{O}(T_0^*\Phi[-1]) \otimes \mathcal{O}(\Phi)$$

where the first line is only heuristic (though it should be possible to make it precise with the machinery of derived algebraic geometry). The functions on the shifted cotangent fibre are easy to describe, because the cotangent fibre is a (dg-) *vector space*. For any Φ we can define $\mathcal{O}(T_0^*\Phi[-1]) \cong \mathrm{Sym}((T_0\Phi)[1])$, so it is only necessary to describe the complex $T_0\Phi$. For instance, if Φ is an abelian variety we have

$$\mathcal{O}(T^*[-1]\Phi) \cong \mathrm{Sym}((\Phi \otimes_{\mathbb{Z}} \mathbb{R})[1]) \otimes \mathcal{O}(\Phi).$$

We'll describe $T_0\Phi$ in the examples that we're interested in this paper in chapter 4. This calculation gives the ring $\mathcal{O}(T^*[-1]\Phi)$ a natural interpretation as a ring of *polyvector fields* on Φ. Indeed, the dg-vector space $T_0\Phi \otimes \mathcal{O}(\Phi)$ precisely describes vector fields on Φ. Placing this space in degree -1 and taking graded symmetric powers (i.e. alternating powers, by the usual sign rule), we produce the algebra of polyvector fields on Φ.

Remark 3.3.1. All algebraic constructions with topological vector spaces in this paper take place in the context of nuclear Frechét spaces (or cochain complexes thereof). For instance, the dual space $V^\vee$ of a vector space V is always the continuous dual equipped with the strong topology, and the tensor product is the completed projective tensor product. Likewise, while we identify the ring of algebraic functions on a space V with the symmetric algebra $\mathrm{Sym}(V^\vee)$, our constructions will all extend to the *completed* symmetric algebra.

Now we introduce the action. Recall that as well as the fields, our Lagrangian field theory data included a map of sheaves from Φ to the sheaf of densities on X. While it is not, in general, possible to integrate the resulting local densities, it *is* possible to define the first variation of this "local action functional". One defines a compactly supported 1-form dS on Φ, i.e. an element of $T_0^*\Phi_c \otimes \mathcal{O}(\Phi) \cong \mathrm{Hom}(T_0\Phi_c, \mathcal{O}(\Phi))$ where $T_0\Phi_c$ denotes the *compactly supported* tangent vectors: that is, $T_0\Phi$ describes a sheaf of cochain complexes on X, and we consider the compactly supported sections on the set U we have in mind. Then dS is the linear map sending a compactly supported tangent direction v to the functional

$$\phi \mapsto \int_U \mathcal{L}(L_v(\phi))$$

where $L_v(\phi)$ denotes the Lie derivative of ϕ along the vector field v. The compact support condition ensures that $\mathcal{L}(L_v(\phi))$ is a compactly supported density, so the integral is well-defined.

The action functional S describes a modified version of the shifted cotangent bundle by modifying the internal differential on the functions. After identifying the functions on the shifted cotangent bundle with polyvector fields the 1-form dS naturally defines a degree one linear operator, namely the interior product

$$\iota_{dS} \colon \mathcal{O}(T^*[-1]\Phi) \to \mathcal{O}(T^*[-1]\Phi).$$

More explicitly, the operator is extended as a derivation from the operator $T_0\Phi_c \otimes \mathcal{O}(\Phi) \to \mathcal{O}(\Phi)$ given by pairing a vector field with the 1-form $dS \in T_0^*\Phi_c \otimes \mathcal{O}(\Phi)$.

Remark 3.3.2. If the fields Φ are described by a cochain complex rather than a more general simplicial or dg-scheme then we can describe the complex $\mathcal{O}(T^*[-1]\Phi)$ of polyvector fields and the classical differential ι_{dS} even more directly. The global functions now form a symmetric algebra

$$\mathcal{O}(T^*[-1]\Phi) \cong \mathrm{Sym}(\Phi^\vee \oplus \Phi[1])$$

generated by linear functions and linear vector fields on Φ. We'll see this later in some examples of free theories, where the action is encoded by a linear operator $\Phi \to \Phi^\vee$ of degree one.

The above discussion took place for a fixed open set $U \subseteq X$. Let's now describe the relationship between the classical observables on different open sets. We describe locality using the machinery of *factorization algebras*, as developed in [**CG15**]. We recall the basic definitions.

Definition 3.3.3. *A* prefactorization algebra $\mathcal{F}$ *on a space* X *taking values in a symmetric monoidal category* $\mathcal{C}$ *with small colimits is a* $\mathcal{C}$*-valued precosheaf on* X *equipped with* S_k*-equivariant isomorphisms*

$$\mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_k) \to \mathcal{F}(U_1 \sqcup \cdots \sqcup U_k)$$

for every collection $U_1, \ldots, U_k \subseteq X$ *of disjoint open sets.*

An open cover $\{U_i\}$ *of a space* X *is called* factorizing *if for every finite subset* $\{x_i, \ldots, x_\ell\}$ *of* U *there is a collection* $U_{i_1}, \ldots, U_{i_\ell}$ *of* pairwise disjoint *sets in the cover such that* $\{x_i, \ldots, x_\ell\} \subseteq U_{i_1} \cup \cdots \cup U_{i_\ell}$.

Given an open cover of X and a precosheaf $\mathcal{F}$ on X we can construct a simplicial object in $\mathcal{C}$ called the Čech complex *of $\mathcal{F}$, defined as*

$$\check{C}(U, \mathcal{F}) = \bigoplus_{k=1}^{\infty} \left(\bigoplus_{U_{i_1}, \dots, U_{i_k}} F(U_{i_1} \cap \dots \cap U_{i_k})[k-1] \right)$$

with the usual Čech maps

A prefactorization algebra is a factorization algebra *if for every open set $U \subseteq M$ and every factorizing cover $\{U_i\}$ of U, the natural map* $\operatorname{colim} \check{C}(U, \mathcal{F}) \to \mathcal{F}(U)$ *is an isomorphism in $\mathcal{C}$.*

Remark 3.3.4. The examples we'll discuss will all be prefactorization algebras taking values in the *homotopy category* of cochain complexes, so isomorphisms are quasi-isomorphisms of complexes.

Definition 3.3.5. *The* factorization algebra of classical observables *associated to the classical Lagrangian theory $(\Phi, \mathcal{L})$ is the factorization algebra* $\mathrm{Obs}^{\mathrm{cl}}_{\Phi}(U)$ *valued in cochain complexes whose sections on U are given by the complex $\mathcal{O}(T^*[-1]\Phi)$ with differential given by the internal differential d_Φ on $\mathcal{O}(\Phi)$ plus the classical differential $-\iota_{dS}$.*

The fact that this forms a factorization, rather than just a prefactorization, is theorem 4.5.1 in [**Gwi12**]. It follows from the fact that Φ forms a sheaf, so the global functions $\mathcal{O}(\Phi)$ forms a cosheaf.

Finally, we need to address the *Poisson structure* on the classical observables. This is definition 2.1.3 in [**Gwi12**].

Definition 3.3.6. *A* P_0-factorization algebra *is a factorization algebra $\mathcal{F}$ valued in cochain complexes, such that each $\mathcal{F}(U)$ is equipped with a commutative product and a degree 1 antisymmetric map $\{,\} \colon \mathcal{F}(U) \otimes \mathcal{F}(U) \to \mathcal{F}(U)$ which is a biderivation for the product, satisfies the identity*

$$d\{x, y\} = \{dx, y\} + (-1)^{|x|}\{x, dy\}$$

and is compatible with the prefactorization structure.

We expect such a structure on the classical observables coming from the *shifted symplectic* structure on the shifted cotangent bundle, but we can recall a familiar concrete description (the well-known Schouten bracket on polyvector fields). Firstly, there's an evaluation map

$$T_0\Phi \otimes \mathcal{O}(\Phi) \to \mathcal{O}(\Phi)$$

taking an element $v \otimes f$ to $df(v) \in \mathcal{O}(\Phi)$ (thinking of the tangent vector $v \in T_0\Phi$ as a constant vector field on Φ). We use this to define the Schouten bracket in low polyvector field degrees:

$$\{1 \otimes f_1, 1 \otimes f_2\} = 0$$

$$\{v_1 \otimes f_1, 1 \otimes f_2\} = 1 \otimes df_2(v_1) \cdot f_1$$

$$\{v_1 \otimes f_1, v_2 \otimes f_2\} = v_2 \otimes df_2(v_1) \cdot f_1 - v_1 \otimes df_1(v_2) \cdot f_2$$

This extends uniquely to an antisymmetric degree 1 pairing on the whole algebra of polyvector fields as a biderivation with respect to, as usual, the wedge product of polyvector fields.

3.4. Quantization of Free Factorization Algebras

From now on we will restrict attention to *free* field theories, where we can use the intuitive, non-perturbative notion of BV quantization described in chapter 3.1. Informally, A classical Lagrangian field theory is free if the action functional is *quadratic*, so the derivative of the action functional is *linear*.

Definition 3.4.1. *A classical Lagrangian field theory is called* free *if the classical differential ι_{dS} increases polynomial degrees by one. That is, if we filter $\mathcal{O}(\Phi)$ by polynomial degree and call the k^{th} filtered piece $F^k\mathcal{O}(\Phi)$, the operator ι_{dS} raises degree by one:*

$$\iota_{dS} \colon \mathrm{Sym}^i(T_0\Phi[1]) \otimes F^j\mathcal{O}(\Phi) \to \mathrm{Sym}^{i-1}(T_0\Phi[1]) \otimes F^{j+1}\mathcal{O}(\Phi).$$

Now, Let $(\Phi, \mathcal{L})$ be a free classical theory, and let $\mathrm{Obs}^{\mathrm{cl}}(\Phi(U))$ be the complex of classical observables on an open set U. We'll quantise the local observables by adding a new term to the differential on this complex: the *BV operator*, which we'll denote by D. This operator is built from the P_0-algebra structure on the classical observables, following the method of deformation quantization for free theories described in [**Gwi12**].

Define the *BV operator* $D \colon \mathcal{O}(T^*[-1]\Phi(U)) \to \mathcal{O}(T^*[-1]\Phi(U))$ by extending an operator built from the Poisson bracket. Set D to be zero on $\mathcal{O}(\Phi(U))$, and to be given by the Poisson bracket in degree 1: $D = \{,\} \colon T_0\Phi(U) \otimes \mathcal{O}(\Phi(U)) \to \mathcal{O}(\Phi(U))$, i.e. the map we described above as "evaluation". We can then extend this to an operator on the whole complex of classical observables according to the formula

$$D(\phi \cdot \psi) = D(\phi) \cdot \psi + (-1)^{|\phi|}\phi \cdot D(\psi) + \{\phi, \psi\}.$$

An algebra with a differential D and Poisson bracket $\{,\}$ satisfying a formula like this is called a *Beilinson-Drinfeld algebra*, or *BD algebra*: Beilinson and Drinfeld constructed in [**BD04**] a family of operads over the formal disk whose fibre at the origin is the P_0 operad. The BD algebra structure given here is a description of an algebra for a generic fibre of the analogous family defined over all of $\mathbb{C}$ rather than just a formal neighbourhood of the origin.

Example 3.4.2. If Φ is a complex of vector spaces, so the classical observables are given by $\mathrm{Sym}(\Phi[1] \oplus \Phi^\vee)$, then we can construct the BV operator even more directly. The Poisson bracket, restricted to $\mathrm{Sym}^{\leq 2}$ is given by the evaluation map $\Phi \otimes \Phi^\vee \to \mathbb{C}$ from Sym^2 to Sym^0, and zero otherwise. This extends uniquely to a degree 1 operator on the whole complex of classical observables, lowering Sym degree by 2 as a BD structure, as above.

Equipped with this operator we can now define the quantum observables.

Definition 3.4.3. *The factorization algebra of* quantum observables *for the free theory* $(\Phi, \mathcal{L})$ *is the factorization algebra with local sections on* U *given by the cochain complex*

$$\mathrm{Obs}^{\mathrm{q}}(U) = (\mathcal{O}(T^*[-1]\Phi(U)), d_\Phi - \iota_{dS} + D)$$

where d_Φ *is the differential coming from the internal differential on* Φ*,* ι_{dS} *is the classical BV differential, and* D *is the quantum BV differential as defined above.*

That this procedure really does define a factorization algebra is proved in [**Gwi12**].

Remark 3.4.4. A more standard thing to write would consider a differential $d_\Phi - \iota_{dS} + \hbar D$, and have the BV operator D defined by a formula like

$$D(\phi \cdot \psi) = D(\phi) \cdot \psi + (-1)^{|\phi|}\phi \cdot D(\psi) + \hbar\{\phi, \psi\}$$

where we'd adjoined a formal parameter $\hbar$ to the algebra of classical observables. We'd then obtain a module flat over $\mathbb{R}[[\hbar]]$ which recovered the classical observables upon setting $\hbar$ to zero. I haven't done this because when working exclusively with free theories it's possible to work completely *non-perturbatively*, i.e. to evaluate at a non-zero value of $\hbar$. The duality phenomena I'm investigating are only visible non-perturbatively (taking into account all topological sectors) so this is necessary, however we lose the ability to consider the quantum observables in any theory which is not free.

3.5. Smearing Observables

In order to describe expectation values in chapter 5 we need to identify a dense subfactorization algebra of the classical observables which are well-behaved. To do so, we restrict to a special setting: where the algebra of functions $\mathcal{O}(\Phi(U))$ on the fields is a *free commutative dg algebra* on a cochain complex of vector spaces. This includes for instance examples where $\Phi(U)$ is modelled by a complex of vector spaces, or the product of a complex of vector spaces and a compact abelian variety. So we can write

$$\mathcal{O}(\Phi(U)) = \mathrm{Sym}(V)$$

for some complex V, hence

$$\mathcal{O}(T^*[-1]\Phi(U)) = \mathrm{Sym}(T_0\Phi(U)[1] \oplus V).$$

Observe that in fact $V \leq T_0^*\Phi$. Indeed, $\mathrm{Sym}(T_0^*\Phi) = \mathcal{O}(T_0\Phi)$ which admits an injective map from $\mathcal{O}(\Phi)$ (taking the ∞-jet of a global algebraic function).

Now, suppose further that the theory is free. Then as an operator on this symmetric algebra the classical differential ι_{dS} is non-increasing in Sym-degree. It can be made to *preserve* Sym-degree by completing the square (up to a constant factor), and therefore it can be described as the extension of its *linear* part, which we'll denote

$$Q\colon T_0\Phi(U) \to V$$

to the whole complex, as a derivation. Thus the classical observables are themselved given by the free cdga on a complex, namely the complex $T_0\Phi(U)[1] \overset{Q}{\to} V$. Let's analyse this complex.

Example 3.5.1. If $\Phi(U)$ is a cochain complex of vector spaces (in degrees ≤ 0), then the complex of classical observables is free on the cochain complex

$$\Phi(U)[1] \overset{Q}{\to} \Phi(U)^\vee.$$

The *smeared* or *smooth* classical observables form a dense subalgebra of $\mathrm{Obs}^{\mathrm{cl}}(U)$, defined using the data of an *invariant pairing* on the fields. That is, we require an antisymmetric map of sheaves of cohomological degree -1

$$\langle -, - \rangle\colon \Phi \otimes \Phi \to \mathrm{Dens}$$

which is non-degenerate as a pairing on the stalks. This pairing defines a non-degenerate pairing $\Phi_c(U)\otimes\Phi_c(U) \to \mathbb{C}$ by integration over U, and hence an embedding $T_0\Phi(U)_c \hookrightarrow T_0^*\Phi(U)$, sending a compactly supported vector field to the functional "pair with that vector field". Since $V \subseteq T_0^*\Phi(U)$ naturally, we can define V^{sm} to be the intersection $V \cap T_0\Phi(U)_c \leq T_0^*\Phi(U)$ (or more precisely, we form the pullback in the category of cochain complexes of vector spaces).

Definition 3.5.2. *The classical* smeared *or* smooth *observables on an open set U are defined to be the free cdga*

$$\mathrm{Obs}_\Phi^{\mathrm{sm,cl}}(U) = \mathrm{Sym}(T_0\Phi_c(U)[1] \overset{Q}{\to} V^{\mathrm{sm}}).$$

If $\mathcal{O}(\Phi(U))$ is a free cdga for every open set $U \subseteq X$ then the smooth observables define a subfactorization algebra of $\mathrm{Obs}_\Phi^{\mathrm{cl}}$, dense in every degree.

Remark 3.5.3. Again, consider the situation where $\Phi(U)$ is a cochain complex of vector spaces. Then the smooth observables include into all classical observables induced by the inclusion of complexes coming from the pairing

$$\left(\Phi(U)_c[1] \overset{Q}{\to} \Phi(U)\right) \hookrightarrow \left(\Phi_c(U)[1] \overset{Q}{\to} \Phi(U)^\vee\right).$$

Definition 3.5.4. *We call a free classical field theory equipped with an invariant pairing* elliptic *if $\mathrm{Obs}^{\mathrm{cl}}(U)$ is a free cdga for every U, and the resulting complex of* linear *smeared observables*

$$\mathcal{E}(X) = T_0\Phi_c(X)[1] \overset{Q}{\to} V^{\mathrm{sm}}$$

on the total space of the manifold X is an elliptic complex.

This is a fairly mild assumption that is satisfied in most realistic free physical theories on compact orientable manifolds (and the theory admits an extension to describe classical observables in interacting theories also, as described in [**CG15**] and [**Cos11a**]). We describe some free examples (which all admit interacting extensions described by Costello and Gwilliam).

Example 3.5.5. (1) **Scalar field theories**

Let $\Phi(U) = C^\infty(U)$, the sheaf of smooth functions on a compact manifold, and let $\mathcal{L}$ be the Lagrangian density for a free scalar field of mass m, namely

$$\mathcal{L}(\phi) = d\phi \wedge *d\phi - m^2\phi \wedge *\phi.$$

The smeared classical observables in this theory are generated by the elliptic complex $C^\infty(X)[1] \overset{Q}{\to} C^\infty(X)$ where $Q = \Delta - m^2$.

(2) **Abelian Chern-Simons theory**

Let T be a torus, let $P \to X^3$ be a principal T bundle on a compact 3-manifold, and let $\Phi(U)$ be the sheaf describing connections on P (where we trivialise the torsor by choosing a fixed reference connection). Chern-Simons theory on this fixed bundle is described by the complex $\Omega^*(U;\mathfrak{t})$, where $\mathfrak{t}$ is the (abelian) Lie algebra of T. Taking a Dolbeault complex on a complex manifold instead of a de Rham complex describes instead *holomorphic* Chern-Simons theory (and indeed, in this language it makes sense to define Chern-Simons theory in any dimension as the theory whose algebra of classical observables are built from this complex).

(3) **Abelian Yang-Mills theory**

Let T be a torus, let $P \to X^4$ be a principal T bundle on a compact Riemannian 4-manifold, and let $\Phi(U)$ be the sheaf describing connections on P as above. Yang-Mills theory on this fixed bundle is described by the shifted cotangent to the Atiyah-Singer-Donaldson complex describing anti-self-dual connections on P, that is, the complex

$$\begin{tikzcd}
\Omega^0(U;\mathfrak{t}) \arrow[r, "d"] & \Omega^1(U;\mathfrak{t}) \arrow[r, "d_+"] & \Omega^2_+(U;\mathfrak{t}) & \\
& \Omega^2_+(U;\mathfrak{t}) \arrow[ur, "1"] \arrow[r, "d"] & \Omega^3(U;\mathfrak{t}) \arrow[r, "d"] & \Omega^4(U;\mathfrak{t})
\end{tikzcd}$$

in degrees -1 to 2. Here $\Omega^2_+(U;\mathfrak{t})$ denotes the space of self-dual 2-forms and d_+ is the composition of d with projection onto this space.

We'll see shortly examples of *p-form* theories that also fit into this framework.

Definition 3.5.6. *The* quantum *smeared observables on an open set U are given by the cochain complex with the same underlying graded abelian group as the classical smeared observables, but with differential* $\mathrm{Sym}(Q) + D$*, where D is the* smeared BV operator *extended from the operator* $\mathrm{Sym}^2(\mathcal{E}(U)) \to \mathrm{Sym}^0(\mathcal{E}(U))$ *given by the invariant pairing restricted to $\mathcal{E}(U)$ according to the BD product formula*

$$D(\phi \cdot \psi) = D(\phi) \cdot \psi + (-1)^{|\phi|}\phi \cdot D(\psi) + \{\phi, \psi\}.$$

The quantum smeared observables embed into the whole complex of quantum observables as a subcomplex, dense in each degree. To see this one just needs to check that the quantum BV operators commute with the inclusion, which follows directly from the definitions.

CHAPTER 4

Generalized Maxwell Theories as Factorization Algebras

Having described the general formalism, I'll explain the specific theories which we'll be studying: the *generalized Maxwell theories*, a family of theories including as its simplest two examples sigma models with target a torus and abelian pure Yang-Mills theories.

4.1. Generalized Maxwell Theories

4.1.1. The Classical Factorization Algebra

We already discussed the fields in generalized Maxwell theories in example 3.2.6 of the previous chapter: we built a sheaf of simplicial abelian algebraic groups Φ_p from an ordinary differential cohomology complex $\mathbb{Z}(p)_\mathcal{D}$. These are the fields in the theories we'll be interested in. The *action* on a compact manifold is defined as the L^2-norm of the *curvature* of a field.

Definition 4.1.1. *The* curvature map *is the map of sheaves of simplicial algebraic groups $F \colon \Phi_p \to \Omega^p_{cl}$ induced from the exterior derivative as a map of sheaves of cochain complexes $\mathbb{Z}(p)_\mathcal{D} \to \Omega^p_{cl}$ by the universal property of the homotopy colimit.*

With this in mind, we define the Lagrangian density of a local field $\phi \in \Phi_p(U)$ to be

$$\mathcal{L}(\phi) = R^2 F(\phi) \wedge *F(\phi)$$

for R a positive real number, using the Riemannian metric on U. If this density is integrable, the resulting integral is just the L^2-norm of $F(\phi)$. We call the number R here a *coupling constant*, and think of it as the radius of the gauge group circle, pre complexification. We could also have produced this scaling by redefining the fields: in our Deligne complex we might have included the lattice $2\pi R\mathbb{Z}$ instead of $\mathbb{Z}$, yielding a circle of radius R in the cohomology.

Remark 4.1.2. We can generalise this setting from a circle (or, in the complexified story, $\mathbb{C}^\times$) to a higher rank torus $T \cong V/L$, where L is a full rank lattice in a real normed vector space V. Let $V_\mathbb{C} = V \otimes_\mathbb{R} \mathbb{C}$ and $T_\mathbb{C}$ denote the complexifications of V and T. The fields in the theory described above generalise immediately by taking algebraic $T_\mathbb{C}$ valued functions mapping into the algebraic de Rham complex with values in $V_\mathbb{C}$. There is now a curvature map taking values in $\Omega^p_{cl} \otimes V_\mathbb{C}$, and we define a Lagrangian density functional by

$$\mathcal{L}(\phi) = \|F(\phi) \wedge *F(\phi)\|^2$$

where $\| - \|$ is the norm on $V_\mathbb{C}$.

In this description, the coupling constants arise from the choice of lattice $L \leq V$: for instance we obtain the theory with coupling constant R above by choosing $2\pi R\mathbb{Z} \leq \mathbb{R}$.

The construction of the previous chapter yields a classical factorization algebra of observables from the above data. I can describe the local sections fairly concretely. We start from the algebra of polyvector fields

$$PV(\Phi_p(U)) = \mathrm{Sym}(T_0\Phi_p(U)) \otimes \mathcal{O}(\Phi_p(U)).$$

We take each term individually. Functions on the Deligne complex $\Phi_p(U)$ were given as an abstract homotopy colimit so aren't easy to describe directly. However, the tangent fibre $T_0\Phi_p(U)$ is much more accessible in that it comes from an actual complex of vector spaces. We find

$$T_0\Phi_p(U) = \Omega^0(U;\mathbb{C})[p-1] \to \Omega^1(U;\mathbb{C})[p-2] \to \cdots \to \Omega^{p-1}(U;\mathbb{C}),$$

i.e. the shifted truncated de Rham complex, or more precisely the associated simplicial algebraic group, as in example 3.2.4. The Poincaré lemma tell us that this is quasi-isomorphic to the sheaf of closed p-forms via the exterior derivative, as a sheaf of cochain complexes.

Now, we describe the classical differential coming from the action functional (or its first variation dS). This is an operator $T_0\Phi_p(U)_c \to \mathcal{O}(\Phi_p(U))$, or using the above description, an operator $\Omega_c^{p-1}(U;\mathbb{C}) \to \mathcal{O}(\Phi_p(U))$ that vanishes on exact forms (i.e. a cochain map out of the shifted truncated de Rham complex). We define such an operator via the curvature map $F\colon \Phi_p(U) \to \Omega^p_{cl}(U)$. This induces a pullback map $F^*\colon \mathcal{O}(\Omega^p_{cl}(U)) \to \mathcal{O}(\Phi_p(U))$. By composing with the curvature map it suffices to define the classical differential as the map $\Omega_c^{p-1}(U;\mathbb{C}) \to \mathcal{O}(\Omega^p_{cl}(U;\mathbb{C})) \xrightarrow{F^*} \mathcal{O}(\Phi_p(U))$

$$\alpha \mapsto \left(\beta \mapsto \int_U \beta \wedge *d\alpha\right) \mapsto \left(A \mapsto \int_U F_A \wedge *d\alpha\right) = \iota_{dS}(\alpha).$$

This functional, sending a field A to $\int_U F_A \wedge *d\alpha$, clearly recovers the first variation of the required action functional.

4.1.2. The Quantum Factorization Algebra

Now, we know abstractly how to quantise this factorization algebra, but we should see what it actually means in this context. There's an evaluation map $\Omega_c^{p-1}(U;\mathbb{C}) \otimes \mathcal{O}(\Phi_p(U)) \to \mathcal{O}(\Phi_p(U))$ which, again, is defined via the curvature map. We'll also identify the 1-forms on the fields $\Omega^1(\Phi_p(U))$ with the quasi-isomorphic complex $\mathcal{O}(\Phi_p(U)) \otimes \Omega^p_{cl}(U;\mathbb{C})^\vee$. The evaluation map is given by pairing a vector field on $\Phi_p(U)$ with a 1-form. We can spell this out in two steps:

(1) Start with $\chi \otimes f \in \Omega_c^{p-1}(U;\mathbb{C}) \otimes \mathcal{O}(\Phi)$. We first take the exterior derivative of both χ and f to yield

$$d\chi \otimes df \in \Omega^p_{c,cl}(U;\mathbb{C}) \otimes \Omega^1(\Phi(U)) \cong \Omega^p_{c,cl}(U;\mathbb{C}) \otimes \mathcal{O}(\Phi_p(U)) \otimes \Omega^p_{cl}(U;\mathbb{C})^\vee.$$

(2) Use the evaluation pairing between $\Omega^p_{cl}(U;\mathbb{C})^\vee$ and $\Omega^p_{c,cl}(U;\mathbb{C})$ (that is, between linear vector fields and linear 1-forms on the fields) to produce a contracted element $df(d\chi) \in \mathcal{O}(\Phi_p(U))$ as required.

This evaluation map then extends to a differential on the whole complex of observables by the BD operator formula: this is the quantum differential in the generalized Maxwell theory.

From now on, when I write $\mathrm{Obs}^{\mathrm{q}}(U)$ I'm referring specifically to the quantum observables in a generalized Maxwell theory (which one will generally be clear from context). I'll write $\mathrm{Obs}^{\mathrm{q}}(U)_0$ to refer specifically to the *gauge invariant degree zero* observables: the part of the cochain complex that refers to actual observables in the usual sense of the word, as opposed to encoding relationships between observables. The notation doesn't refer to the entire degree zero part of the cochain complex, but rather to the subcomplex $\mathcal{O}(H^0(\Phi(U))) \leq \mathrm{Obs}^{\mathrm{q}}_\Phi(U)$, where the projection $\Phi(U) \to H^0(\Phi(U))$ induces a pullback map $\mathcal{O}(H^0\Phi(U))) \to \mathcal{O}(\Phi(U))$.

For generalized Maxwell theories specifically, the local degree zero observables are given by functions on the 0^{th} hypercohomology of the Deligne complex (which, with our degree conventions, is the degree p differential cohomology group $\hat{H}^p(U))$. We compute this using the long exact sequence on hypercohomology associated to the short exact sequence of sheaves

$$0 \to \tau_{<p}\Omega^*_{\mathbb{C}}[p-1] \to \mathbb{Z}(p)_{\mathcal{D},\mathbb{C}} \to 2\pi R\mathbb{Z}[p] \to 0$$

yielding $\mathbb{H}^0(U;\mathbb{Z}(p)_{\mathcal{D}})$ isomorphic to the product of a torus (on which there are no non-constant global functions, so we can safely ignore it) and the group $\Omega^p_{cl,\mathbb{Z}}(U;\mathbb{C})$ of closed p-forms whose cohomology class lies in the subgroup $H^p(U;2\pi R\mathbb{Z}) \leq H^p(U;\mathbb{C})$. The calculation is described in [**Bry93**], theorem 1.5.3.

4.2. Free Theories from p-forms

In order to do calculations with Maxwell theories we will relate them to much easier free field theories where the fields are sheaves of p-forms. This will correspond, intuitively, to considering observables that factor through the curvature map $\hat{H}^p(U) \to \Omega^p_{cl}(U)$. These theories will be especially easy in that the action functional will involve no derivatives at all, so the classical BV operator is just a scalar.

Definition 4.2.1. *Fix $0 < p < n$ as before. The* free p-form theory *on X with coupling constant R is the Lagrangian field theory with sheaf of fields given by the sheaf of vector spaces Ω^p and action functional*

$$S_R(\alpha) = R^2 \|\alpha\|_2^2 = R^2 \int_X \alpha \wedge *\alpha$$

given by the L^2-norm. The free closed p-form theory *with coupling constant R is the subtheory with sheaf of fields Ω^p_{cl} – the sheaf of closed p-forms with the same action functional.*

We can build classical and quantum factorization algebras directly from this data as a very easy application of the BV formalism described above: I'll denote them by $\mathrm{Obs}^{\mathrm{cl}}_{\Omega^p}$, $\mathrm{Obs}^{\mathrm{q}}_{\Omega^p}$ etc, with the choice of R surpressed. For the general p-form theory we compute

$$\mathrm{Obs}^{\mathrm{cl}}_{\Omega^p}(U) = \mathrm{Sym}(\Omega^p_c(U)[1] \overset{R^2\iota}{\to} \Omega^p(U)^\vee)$$

where ι is the inclusion of $\Omega^p_c(U)$ into the dual space given by the L^2-pairing. This follows directly from the construction of the action functional given in chapter 3.3: this operator describes the first variation of the action functional.

The quantum BV operator D is induced from the evaluation pairing $\Omega^p_c(U) \otimes \Omega^p(U)^\vee \to \mathbb{R}$. We can produce a smeared complex of quantum observables using the standard L^2 pairing on p-forms coming from the Riemannian metric on X. That is, we have local smeared quantum observables given by

$$(\mathrm{Sym}(\Omega^p_c(U)[1] \oplus \Omega^p_c(U)), \mathrm{Sym}(\cdot R^2) + D)$$

where $\cdot R^2$ is now just a scalar multiplication operator, and D is the operator extended from the L^2 pairing as a map $\mathrm{Sym}^2 \to \mathrm{Sym}^0$ according to the usual BD formula. This complex is quasi-isomorphic to $\mathbb{R}$ for every U. Indeed, by a spectral sequence argument it suffices to check this for the classical observables, where we're computing Sym of a contractible complex.

The closed p-form theory is similar until we smear. That is, the quantum factorization algebra is given as the complex

$$\mathrm{Obs}^{\mathrm{q}}_{\Omega^p_{cl}}(U) = (\mathrm{Sym}(\Omega^p_{c,cl}(U)[1] \oplus \Omega^p_{cl}(U)^\vee), \mathrm{Sym}(R^2\iota) + D)$$

where D is induced from the evaluation pairing as above. The theory is not quite the same after smearing, in particular it is no longer locally contractible. The smeared version of the factorization algebra is now

$$\mathrm{Obs}^{\mathrm{sm}}_{\Omega^p_{cl}}(U) = (\mathrm{Sym}(\Omega^p_{c,cl}(U)[1] \oplus \Omega^p_c(U)/d^*\Omega^{p+1}_c(U)), \mathrm{Sym}(R^2\pi) + D)$$

where now π is the projection $\Omega^p_c(U) \to \Omega^p_c(U)/d^*\Omega^{p+1}_c(U)$, and where D is induced by the L^2-pairing. The embedding $\Omega^p_c(U)/d^*\Omega^{p+1}_c(U) \to \Omega^p_{cl}(U)^\vee$ is given, as usual, by the metric, and we use the fact that local closed and local coexact forms are orthogonal with respect to the L^2-pairing. This complex is only quasi-isomorphic to $\mathbb{R}$ *globally*, i.e. on compact U where we can use Hodge theory to identify closed forms as an orthogonal complement to coexact forms

This story all proceeds identically for complex-valued forms, which we'll use from now on. The complexification of the closed p-form theory is – by design – closely related to the generalized Maxwell theory of order p: the complexes of observables are isomorphic on certain open sets, and one has a natural map on *degree zero* observables $\mathrm{Obs}^{\mathrm{q}}_{\Omega^p_{cl}}(U;\mathbb{C})_0 \to \mathrm{Obs}^{\mathrm{q}}_p(U)_0$ for any U. This map should be thought of as the inclusion of those observables that factor through the curvature map. Concretely, the map is induced from the map of sheaves of cochain complexes

$$F \colon \mathbb{Z}(p)_{\mathcal{D}}(A) \to \Omega^p_{cl} \otimes_{\mathbb{C}} A$$

(the *curvature* map) for a commutative $\mathbb{C}$-algebra A induced by the derivative. This gives a map of sheaves of simplicial algebraic groups upon taking homotopy colimits. If $U \subseteq X$ is contractible then the map of complexes F on the open set U is a quasi-isomorphism, and so the induced map of simplicial algebraic groups is a homotopy equivalence.

The map F can be promoted to a morphism of factorization algebras. Indeed, we first extend the degree zero map to a map on all classical observables by pulling back polyvector fields. Concretely this is the map

$$F\colon \mathrm{Sym}(\Omega^p_{c,cl}(U;\mathbb{C})[1]) \otimes \mathrm{Sym}(\Omega^p_{cl}(U;\mathbb{C})^\vee) \to \mathrm{Sym}(T_0\Phi_c(U)[1]) \otimes \mathcal{O}(\Phi(U))$$

given by F^* in the second factor and a quasi-inverse d^{-1} to the quasi-isomorphism of cosheaves of cochain complexes $d\colon T_0\Phi_c \to \Omega^p_{c,cl}$ in the second factor. Specifically, on global sections we can define $d^{-1}\colon \Omega^p_{cl}(X;\mathbb{C}) \to T_0\Phi(X)$ using the Hodge decomposition. This preserves the support of a form, so restricts to the compactly supported local sections on an open set U, and defines a quasi-inverse to the quasi-isomorphism of complexes on contractible open sets.

One needs to check that this is compatible with the classical BV operator Q, which requires observing that the square

$$\begin{tikzcd}
\Omega^p_{c,cl}(U;\mathbb{C}) \arrow[r] \arrow[d, "d^{-1}"'] & \mathcal{O}(\Omega^p_{cl}(U;\mathbb{C})) \arrow[d, "F^*"] \\
T_0\Phi_c(U) \arrow[r] & \mathcal{O}(\Phi(U))
\end{tikzcd}$$

commutes, where the horizontal arrows are those maps defining the Poisson brackets. If, further, the open set U is contractible then this map defines a *quasi-isomorphism* of classical observables.

We then need to check that our map commutes with the quantum BV operator; that is, we check that the triangle

$$\begin{tikzcd}
\Omega^p_{c,cl}(U;\mathbb{C}) \otimes \Omega^p_{cl}(U;\mathbb{C}) \arrow[r, "\langle -,- \rangle"] \arrow[d, "d^{-1}\otimes F^*"'] & \mathbb{C} \\
T_0\Phi_p(U) \otimes \mathcal{O}(\Phi(U)) \arrow[ur, "D"] &
\end{tikzcd}$$

commutes, for D the quantum BV operator in the generalized Yang-Mills theory. This is clear from the definition of the map D, which first applies the exterior derivative to the linear vector field and the linear functional, then pairs the result via the L^2-pairing. Again, if U is in fact contractible then the resulting map on local observables is actually a quasi-isomorphism.

CHAPTER 5

Expectation Values

In this chapter we'll explain how to compute vacuum expectation values of observables in free theories where the fields are given by a cochain complex of vector spaces, such as the closed p-form theories introduced in the previous chapter. While the method does not apply directly to free theories with a more complicated space of fields, we show that the expectation value can be computed by functional integrals, which *will* generalise to more complicated settings like that of generalized Maxwell theories.

5.1. Expectation Values from Free Quantum Factorization Algebras

For an elliptic theory $(\Phi, \mathcal{L})$, consider the complex of global classical observables $\mathrm{Obs}(X)$ with underlying graded vector space $\mathcal{O}(T^*[-1]\Phi)$. This cdga is freely generated by a cochain complex, and its smeared version is assumed to be freely generated by an elliptic complex $\mathcal{E}$ as described in chapter 3.5. Now, Hodge theory gives us a Laplacian operator $\Delta : \mathcal{E} \to \mathcal{E}$ and a splitting in each degree: $\mathcal{E}_i = \mathcal{H}_i \oplus \mathcal{H}_i^\perp$, where $\mathcal{H}_i$ denotes the finite-dimensional vector space of *harmonic* elements in degree i. In particular, we can apply this to the degree zero elements $\mathcal{E}_0$, which represent a linearization of the space of the (degree zero) *fields* in our theory.

Definition 5.1.1. *The Hodge decomposition defines a splitting*

$$\mathcal{E}_0 = \mathcal{M} \oplus \mathcal{M}^\perp$$

where $\mathcal{M}$ denotes the finite-dimensional space of harmonic fields. We call $\mathcal{M}$ the space of massless modes *of the theory, and $\mathcal{M}^\perp$ the space of* massive modes*.*

Remark 5.1.2. In the terminology of Costello's work on perturbation theory, massless modes provide an example of non-propagating degrees of freedom, as in [**Cos11b**, Definition 13.1.1].

The terminology comes from the example of the free scalar field.

Example 5.1.3. Let $\Phi(U) = C^\infty(U)$ with the action for a free scalar field of mass $m \geq 0$, i.e.

$$S(\phi) = \int_X \left(\phi \Delta \phi - m^2 |\phi|^2\right) \mathrm{dvol}\,.$$

The complex of smeared quantum global observables here, as remarked upon in 3.5.5 has underlying graded vector space $\mathrm{Sym}(C^\infty(X)[1] \oplus C^\infty(X))$, classical differential $\mathrm{Sym}(\Delta - m^2)$ and quantum differential induced from the L^2-pairing on functions. The relevant elliptic complex then is the two-step complex

$$C^\infty(X) \overset{\Delta - m^2}{\to} C^\infty(X)$$

in degrees -1 and 0. The cohomology of this complex is finite-dimensional since X is assumed to be compact, so the m^2-eigenspace of the Laplacian is finite-dimensional (or just by Hodge theory, since the complex is elliptic).

Focusing on the case $m = 0$, the Hodge decomposition splits $C^\infty(X)$ in each degree as a sum of eigenspaces for Δ^2, or equivalently as a sum of eigenspaces for Δ. The eigenspaces are the spaces of solutions to $\Delta\phi = \lambda\phi$, which we think of as the energy $\sqrt{\lambda}$ pieces of the space of fields (by analogy with the Klein-Gordon equation in Lorentzian signature). The cohomology is then represented by the harmonic / massless piece.

From now on we'll always suppose that the complex $\mathcal{E}$ has no cohomology outside of degree zero. We don't expect this hypothesis to be necessary, but including it makes the argument below simpler, and is satisfied by the theories we study in this paper. With this condition relaxed, the expectation value would involve projecting to the subspace $\mathrm{Sym}(H^0(\mathcal{E})) \leq H^0(\mathrm{Sym}\,\mathcal{E})$ corresponding to global degree zero observables.

Suppose now that we're considering a theory with no massless modes; for instance, we might restrict to the subcomplex with $\mathcal{M}^\perp$ in degree zero. Then the elliptic complex $\mathcal{E}$ is contractible, therefore the cohomology of $\mathrm{Obs}^{\mathrm{sm}} = \mathrm{Sym}(\mathcal{E})$ is isomorphic to $\mathrm{Sym}(0) = \mathbb{C}$ in degree zero. To see this is also true for the smeared *quantum* observables we use a simple spectral sequence argument, using the filtration of the complex by Sym degree. The BV operator is extended from the map from Sym^2 to Sym^0 by the L^2 pairing, so in general lowers Sym-degree by two. The E_1 page of the spectral sequence computes the cohomology of the classical complex of smeared observables (i.e. the cohomology with respect to only the Sym degree 0 part of the differential), and the spectral sequence converges to the cohomology of the complex of smooth quantum observables (i.e. the cohomology with respect to the entire differential). Since the E_1 page is quasi-isomorphic to $\mathbb{C}$ in degree 0, so must be the E_∞ page. Finally we observe that there is a *unique* quasi-isomorphism from this complex of smooth observables on U to $\mathbb{C}$ characterized by the property that 1 in Sym^0 maps to 1. We call this the *expectation value* map. It takes a degree zero smeared quantum observable and returns a number.

Remark 5.1.4. The reader might find this assumption surprising, since generalized Maxwell theories are far from being of this type. However, there is a map onto the generalized Maxwell theory from a closed p-form theory, which is an elliptic theory with no massless modes, and the image of this map is dense on contractible open sets. In the last part of this paper we'll define expectation values and duals for the observables in the image of this map, using the above assumptions for the closed p-form theory.

Remark 5.1.5. With the setup we've been using, having no massless modes was essential. The massless modes correspond to the locus in $\mathcal{E}_0$ where the action functional vanishes, so the locus where the exponentiated action is *degenerate*. Since we'll be computing expectation values as a limit of finite dimensional Gaussian integrals it will be important to ensure that there are no massless modes so that the Gaussian is non-degenerate, and so the finite dimensional Gaussian integrals give finite answers.

It might be possible, in a somewhat different formalism, to work with a non-linear space of fields splitting into a *linear* space of massive modes and a *compact finite-dimensional* moduli space of massless modes. One could then describe an expectation value by integrating out the space of massive modes over each point in the moduli space of massless modes (using the formalism we will describe below) to produce a section of a rank 1 local system. If this local system was actually trivialisable then such a section could be integrated to give a number. Failure of trivialisability would be an example of an *anomaly* for a free field theory.

5.2. Computing Expectation Values

The ideas of this chapter are not original, but are merely a recollection of familiar physical techniques, whose analysis is well understood, in the present context. A modern mathematical account of the relationship between functional integrals, Feynman diagrams and homological algebra can be found in [**GJF12**].

5.2.1. Feynman Diagrams for Free Theories

So, let's fix a free elliptic theory with fields Φ and no massless modes: for instance a p-form or closed p-form theory on a compact manifold X. The idea of the Feynman diagram expansion is to compute expectation values of observables in our theory combinatorially. The crucial idea that we'll use in order to check that we can do this is that –for smeared observables – the expectation value map is *uniquely characterized*. That is, for smeared observables there is a unique quasi-isomorphism from global smeared observables to $\mathbb{C}$ that sends 1 to 1. Therefore to check that a procedure for computing expectation values is valid it suffices to check that it is a non-trivial quasi-isomorphism, then rescale so the map is appropriately normalized.

Take a global degree zero smeared observable $\mathcal{O} \in \mathrm{Sym}(\Phi(X)_0)$ which is *gauge invariant*. That is, we consider observables that can be written as a product of linear observables

$$\mathcal{O} = \mathcal{O}_1^{n_1}\mathcal{O}_2^{n_2}\cdots\mathcal{O}_k^{n_k}$$

where $\mathcal{O}_1, \ldots \mathcal{O}_k$ are linearly independent linear smeared observables in $\Phi(X)_0$ (not necessarily gauge invariant themselves), and where $\mathcal{O}$ is *closed* in the classical (or equivalently quantum) complex of smeared observables. This corresponds to gauge invariance because the only differential *from* the degree zero observables in either the classical or quantum complexes of smeared observables comes from Sym of the underlying differential on Φ. Closed elements are observables in the kernel of $\mathrm{Sym}(d^\vee)$ where $d \colon \Phi_{-1} \to \Phi_0$ is the underlying differential, and where the adjoint $d^\vee$ is defined on Φ_0 using the invariant pairing on fields. The kernel of $d^\vee$ corresponds exactly to the cokernel of d, which is the space of degree zero observables we think of as invariant under gauge transformations.

Generally gauge invariant polynomial observables are sums of monomial observables of this form, and we extend the procedure of computing duals linearly, so it suffices to consider $\mathcal{O}$ of this form.

We compute the expectation value of $\mathcal{O}$ combinatorially as follows. Depict $\mathcal{O}$ as a graph with k vertices, and with n_i half edges attached to vertex i. The expectation value $\langle\mathcal{O}\rangle$ of $\mathcal{O}$ is computed as a sum of terms constructed by gluing edges onto this frame in a prescribed way. Specifically, we attach *propagator edges* – which connect together two of these half-edges – in order to leave no free half-edges remaining. A propagator between linear observables $\mathcal{O}_i$ and $\mathcal{O}_j$ receives weight via the pairing

$$\frac{1}{2}\int_X \langle \mathcal{O}_i, Q^{-1}\mathcal{O}_j\rangle$$

where Q is the classical BV operator, Q^{-1} is defined by inverting Q on each eigenspace for the corresponding Laplacian (using the non-existence of massless modes), and $\langle -, -\rangle$ is the invariant pairing on smeared observables. A diagram is weighted by the product of all these edge weights. The expectation value is the sum of these weights over all such diagrams.

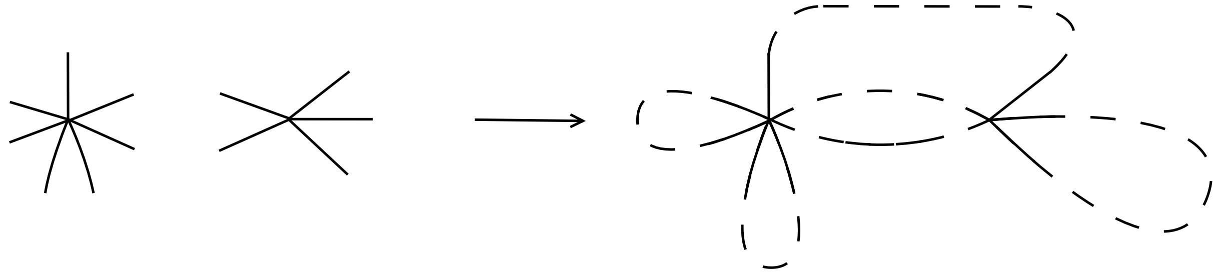

Figure 5.1. One of the terms in the Feynman diagram expansion computing the expectation value of an observable of form $\mathcal{O}_1^7\mathcal{O}_2^5$. On the left we see the starting point, with half-edges, and on the right we see one way of connecting these half-edges with propagator edges (indicated by dashed lines).

To check that this computes the expectation value, we must show that it is non-zero, and that it vanishes on the image of the differential in the complex of quantum observables. The former is easy: the observable 1 has expectation value 1 (so we're also already appropriately normalized). For the latter, we'll show that the path integral computation for degree zero global observables in $\mathrm{Obs}^{\mathrm{q}}(U)_0$ arise as a limit of finite-dimensional Gaussian integrals, and that the images of the quantum BV differential are all divergences, so vanish by Stokes' theorem.

5.2.2. Regularization and the Path Integral

The classical complex of linear observables in our theory is elliptic, so induces a Laplacian operator Δ acting on $\mathcal{E}_0$ with discrete spectrum $0 < \lambda_1 < \lambda_2 < \cdots$ and finite-dimensional eigenspaces. Let $F_kH^0(\Phi(X))$ denote the sum of the first k eigenspaces: this defines a filtration of the global degree zero (linearized) fields by finite-dimensional subspaces. We recall a standard result about infinite dimensional Gaussian integrals.

Proposition 5.2.1. *Let $\mathcal{O}$ be a smeared global observable. The finite-dimensional Gaussian integrals*

$$\frac{1}{Z_k}\int_{F^kH^0(\Phi(X))}\mathcal{O}(a)e^{-S(a)}da,$$

where Z_k is the volume $\int_{F^kH^0(\Phi(X))}e^{-S(a)}da$, converge to a real number $I(\mathcal{O})$ as $k\to\infty$, and this number agrees with the expectation value computed by the Feynman diagrammatic method.

Proof. We check that for each k the Gaussian integral admits a diagrammatic description, and observe that the expressions computed by these diagrams converge to the expression we want. We may assume as usual that $\mathcal{O}$ splits as a product of linear smeared observables $\mathcal{O} = \mathcal{O}_1^{n_1}\mathcal{O}_2^{n_2}\cdots\mathcal{O}_\ell^{n_\ell}$. The $\mathcal{O}_i$ describe linear operators on the filtered pieces. We can write the Gaussian integral using a generating function as

$$\int_{F^kH^0(\Phi(X))}\mathcal{O}(a)e^{-S(a)}da = \frac{\partial^{n_1+\cdots+n_\ell}}{\partial t_1^{n_1}\cdots\partial t_\ell^{n_\ell}}\bigg|_{t_1=\cdots=t_\ell=0}\int_{F^kH^0(\Phi(X))}e^{-\int_X\langle a,Qa\rangle+t_1\mathcal{O}_1(a)\cdots+t_\ell\mathcal{O}_\ell(a)}da,$$

provided that k is large enough that upon projecting to $F^kH^0(\Phi(X))$ the $\mathcal{O}_i$ are linearly independent. Call this projection $\mathcal{O}_i^{(k)}$. This expression is further simplified by completing the square, yielding

$$Z_k\,\frac{\partial^{n_1+\cdots+n_\ell}}{\partial t_1^{n_1}\cdots\partial t_\ell^{n_\ell}}\bigg|_{t_1=\cdots=t_\ell=0}e^{\frac{1}{2}\int_X\langle(t_1\mathcal{O}_1^{(k)}+\cdots t_\ell\mathcal{O}_\ell^{(k)}),Q^{-1}(t_1\mathcal{O}_1^{(k)}+\cdots t_\ell\mathcal{O}_\ell^{(k)})\rangle}$$

where we've identified the linear smeared observables with differential forms. We can now compute the Gaussian integral diagrammatically. The $t_1^{n_1}\cdots t_\ell^{n_\ell}$-term of the generating function is the sum over Feynman diagrams as described above, where a diagram is weighted by a product of matrix elements $\frac{1}{2}\int_X\langle\mathcal{O}_i^{(k)},Q^{-1}\mathcal{O}_j^{(k)}\rangle$ corresponding to the edges. We see that as $k\to\infty$ this agrees with the weight we expect. $\square$

Now we can justify why the expectation value vanishes on the image of the quantum differential. Let $\mathcal{O}\in\mathrm{Obs}_\Phi^{\mathrm{sm}}(X)_0$ be a smeared degree 0 global observable, and suppose $\mathcal{O} = d_\Phi V + (D-\iota_{dS})W$ is in the image of the quantum differential. The exact term $d_\Phi V$ is zero in H^0 of the fields, so it suffices to consider the W piece. The restriction of W to a filtered piece is a vector field on the vector space $F^kH^0(\Phi(X))$, and we can compute the divergence

$$\mathrm{div}(e^{-S(a)}W) = \mathcal{O}e^{-S(a)},$$

where the restriction to the filtered piece is suppressed in the notation. So the expectation value of $\mathcal{O}$ is a limit of integrals of divergences, which vanish by Stokes' theorem, and the expectation value is zero. This implies that the procedure described above does indeed compute the cohomology class of a global smeared observable in the canonically trivialized cohomology.

CHAPTER 6

Fourier Duality for Polynomial Observables

In its simplest form, Fourier duality is an isomorphism on degree 0 observables in the free p-form theories: $\mathrm{Obs}^{\mathrm{q}}_{\Omega^p,R}(U)_0 \cong \mathrm{Obs}^{\mathrm{q}}_{\Omega^{n-p},1/2R}(U)_0$. It will not extend to any kind of cochain maps in these theories, and in particular will not be compatible with the expectation value maps, but we'll show that it *is* compatible with the expectation values after the restriction $\mathrm{Obs}^{\mathrm{q}}_{\Omega^p,R} \to \mathrm{Obs}^{\mathrm{q}}_{\Omega^p_{cl},R}$ to the closed p-form theories.

6.1. Feynman Diagrams for Fourier Duality

We'll construct the Fourier transform in an explicit combinatorial way using Feynman diagrams extending the Feynman diagram expression computing expectation values. Take a smeared monomial observable $\mathcal{O} \in \mathrm{Obs}^{\mathrm{sm}}_{\Omega^p,R}(U)_0$. As above, we write $\mathcal{O}$ as

$$\mathcal{O}_1^{n_1}\mathcal{O}_2^{n_2}\cdots\mathcal{O}_k^{n_k}$$

where $\mathcal{O}_1, \ldots \mathcal{O}_k$ are linearly independent linear smeared observables in $\Omega^p_c(U)$.

We compute the Fourier dual of $\mathcal{O}$ in a similar diagrammatic way to the method we used to compute expectation values. Depict $\mathcal{O}$ as a graph with k vertices, and with n_i half edges attached to vertex i. Now, we can attach any number of *propagator edges* as before, and also any number of *source terms* – which attach to an initial half-edge and leave a half-edge free – in such a way as to leave none of the original half-edges unused. The source terms have the effect of replacing a linear term $\mathcal{O}_i$ with its Hodge dual $*\mathcal{O}_i$, i.e. the observable obtained by precomposition with the

Hodge star operator. The result is a new observable

$$(*\mathcal{O}_1)^{m_1}(*\mathcal{O}_2)^{m_2}\cdots(*\mathcal{O}_k)^{m_k}$$

where m_i is the number of source edges connected to vertex i, now thought of as a degree zero observable in $\mathrm{Obs}^{\mathrm{sm}}_{\Omega^{n-p},1/2R}(U)_0$.

Definition 6.1.1. *The total Fourier dual observable* $\widetilde{\mathcal{O}}$ *is the sum of these observables over all such graphs, where each observable is weighted by the product over all edges of the corresponding graph of the following weights.*

- *A propagator between linear observables* $\mathcal{O}_i$ *and* $\mathcal{O}_j$ *receives weight*

$$\frac{1}{2R^2}\langle \mathcal{O}_i, \mathcal{O}_j\rangle = \frac{1}{2R^2}\int_X \mathcal{O}_i \wedge *\mathcal{O}_j.$$

- *A source term attached to a linear observable* $\mathcal{O}$ *receives weight* $i/2R^2$.

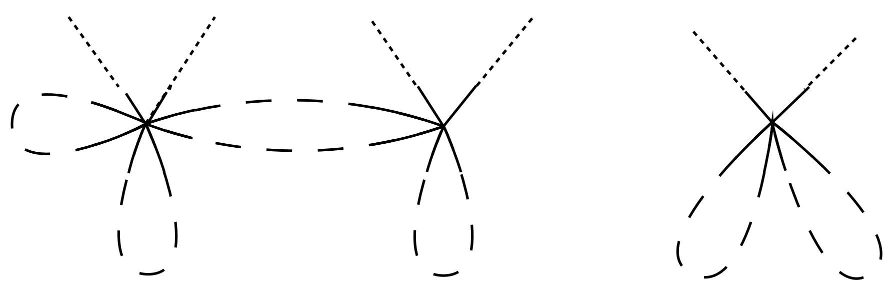

Figure 6.1. The Feynman diagram corresponding to a degree 6 term in the Fourier dual of an observable of form $\mathcal{O}_1^8\,\mathcal{O}_2^6\,\mathcal{O}_3^6$. Propagators are indicated by dashed lines and sources by dotted lines.

From a path integral perspective, these terms have natural interpretations. The Fourier transform of $\mathcal{O}$ can be thought of as the expectation value of an observable of form $\mathcal{O}e^{i\langle a,\widetilde{a}\rangle}$ where a is a field and $\widetilde{a}$ is its Fourier dual variable. Alternatively, this can be thought of as a functional derivative of an exponential of form $e^{-S(a)+i\langle a,\widetilde{a}\rangle}$. The propagator terms arise from applying a functional derivative to the action term, while the source terms arise from applying it to the second term, implementing the Fourier dual. The Hodge star in the source term arises from the specific pairing in the p-form theory, namely the L^2 pairing $\int a\wedge *\widetilde{a}$.

Example 6.1.2. To demonstrate the idea, we compute the Fourier dual observable to $\mathcal{O}^4$ for $\mathcal{O}$ a linear smeared observable. There is one term with no propagator edges and four sources, six with one propagator edge and two sources, and three with two propagator edges and no sources. The dual is therefore

$$\widetilde{\mathcal{O}^4} = \frac{1}{16R^8}(*\mathcal{O})^4 - \frac{6}{8R^6}\|\mathcal{O}\|^2(*\mathcal{O})^2 + \frac{3}{4R^4}\|\mathcal{O}\|^4.$$

If $R^2 = 1/2$ and $\|\mathcal{O}\| = 1$ this recovers the fourth Hermite polynomial $\mathrm{He}_4(*\mathcal{O})$.

We can compute the dual of a general global observable by smearing first, then dualizing: the result is that an observable has a uniquely determined smeared dual for each choice of smearing. In order to compare expectation values of an observable and its dual, the crucial tool that we'll use is Plancherel's formula, which we can rederive in terms of Feynman diagrams. The first step is to prove a Fourier inversion formula in this language. In doing so we'll need to remember that after dualizing once, the new observable lives in the *dual* theory, with a different action: therefore the weights assigned to edges will be different, corresponding to a different value of the parameter R.

We'll also use the convention that the second application of the Fourier transform is the inverse Fourier transform, which assigns weight $-i/2R^2$ to a source edge, but is otherwise identical.

Proposition 6.1.3. *A smeared observable $\mathcal{O}$ is equal to its Fourier double dual $\widetilde{\widetilde{\mathcal{O}}}$.*

Proof. Let $\mathcal{O} = \mathcal{O}_1^{n_1}\mathcal{O}_2^{n_2}\cdots\mathcal{O}_k^{n_k}$ as above. The Fourier double dual of $\mathcal{O}$ is computed as a sum over diagrams with two kinds of edges: those coming from the first dual and those coming from the second. We'll show that these diagrams all naturally cancel in pairs apart from the diagram with no propagator edges. We depict such diagrams with solid edges coming from the first dual, and dotted edges coming from the second dual.

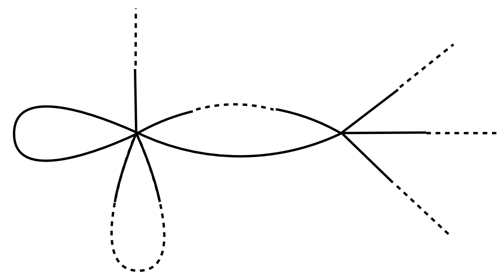

Figure 6.2. A diagram depicting a summand of the Fourier double dual of an observable of form $\mathcal{O}_1^7\,\mathcal{O}_2^5$.

So choose any diagram D with at least one propagator, and choose a propagator edge in the diagram. We produce a new diagram D' by changing this propagator edge from solid to dotted or from dotted to solid.

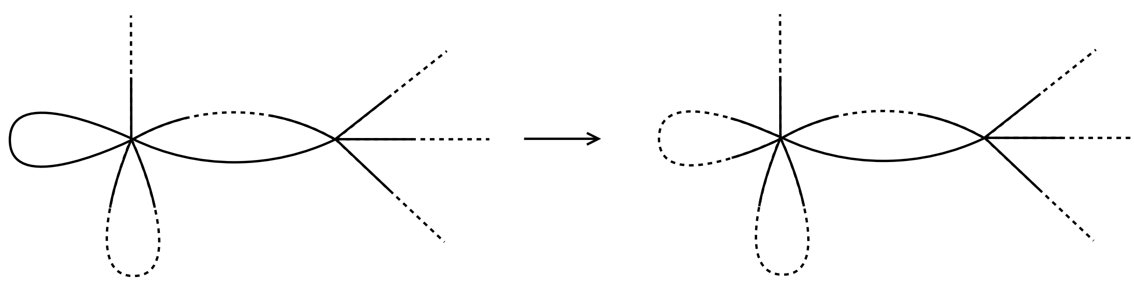

Figure 6.3. In this diagram we chose the solid leftmost propagator loop (coming from the first dual), and replaced it by two source terms (solid lines) connected with a dotted propagator loop coming from the second dual.

It suffices to show that the weight attached to this new diagram is -1 times the weight attached to the original diagram, so that the two cancel. This is easy to see: the propagator from the first term contributes a weight $\frac{1}{2R^2}\int_X \mathcal{O}_i \wedge *\mathcal{O}_j$. In the second dual, the weights come from the source terms in the original theory, but the propagator in the *dual* theory, which contributes a weight using the dual theory. So the total weight is

$$2R^2\left(\frac{i}{2R^2}\right)^2\int_X \mathcal{O}_i \wedge *\mathcal{O}_j$$

Which is -1 times the weight of the other diagram, as required.

Finally, we note that the weight assigned to the diagram with no propagator edges in the Fourier double dual is 1. Indeed, at each free edge, we have a composite of two source terms, contributing a factor of $\left(\frac{i}{2R^2}\right)\left(\frac{-i}{2(1/2R)^2}\right) = 1$. □

For further justification for these choices of weights, we should compare this combinatorial Fourier dual to one calculated using functional integrals (for global smeared observables). We'll perform such a check by defining a sequence of Gaussian integrals on filtered pieces, and checking that they converge to a Fourier dual that agrees with the one combinatorially described above. As before, $F^k\Omega^p(X)$ refers to the filtration by eigenspaces of the Laplacian, this time on the space of all p-forms, not just closed p-forms.

Proposition 6.1.4. *Let $\mathcal{O}$ be a smeared global observable. The finite-dimensional Gaussian integrals*

$$\widetilde{\mathcal{O}}(\widetilde{a}) = \left(\frac{1}{Z_k}\int_{F^k\Omega^p(X)} \mathcal{O}(a)e^{-S_R(a)+i\int_X \widetilde{a}\wedge a}da\right)e^{S_{1/2R}(\widetilde{a})}$$

where $\widetilde{a}$ is an $(n-p)$-form, converge to a smeared global observable which agrees with the Fourier dual observable computed by the Feynman diagrammatic method.

Proof. We use the same method of proof as for 5.2.1, writing the integral as a derivative of a generating function. Specifically, for $\mathcal{O} = \mathcal{O}_1^{n_1}\mathcal{O}_2^{n_2}\cdots\mathcal{O}_\ell^{n_\ell}$ we expand

$$\begin{aligned}
&\frac{1}{Z_k}\int_{F^k\Omega^p(X)} \mathcal{O}(a)e^{-S_R(a)+i\int_X \widetilde{a}\wedge a}da \\
&\qquad = \left.\frac{\partial^{n_1+\cdots n_\ell}}{\partial t_1^{n_1}\cdots\partial t_\ell^{n_\ell}}\right|_{t_1=\cdots=t_\ell=0} \frac{1}{Z_k}\int_{F^k\Omega(X)} e^{S_R(a)+\sum t_i\int_X \mathcal{O}_i\wedge *a+i\int_X \widetilde{a}\wedge a}da \\
&\qquad = \left.\frac{\partial^{n_1+\cdots n_\ell}}{\partial t_1^{n_1}\cdots\partial t_\ell^{n_\ell}}\right|_{t_1=\cdots=t_\ell=0} \\
&\qquad\qquad e^{-S_{1/2R}(\widetilde{a})}e^{\frac{1}{4R^2}\int_X (t_1\mathcal{O}_1^{(k)}+\cdots+t_\ell\mathcal{O}_\ell^{(k)})\wedge*(t_1\mathcal{O}_1^{(k)}+\cdots+t_\ell\mathcal{O}_\ell^{(k)})+\frac{i}{2R^2}\int_X (t_1\mathcal{O}_1^{(k)}+\cdots+t_\ell\mathcal{O}_\ell^{(k)})\wedge\widetilde{a}}
\end{aligned}$$

(by completing the square) and extract the $t_1^{n_1}\cdots t_\ell^{n_\ell}$-term. Once again we're denoting by $\mathcal{O}_i^{(k)}$ the projection of $\mathcal{O}_i$ onto the k^{th} filtered piece $F^k\Omega^p(X)$. We choose the level in the filtration large enough so that the upon projecting to the filtered piece the forms $\mathcal{O}_i$ are linearly independent. One then observes that in the limit as $k \to \infty$ the relevant term is given by a sum over diagrams as described with the correct weights. □

Now, for any open set $U \subseteq X$ we have a restriction map of degree zero local observables

$$r(U)\colon \mathrm{Obs}^{\mathrm{sm}}_{\Omega^p}(U)_0 \to \mathrm{Obs}^{\mathrm{sm}}_{\Omega^p_{cl}}(U)_0$$

induced by the projection $\Omega^p_c(U) \to \Omega^p_c(U)/d^*\Omega^{p+1}_c(U)$. This gives us a candidate notion of duality in the closed p-form theory. So, we might take a degree 0 observable in the image of $r(U)$, choose a preimage, compute the dual then restrict once more. Of course, this is not quite canonical, because the map $r(U)$ is not injective: the resulting dual observable might depend on the choice of preimage we made. However, in certain circumstances we might be able to choose a consistent scheme for choosing such a preimage, therefore a canonical duality map. We'll give such an example in chapter 6.3, but first we'll prove that for *any* choice of lift, the resulting dual observable in the Ω^p_{cl} theory has the same expectation value as the original theory.

Remark 6.1.5. We can also consider duality for closed p-form theories with coefficients in a vector space V, and – as we'll observe shortly – generalized Maxwell theories with gauge group a higher rank torus $T = V/L$, as mentioned in chapter 4. The theory generalises in a natural way, with a p-form theory with gauge group T dual to an $(n-p)$-form theory with gauge group $\hat{T}$, the dual torus. Indeed, there is an identical relationship between the generalized Maxwell theory with gauge group a higher rank torus and a closed p-form theory where the forms have coefficients in a vector bundle, and where the classical BV operator is given by the matrix describing the lattice L. By diagonalizing this matrix the system separates into a sum of rank one theories, with monomial smeared observables likewise splitting into products of monomial observables in rank one theories which one can dualise individually.

6.2. Fourier Duality and Expectation Values

At this point we have two equivalent ways of thinking about both the Fourier transform and the expectation value map for smeared observables: by Feynman diagrams (which allowed us to describe the dual locally) and by functional integration (which allow us to perform calculations, but only globally). We'll compare the expectation values of dual observables using a functional integral calculation, in which the restriction to *closed* p-forms will be crucial.

For an actual equality of expectation values as described above we'll have to restrict to observables on a contractible open set U. Recall this is the setting where the local observables in the closed p-form theory agree with observables in the original generalized Maxwell theory. On more general open sets connections on higher circle bundles are related to only those closed p-forms with *integral periods*. As such there is a map on degree zero observables $\mathrm{Obs}^{\mathrm{sm}}_{\Omega^p_{cl},R}(U)_0 \to \mathrm{Obs}^{\mathrm{q}}_R(U)_0$ for *any* U, which sends a compactly supported closed p-form a to the local observable

$$A \mapsto \int_U F_A \wedge *a.$$

However, this is generally not an isomorphism. Still, from a functional integral point-of-view we can define the expectation value of such an observable in the generalized Maxwell theory, even if X has non-vanishing degree p cohomology. Given $\mathcal{O} \in \mathrm{Obs}^{\mathrm{sm}}_{\Omega^p_{cl},R}(U)_0$ we extend $\mathcal{O}$ to a global degree zero obervable, and define its *expectation value* to be

$$\langle \mathcal{O} \rangle_R = \lim_{k\to\infty} \int_{F^k\Omega^p_{cl,\mathbb{Z}}(X)} \mathcal{O}(a) e^{-S_R(a)} da$$

Recall here that $\Omega^p_{cl,\mathbb{Z}}(X)$ is our notation for the closed p-forms with integral periods. This is the product of a finite-rank lattice with a vector space, and our filtration is the intersection of this subgroup with the filtration $F^k\Omega^p_{cl}(X)$ defined previously; this intersection is well-behaved since the lattice part is contained in the harmonic forms, thus in the intersection of all the filtered pieces. We notice that if $H^p(U) = 0$ then this definition agrees with the one we used in chapter 5.2.1, so in particular the limit converges. In general, the integrand is dominated in absolute value by the integrand over all closed p-forms, which we already know converges (since the proof of 5.2.1 still applies with $\mathcal{O}_i$ replaced by $|\mathcal{O}_i|$).

Using this definition (and bearing in mind its relationship to the notion of expectation value considered above on certain open sets), we'll prove the main compatibility with duality.

Theorem 6.2.1. *Let $\mathcal{O}$ be a local observable in $\mathrm{Obs}^{\mathrm{sm}}_{\Omega^p,R}(U)_0$, and let $\widetilde{\mathcal{O}} \in \mathrm{Obs}^{\mathrm{sm}}_{\Omega^{n-p},1/2R}(U)_0$ be its Fourier dual observable. Let $r(\mathcal{O})$ and $r(\widetilde{\mathcal{O}})$ be the restrictions to local observables in $\mathrm{Obs}^{\mathrm{sm}}_{\Omega^p_{cl}}(U)_0$ and $\mathrm{Obs}^{\mathrm{sm}}_{\Omega^{n-p}_{cl}}(U)_0$ respectively. Then, computing the expectation values of $r(\mathcal{O})$ and $r(\widetilde{\mathcal{O}})$, we find*

$$\langle r(\mathcal{O})\rangle_R = \langle r(\widetilde{\mathcal{O}})\rangle_{\frac{1}{2R}}.$$

Remark 6.2.2. Using the map discussed above we can just as well consider the observables $r(\mathcal{O})$ and $r(\widetilde{\mathcal{O}})$ as observables in the appropriate generalized Maxwell theories.

Proof. We know by 6.1.3 that $\mathcal{O} = \widetilde{\widetilde{\mathcal{O}}}$, so in particular $\langle r(\mathcal{O})\rangle_R = \langle r(\widetilde{\widetilde{\mathcal{O}}})\rangle_R$. By the calculation in Proposition 6.1.4 we can write this expectation value as the limit as $k \to \infty$ of the Gaussian integrals

$$\begin{aligned}
\frac{1}{Z_k}\int_{F^k\Omega^p_{cl,\mathbb{Z}}(X)} \mathcal{O}(a)e^{-S_R(a)}da &= \frac{1}{Z_k}\int_{F^k\Omega^p_{cl,\mathbb{Z}}(X)} \widetilde{\widetilde{\mathcal{O}}}(a)e^{-S_R(a)}da \\
&= \frac{1}{Z_k}\int_{F^k\Omega^p_{cl,\mathbb{Z}}(X)}\int_{F^k\Omega^{n-p}(X)} \widetilde{\mathcal{O}}(\widetilde{a})e^{-S_{1/2R}(\widetilde{a})-i\int_X \widetilde{a}\wedge a}d\widetilde{a}\, da \\
&= \frac{1}{Z_k}\int_{F^k\Omega^p(X)}\int_{F^k\Omega^{n-p}(X)} \widetilde{\mathcal{O}}(\widetilde{a})e^{-S_{1/2R}(\widetilde{a})-i\int_X \widetilde{a}\wedge a}\,\delta_{\Omega^p_{cl,\mathbb{Z}}(X)}(a)\, d\widetilde{a}\, da
\end{aligned}$$

The last line needs a little explanation. The distribution $\delta_{\Omega^p_{cl,\mathbb{Z}}(X)}$ is the delta-function on the closed and integral p-forms sitting inside all p-forms (restricted to the filtered piece): pairing with this distribution and integrating over all p-forms in the filtered piece is the same as integrating only over the relevant subgroup.

Now, for a fixed value of k, we can reinterpret the final integral above by changing the order of integration. This computes the Fourier dual of the delta function $\delta_{\Omega^p_{cl,\mathbb{Z}}(X)}$ and then pushes forward along the Hodge star. The Fourier dual of the delta function is $\delta_{\Omega^p_{cocl,\mathbb{Z}}(X)}$, the delta function on the group of *coclosed* p-forms with integral d^* cohomology class. That is, the external product $\delta_{d^*\Omega^{p+1}(X)} \boxtimes \delta_{\mathcal{H}^p_{\mathbb{Z}}}$ where $\mathcal{H}^p_{\mathbb{Z}}$ is the lattice in the space of harmonic p-forms corresponding to the integral cohomology via Hodge theory. Pushing this distribution forward along the Hodge star yields the delta function $\delta_{\Omega^{n-p}_{cl,\mathbb{Z}}(X)}$ on the closed $(n-p)$-forms with integral periods. Therefore

$$\begin{aligned}
\langle r(\mathcal{O})\rangle_R &= \lim_{k\to\infty} \frac{1}{Z_k} \int_{F^k\Omega^{n-p}(X)} \widetilde{\mathcal{O}}(\widetilde{a}) e^{-S_{1/2R}(\widetilde{a})} \delta_{\Omega^{n-p}_{cl,\mathbb{Z}}(X)} d\widetilde{a} \\
&= \lim_{k\to\infty} \frac{1}{Z_k} \int_{F^k\Omega^{n-p}_{cl,\mathbb{Z}}(X)} \widetilde{\mathcal{O}}(\widetilde{a}) e^{-S_{1/2R}(\widetilde{a})} d\widetilde{a} \\
&= \langle r(\widetilde{\mathcal{O}})\rangle_{\frac{1}{2R}}
\end{aligned}$$

as required. □

So to summarise, duality gives the following structure to the factorization algebra of quantum observables in our theories.

- For each open set U, we have a subalgebra $\mathrm{Obs}^{\mathrm{sm}}_{\Omega^p_{cl},R}(U)_0 \leq \mathrm{Obs}^{\mathrm{q}}_R(U)_0$ of the space of degree 0 local observables. If U is contractible (for instance for local observables in a small neighbourhood of a point) this subalgebra is dense.
- For a local observable $\mathcal{O}$ living in this subalgebra we can define a *Fourier dual* observable in $\mathrm{Obs}^{\mathrm{sm}}_{\Omega^{n-p}_{cl},1/2R}(U)_0$. This depends on a choice of extension of $\mathcal{O}$ to a functional on all p-forms, rather than just closed p-forms.
- For any choice of dual observable, we can compute their expectation values in the original theory and its dual, and they agree. If $H^p(U) = 0$ then this expectation value map agrees with a natural construction from the point of view of the factorization algebra.

We can rephrase the theorem in the language of factorization algebras. Note that $\mathrm{Obs}^{\mathrm{sm}}_{\Omega^p}(U)_0$ and $\mathrm{Obs}^{\mathrm{sm}}_{\Omega^p_{cl}}(U)_0$ form factorization algebras themselves as U varies, concentrated in degree zero. The inclusion maps $\mathrm{Obs}^{\mathrm{sm}}_{\Omega^p}(U)_0 \to \mathrm{Obs}^{\mathrm{sm}}_{\Omega^p}(U)$ and $\mathrm{Obs}^{\mathrm{sm}}_{\Omega^p_{cl}}(U)_0 \to \mathrm{Obs}^{\mathrm{sm}}_{\Omega^p_{cl}}(U)$ are cochain maps since the target complexes are concentrated in non-positive degrees, and factorization algebra maps because the factorization algebra structure maps preserve the degree zero piece. Likewise, the restriction maps $\mathrm{Obs}^{\mathrm{sm}}_{\Omega^p}(U)_0 \to \mathrm{Obs}^{\mathrm{sm}}_{\Omega^p_{cl}}(U)_0$ clearly commute with the factorization algebra structure maps, so define a factorization algebra map.

It's also easy to observe that the Fourier duality map $\mathrm{Obs}^{\mathrm{sm}}_{\Omega^p,R}(U)_0 \to \mathrm{Obs}^{\mathrm{sm}}_{\Omega^{n-p},\frac{1}{2R}}(U)_0$ defines a factorization algebra map: the degree zero observables are just the free cdga on the local sections of a cosheaf of vector spaces, so the structure maps are just the maps induced on the free algebra from the cosheaf structure maps. The structure maps are therefore given by the ordinary product

$$\begin{aligned}\mathrm{Sym}(\Omega^p_c(U_1)/d^*\Omega^{p+1}_c(U_1)) \otimes \mathrm{Sym}(\Omega^p_c(U_2)/d^*\Omega^{p+1}_c(U_2)) &\to \mathrm{Sym}(\Omega^p_c(V)/d^*\Omega^{p+1}_c(V))\\ (\mathcal{O}_1\cdots\mathcal{O}_n)\otimes(\mathcal{O}'_1\cdots\mathcal{O}'_m) &\mapsto \mathcal{O}_1\cdots\mathcal{O}_n\cdot\mathcal{O}'_1\cdots\mathcal{O}'_m\end{aligned}$$

where $\mathcal{O}_i$ and $\mathcal{O}'_j$ are local linear smeared observables: sections of $\Omega^p_c/d^*\Omega^{p+1}_c$ with compact support in disjoint open sets U_1 and U_2 respectively. The Fourier transform of a product of observables with disjoint support is the product of the Fourier transforms, since if $\mathcal{O}$ and $\mathcal{O}'$ have disjoint support then their L^2 inner product is zero, so all Feynman diagrams with propagator edges between their vertices contribute zero weight. Therefore duality gives a factorization algebra map on the factorization algebra consisting of degree zero observables only.

Combining all of these statements we have a correspondence of factorization algebras of form

$$\begin{tikzcd}
 & & (\mathrm{Obs}^{\mathrm{sm}}_{\Omega^p,R})_0 \arrow[r, leftrightarrow, "\sim"] \arrow[dl, "r_R"'] & (\mathrm{Obs}^{\mathrm{sm}}_{\Omega^{n-p},\frac{1}{2R}})_0 \arrow[dr, "r_{\frac{1}{2R}}"] & & \\
\mathrm{Obs}^{\mathrm{q}}_{p,R} & \mathrm{Obs}^{\mathrm{sm}}_{\Omega^p_{cl},R} \arrow[l, "F_R"'] & & & \mathrm{Obs}^{\mathrm{sm}}_{\Omega^{n-p}_{cl},\frac{1}{2R}} \arrow[r, "F_{\frac{1}{2R}}"] & \mathrm{Obs}^{\mathrm{q}}_{n-p,\frac{1}{2R}}
\end{tikzcd}$$

where the top arrow is the isomorphism given by Fourier duality, and the diagonal arrows are given by restriction, then inclusion of degree zero observables into all observables. We say a pair of local observables $\mathcal{O}$, $\mathcal{O}'$ in $\mathrm{Obs}^{\mathrm{q}}_{p,R}(U)$ and $\mathrm{Obs}^{\mathrm{q}}_{n-p,\frac{1}{2R}}(U)$ respectively are *incident* if they are the images under the restriction maps of Fourier dual degree zero observables. In this language, theorem 6.2.1 can be rephrased in the following way.

Corollary 6.2.3. *If $\mathcal{O}$ and $\mathcal{O}'$ are incident local observables in dual generalized Maxwell theories then $\langle \mathcal{O} \rangle_R = \langle \mathcal{O}' \rangle_{\frac{1}{2R}}$.*

At the end of the next chapter we'll observe that it is not possible to improve this statement from a correspondence to a *map* of factorization algebras (even of the factorization algebras in the closed p-form theories) in any natural way.

6.3. Wilson and 't Hooft Operators

In this chapter we'll give a concrete example of observables that admit canonical duals in generalized Maxwell theories, corresponding to familiar observables in the usual Maxwell theory, i.e. the case $p = 2$. For Wilson and 't Hooft operators in dimension 4 specifically, the behaviour under abelian duality is discussed in a paper of Kapustin and Tikhonov [**KT09**].

Wilson and 't Hooft operators can be defined classically in Yang-Mills theory with any compact gauge group G, just as functionals on the space of fields (from which we will – for an abelian gauge group – construct classical and quantum observables). So let X, for the moment, be a Riemannian 4-manifold.

Definition 6.3.1. *Let ρ be an irreducible representation of G. The* Wilson operator $W_{\gamma,\rho}$ *around an oriented loop γ in X is the functional on the space of connections on principal G-bundles sending a connection A to*

$$W_{\gamma,\rho}(A) = \mathrm{Tr}(\rho(\mathrm{Hol}_\gamma(A))),$$

where $\mathrm{Hol}_\gamma(A)$ *denotes the holonomy of the connection around the loop* γ*. Equivalently we can compute the Wilson operator as a path-ordered exponential*

$$W_{\gamma,\rho}(A) = \mathrm{Tr}(\rho(\mathcal{P}e^{i\oint_\gamma A})).$$

Suppose γ bounds a disk D. In this case there is a candidate dual observable to the Wilson operator.

Definition 6.3.2. *Let* $\mu\colon U(1) \to G$ *be a cocharacter for the group* G*. The* 't Hooft operator $T_{\gamma,\mu}$ *around the loop* γ *in* X *is the functional on the space of connections on principal* G*-bundles sending a connection* A *to*

$$T_{\gamma,\mu}(A) = e^{i\int_D \mu^*(*F_A)}$$

where F_A *is the curvature of* A*,* $*F_A$ *is its Hodge star, and where* $\mu^*\colon \Omega^2(X;\mathfrak{g}^*) \to \Omega^2(X)$ *is the pullback along the cocharacter.*

The relationship between these two kinds of operator is clearest in the abelian case, so let $G = U(1)$ (or, with minor modifications, any torus). The irreducible representations of $U(1)$ are given by the n-power maps $z \mapsto z^n$ for $n \in \mathbb{Z}$, so we can write our Wilson operators as

$$W_{\gamma,n}(A) = e^{in\oint_\gamma A} = e^{in\int_D F_A}$$

assuming $\partial D = \gamma$ as above. Cocharacters are also indexed by integers, so similarly we can describe the 't Hooft operators as

$$T_{\gamma,m}(A) = e^{im\int_D *F_A}.$$

As described above we compute the dual of an observable in abelian Yang-Mills theory by taking its Fourier dual as a functional on all 2-forms, then precomposing with the Hodge star. The Fourier dual of a plane wave is a plane wave, so we should expect Wilson and 't Hooft observables to be dual to one another. In the rest of this chapter we'll prove this, and generalise it to higher degree theories.

Consider the degree p generalized Maxwell theory with coupling constant R on an n-manifold X. I'll first describe degree zero gauge invariant observables associated to a complex number r and a singular chain $C \in C_p(U)$, for $U \subseteq X$ an open set. Recall that the local degree zero observables are given by $\mathcal{O}(H^0(\Phi(U))) \leq \mathcal{O}(T^*[-1]\Phi(U))$, and that in this case the degree zero cohomology $H^0(\Phi)$ is given by the group $\Omega^p_{cl,\mathbb{Z}}(U;\mathbb{C})$ of closed p-forms with integral periods (periods in the lattice $H^p(U; 2\pi R\mathbb{Z})$). So, analogously to the above we define a *Wilson*-type operator by

$$W_{C,r}(\alpha) = e^{ir \int_C \alpha}.$$

Similarly, if C is instead a chain in $C_{n-p}(U)$, we can define an *'t Hooft*-type operator by first applying the Hodge star:

$$T_{C,r}(\alpha) = e^{ir \int_C *\alpha}.$$

Remark 6.3.3. These operators don't quite arise from our definitions: they aren't polynomial functions in linear observables. However, they can be arbitrarily well approximated by polynomials by taking a finite number of terms in the Taylor series. We should either note that our constructions, in which the observables are described by a symmetric algebra, extend to completed symmetric algebras, or equivalently just interpret claims about duality for these observables as claims about these polynomial approximations at every degree.

Now, let's investigate duality for these observables. Firstly, suppose U is an open set with $H^p(U) = 0$, so that the condition of having integral periods is trivial. Then the observables defined above immediately lift to observables in the closed p-form theory, and admit canonical extensions to observables in the theory where fields are *all* p-forms (given by precisely the same formula). We can also investigate approximations for these observables by *smeared* observables. Integration over a p-chain C can be written as the L^2-pairing with a particular current: the delta function δ_C. This current can, in turn, be approximated in L^2 by p-forms supported on small neighbourhoods of C.

So let's investigate the Fourier dual of the smeared observable

$$\mathcal{O}_\beta(a) = e^{ir\int_X a\wedge *\beta}$$

where β is a p-form.

We'll compute this dual using the functional integral at a regularized level:

$$\begin{aligned}
\widetilde{\mathcal{O}_{\beta,r}}(\widetilde{a}) &= \lim_{k\to\infty}\frac{1}{Z_k}\left(\int_{F^k\Omega^p(X)} e^{-S_R(a)+i\int_X \widetilde{a}\wedge a+ir\int_X a\wedge *\beta}da\right)e^{S_{1/2R}(\widetilde{a})}\\
&= e^{-\frac{r^2}{4R^2}\|\beta\|^2}e^{\frac{-r}{2R^2}\int_X \widetilde{a}\wedge\beta}\\
&= e^{-\frac{r^2}{4R^2}\|\beta\|^2}\cdot\mathcal{O}_{*\beta,ir/2R^2}(\widetilde{a}).
\end{aligned}$$

This calculation allows us to produce the dual of the original Wilson operator by dualizing increasingly good smooth approximations. We find

$$\widetilde{W_{C,r}} = e^{-\frac{r^2}{4R^2}\|C\|^2}T_{C,ir/2R^2}$$

where $\|C\|$ is the L^2-norm of the chain C: the usual L^2 norm with respect to the metric of its image under Poincaré duality.

To summarise, duality for (generalized) Wilson and 't Hooft operators tells us the following.

Corollary 6.3.4. *There is an equality of expectation values in generalized Maxwell theories*

$$\langle W_{C,r}\rangle_R = e^{-\frac{r^2}{4R^2}\|C\|^2}\langle T_{C,ir/2R^2}\rangle_{\frac{1}{2R}}.$$

Remark 6.3.5. We described a canonical dual for Wilson and 't Hooft operators, using a natural choice of lift from operators acting on closed p-forms to operators acting on all p-forms. It's natural to ask whether it's possible to do this for all observables, thus promoting abelian duality from a correspondence to a genuine map of factorization algebras. It turns out however that this is impossible. We'll demonstrate this in a specific example.

Let X be a $2p$-manifold satisfying $H^p(X) = 0$, so $\Omega^p(X)$ splits as $d\Omega^{p-1}(X) \oplus d^*\Omega^{p+1}(X)$. We'll discuss global linear smeared observables in the closed p-form theory on X. Such an observable is an element of $d\Omega^{p-1}(X))^\vee$ given by L^2-pairing with an exact p-form. Prescribing an extension of an observable in the closed p-form theory to an observable in the *full* p-form theory is equivalent to prescribing the action of the lifted observable on coexact p-forms, so a choice of such an extension for all linear smeared observables is a map

$$f \colon d\Omega^{p-1}(X) \to d^*\Omega^{p+1}(X).$$

Having specified such a map we obtain a canonical dual observable for every linear observable in the closed p-form theory. Applying this duality procedure twice should bring us back to the observable we started with, i.e. $*f*f = \mathrm{id}$, which means in particular that f must be an isomorphism.

However, we also need compatibility with the canonical duals constructed above for Wilson and 't Hooft operators. Let I be a linear observable of form "integrate over a p-cycle", and let $\alpha_i \to I$ be a sequence of smeared observables approximating I. The duals of Wilson observables of form e^I are determined by the duals of linear observables since e^x is Fourier self-dual, and to obtain the required dual for our Wilson operator we need to choose the trivial lift for I, i.e. we need $f(\alpha_i) \to 0$. But then $*f*f(\alpha_i) \to 0$, so $\alpha_i \to 0$ which is false. So there is no possible canonical lift f compatible with the natural duals for Wilson and 't Hooft operators, and in particular no way of improving abelian duality to a genuine map rather than just a correspondence.

Part 2

Twists of $N = 4$ Supersymmetric Gauge Theories

CHAPTER 7

Introduction

The material in this part of the thesis is taken from my joint paper "Geometric Langlands Twists of $N = 4$ Gauge Theory from Derived Algebraic Geometry" [**EY15**] with Philsang Yoo.

In this part of the thesis, we engage directly with the work [**KW06**] of Kapustin and Witten, and construct the classical geometric Langlands twists of $N = 4$ supersymmetric gauge theories. In order to do so, we develop machinery, using techniques from derived algebraic geometry developed by Gaitsgory and Rozenblyum to investigate what it means to twist a classical supersymmetric field theory, on the level of the derived space of solutions to the classical equations of motion.

With some technical machinery in hand, we will prove the following theorem.

Theorem 7.0.6. *The classical $N = 4$ supersymmetric Yang-Mills theory admits a holomorphic twist defined on any complex algebraic surface X, whose moduli space of solutions to the equations of motion has the form*

$$\mathrm{EOM}_{\mathrm{hol}}(X) \cong T^*_{\mathrm{form}}[-1]\,\mathrm{Bun}_G(X).$$

This theory admits a $\mathbb{CP}^1$ of topological twists. In particular, the B-twist has classical moduli space equivalent to

$$\mathrm{EOM}_{\mathrm{B}}(X) \cong T^*_{\mathrm{form}}[-1]\,\mathrm{Loc}_G(X)$$

and the A-twist has classical moduli space equivalent to

$$\mathrm{EOM}_A(X) \cong \mathrm{Bun}_G(X)_{\mathrm{dR}}.$$

The A-twisted theory has some additional structure: it arises as the special point in a family of "holomorphic-topological twists". We use this structure to identify the moduli space on a product $\Sigma_1 \times \Sigma_2$ of algebraic curves in a different way. In particular, we use this to give concrete descriptions of the Hilbert space of each of the topologically twisted theories.

CHAPTER 8

Classical $N = 4$ Theories and their Twists

In this chapter we'll discuss the foundational constructions of supersymmetric gauge theories, and the general formalism of "twisting" supersymmetric theories. For simplicity, from chapter 8.1.1 onwards we'll stick to considering 4-dimensional theories in Riemannian signature, but many of the constructions we discuss (particularly those purely algebraic constructions involving supersymmetry algebras) have natural analogues in other dimensions. For instance, the construction of $N = 4$ supersymmetric gauge theories in four-dimensions by dimensional reduction fits into a natural family of constructions using the theory of normed division algebras. This is beautifully explained by Anastasiou, Borsten et al [**ABD⁺13**]. Throughout this chapter we'll refer to appendix A for general constructions with supersymmetry algebras.

8.1. Holomorphic and Topological Twists

The idea of a *twist* of a supersymmetry algebra, or of a supersymmetric field theory, originated in [**Wit88a**] as a procedure for constructing topological "sectors" of general supersymmetric field theories, but one can make sense of twists in much greater generality. One can form a twist of a supersymmetry algebra $\mathcal{A}$ – and a twist of a theory on which it acts – from any supercharge Q (i.e. fermionic element of the supersymmetry algebra) such that $[Q, Q] = 0$. The definition of the twisted supersymmetry algebra is straightforward.

Let $\mathcal{A}$ be the complexified supersymmetry algebra in dimension n associated to a spinorial complex representation Σ of $\mathrm{Spin}(n)$, a non-degenerate pairing $\Gamma\colon \Sigma\otimes\Sigma \to V_{\mathbb{C}}$ where $V_{\mathbb{C}}$ is the n-dimensional vector representation, and a subalgebra $\mathfrak{g}_R$ of R-symmetries. The example that we'll be most concerned with is the 4d supersymmetry algebra associated to a finite-dimensional complex vector space W, given by

$$\mathcal{A}^W = (\mathfrak{so}(4;\mathbb{C}) \ltimes \mathbb{C}^4) \oplus \mathfrak{g}_R \oplus \Pi((S_+ \otimes W) \oplus (S_- \otimes W^*))$$

where $\mathfrak{g}_R = \mathfrak{sl}(4;\mathbb{C})$, as described in appendix A.

Definition 8.1.1. *The* twisted supersymmetry algebra *associated to a fermionic element $Q \in \mathcal{A}$ with $[Q,Q] = 0$ is the cohomology of $\mathcal{A}$ with respect to the differential $[Q,-]$.*

A more subtle notion is that of a twist of a supersymmetric field theory, which should be thought of as the derived Q-invariants of the original theory, admitting an action of the twisted supersymmetry algebra. Such twisted theories inherit properties (invariance under certain natural symmetries) from properties of the supercharge Q. We'll discuss two such properties: topological and holomorphic invariance.

Perhaps the most important types of twist are *topological twists.* In the literature, these are defined as coming from supercharges $Q \in \Pi(\Sigma)$ which are $\mathrm{Spin}(n)$-invariant. Of course, there are generally no such Q; for instance in 4 dimensions the odd part of the $N = 1$ supersymmetry algebra decomposes as a sum of irreducible two-complex dimensional $\mathrm{Spin}(4)$-representations. However, it suffices to find Q that is $\mathrm{Spin}(n)$-invariant *after* modifying the action of the complexified rotations $\mathfrak{so}(n;\mathbb{C})$ on the space of supercharges. Let's make this more precise by first giving a more natural definition, then showing why the above notion implies the more natural condition.

Definition 8.1.2. *A supercharge Q with $[Q,Q] = 0$ is called* topological *if the map*

$$[Q,-] \colon \Sigma \to V_{\mathbb{C}}$$

is surjective.

Remark 8.1.3. The above definition also makes sense for theories with an action of an uncomplexified supersymmetry algebra. A real supercharge Q is likewise called topological if the map $[Q,-]$ is surjective onto the space $V_{\mathbb{R}}$ of real translations.

We'll see shortly that this implies that all translations act trivially on Q-twisted theories for a topological supercharge Q. Now, let's recover the classical notion of a topological twist. If $\phi\colon \mathfrak{so}(n;\mathbb{C}) \to \mathfrak{g}_R$ is a Lie algebra homomorphism, we can define a ϕ-twisted action of $\mathfrak{so}(n;\mathbb{C})$ on Σ. Indeed, Σ always takes the form $S \otimes W$ (in odd dimensions), or $S_+ \otimes W_1 \oplus S_- \otimes W_2$ (in even dimensions) where W, W_1 and W_2 are finite-dimensional vector spaces acted on by the R-symmetries. With this in mind we define the twisted action of $X \in \mathfrak{so}(n;\mathbb{C})$ by

$$\begin{aligned} X(s \otimes w) &= X(s) \otimes \phi(X)(w) \\ \text{or } X(s_+ \otimes w_1 + s_- \otimes w_2) &= X(s_+) \otimes \phi(X)(w_1) + X(s_-) \otimes \phi(X)^*(w_2) \end{aligned}$$

depending on the dimension.

Proposition 8.1.4. *Let Q be a non-zero supercharge in n dimensions such that $[Q,Q] = 0$, and such that there exists a homomorphism $\phi\colon \mathfrak{so}(n;\mathbb{C}) \to \mathfrak{g}_R$ making Q invariant under the ϕ-twisted action of $\mathfrak{so}(n;\mathbb{C})$. Then Q is topological.*

Proof. We can replace the supersymmetry algebra with the supersymmetry algebra *twisted by ϕ*, with brackets modified as follows:

- The rotations $\mathfrak{so}(n;\mathbb{C})$ act on Σ according to the ϕ-twisted action.
- Rotations bracket with elements of $\mathfrak{g}_R$ as their image under the embedding ϕ.

The bracket of two odd elements is unchanged, so it suffices to check that Q is topological in this twisted algebra. In this algebra, since Q spans an irreducible $\mathfrak{so}(n;\mathbb{C})$ representation, the image of $[Q,-]$ in $V_\mathbb{C}$ should be itself an irreducible subrepresentation, so either 0 or $V_\mathbb{C}$ itself. Since the pairing Γ is non-degenerate, the map $[Q,-]$ is never 0 when $Q \neq 0$, so its image is all of $V_\mathbb{C}$ as required. $\square$

Remark 8.1.5. The converse to this proposition is false in general. For a counterexample, we consider the case of the $N = 1$ supersymmetry algebra in dimension $n = 8$, where the positive helicity Weyl spinor representation is related to the vector representation by triality (i.e. by precomposing by an outer automorphism of $\mathfrak{so}(8;\mathbb{C})$). The R-symmetry group is just $\mathbb{C}^\times$, so twisting homomorphisms are just characters, and we observe that there are no non-zero invariant vectors for the vector representation of $\mathfrak{so}(8;\mathbb{C})$ twisted by a character, and similarly for the twisted Weyl spinor representation. However, there *are* topological supercharges in the positive Weyl spinor representation in dimension 8. In dimension 8 any Weyl spinor Q_+ pairs with itself to 0 under the Γ-pairing, and if Q_+ is not *pure* – i.e. if its nullspace in $\mathbb{C}^8$ under Clifford multiplication is not of dimension 4 – then the map $\Gamma(Q_+, -)\colon S_{8-} \to \mathbb{C}^8$ is surjective.

Remark 8.1.6. In dimension 4 – the case we'll principally be interested in in this paper – there is a classification of twisting homomorphisms ϕ that yield topological twists by this procedure [**Loz99**]. We'll investigate twists coming from the so-called "Kapustin-Witten" twisting homomorphism, which we'll define at the beginning of the next subsection.

The notion of a topological twist suggests a natural definition for a *holomorphic* twist. We should ask the image of the map $[Q, -]$ from the odd to the even part of the supersymmetry algebra to contain *exactly half* of all translations. In order for this to make sense, suppose n is even.

Definition 8.1.7. *A supercharge Q with $[Q,Q] = 0$ is called* holomorphic *if there exists a $\mathbb{C}$-linear isomorphism between $V_\mathbb{C}$ and $\mathbb{C}^{n/2} \otimes_\mathbb{R} \mathbb{C}$ such that the image of $[Q, -]$ in $V_\mathbb{C}$ spans the holomorphic subspace $\mathbb{R}^{n/2} \otimes_\mathbb{R} \mathbb{C}$.*

To put it another way, Q is holomorphic if we can choose a splitting of the algebra of translations into holomorphic and anti-holomorphic directions such that the image of $[Q, -]$ is precisely the anti-holomorphic piece. There's a natural procedure for constructing holomorphic twists analogous to the procedure for topological twists above, which is straightforward to describe in four dimensions. The procedure depends on a choice of embedding $\mathrm{SU}(2) \to \mathrm{SU}(2)_+ \times \mathrm{SU}(2)_-$, or on the level of complexified Lie algebras $\mathfrak{so}(3;\mathbb{C}) \to \mathfrak{so}(4;\mathbb{C})$. This defines an action of $\mathrm{SU}(2)$ on $V_\mathbb{C}$ by restricting the tensor product action on $S_+ \otimes S_-$, and thus a subspace of $V_\mathbb{C}$ by taking invariant vectors. We want this to give a real subspace (i.e. a half-dimensional subspace), so we must restrict attention to the inclusions ι_1 and ι_2 of the two factors.

Proposition 8.1.8. *Let Q be a non-zero supercharge Q with $[Q,Q]=0$, and suppose there exists a homomorphism $\phi\colon \mathfrak{so}(4;\mathbb{C}) \to \mathfrak{g}_R$ making Q invariant under the ϕ-twisted action of $\iota_i(\mathfrak{so}(3;\mathbb{C}))$, where $i=1$ or 2. Then Q is either a holomorphic or a topological twist.*

Proof. This is very similar to the proof of proposition 8.1.4 above. Again we can replace the supersymmetry algebra by its ϕ-twisted version, but now the image of $[Q,-]$ in the translations is a $\iota_i(\mathfrak{so}(3;\mathbb{C}))$-subrepresentation of $V_{\mathbb{C}}$. As a module for this algebra $V_{\mathbb{C}}$ decomposes as the sum of two two-dimensional irreducible representations. Thus the image of $[Q,-]$ is zero, half-dimensional or full-dimensional. As before, non-degeneracy of Γ ensures that it's non-zero, so Q is either holomorphic or topological. $\square$

8.1.1. Twists of the $N=4$ Supersymmetry Algebra

For most of the rest of this paper, we'll specialize to the 4-dimensional setting and the case where $W=\mathbb{C}^4$, i.e. to $N=4$ supersymmetry. We'll take the R-symmetry algebra to be $\mathfrak{g}_R=\mathfrak{sl}(4;\mathbb{C}) \subseteq \mathfrak{gl}(4;\mathbb{C})$; this is the R-symmetry algebra that'll act on supersymmetric gauge theories, since the theories we'll define will require fixing a choice of trivialization of $\det \mathbb{C}^4$. We'll consider several holomorphic and topological twists of an $N=4$ supersymmetric gauge theory, so let's discuss these twists at the level of the supersymmetry algebra

$$\mathcal{A}^{N=4} = (\mathfrak{so}(4;\mathbb{C}) \oplus \mathfrak{g}_R \oplus V_{\mathbb{C}}) \oplus \Pi\,(S_+ \otimes W \oplus S_- \otimes W^*)$$

where $W=\mathbb{C}^4$, and where $\mathfrak{g}_R$ acts on W by its fundamental representation.

We'll first analyse a family of holomorphic twists of this supersymmetry algebra. We'll fix a particular twisting homomorphism ϕ, the *Kapustin-Witten twist*, defined to be the composite

$$\phi_{\mathrm{KW}} \colon \mathfrak{so}(4;\mathbb{C}) \cong \mathfrak{sl}(2;\mathbb{C}) \oplus \mathfrak{sl}(2;\mathbb{C}) \to \mathfrak{sl}(4;\mathbb{C})$$

where the first map is the exceptional isomorphism in dimension 4, and the second map is the block diagonal embedding. We'll get a space of holomorphic supercharges for each factor of $\mathrm{SU}(2)_+ \times \mathrm{SU}(2)_-$, which we'll describe concretely. Choose a $\mathbb{C}$-basis for the space of supercharges by choosing bases for its constituent pieces as follows:

$$\begin{aligned}
S_+ &= \langle \alpha_1, \alpha_2 \rangle \\
S_- &= \langle \alpha_1^\vee, \alpha_2^\vee \rangle \\
W &= \langle e_1, e_2, f_1, f_2 \rangle \\
W^* &= \langle e_1^*, e_2^*, f_1^*, f_2^* \rangle
\end{aligned}$$

where $\mathfrak{so}(4;\mathbb{C})$ acts on W via the ϕ_{KW}-twist so that $\{e_i\}$ and $\{f_i\}$ are bases for the two semispin factors (i.e. the summands on which $\mathrm{SU}(2)_+$ and $\mathrm{SU}(2)_-$ act), and where the basis given for W^* is the dual basis to the one for W. Tensor products of basis elements yield a basis for $S_+ \otimes W \oplus S_- \otimes W^*$. Consider the embedding $\iota_2 \colon \mathrm{SU}(2) \to \mathrm{SU}(2)_+ \times \mathrm{SU}(2) -$ by inclusion of the second factor. The resulting invariant supercharges are those in $S_+ \otimes \langle e_1, e_2 \rangle$. From now on we'll fix a reference holomorphic supercharge

$$Q_{\mathrm{hol}} = \alpha_1 \otimes e_1.$$

Now, let's compute the Q_{hol}-cohomology of the $N = 4$ supersymmetry algebra; that is, the cohomology of the cochain complex

$$\mathfrak{so}(4;\mathbb{C}) \oplus \mathfrak{g}_R \xrightarrow{[Q_{\mathrm{hol}}, -]} \Pi(S_+ \otimes W \oplus S_- \otimes W^*) \xrightarrow{[Q_{\mathrm{hol}}, -]} V_{\mathbb{C}} \ .$$

Consider the terms sequentially.

- In the translation term we expect to find a half-dimensional family of "antiholomorphic" translations as the cokernel of $[Q_{\mathrm{hol}}, -]$. Indeed, the image in the translations is the span of $\Gamma(\alpha_1, \alpha_1^\vee)$ and $\Gamma(\alpha_1, \alpha_2^\vee)$, which are linearly independent. From now on we'll work in coordinates on $V_{\mathbb{C}}$ defined by

$$\frac{\partial}{\partial \overline{z}_i} = \Gamma(\alpha_1, \alpha_i^\vee), \frac{\partial}{\partial z_i} = \Gamma(\alpha_2, \alpha_i^\vee).$$

- In the remaining bosonic term, the kernel of $[Q_{\mathrm{hol}}, -]$ is spanned by $\mathfrak{so}(3;\mathbb{C})_- = \mathfrak{su}(2)_- \otimes \mathbb{C}$ and $\mathrm{Ann}(e_1) \cong \mathfrak{p}$, a parabolic subalgebra of $\mathfrak{sl}(4;\mathbb{C})$ with Levi subalgebra $\mathfrak{sl}(3;\mathbb{C})$.
- In the fermionic term, consider the two summands separately. First look at $S_+ \otimes W$. These elements are all $[Q_{\mathrm{hol}}, -]$-closed, and the exact elements are just the five-dimensional subspace generated by $S_+ \otimes \langle e_1 \rangle$ and $\langle \alpha_1 \rangle \otimes W$, leaving

$$\langle \alpha_2 \otimes e_2, \alpha_2 \otimes f_1, \alpha_2 \otimes f_2 \rangle$$

as the cohomology. Finally, look at $S_- \otimes W^*$. There are no exact elements in this subspace, and the closed elements are given by

$$S_- \otimes \langle e_2^*, f_1^*, f_2^* \rangle.$$

So overall, the twisted supersymmetry algebra has form

$$\left(\mathfrak{so}(3;\mathbb{C}) \oplus \mathfrak{p} \oplus \left\langle \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2} \right\rangle\right) \oplus \Pi\Big(\langle \alpha_2 \otimes e_2, \alpha_2 \otimes f_1, \alpha_2 \otimes f_2 \rangle \oplus S_- \otimes \langle e_2^*, f_1^*, f_2^* \rangle\Big)$$

where $\mathfrak{so}(3;\mathbb{C})$ acts on S_- by its spin representation, and $\mathfrak{sl}(3;\mathbb{C}) \subseteq \mathfrak{p}$ acts on $\langle e_2, f_1, f_2 \rangle$ and its dual space by the fundamental and anti-fundamental representations respectively.

Now, the twists we'll really be concerned with will all be *further twists* of such a holomorphic twist. That is, they'll be determined by supercharges $Q = Q_{\mathrm{hol}} + Q'$ where Q' commutes with Q_{hol} but is not obtained from Q_{hol} by the action of some symmetry, so survives in the Q_{hol} twist. All such supercharges are holomorphic or stronger (i.e. at least half the translations are Q-exact); indeed, the image of $[Q,-]$ in $V_{\mathbb{C}}$ contains the image of $[Q_{\mathrm{hol}},-]$.

Remark 8.1.9. For our further twists, we have an isomorphism

$$H^\bullet(\mathcal{A}^{N=4}; Q_{\mathrm{hol}} + Q') \cong H^\bullet(H^\bullet(\mathcal{A}^{N=4}; Q_{\mathrm{hol}}); Q').$$

This is clear for Q' contained entirely in the S_- summand of space of supersymmetries, this follows from the degeneration of the spectral sequence of the double complex for $\mathcal{A}^{N=4}$ where S_+ is placed in bidegree $(1,0)$ and S_- is placed in bidegree $(0,1)$. If instead Q' is contained entirely in the S_+ summand, the complexes $(\mathcal{A}^{N=4}, Q_{\mathrm{hol}} + Q')$ and $(H^\bullet(\mathcal{A}^{N=4}, Q_{\mathrm{hol}}), Q')$ in degrees 0, 1 and 2 split as the sum of two two-step complexes. The claim follows for further twists of form $Q' = \alpha_2 \otimes w$ where $w \in W$ by examining the cohomology of each of these two-step complexes.

We'll investigate which such supercharges Q are topological. Using the same twisting homomorphism ϕ_{KW} as above, we need to check which supercharges in the cohomology above are invariant under the twisted $\mathrm{SU}(2)_-$-action (since they're already $\mathrm{SU}(2)_+$ invariant). In $S_+ \otimes W$ these are just multiples of $\alpha_2 \otimes e_2$. In the other factor, $S_- \otimes W^*$, the group $\mathrm{SU}(2)_-$ acts on the $[Q_{\mathrm{hol}},-]$-cohomology as the module $S_- \otimes (\mathbb{C} \oplus S_-) \cong \mathrm{Sym}^2(S_-) \oplus \mathbb{C}$. The invariant factor is generated by the supercharge $\alpha_1^\vee \otimes f_1^* - \alpha_2^\vee \otimes f_2^*$. As such, the $\mathbb{CP}^1$-family of supercharges

$$Q_{(\mu:\nu)} = Q_{\mathrm{hol}} + (\mu(\alpha_1^\vee \otimes f_1^* - \alpha_2^\vee \otimes f_2^*) + \nu(\alpha_2 \otimes e_2)), \text{ for } (\mu:\nu) \in \mathbb{CP}^1$$

are all topological. This is the *Kapustin-Witten family of topological twists* considered in [**KW06**]. We'll be most interested in the cases where $(\mu:\nu) = (0:1)$ and $(1:0)$. We call these twists the *A-twist* Q_A and the *B-twist* Q_B respectively.

Finally, we'll be interested in a family of supercharges approximating Q_A which are somewhere in between topological and holomorphic; a *three-dimensional* family of translations will be exact for the action of these supercharges. Specifically, we can consider the supercharge

$$Q_\lambda = Q_{\mathrm{hol}} + \lambda(\alpha_2^\vee \otimes f_2^*) + (\alpha_2 \otimes e_2)$$

for each $\lambda \in \mathbb{C}$. These *holomorphic-topological* twists (so called because we think of them as being holomorphic in two real dimensions – i.e. one complex dimension – and topological in the remaining two) converge to Q_A as $\lambda \to 0$. Twists of this form were originally studied by Kapustin [**Kap06**].

8.1.2. Superspace Formalism

The above formalism will allow us to define the action of a supersymmetry algebra on certain theories over $\mathbb{R}^4$, and to produce topologically and holomorphically twisted versions with desirable symmetry properties. However, it'll be important for us to generalize these theories to theories defined on more general manifolds than $\mathbb{R}^4$. We'll do this by *globalizing* the twisted supersymmetry algebras, i.e. realizing them as acting locally on the total spaces of certain *super vector bundles* over our manifolds by infinitesimal symmetries. To set up this so-called "superspace formalism" we'll need some language from supergeometry. By a *super-ring*, we'll just mean a $\mathbb{Z}/2\mathbb{Z}$-graded commutative ring. We'll consider suitable "superspaces" whose local functions form such a super-ring.

Definition 8.1.10. *A supermanifold of dimension $n|m$ is a ringed space (M, C^∞_M) which is locally isomorphic to $(\mathbb{R}^n, C^\infty(\mathbb{R}^n; \mathbb{C})[\varepsilon_1, \ldots, \varepsilon_m])$, where the ε_i are odd variables.*

Remark 8.1.11. Note that we're defining a supermanifold to have a structure sheaf consisting of *complex valued* functions. Such an object is sometimes called a *complex supersymmetric* (or *cs*) supermanifold, for instance by Witten [**Wit12**].

A typical example of the kind of supermanifold we are going to consider is the total space of an odd vector bundle. We can define this as follows.

Example 8.1.12. Let M be a real manifold, and let E be a complex vector bundle on M. Then we define a supermanifold $(\Pi E, C^\infty_{\Pi E})$ by setting $C^\infty_{\Pi E}(U) = C^\infty(U, \wedge^\bullet E^*)$ for each open set $U \subset M$. In particular, if $E = T_M$ is the tangent bundle of M, then the sheaf of smooth functions on $U \subset \Pi E$ is the space $\Omega^\bullet(U)$ of smooth differential forms on U. Supermanifolds diffeomorphic to a supermanifold of this form are called *split*.

We can also define an algebraic analogue.

Definition 8.1.13. *A supervariety of dimension $n|m$ is a ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to $(\operatorname{Spec} R, R[\varepsilon_1, \cdots, \varepsilon_m])$ for a reduced $\mathbb{C}$-algebra R of Krull dimension n.*

Note that every smooth supervariety naturally yields a supermanifold. Our vector bundle example still makes sense in an algebraic sense.

Example 8.1.14. Let X be a smooth complex algebraic variety and E be an algebraic vector bundle on X. Then we define a supervariety $(\Pi E, \mathcal{O}_{\Pi E})$ by setting $\mathcal{O}_{\Pi E}(U) = \mathcal{O}(U, \wedge^\bullet E^*)$. Supervarieties isomorphic to the ones of this form are called *split* supervarieties.

Remark 8.1.15. There is a fundamental difference between the smooth and algebraic settings. In the smooth setting, a theorem of Batchelor [**Bat79**] says that all supermanifolds are split. In the complex algebraic setting this is very much not true, and there are many non-split supervarieties. Luckily, all the examples we'll need to deal with in what follows will be split, so this subtlety will not play a role.

Example 8.1.16. An example of a natural supervariety of this form is the complex *super projective space* $\mathbb{CP}^{n|m}$, modelling the quotient of the supermanifold $\mathbb{C}^{n+1|m} \setminus \{0\}$ under the action of $\mathbb{C}^\times$ by rescaling. Concretely, $\mathbb{CP}^{n|m}$ is the total space of the odd algebraic vector bundle $\Pi(\mathcal{O}(1) \otimes \mathbb{C}^m)$ over $\mathbb{CP}^n$, as one can readily check by analysing the transition functions for the odd coordinates between affine charts.

If we want to do calculus on supermanifolds, we need an analogue of the *canonical bundle* for a supermanifold.

Definition 8.1.17. *For a split supermanifold* ΠE *for* $E \to M$*, we define the* Berezinian *to be the super vector bundle* $\mathrm{Ber}_{\Pi E} = \det(T^*_M \oplus E^*)$ *over* ΠE*. Similarly, for a split supervariety* ΠE *for* $E \to X$*, we define the Berezinian to be the algebraic super vector bundle* $\mathrm{Ber}_{\Pi E} = \det(T^*_X \oplus E^*)$ *over* ΠE*, where* T_X *denotes the algebraic tangent bundle of* X*.*

Example 8.1.18. Let Σ be a smooth curve and L be a line bundle over Σ. For the supervariety ΠL over Σ with projection map $p \colon \Pi L \to \Sigma$, its Berezinian is the bundle $\mathrm{Ber}_{\Pi L} = p^*(K_\Sigma \otimes L^*)$ on ΠL.

Definition 8.1.19. *A* Calabi-Yau structure *on a supervariety* X *is a trivialization of the Berezinian, i.e. a complex vector bundle isomorphism from* Ber_X *to the trivial bundle.*

Now let us globalise the Kapustin-Witten family of topological twists in the language of supergeometry. To do this, we'll find an odd vector bundle ΠE over $\mathbb{C}^2$ and an action of the Q_{hol}-cohomology of the supersymmetry algebra on ΠE extending the natural action of the bosonic symmetries $\mathfrak{so}(3;\mathbb{C}) \oplus \left\langle \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2} \right\rangle$. Since the space of odd symmetries is 9-dimensional, a natural choice for ΠE is the superspace $\mathbb{C}^{2|3} \to \mathbb{C}^2$ (which has a 9-dimensional space of odd vector fields). Choose coordinates $(z_1, z_2, \varepsilon, \varepsilon_1, \varepsilon_2)$ for this superspace, where the complexified rotations $\mathfrak{so}(3;\mathbb{C})$ act on the bosonic coordinates by its spin representation, and the R-symmetries $\mathfrak{sl}(3;\mathbb{C})$ act on the fermionic coordinates. In these coordinates, we define the action of the supersymmetries by the following odd vector fields.

$$
\begin{aligned}
\alpha_2 \otimes e_2 &= \frac{\partial}{\partial \varepsilon} \\
\alpha_2 \otimes f_i &= (-1)^{i+1} \frac{\partial}{\partial \varepsilon_i} \\
\alpha_j^\vee \otimes e_2^* &= \varepsilon \frac{\partial}{\partial z_j} \\
\text{and} \quad \alpha_j^\vee \otimes f_i^* &= (-1)^{i+1} \varepsilon_i \frac{\partial}{\partial z_j}
\end{aligned}
$$

for $i, j \in \{1, 2\}$. This does indeed define an action of the super Lie algebra, i.e. the vector fields satisfy the correct commutation relations. In this notation, the topological supercharges act by the vector fields

$$Q_{(\mu:\nu)} = \left(\mu \left(\varepsilon_1 \frac{\partial}{\partial z_1} + \varepsilon_2 \frac{\partial}{\partial z_2} \right) + \nu \frac{\partial}{\partial \varepsilon} \right).$$

Note that we abuse the notation $Q_{(\mu:\nu)}$ to mean the one in the previous subsection after taking Q_{hol}-cohomology.

It remains to extend these local vector fields to global vector fields on a 4-manifold X. We'll be able to do this if X has the structure of a complex surface. Since $\mathrm{SU}(2)_-$ acts on S_- as the fundamental representation, one can identify $\varepsilon_i = dz_i$ and hence simply write $\varepsilon_1 \frac{\partial}{\partial z_1} + \varepsilon_2 \frac{\partial}{\partial z_2} = \partial$. On the other hand, ε belongs to the trivial representation, and hence should be a trivial odd line bundle. Namely, for a given complex surface X, the global superspace we end up with after the holomorphic twist is $Y = \Pi TX \times \mathbb{C}^{0|1}$, where further twists are described by the algebraic vector fields $\lambda \partial + \mu \frac{\partial}{\partial \varepsilon}$.

If, furthermore, X splits as the product of two smooth algebraic curves $X = \Sigma_1 \times \Sigma_2$, we can globalise the action of the holomorphic-topological twists Q_λ. In the coordinates above, these twists act locally by

$$Q_\lambda = \lambda \varepsilon_2 \frac{\partial}{\partial z_2} + \frac{\partial}{\partial \varepsilon}$$

which, by the argument above, describes the local action of the odd vector field $\lambda \partial_2 + \frac{\partial}{\partial \varepsilon}$ where ∂_2 is the algebraic de Rham operator on Σ_2 only.

8.2. Twisted Supersymmetric Field Theories

Now, let's discuss what we'll mean by a classical field theory, and what it means to twist such an object. The definitions in this chapter will build on the perturbative definitions given by Costello in [**Cos11a**], but extended to a global, non-perturbative setting. In doing so we'll find that, indeed, topological and holomorphic twists give rise to topological and holomorphic field theories respectively, justifying their names (by holomorphic field theories, we mean those where observables depend only on a choice of complex structure on spacetime, not on a choice of metric. In two dimensions this will coincide with the notion of a (chiral) conformal field theory). The supercharge Q with which we wish to twist generates a one-odd-dimensional abelian superalgebra $\mathbb{C}Q$, and the twisted theory will be – perturbatively – defined as something very close to the derived $\mathbb{C}Q$-invariants of the untwisted theory.

Globally, we can define a twist with respect Q as a family of derived stacks over $\mathbb{A}^1$ so that, on the relative tangent bundle to a section, we recover a perturbative twist of the fiber at 0 by Q. In general there is no reason that such global twists should be unique, but in many examples we'll see that there exists a natural choice provided by theorems of Gaitsgory and Rozenblyum.

8.2.1. Classical Field Theories

Costello and Gwilliam [**CG15**] give a beautiful axiomatization of the notion of a perturbative classical field theory amenable to quantization and explicit calculation. The definition we'll give will be a global extension of this definition, but to perform any calculations (especially for quantization) we'll restrict to the world of perturbation theory, and to their language. One should view our definition as encoding the *moduli space of solutions to the equations of motion* in a theory, and Costello and Gwilliam's definition as describing the formal neighborhood of a point in this moduli space. We'll begin by briefly recalling the definition of a perturbative classical field theory.

Remark 8.2.1. In this chapter, by "vector spaces" we'll mean cochain complexes of nuclear Fréchet spaces. We'll use $E^\vee$ to denote the strong dual of a vector space, and $E \otimes F$ will denote the completed projective tensor product. We'll write $\widehat{\mathrm{Sym}}(E)$ for the completed symmetric algebra built using this tensor product.

For a vector bundle E on a space X, we'll use the calligraphic letter $\mathcal{E}$ for its sheaf of sections, and we'll denote by $\mathcal{E}_c$ the corresponding sheaf of *compactly supported* sections. We'll write $E^!$ for the twisted dual bundle $E^\vee \otimes \mathrm{Dens}_X$ where Dens_X is the bundle of *densities*, so there's a natural pairing $E \otimes E^! \to \mathrm{Dens}_X$ of vector bundles.

Definition 8.2.2. *An* elliptic L_∞ algebra E *on a topological space* X *is a local* L_∞ *algebra (as in appendix B) over* X *which is elliptic as a cochain complex. A* perturbative classical field theory *is an elliptic* L_∞ *algebra* E *equipped with a non-degenerate, invariant, symmetric bilinear pairing*

$$\langle -,-\rangle \colon E \otimes E[3] \to \mathrm{Dens}_X$$

where Dens_X *denotes the bundle of densities on* X*. Here* invariant *means that the induced pairing on the sheaf of compactly supported sections*

$$\int_X \langle -,-\rangle \colon \mathcal{E}_c \otimes \mathcal{E}_c[3] \to \mathbb{C}$$

is invariant.

From a perturbative classical field theory in this sense, we can produce a more geometric object. Indeed, the fundamental theorem of deformation theory (as described in appendix B) allows us to associate to a local L_∞ algebra E a sheaf of formal moduli problems BE, and this correspondence provides an equivalence of categories. If the L_∞ algebra E is equipped with a degree k pairing then we say the formal moduli problem BE inherits a *presymplectic form* of degree $k+2$. We use this to motivate a general definition in the language of derived algebraic geometry, using a theory of shifted symplectic structures that is applicable in great generality.

In their 2013 paper [**PTVV13**], Pantev, Toën, Vaquié and Vezzosi define the notion of a *shifted symplectic structure* on a derived Artin stack. We refer to their paper and the paper [**Cal13**] of Calaque for details, but we should note that a k-symplectic structure on $\mathcal{M}$ induces a non-degenerate degree k pairing on the tangent complex $\mathbb{T}_{\mathcal{M}}$, and thus a degree $k-2$ pairing on the shifted tangent complex $\mathbb{T}_{\mathcal{M}}[-1]$. In the recent sequel [**CPT$^+$15**], Calaque, Pantev, Toën, Vaquié and Vezzosi generalize this notion to that of a *shifted Poisson structure*, and prove that this recovers the notion of a shifted symplectic structure when a non-degeneracy condition is imposed (a different proof for Deligne-Mumford stacks only also appeared in an earlier preprint of Pridham [**Pri15**]).

We'll begin by giving an *ideal* definition of a non-perturbative classical field theory that we believe best captures the structure of local classical solutions to the equations of motion.

Definition 8.2.3. *A* classical field theory *on a smooth manifold X is a sheaf $\mathcal{M}$ of (-1)-shifted Poisson derived stacks such that for each open set $U \subset X$, the shifted tangent complex $\mathbb{T}_p[-1]\mathcal{M}(U)$ for a closed point $p \in \mathcal{M}(U)$ is homotopy equivalent to a perturbative classical field theory when equipped with the degree -3 pairing induced from the shifted Poisson bracket.*

Remark 8.2.4. We assume that Costello's assumption of ellipticity is always satisfied in an algebraic setting, in view of the main example of de Rham forms $\Omega^\bullet_{\mathrm{alg}}(X)$ becoming elliptic in the analytic topology by the Dolbeault resolution. It is possible that one needs a more careful definition of ellipticity in an algebraic setting for a treatment of the quantization of algebraic perturbative theories, but this is beyond the scope of the present paper.

In practice, in this paper we'll need to use a modified, algebraic version of this definition. There are several reasons for this.

(1) Since we hope to eventually describe the moduli spaces of interest in the geometric Langlands program as local solutions in a classical field theory, we'll need a model that depends on an *algebraic structure* on the spacetime manifold. As such we won't be able to make sense of classical solutions on a general analytic open set. Instead we'll need to work with a topology whose open embeddings are algebraic maps.
(2) The theories we'll construct will be built using mapping spaces out of spacetime. In general, if a spacetime patch U is not proper, these mapping spaces will be of infinite type, and so it will be technically difficult to describe shifted Poisson structures on them. Rather than getting bogged down in these functional analysis issues we'll simply ask for

a shifted symplectic structure on the *global* sections (with the understanding that a more sophisticated analysis should also recover a global version of the local Poisson bracket used by Costello and Gwilliam).

Remark 8.2.5. We expect that an alternative version of the theory should exist in the analytic topology, using a suitable notion of analytic derived stacks, for example based on the C^∞ dg-manifolds of Carchedi and Roytenberg [**CR12**], the d-manifolds of Joyce [**Joy11**], or on a formalism of Ben-Bassat and Kremnizer (to appear) generalizing to a complex analytic setting their non-Archimedean analytic geometry [**BBK13**].

Definition 8.2.6. *An* algebraic classical field theory *on a smooth proper algebraic variety X is an assignment of a derived stack $\mathcal{M}(U)$ to each Zariski open set $U \subseteq X$, with a (-1)-shifted symplectic structure on the space $\mathcal{M}(X)$ of global sections whose shifted tangent complex $\mathbb{T}_{\mathcal{M}(X)}[-1]$ is homotopy equivalent to the global sections of a perturbative classical field theory when equipped with the degree -3 pairing induced from the shifted symplectic pairing.*

Remarks 8.2.7. (1) We've deliberately left the nature of the "assignment" in the definition imprecise, although we expect that the correct definition is a sheaf of derived stacks. Constructing the restriction maps and finding a symplectic structure – much like investigating the shifted Poisson structure on open sets – will involve subtle functional analytic issues involving Verdier duality on infinite-dimensional stacks which is beyond the scope of the present work. The main theorems of this paper involve a determination of the global sections of a classical field theory on a smooth proper variety, and are expected to need adjustment to extend to sheaves of derived stacks. We hope to discuss this issue elsewhere.

(2) In what follows, we sometimes consider theories defined on not necessarily proper varieties, for instance $\mathbb{C}^n$. We will informally refer to assignments of derived stacks in this general setting also as algebraic classical field theories, even without an analysis of shifted Poisson structures.

(3) We could just as readily have made this definition using a finer topology, the étale topology for instance, but Zariski open sets will be sufficiently general for the examples in the present paper.

The intuition behind this definition is – as we already stated – to encode the idea of the derived moduli spaces of solutions to the equations of motion. Globally, given a space of fields and an action functional we can produce a shifted symplectic derived stack by taking the derived critical locus of the action functional. Locally there are subtleties due to the existence of a boundary (as discussed for instance by Deligne and Freed in their notes on classical field theories [**DF99**]): one can still determine the equations of motion but the space of derived solutions will at best have a shifted Poisson structure.

In what follows we'll single out a special family of algebraic classical field theories which is adapted for discussion of twists of supersymmetric Yang-Mills theories. These will model theories whose classical fields include a 1-form field, which is constrained to describe an algebraic structure on a G-bundle on-shell, and where the rest of the fields are all determined by formal data.

Definition 8.2.8. *A* formal algebraic gauge theory *on a smooth variety X is an algebraic classical field theory $\mathcal{M}$ on X with a map $\sigma\colon \mathrm{Bun}_G(U) \to \mathcal{M}(U)$ for each Zariski open set $U \subseteq X$, such that σ is inf-schematic and induces an equivalence $\mathrm{Bun}_G(U)^{\mathrm{red}} \to \mathcal{M}(U)^{\mathrm{red}}$ of their reduced parts. If a formal algebraic gauge theory $\mathcal{M}$ additionally admits such a map $\pi\colon \mathcal{M}(U) \to \mathrm{Bun}_G(U)$ for each U such that σ is a section of π, then we call $\mathcal{M}$* fiberwise formal.

Remark 8.2.9. We'll see in our examples that there are natural twists of supersymmetric gauge theories that are not of this formal nature, for instance twists that form the total space of a (dg) vector bundle over Bun_G. We'll motivate the appearance of such example by viewing them as natural extensions of formal algebraic gauge theories, but they do not intrinsically fit into the above definition. We think of the definition as a tool that allows us to compute twists of supersymmetric gauge theories.

Example 8.2.10. Given any sheaf $\mathcal{M}$ of derived stacks with elliptic tangent complex and where $\mathcal{M}(X)$ is finitely presented we obtain an algebraic classical field theory by taking the *formal shifted cotangent space* $T^*_{\mathrm{form}}[-1]\mathcal{M}$. At the perturbative level, if E is an elliptic L_∞ algebra this corresponds to taking the direct sum of L_∞-algebras $E \oplus E^![-3]$, with invariant pairing induced from the evaluation pairing $E \otimes E^! \to \mathrm{Dens}_X$. If $\mathcal{M}$ admits a map $\sigma \colon \mathrm{Bun}_G \to \mathcal{M}$ satisfying the hypotheses of definition 8.2.8, then $T^*_{\mathrm{form}}[-1]\mathcal{M}$ defines a formal algebraic gauge theory, using the zero section map associated to the formal shifted cotangent space. Likewise, if $\mathcal{M}$ also admits a map $\pi \colon \mathcal{M} \to \mathrm{Bun}_G$, so that σ is a section as in definition 8.2.8 then the projection map makes $T^*_{\mathrm{form}}[-1]\mathcal{M}$ into a fiberwise formal algebraic gauge theory.

Having given a definition of a classical field theory, let's investigate what it means to *twist* such objects. We'll begin by explaining what it means to twist a perturbative classical field theory, then use this to give a non-perturbative definition of a twist of a formal algebraic gauge theory which will suffice for our examples.

8.2.2. Perturbative Twisting

Definition 8.2.11. *A classical field theory E on a space X with an action of the super Poincaré algebra (such as $\mathbb{R}^n$) is called* supersymmetric *if it admits an action by the super Lie algebra $\mathfrak{so}(n,\mathbb{C}) \ltimes \mathbb{C}^4 \oplus \Pi((S_+ \otimes W) \oplus (S_- \otimes W^*))$ extending the natural action of the Poincaré algebra for some vector space W (for a definition of a superalgebra action on a local L_∞ algebra, see the appendix, definition B.2).*

We'll be interested in supersymmetric field theories where the action extends to an action of the full supersymmetry algebra for some choice of R-symmetries. In our examples for $N = 4$, this will be the case with the subalgebra $\mathfrak{sl}(4;\mathbb{C}) \subseteq \mathfrak{gl}(4;\mathbb{C})$ of (complexified) R-symmetries preserving a trivialization of the determinant bundle.

The data required to twist a classical field theory is the action of a certain supergroup. Define a supergroup

$$H = \mathbb{C}^\times \ltimes \Pi\mathbb{C}$$

where $\mathbb{C}^\times$ acts with weight 1. This group arises as the group of automorphisms of the odd complex line.

Definition 8.2.12. Twisting data *for a classical field theory Φ on a space X is a local action (α, Q) of H on $\Phi(U)$ for all U. That is, in the perturbative case Φ is a sheaf of L_∞ algebras with H-module structure, and in the non-perturbative case Φ is a family of derived stacks with H-action. In our notation, α is a $\mathbb{C}^\times$ action, and Q is an odd infinitesimal symmetry with α-weight 1.*

An important source of twisting data is a supersymmetry action. Let Q be a supercharge such that $[Q, Q] = 0$, and let α be a $\mathbb{C}^\times$ action such that Q has weight one (we can always find such an action by choosing a suitable $\mathbb{C}^\times$ in the group of R-symmetries, after choosing an exponentiation of the action of the R-symmetry algebra to an action of an R-symmetry *group*.) Since $[Q, Q] = 0$, the supercharge Q generates a subalgebra isomorphic to $\Pi\mathbb{C}$ acting on any theory with the appropriate supersymmetry action, and along with α this defines an action of the supergroup H.

Lemma 8.2.13. *There is an equivalence of categories*

$$\{\textit{super vector spaces with an } H\textit{-action}\} \cong \{\textit{super cochain complexes}\}.$$

Here the grading is given by the weight under the action of $\mathbb{C}^\times$ and the differential is given by the action of $\Pi\mathbb{C}$. We use this fact to define a twisted theory for the data (α, Q).

Definition 8.2.14. *Let E be a perturbative classical field theory with an action of the supergroup H. The* twisted theory *E^Q (where Q is a generator of $\Pi\mathbb{C}$) is the theory obtained by introducing a new differential graded structure on E in accordance with the previous lemma and taking the total complex with respect to this new grading and the cohomological grading.*

Remark 8.2.15. The twisted theory E^Q fits into a family of classical field theories deforming E – i.e. a sheaf of perturbative field theories over the line $\mathbb{A}^1$ – whose fiber at λ is the theory obtained by applying the twisting construction with respect to the dilated twisting data $(\lambda Q, \alpha)$.

Remarks 8.2.16. This definition needs some unpacking. We should explain what we want to do intuitively, in particular the role of the action α.

- On the level of functions – that is, observables – our first idea is to take the Q-coinvariants. By identifying observables with their orbits under Q we force all Q-exact symmetries to act trivially, so if we choose a holomorphic or topological supercharge we impose strong symmetry conditions on the observables in the twisted theory. The naïve thing to do to implement this procedure would be to take the derived invariants of our classical field theory with respect to the group $\Pi\mathbb{C}$ generated by Q.
- This is all well and good, but recall what a $\Pi\mathbb{C}$-action actually means: the data of a family of classical field theories over the space $B(\Pi\mathbb{C})$ whose fiber over zero is E. That is, a module over $\mathbb{C}[[t]]$, where t is a fermionic degree 1 parameter. One really wants to restrict interest to a generic fiber of this family.
- To do this we restrict to the odd formal *punctured* disc, or equivalently invert the parameter t, then take invariants for an action α of $\mathbb{C}^\times$ for which t has weight 1, thus extracting a "generic" fiber instead of the special fiber at 0. This is an instance of the Tate construction for the homotopy $\Pi\mathbb{C}$ action Q. It's important to restrict to the formal punctured disc, since not all these invariant fields extend across zero: if we just took $\mathbb{C}^\times$ invariants in $\mathcal{E}[[t]]$ we'd obtain elements of $\mathcal{E}$ of the form ϕt^k where ϕ had weight $-k$. In particular we'd find ourselves throwing away everything of positive $\mathbb{C}^\times$ weight in $\mathcal{E}$.
- Now, this procedure is exactly the same as the definition we gave above. Taking derived Q invariants corresponds to taking the complex $\mathcal{E}[[t]]$ with differential $d_\mathcal{E} + tQ$. Inverting t and taking invariants under the action α is then the same as adding the α weight to the original grading, and adding the operator Q to the original differential $d_\mathcal{E}$, just as in our definition.

Proposition 8.2.17. *The twisted theory E^Q is still a classical field theory when equipped with a pairing inherited from E.*

Proof. First note that E^Q is still an elliptic L_∞ algebra. The complex obtained as the $\Pi\mathbb{C}$ invariants of the theory – the complex $(\mathcal{E}[[t]], d_{\mathcal{E}} + tQ)$ – is required to have the structure of an elliptic L_∞ algebra by the definition of a group action on a field theory. Inverting t preserves this structure, as does taking $\mathbb{C}^\times$-invariants, again because α is a local L_∞-action.

It remains to construct an invariant pairing on E^Q of the correct degree (we'll follow Costello [**Cos11a**, 13.1]). The pairing on $\mathcal{E}$ induces a degree -3 pairing of form

$$\langle -, - \rangle_Q \colon E[[t]] \otimes E[[t]][3] \to \mathrm{Dens}_X[[t]]$$

by $\langle e_1 t^{k_1}, e_2 t^{k_2} \rangle_Q = \langle e_1, e_2 \rangle t^{k_1+k_2}$. We only need to check that this is compatible with the differential $d_{\mathcal{E}} + tQ$, i.e. that exact terms on the left vanish under the pairing map, or more precisely that

$$\left(\langle d_{\mathcal{E}} f_1, f_2 \rangle + \langle f_1, d_{\mathcal{E}} f_2 \rangle \right) t^{k_1+k_2} + \left(\langle Q f_1, f_2 \rangle + \langle f_1, Q f_2 \rangle \right) t^{k_1+k_2+1} = 0.$$

The first term vanishes because of compatibility of $d_{\mathcal{E}}$ with the pairing, and the second term vanishes because Q is a symmetry of the classical field theory. This pairing yields an invariant $\mathrm{Dens}_X((t))$ valued pairing after inverting t. By construction these pairings are equivariant with respect to the action of $\mathbb{C}^\times$ by rescaling t, so descends to a pairing

$$\langle -, - \rangle_Q \colon \left(E((t)) \otimes E((t))[3] \right)^{\mathbb{C}^\times} \to \mathrm{Dens}_X((t))^{\mathbb{C}^\times} = \mathrm{Dens}_X\,.$$

This pairing is still invariant, so gives E^Q the structure of a classical field theory. $\square$

8.2.3. Global Twisting

Now, let $\mathcal{M}$ be a non-perturbative algebraic classical field theory on $\mathbb{C}^n$, and suppose $\mathcal{M}$ admits an action of a supersymmetry algebra extending the action of the translations. As above, choose a supercharge Q satisfying $[Q,Q]=0$, and an action α of $\mathbb{C}^\times$ on $\mathcal{M}$ so that Q has α-weight one.

Definition 8.2.18. *A* deformation *of a derived stack $\mathcal{X}$ is a derived stack $\pi\colon \mathcal{X}' \to \mathbb{A}^1$ flat over the affine line along with an immersion $\mathcal{X} \hookrightarrow \mathcal{X}'_0$, and an equivalence $\mathcal{X}'|_{\mathbb{G}_m} \cong \mathcal{X}'_1 \times \mathbb{G}_m$, where $\mathcal{X}'_t$ is the fiber over the point t.*

We'll begin with a prototypical example of a deformation, presented somewhat informally for motivation. We'll provide a more conceptual and general treatment of the example later in 8.2.28.

Example 8.2.19. Consider a smooth proper variety X. We define a ringed space X_{Dol} by $X_{\mathrm{Dol}} := (X, \mathcal{O}_{T[1]X})$, where the structure sheaf $\mathcal{O}_{T[1]X}$ is equivalent to $\mathrm{Sym}_X(\mathbb{L}_X[-1]) = \Omega^\bullet_{\mathrm{alg},X}$. As one has a quasi-isomorphism $\Omega^p_{\mathrm{alg},X} \simeq (\mathcal{A}^{p,\bullet}_X, \overline{\partial})$ in the analytic topology, X_{Dol} is justifiably called the *Dolbeault stack* of X. Similarly, one defines $X_{\lambda\text{-dR}}$ to be the ringed space $(X, (\Omega^\bullet_{\mathrm{alg},X}, \lambda\partial))$. Of course, $X_{\mathrm{dR}} := X_{1\text{-dR}}$ is called the *de Rham stack* of X because one has $(\Omega^\bullet_{\mathrm{alg},X}, \partial) \simeq (\mathcal{A}^{\bullet;\bullet}_X, \partial + \overline{\partial}) \simeq (\mathcal{A}^\bullet_X, d)$ in the analytic topology. It will sometimes be convenient to write $X_{0\text{-dR}}$ for X_{Dol}.

There exists a ringed space X_{Hod} and a map $X_{\mathrm{Hod}} \to \mathbb{A}^1$ such that the fiber over λ is $(X, (\Omega^\bullet_{\mathrm{alg},X}, \lambda\partial))$. That is, both squares in the following diagram are fiber product squares.

$$\begin{tikzcd} X_{\mathrm{Dol}} \arrow[r] \arrow[d] & X_{\mathrm{Hod}} \arrow[d] & X_{\lambda\text{-dR}} \arrow[l] \arrow[d] \\ \{0\} \arrow[r] & \mathbb{A}^1 & \{\lambda\} \arrow[l] \end{tikzcd}$$

In particular, X_{Hod} is a deformation of X_{Dol}.

Now we would like to write down this information in a way that can be easily generalized to other situations. First, observe that as X_{Dol} and $X_{\lambda\text{-dR}}$ have the same closed points, they differ only by an infinitesimal thickening from the original space X. In order to write this more carefully, let us introduce the canonical map $\sigma_\lambda \colon X \to X_{\lambda\text{-dR}}$. Then we would like to compare $\mathbb{T}_{\sigma_0(x)} X_{\mathrm{Dol}}$ and $\mathbb{T}_{\sigma_\lambda(x)} X_{\lambda\text{-dR}}$ for every $x \in X$. A way to compare them is to find a section $s \colon \mathbb{A}^1 \to X_{\mathrm{Hod}}$ so that both of them are realized as fibers of $s^* \mathbb{T}_{X_{\mathrm{Hod}}/\mathbb{A}^1}$. If that is the case, then one declares X_{dR} to be a *twist* of X_{Dol}.

On the other hand, in general, one might not have a map playing the role of σ_λ, even if we started with a map σ_0 which is an equivalence at the level of closed points. Then it would be reasonable to ask for compatibility for every point $x_1 \in X_{\mathrm{dR}}$. Namely, for a closed point $x_1 \in X_{\mathrm{dR}}$, we ask the existence of a section $s \colon \mathbb{A}^1 \to X_{\mathrm{Hod}}$ such that

(1) $s(0) = \sigma_0(x)$ for some $x \in X$,

(2) $s(\lambda) = x_\lambda$ for some $x_\lambda \in X_{\lambda\text{-dR}}$, and

(3) $s^* \mathbb{T}_{X_{\mathrm{Hod}}/\mathbb{A}^1}$ is a deformation of $\mathbb{T}_{\sigma_0(x)} X_{\mathrm{Dol}}$.

Even only with this weaker requirement, we think of X_{dR} as a twist of X_{Dol}.

When we define a twist of a formal algebraic gauge theory, there are two additional small complications to be introduced. Firstly, given twisting data (α, Q), before twisting by Q we need to deal with modifying the gradings by the $\mathbb{C}^\times$-weight under α.

Definition 8.2.20. *A* regrading *of a formal algebraic gauge theory $\mathcal{M}$ with respect to a $\mathbb{C}^\times$ action α such that $\sigma \colon \mathrm{Bun}_G \to \mathcal{M}$ is equivariant for the trivial action on Bun_G is a formal algebraic gauge theory $\sigma_\alpha \colon \mathrm{Bun}_G \to \mathcal{M}^\alpha$ such that the restricted tangent complex $\sigma_\alpha^* \mathbb{T}_{\mathcal{M}^\alpha}[-1]$ is equivalent to the restricted tangent complex of $\mathcal{M}$ with degrees modified by adding the α-weight to the cohomological degree and the α-weight mod 2 to the fermionic degree, as a sheaf of Lie algebras.*

The second complication is that a perturbative classical field theory consists of more data than just a cochain complex, and our twist must preserve this additional information on the level of each tangent complex, in the sense discussed in the previous chapter on twists of perturbative field theorys.

Bearing these two points in mind, by mimicking the motivating example with X replaced by Bun_G, we obtain the following definition.

Definition 8.2.21. *A classical non-perturbative field theory $\mathcal{M}^Q$ is a* twist *of a formal algebraic gauge theory $\mathcal{M}$ with respect to twisting data (α, Q) if there is a deformation $\pi \colon \mathcal{M}' \to \mathbb{A}^1$ of the regrading $\mathcal{M}^\alpha$, whose generic fiber is equivalent to $\mathcal{M}^Q$, such that for every closed point $x_1 \in \mathcal{M}^Q$, there is a section $s \colon \mathbb{A}^1 \to \mathcal{M}'$ of the map π such that*

(1) *$s(0) = \sigma_\alpha(x)$ for some $x \in \mathrm{Bun}_G$,*

(2) *$s(\lambda) = x_\lambda$ for some $x_\lambda \in \mathcal{M}^{\lambda Q}$, and*

(3) *$s^*\mathbb{T}_{\mathcal{M}'/\mathbb{A}^1}$ is a perturbative twist of $\mathbb{T}_{\sigma(x)}\mathcal{M}$ with respect to the given twisting data as in remark 8.2.15.*

Remark 8.2.22. One could define twists of more general classical field theories as long as they could be viewed as formal extensions of some fixed base stack (playing the role of Bun_G in the above definition). For example, one could replace Bun_G by maps into a target other than BG to describe twists of supersymmetric sigma models, or if $\mathcal{M}^Q = T^*[-1]\mathcal{B}$ was a cotangent theory one might use the base space $\mathcal{B}$.

Remark 8.2.23. One ought to be able to produce twisted field theories explicitly from a functor-of-points perspective, along the lines of a construction explained by Grady and Gwilliam [**GG14**]. Let $\mathcal{L}$ be an L_∞ space (we refer the reader to Grady-Gwilliam or Costello [**Cos11a**] for details concerning the theory of L_∞ spaces) over a scheme M whose fibers are finitely generated and concentrated in non-negative degrees, and let $\mathcal{L}$ be equipped with a degree -3 invariant pairing on its fibers making it into a sheaf of perturbative classical field theories. Then we can attempt to build a non-perturbative classical field theory out of $\mathcal{L}$ as follows. Let $\mathcal{L}_{>0}$ be the truncation in positive degrees: a *nilpotent* L_∞ space, and let L_0 be the degree 0 piece: a sheaf of Lie algebras. We can attempt to construct a sheaf $\mathcal{M}$ of derived stacks over M by a Maurer-Cartan procedure. To do so, choose an exponentiation of L_0 to a sheaf of algebraic groups G. Define, for a cdga R

concentrated in non-positive degrees, the R-points of $\mathcal{M}$ over U by

$$\mathcal{M}(U)(R) = \mathrm{MC}(\mathcal{L}_{>0}(U) \otimes R)/G(U)(R).$$

We can easily compute the shifted tangent complex at a point $p \in \mathcal{M}$, since

$$\begin{aligned}\mathbb{T}_p[-1]\mathcal{M} &= \mathbb{T}_p[-1](\mathrm{MC}(\mathcal{L}_{>0})/G) \\ &\cong (\mathcal{L}_0)_p \to (\mathcal{L}_{>0})_p \\ &= \mathcal{L}_p,\end{aligned}$$

so we recover the perturbative theory. Grady and Gwilliam [**GG14**] prove that this construction satisfies a descent condition, albeit a weaker condition than the condition we've demanded for derived stacks. We anticipate that applying this construction to the twist of a perturbative classical field theory will yield a non-perturbative twisted theory, compatibly with the examples we construct elsewhere in the paper.

We'll construct twists of the $N = 4$ theories of interest to us in chapter 9 below, but why should the twisted theory with respect to specified twisting data be well-defined? Well, for many theories of the type we're considering it is possible to recover the full non-perturbative theory from a family of perturbative theories parametrized by Bun_G. This follows from a theorem of Gaitsgory and Rozenblyum [**GRd**]. Even when this formal procedure fails we'll see that the Gaitsgory-Rozenblyum correspondence often provides a natural choice of twist.

The following definition, also due to Gaitsgory and Rozenblyum [**GRc**], models in derived algebraic geometry a family of formal moduli problems as described in appendix B over a base derived stack $\mathcal{X}$, coherently equipped with base points.

Definition 8.2.24. *A* pointed formal moduli problem $\mathcal{Y}$ *over a derived stack* $\mathcal{X}$ *is an inf-schematic morphism* $\pi\colon \mathcal{Y} \to \mathcal{X}$ *of prestacks with an inf-schematic section* $\sigma\colon \mathcal{X} \to \mathcal{Y}$ *such that the induced* *map* $\pi^{\mathrm{red}}\colon \mathcal{Y}^{\mathrm{red}} \to \mathcal{X}^{\mathrm{red}}$ *is an isomorphism. We'll denote the category of pointed formal moduli problems over* $\mathcal{X}$ *by* $\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$*.*

Theorem 8.2.25 (Gaitsgory-Rozenblyum [**GRc**, 1.6.4] [**GRd**, 3.1.4])**.** *For a derived stack* $\mathcal{X}$ *which is locally almost of finite type there is an equivalence*

$$F\colon \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}) \to \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})),$$

where $\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))$ *is the category of Lie algebra objects in ind-coherent sheaves on* $\mathcal{X}$*.*

We can now more succinctly say that a fiberwise formal algebraic gauge theory is an assignment to open sets in X of pointed formal moduli problems over Bun_G, with the structure of an algebraic classical field theory on its total space. Theorem 8.2.25 therefore says that fiberwise formal algebraic gauge theories are completely determined by Lie algebra objects in sheaves over Bun_G. We'll take advantage of this, and define the twist of a fiberwise formal algebraic gauge theory using this sheaf of Lie algebras.

It will be useful to unpack what exactly the functor in the theorem is. It is constructed as a composition of two equivalences

$$\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}) \xrightarrow{\Omega_{\mathcal{X}}} \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}) \xrightarrow{\mathrm{Lie}} \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})),$$

where $\mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}})$ stands for the category of group objects in $\mathrm{FormMod}_{/\mathcal{X}}$. Here $\Omega_{\mathcal{X}}$ is the based loop space functor $\mathcal{Y} \mapsto \Omega_{\mathcal{X}}\mathcal{Y} = \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ and Lie is the functor given by $\mathcal{H} \mapsto \mathbb{T}_{\mathcal{H}/\mathcal{X}}|_{\mathcal{X}}$, so that the composition in terms of the underlying ind-coherent sheaf is simply $\mathcal{Y} \mapsto \mathbb{T}_{\mathcal{Y}/\mathcal{X}}|_{\mathcal{X}}[-1]$. In other words, one can write $F = \sigma^*\mathbb{T}_{/\mathcal{X}}[-1]$, the restricted relative shifted tangent complex.

Now, we'll discuss a construction of twists of fiberwise formal algebraic gauge theories. In order to give as general a construction as possible we'll need to consider a stronger form of the Gaitsgory-Rozenbylum correspondence than theorem 8.2.25, also due to Gaitsgory-Rozenblyum. This is because a fiberwise formal algebraic gauge theory does not necessarily remain fiberwise formal when we twist: in general the twisting data will not preserve the fibers of the projection map π, so this structure is lost upon twisting.

Consider the commutative diagram:

$$\begin{tikzcd} \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}) \arrow[r, "\Omega_{\mathcal{X}}"] \arrow[d, "\mathrm{forget}"'] & \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}) \arrow[r, "\mathrm{Lie}"] \arrow[d, "\mathrm{forget}"] & \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})) \arrow[d] \\ \mathrm{FormMod}_{\mathcal{X}/} \arrow[r, "\Omega_{\mathcal{X}}"] & \mathrm{FormGrpoid}(\mathcal{X}) \arrow[r, "\mathrm{Lie}"] & \mathrm{LieAlgebroid}(\mathcal{X}). \end{tikzcd}$$

Here $\mathrm{FormMod}_{\mathcal{X}/}$ stands for the category of formal moduli problems under $\mathcal{X}$, so that a formal algebraic gauge theory is exactly an algebraic classical field theory – given by a family of formal moduli problems – under $\mathcal{X} = \mathrm{Bun}_G$. The other categories are also defined in Gaitsgory-Rozenblyum, but for our purposes it will suffice to note that the abusive notations $\Omega_{\mathcal{X}}$ and Lie still realize equivalences and that the forgetful functor from $\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$ to $\mathrm{FormMod}_{\mathcal{X}/}$ is given by the natural identification $\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}) = (\mathrm{FormMod}_{\mathcal{X}/})_{/\mathcal{X}}$. We'll now state the necessary generalization of theorem 8.2.25.

Theorem 8.2.26 (Gaitsgory-Rozenblyum [**GRc**, 2.3.2] [**GRe**, 2.1])**.** *The functor*

$$\mathrm{Lie} \circ \Omega_{\mathcal{X}} \colon \mathrm{FormMod}_{\mathcal{X}/} \to \mathrm{LieAlgebroid}(\mathcal{X})$$

is an equivalence for any derived stack $\mathcal{X}$ locally almost of finite type.

We don't define the general notion of Lie algebroids here, referring the reader instead to Gaitsgory-Rozenblyum [**GRe**] for details. In the present paper essentially only two types of examples of Lie algebroids will appear, the initial object and the terminal object in the category $\mathrm{LieAlgebroid}(\mathcal{X})$, so we'll use a more concrete way to think about them in terms of an anchor map. Namely, we use the forgetful functor

$$\mathrm{Anch} \colon \mathrm{LieAlgebroid}(\mathcal{X}) \to \mathrm{IndCoh}(\mathcal{X})_{/\mathbb{T}_{\mathcal{X}}}$$

defined by sending the formal moduli problem $\mathcal{X} \to \mathcal{Y}$, which we identify with a Lie algebroid by theorem 8.2.26, to $\mathbb{T}_{\mathcal{X}/\mathcal{Y}} \to \mathbb{T}_{\mathcal{X}}$, where the map is induced from the identity $\mathbb{T}_{\mathcal{X}} \to \mathbb{T}_{\mathcal{X}}$. In particular we have $\mathrm{Anch}(\mathcal{X} \to \mathcal{X}) = (0 \to \mathbb{T}_{\mathcal{X}})$, which we call the *zero Lie algebroid*, and $\mathrm{Anch}(\mathcal{X} \to \mathcal{X}_{\mathrm{dR}}) = (\mathrm{id}\colon \mathbb{T}_{\mathcal{X}} \to \mathbb{T}_{\mathcal{X}})$, which we call the *tangent Lie algebroid*.

At this point we'll introduce our main example: the de Rham prestack arising as a deformation of the formal 1-shifted tangent bundle. Before we do so we'll introduce some relevant geometric objects originally constructed by Simpson [**Sim97**, **Sim98**, **Sim09**].

Definition 8.2.27. *A* λ-connection *on an algebraic G-bundle P over a smooth complex variety X is a map*

$$\partial_\lambda \colon \Omega^0_{\mathrm{alg}}(X; \mathfrak{g}_P) \to \Omega^1_{\mathrm{alg}}(X; \mathfrak{g}_P)$$

such that $\partial_\lambda(f \cdot s) = \lambda(\partial f)s + f\partial_\lambda s$ for $f \in \mathcal{O}_X$ and $s \in \Omega^1_{\mathrm{alg}}(X; \mathfrak{g}_P)$. A λ-connection ∂_λ is called flat *if $\partial_\lambda^2 = 0$, where ∂_λ naturally extends to a map $\Omega^i_{\mathrm{alg}}(X; \mathfrak{g}_P) \to \Omega^{i+1}_{\mathrm{alg}}(X; \mathfrak{g}_P)$ for all i.*

In particular, if $\lambda \neq 0$ and ∂_λ is a flat λ-connection, then $\lambda^{-1}\partial_\lambda$ is an algebraic flat connection on an algebraic G-bundle. If $\lambda = 0$ then a flat λ-connection is a section ϕ of $\Omega^1_{\mathrm{alg}}(X; \mathfrak{g}_P)$ satisfying $[\phi, \phi] = 0$: a Higgs field.

Example 8.2.28. Let $\mathcal{X}$ be a derived Artin stack. We can define a prestack $\mathcal{X}_{\mathrm{Hod}}$, the *Hodge prestack* of $\mathcal{X}$, as a deformation of the formal 1-shifted tangent bundle $T_{\mathrm{form}}[1]\mathcal{X}$. Such a deformation is – by definition – a flat morphism $\pi\colon \mathcal{Y} \to \mathbb{A}^1$ with $\mathcal{Y}_0 = T_{\mathrm{form}}[1]\mathcal{X}$ and $\mathcal{Y}|_{\mathbb{G}_m} \cong \mathcal{Y}_1 \times \mathbb{G}_m$. We first construct a formal moduli under $X \times \mathbb{A}^1$. Having $T_{\mathrm{form}}[1]\mathcal{X}$ as an object of $\mathrm{FormMod}_{\mathcal{X}/}$ using theorem 8.2.26, whose associated Lie algebroid is $0\colon \mathbb{T}_{\mathcal{X}} \to \mathbb{T}_{\mathcal{X}}$, one can easily think of its deformation λQ parametrized by $\lambda \in \mathbb{A}^1$ with $Q = \mathrm{id}\colon \mathbb{T}_{\mathcal{X}} \to \mathbb{T}_{\mathcal{X}}$ in the category of Lie algebroids: this gives rise to a formal moduli problem under $\mathcal{X} \times \mathbb{A}^1$. It remains to construct a map down to $\mathbb{A}^1$ for which we refer to Gaitsgory-Rozenblyum [**GRf**], where this map is constructed as an example of a more general "scaling" construction, applied to the prestack $\mathcal{X}_{\mathrm{dR}}$.

We denote the fiber of X_{Hod} over a point $\lambda \in \mathbb{C}$ by $X_{\lambda\text{-dR}}$. The fiber over $\lambda = 1$ is the usual de Rham prestack X_{dR} – since the formal moduli problem $\mathcal{X}_{\mathrm{dR}}$ under $\mathcal{X}$ corresponds to the tangent Lie algebroid $\mathrm{id}\colon \mathbb{T}_{\mathcal{X}} \to \mathbb{T}_{\mathcal{X}}$ – and the fiber over $\lambda = 0$ is also called the *Dolbeault stack*, and denoted X_{Dol}. We denote the mapping stack into BG by

$$\underline{\mathrm{Map}}(X_{\lambda\text{-dR}}, BG) = \mathrm{Loc}_G^{\lambda}(X).$$

It represents flat λ-connections on X when X is a smooth variety. When $\lambda = 0$ we recover the moduli stack of Higgs bundles on X for the group G.

Remark 8.2.29. Simpson [**Sim09**] originally gave a different definition in the case where X is a scheme, modelling X_{Hod} as a groupoid in schemes living over $\mathbb{A}^1$. First form the deformation to the normal cone of the diagonal map $\Delta\colon X \hookrightarrow X \times X$. This is a $\mathbb{G}_m$-equivariant scheme living over $\mathbb{A}^1$ whose fiber over $\lambda \neq 0$ is just $X \times X$ with X included diagonally, and whose fiber over 0 is the tangent space TX with X included as the zero section. Form the formal completion of $X \times \mathbb{A}^1$ inside this total space. This admits two maps to $X \times \mathbb{A}^1$ inherited from the two projections $X \times X \to X$,

$$\mathrm{Def}(\Delta)^{\wedge}_{X \times \mathbb{A}^1} \rightrightarrows X \times \mathbb{A}^1.$$

The Hodge prestack X_{Hod} is equivalent to the coequalizer of these arrows in the category of stacks. For $\lambda = 1$, it coincides with the usual definition of the de Rham prestack X_{dR}. For $\lambda = 0$, the coequalizer of the trivial action $T_{\mathrm{form}}X \rightrightarrows X$ is the relative classifying space $B_X T_{\mathrm{form}}X$ of the sheaf $T_{\mathrm{form}}X$ of formal groups over X, which in turn is the same as $T_{\mathrm{form}}[1]X$ by the discussion below the theorem 8.2.25: the two prestacks arise from the same Lie algebra.

With this apparatus in hand, one can construct twists of fiberwise formal algebraic gauge theories, as long as the twisting data is compatible with the structure map $\sigma\colon \mathrm{Bun}_G \to \mathcal{M}$, so that a twist exists within the category of formal algebraic gauge theories. Let $\mathcal{M}$ be a fiberwise formal algebraic gauge theory acted on by twisting data (α, Q) preserving the fibers of the map σ. This condition will be necessary for a natural twist to exist within formal algebraic gauge theories. Let's be clear about precisely what compatibility we require between the structure maps of out formal algebraic gauge theories and the H-action.

Definition 8.2.30. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of derived stacks, and suppose that the supergroup H acts on $\mathcal{Y}$. We say that the H-action* preserves the fibers *of the map f if the image of the map $df\colon \mathbb{T}_{\mathcal{X}} \to f^*\mathbb{T}_{\mathcal{Y}}$ is invariant under the H-action. In particular this makes the relative tangent complex $\mathbb{T}_{\mathcal{X}/\mathcal{Y}}$ into a sheaf of H-representations.*

We will proceed by defining the canonical twist for the case of σ and π both being preserved by the twisting data and of σ being preserved independently first and show that these two are compatible.

Definition 8.2.31 (Twisting a fiberwise formal algebraic gauge theory)**.** *Let $\mathcal{M}$ be a fiberwise formal algebraic gauge theory with $\sigma\colon \mathrm{Bun}_G \to \mathcal{M}$ and $\pi\colon \mathcal{M} \to \mathrm{Bun}_G$. We always assume that the action of H on Bun_G is trivial.*

(1) *Suppose that the twisting data (α, Q) preserves the fibers of both σ and π. Then $\mathcal{M}$ – as a Lie algebra object in $\mathrm{IndCoh}(\mathrm{Bun}_G)$ by theorem 8.2.25 – has a twist $\mathcal{M}^Q$ in the same category by proposition 8.2.17, which in turn can be identified with a fiberwise formal algebraic gauge theory by theorem 8.2.25.*

(2) *Suppose that the twisting data (α, Q) preserves the fibers of σ. An H-equivariant map $\sigma\colon \mathrm{Bun}_G \to \mathcal{M}$ gives an ind-coherent sheaf $\mathbb{T}_{\mathrm{Bun}_G/\mathcal{M}}$ with H-action, while an H-equivariant map $\mathcal{M} \to (\mathrm{Bun}_G)_{\mathrm{dR}}$ under Bun_G gives a map $\mathbb{T}_{\mathrm{Bun}_G/\mathcal{M}} \to \mathbb{T}_{\mathrm{Bun}_G}$ of ind-coherent sheaves with H-action by theorem 8.2.26. Hence we can define the twisted anchor map as the twist of $\mathrm{anch}(\mathbb{T}_{\mathrm{Bun}_G/\mathcal{M}})$ which is still an object of $\mathrm{IndCoh}(\mathrm{Bun}_G)_{/\mathbb{T}_{\mathrm{Bun}_G}}$.*

Note that in the first case, one retains a Lie algebra structure, which by theorem 8.2.25 gives rise to a pointed formal moduli over $\mathrm{Bun}_G \times \mathbb{A}^1$. Note that the projection down to $\mathbb{A}^1$ supplies the structure of a twist in the sense of 8.2.21; the necessary section is given by composing the pointing with the map $\mathbb{A}^1 \to \mathbb{A}^1 \times \mathrm{Bun}_G$ associated to a closed point of Bun_G. In the second case we only obtain an ind-coherent sheaf with an anchor map to $\mathbb{T}_{\mathrm{Bun}_G}$. These two definitions of twist are compatible.

Proposition 8.2.32. *Given a fiberwise formal algebraic gauge theory $\mathcal{M}$ with twisting data preserving the fibers of both σ and π, the anchor map of the twisted theory* $\mathrm{anch}(\mathbb{T}_{\mathrm{Bun}_G/\mathcal{M}^Q})$ *is equivalent to the twist of the anchor* $\mathrm{anch}(\mathbb{T}_{\mathrm{Bun}_G/\mathcal{M}})$.

Proof. Because the twist $\mathcal{M}^Q$ is still a fiberwise formal algebraic gauge theory, its anchor map is zero. The underlying ind-coherent sheaves of both the theory obtained by applying the functor anch to the twisted theory $\mathcal{M}^Q$, and the theory obtained by twisting $\mathrm{anch}(\mathbb{T}_{\mathrm{Bun}_G/\mathcal{M}})$ coincide, and hence we must only check that if our twisting data is equivariant for π then the twisted anchor map defined in definition 2 is zero. In this case we can factor the anchor map $\mathbb{T}_{\mathrm{Bun}_G/\mathcal{M}} \to \mathbb{T}_{\mathrm{Bun}_G}$ through zero as maps of H-representations, by applying the functor of theorem 8.2.26 to the diagram

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "\pi"] & \mathrm{Bun}_G \arrow[r] & (\mathrm{Bun}_G)_{\mathrm{dR}} \\
\mathrm{Bun}_G \arrow[u, "\sigma"] \arrow[ur, equal] \arrow[urr] & &
\end{tikzcd}$$

in formal moduli problems under Bun_G. Because these maps are H-equivariant the twisted anchor map from the twist of $\mathbb{T}_{\mathrm{Bun}_G/\mathcal{M}}$ still factors through the zero bundle, so is the zero map. $\square$

With this proposition in mind, we'll abuse notation and always refer to the twisted anchor map as $\mathbb{T}_{\mathrm{Bun}_G/\mathcal{M}^Q}$, even if the twisting data does not preserve the fibers of π. In some examples we can promote this anchor map to a unique Lie algebroid, and therefore to a unique formal algebraic gauge theory.

Definition 8.2.33. *A deformation L' of a Lie algebroid L on a derived stack $\mathcal{X}$ is a Lie algebroid on $\mathcal{X} \times \mathbb{A}^1$ such that the moduli problem under $\mathcal{X}$ corresponding to L via theorem 8.2.26 and the moduli problem under $\mathcal{X}$ obtained by restricting the moduli problem associated to L' to $\mathcal{X} \times \{0\}$ coincide.*

Lemma 8.2.34. *If the twisted family* $\mathbb{T}_{\mathrm{Bun}_G/\mathcal{M}^{\lambda Q}} \in \mathrm{IndCoh}(\mathrm{Bun}_G)_{/\mathbb{T}_{\mathrm{Bun}_G}}$ *for* $\lambda \in \mathbb{A}^1$ *is the image under the functor* Anch *of a deformation in* LieAlgebroid(Bun_G)*, deforming the Lie algebroid corresponding to* $\mathcal{M}$ *then there exists a formal moduli problem* $\mathcal{M}'$ *under* $\mathcal{M} \times \mathbb{A}^1$*, corresponding to a deformation of a Lie algebroid, with respect to the twisting data* (α, Q)*. If this object* $\mathrm{Anch}^{-1}(\mathbb{T}_{\mathrm{Bun}_G/\mathcal{M}^Q})$ *is unique up to equivalence then so is the twisted derived stack* $\mathcal{M}^Q$*, among formal algebraic gauge theories.*

Proof. This is a direct application of theorem 8.2.26. □

Remark 8.2.35. If, in addition, one can find a map $\mathcal{M}' \to \mathbb{A}^1$ so that the composite $\mathbb{A}^1 \to \mathbb{A}^1 \times \mathrm{Bun}_G \to \mathcal{M}' \to \mathbb{A}^1$ is the identity for every closed point P of Bun_G, then $\mathcal{M}'$ has the structure of a twist as in definition 8.2.21. We observed, following 8.2.31 that there is automatically such a map when $\mathcal{M}$ is fiberwise formal and the twisting data preserves the fibers of σ. There will also naturally be such a map for examples built from the Hodge stack. We will call such twists – when they exist and are essentially unique – *canonical twists.*

For reference later, we should spell out exactly what we've shown for fiberwise formal theories – i.e. in situations where we twist a Lie algebra object, and the twisted theory does not develop a non-trivial anchor map.

Corollary 8.2.36. *If* $\mathcal{M}$ *is a fiberwise formal algebraic gauge theory acted on by twisting data* (α, Q) *preserving the fibers of the map* $\pi\colon \mathcal{M} \to \mathrm{Bun}_G$*, then there exists a canonical twist* $\mathcal{M}^Q$*, which is itself a fiberwise formal algebraic gauge theory.*

As well as fiberwise formal theories and twisting data preserving the fibers of π, we'll use lemma 8.2.34 for the following simple example. A deeper understanding of the anchor map functor would allow for a more general theorem ensuring the existence of canonical twists of fiberwise formal algebraic gauge theories: i.e. twists of sheaves of Lie algebras into Lie algebroids.

Example 8.2.37. If $\mathcal{M} = T[1]\operatorname{Bun}_G$ and the twisting data acts as a non-vanishing degree 1 vector field, then $\mathcal{M}^Q$ is $(\operatorname{Bun}_G)_{\mathrm{dR}}$. This follows because the vector field amounts to $\mathrm{id}\colon \mathbb{T}_{\operatorname{Bun}_G} \to \mathbb{T}_{\operatorname{Bun}_G}$ as ind-coherent sheaves over Bun_G. Note that this object is the terminal object in $\operatorname{IndCoh}(\operatorname{Bun}_G)_{/\mathbb{T}_{\operatorname{Bun}_G}}$ so is the image under the functor Anch of a unique Lie algebroid. In this case there is a natural map $(\operatorname{Bun}_G)_{\mathrm{Hod}} \to \mathbb{A}^1$, realizing $(\operatorname{Bun}_G)_{\mathrm{dR}}$ as a twist of $T[1]\operatorname{Bun}_G$.

Having defined a twisting procedure for fiberwise formal algebraic gauge theories, let's investigate the properties enjoyed by these twisted theories. The twisted theory $\mathcal{M}^Q$ retains only a limited amount of supersymmetry: it is acted on by the Q-cohomology of the full supersymmetry algebra. More precisely, we have the following statement at the perturbative level, which immediately implies an analogous result non-perturbatively.

Proposition 8.2.38. *Suppose twisting data (α, Q) comes from the action of a supersymmetry algebra $\mathcal{A}$. The action of the Chevalley-Eilenberg cochains $C^\bullet(\mathcal{A})$ on the theory $\mathcal{E}$ defines an action of $C^\bullet(H^\bullet(\mathcal{A}, Q))$ on the twisted theory $\mathcal{E}^Q$, where we think of Q as a fermionic endomorphism of cohomological degree 0 acting on $\mathcal{A}$, and hence on $C^\bullet(\mathcal{A})$. Furthermore the action of the translation algebra factors through the action of this algebra.*

Remark 8.2.39. In particular, this tells us that Q-exact translations act trivially in the twisted theory.

Proof. We use the fact that, since $\mathcal{A}$ acts by symmetries, $[A, B](\phi) = A(B(\phi)) - B(A(\phi))$. First we'll show that the $\mathcal{A}$ action on $\mathcal{E}$ induces an $\mathcal{A}$-action on $\mathcal{E}^Q$ which is well-defined up to Q-exact symmetries. Let ϕ and $\phi + Q\psi$ be equivalent fields in $\mathcal{E}^Q$, and let $A \in C^1(\mathcal{A})$ be a symmetry. The action of A on $\phi + Q\psi$ is by

$$\begin{aligned} A(\phi + Q\psi) &= A\phi + AQ\psi \\ &= A\phi + QA\psi - [Q, A]\psi \\ &= A\phi - [Q, A]\psi \end{aligned}$$

since $QA\psi = 0$ in $\mathcal{E}^Q$. This expression in turn equals $A\phi$ up to Q-exact elements of the supersymmetry algebra, so this yields a well-defined action of the Q-closed symmetries in $\mathcal{A}$.

Now, let $A = [Q, \lambda] \in C^1(\mathcal{A})$ be a Q-exact symmetry. The action of A on a field $[\phi]$ in $\mathcal{E}^Q$ is by

$$\begin{aligned} A\phi &= [Q, \lambda][\phi] \\ &= Q\lambda[\phi] - \lambda Q[\phi] \\ &= 0 - \lambda(0) = 0 \end{aligned}$$

since $Q\lambda\phi$ and $Q\phi$ vanish in $\mathcal{E}^Q$. Note that here we're using the well-defined action of Q-closed symmetries on $\mathcal{E}^Q$ from the previous paragraph, so if $\phi \in \mathcal{E}$ has Q-cohomology class $[\phi]$ then $[\lambda[\phi]] = \lambda[\phi]$. In particular $Q[\phi] = [Q\phi] = [0]$. Thus we've shown that Q-exact symmetries act trivially, which means we have a well-defined action of $H^\bullet(\mathcal{A}, Q)$ on $\mathcal{E}^Q$ as required.

For the last statement we only need to note that the action of translations on $\mathcal{E}^Q$ by pushing forward along infinitesimal symmetries of spacetime agrees with the action of translations given here (which is well-defined since all translations are Q-closed) by construction of the twist. $\square$

We focus now on the two types of twist that we're principally interested in: holomorphic and topological twists.

Definition 8.2.40. *A classical perturbative field theory E on $\mathbb{R}^n$ is called* topological *if it is translation invariant; That is if the action of the Lie algebra $\mathbb{R}^n$ on the sheaf $\mathcal{E}$ by translations is homotopically trivial. The theory E is called* holomorphic *if the analogous condition holds for the Lie algebra of holomorphic vector fields for a specified complex structure on $\mathbb{R}^n$.*

Proposition 8.2.41. *If Q is a topological (resp. holomorphic) supercharge, then the twisted perturbative theory E^Q is topological (resp. holomorphic).*

Proof. If Q is topological, then by definition all translations are Q-exact, so vanish in the Q-cohomology. The action of translations is given by a cochain map from the Chevalley-Eilenberg cohomology

$$a\colon C^\bullet(\mathbb{C}^n) \to \mathrm{End}(\mathcal{E}^Q(\mathbb{R}^n)).$$

This action factors through the action of the full supersymmetry algebra, i.e. through the map $C^\bullet(\mathbb{C}^n) \to C^\bullet(\mathcal{A})$ induced by projection onto the translations in the supersymmetry algebra. Now apply proposition 8.2.38, and note that all translations must act trivially.

The holomorphic case proceeds identically. $\square$

CHAPTER 9

Constructing Supersymmetric Gauge Theories

We'll discuss two procedures for constructing supersymmetric gauge theories in four dimensions: *dimensional reduction from 10 dimensions* and *compactification from a supertwistor space*. In this chapter we'll review both constructions for $N = 4$ theories (though analogous constructions also give rise to theories in dimensions other than 4, and theories with less supersymmetry). The idea of dimensional reduction was developed by Cremmer and Scherk [**CS76**] and by Scherk and Schwarz [**SS79**] in the 1970's, and the application we're most concerned with is the construction of $N = 4$ supersymmetric gauge theory in four dimensions from $N = 1$ gauge theory in ten dimensions by Brink, Schwarz and Scherk [**BSS77**]. We currently don't have a fully rigorous definition of dimensional reduction for our notion of classical field theories, so the construction via dimensional reduction from 10 dimensions should be thought of as motivational, while the construction via compatification from twistor space should be thought of as a true definition.

9.1. Compactification and Dimensional Reduction

Before getting into the specifics we'll recall the general ideas behind compactification and dimensional reduction for classical field theories. Throughout this chapter a classical field theory $\mathcal{M}$ will be a family of derived stacks with a shifted symplectic structure on the global section as in definition 8.2.6.

Definition 9.1.1. *If $p\colon X \to Y$ is a smooth and proper map of smooth complex varieties, then the* compactification *of the theory along p of a classical field theory $\mathcal{M}$ on X is the pushforward assignment $p_*\mathcal{M}$.*

Proposition 9.1.2. *The compactification of a classical field theory $\mathcal{M}$ is still a classical field theory.*

Proof. We just have to note that the global sections of compactified theories still carry shifted symplectic structures compatibly with the structure maps, and that the shifted tangent complex at a point is still a perturbative classical field theory. The survival of the shifted symplectic structure under the compactification along $p\colon X \to Y$ is obvious, since $p_*\mathcal{M}(Y) = \mathcal{M}(p^{-1}Y) = \mathcal{M}(X)$ by definition. The shifted tangent complex certainly retains its invariant pairing coming from this symplectic pairing, and it retains the structure of an elliptic L_∞ algebra, so it forms a perturbative field theory. $\square$

Definition 9.1.3 (Definition sketch)**.** *The* dimensional reduction *of a classical field theory $\mathcal{M}$ on a smooth variety X along a fiber bundle $p\colon X \to Y$ whose fiber is a homogeneous space for an algebraic group G is the classical field theory on Y whose sections on an open set $U \subseteq Y$ are the G-invariants in $\mathcal{M}(p^{-1}U)$ under the action induced from the G-action on the fibers of p.*

This definition is currently unsatisfactory; we expect to have to impose additional conditions on the theory and the fibration for the theory obtained by taking invariants to remain a classical theory. As such, we'll refer to dimensional reduction purely in an informal sense.

Remark 9.1.4. Costello [**Cos11a**, 19.2.1] uses the term "dimensional reduction" for what we call "compactification", and he requires an additional piece of structure. He requires perturbative classical field theories to arise as the sections of a finitely generated complex of vector bundles, which is broken by the pushforward. Thus he defines the compactification to consist of a finitely generated complex of vector bundles whose sections carry the structure of a perturbative classical field theory as we define it, along with a homotopy equivalence to the pushforward of a perturbative classical field theory on X. For our purposes we won't need this finiteness condition, so this subtelty won't arise.

It'll also be important to understand how compactification and twisting relate to one another. If the compactified theory $p_*\mathcal{M}$ is locally supersymmetric as in section 8.1.2 then the original theory $\mathcal{M}$ also admits an action of the supersymmetry algebra by four-dimensional local isometries fixing the fibers. If the theory $\mathcal{M}$ was a fiberwise formal algebraic gauge theory then the compactification $p_*\mathcal{M}$ still defines a family of pointed formal moduli problems over Bun_G, i.e. there are a pair of maps $p_*\mathcal{M}(U) \rightleftarrows \mathrm{Bun}_G(p^{-1}U)$ satisfying the hypotheses of definition 8.2.8.

Therefore if we have twisting data (α, Q) for $\mathcal{M}$ then it makes sense to twist *either* the original theory *or* the compactified theory. Denote these twisted theories by $\mathcal{M}^Q$ and $(p_*\mathcal{M})^Q$ respectively. We'll describe the relationship perturbatively.

Lemma 9.1.5. *If $\sigma\colon \mathrm{Bun}_G \rightleftarrows \mathcal{M} \colon \pi$ is a fiberwise formal algebraic gauge theory and $\mathcal{M}^Q$ is a twist of $\mathcal{M}$ with respect to twisting data that preserves the fibers of π and σ, then there exists a quasi-isomorphism of classical field theories*

$$p_*(\mathcal{M}^Q) \cong (p_*\mathcal{M})^Q.$$

Proof. By corollary 8.2.36 it suffices to check this at the level of perturbative field theories on Y, i.e. taking the shifted relative tangent complexes as sheaves of dg Lie algebras on Bun_G over Y. Write $p_*(\mathcal{E}^Q)$ and $(p_*\mathcal{E})^Q$ for these two sheaves. Fixing an open set $U \subseteq Y$ in the base, by definition $p_*(\mathcal{E}^Q)(U)$ is obtained as the local sections $\mathcal{E}^Q(p^{-1}U)$. Likewise, $(p_*\mathcal{E})^Q(U)$ is obtained by taking the space of local sections $\mathcal{E}(p^{-1}U)$ and applying the twisting procedure with respect to the specified twisting data, which also recovers the space of local sections $\mathcal{E}^Q(p^{-1}U)$, so the two sheaves coincide, thus so do the global derived stacks. □

9.2. $N = 1$ Super-Yang-Mills in Ten Dimensions

We'll now give an informal description of a supersymmetric ten-dimensional field theory in terms of fields and an action functional, while explaining the action of the supersymmetry algebra (as described in appendix A) as clearly as possible. Let G be a complex reductive group with Lie algebra $\mathfrak{g}$ (we'll describe a complexification of the usual super Yang-Mills theory). There are two fields A and Ψ, where A is identified with a $\mathfrak{g}$-valued 1-form and Ψ is a Weyl fermion: a section of the bundle $S_{10+} \otimes \mathfrak{g}$. The Lagrangian density can be identified with

$$\mathcal{L}(A, \Psi) = \mathrm{Tr}\left(\frac{1}{2} F_A \wedge *F_A + \Psi \wedge * \not{D}_A \Psi\right)$$

where $F_A = dA + \frac{1}{2}[A, A]$, $D_A \Psi = d\Psi + [A, \Psi]$, and where we define the Dirac operator $\not{D}_A$ using Clifford multiplication. Here the trace is defined by means of a specified faithful finite-dimensional representation of G. Define ρ to be the Clifford multiplication map thought of as a map of vector bundles $T^*\mathbb{C}^{10} \otimes S_{10+} \otimes \mathfrak{g} \to S_{10-} \otimes \mathfrak{g}$, using the metric to identify the tangent and cotangent bundles. We define $\not{D}_A = \rho \circ D_A$. The trace pairing here implicitly includes both the invariant pairing on the Lie algebra and the ten-dimensional spinor pairing $S_{10-} \otimes S_{10+} \to \mathbb{C}$.

One can describe $N = 1$ super Yang-Mills in the homological formalism of chapter 8.2, expanding a more familiar definition for Yang-Mills in the second order formalism to an $N = 1$ vector multiplet. Consider the elliptic complex

$$\Omega^0_{\mathbb{C}}(\mathbb{R}^{10}; \mathfrak{g}_P) \xrightarrow{\quad d \quad} \Omega^1_{\mathbb{C}}(\mathbb{R}^{10}; \mathfrak{g}_P) \xrightarrow{\quad d*d \quad} \Omega^9_{\mathbb{C}}(\mathbb{R}^{10}; \mathfrak{g}_P) \xrightarrow{\quad d \quad} \Omega^{10}_{\mathbb{C}}(\mathbb{R}^{10}; \mathfrak{g}_P)$$

$$\Omega^0_{\mathbb{C}}(\mathbb{R}^{10}; S_{10+} \otimes \mathfrak{g}_P) \xrightarrow{*\not{d}} \Omega^{10}_{\mathbb{C}}(\mathbb{R}^{10}; S_{10-} \otimes \mathfrak{g}_P)$$

in degrees $0, 1, 2$ and 3, where we write $\Omega^i_{\mathbb{C}}(\mathbb{R}^{10})$ for the complexification $\Omega^i(\mathbb{R}^{10}) \otimes_{\mathbb{R}} \mathbb{C}$. This complex admits an invariant pairing built from the wedge-and-integrate pairing on forms and the ten-dimensional spinor pairing between S_{10+} and S_{10-}. There is a natural L_∞-structure coming from the action, for which the pairing is invariant. The only non-trivial brackets are given by the action of $\Omega^0(\mathbb{C}^{10};\mathfrak{g}_P)$ on everything, the degree two brackets

$$\begin{aligned}\ell_2^{\mathrm{Bos}}\colon \Omega^1_{\mathbb{C}}(\mathbb{R}^{10};\mathfrak{g}_P)\otimes\Omega^1_{\mathbb{C}}(\mathbb{R}^{10};\mathfrak{g}_P) &\to \Omega^9_{\mathbb{C}}(\mathbb{R}^{10};\mathfrak{g}_P)\\ (A\otimes B) &\mapsto A\wedge *dB\\ \ell_2^{\mathrm{Fer}}\colon \Omega^1_{\mathbb{C}}(\mathbb{R}^{10};\mathfrak{g}_P)\otimes\Omega^0_{\mathbb{C}}(\mathbb{R}^{10};S_{10+}\otimes\mathfrak{g}_P) &\to \Omega^{10}_{\mathbb{C}}(\mathbb{R}^{10};S_{10-}\otimes\mathfrak{g}_P)\\ (A\otimes\Psi) &\mapsto *\not{A}\Psi\end{aligned}$$

and the degree three bracket

$$\begin{aligned}\ell_3\colon \Omega^1_{\mathbb{C}}(\mathbb{R}^{10};\mathfrak{g}_P)\otimes\Omega^1_{\mathbb{C}}(\mathbb{R}^{10};\mathfrak{g}_P)\otimes\Omega^1_{\mathbb{C}}(\mathbb{R}^{10};\mathfrak{g}_P) &\to \Omega^9_{\mathbb{C}}(\mathbb{R}^{10};\mathfrak{g}_P)\\ (A\otimes B\otimes C) &\mapsto A\wedge *(B\wedge C).\end{aligned}$$

Now, we must define the action of the supersymmetry algebra. The bosonic piece acts by isometries on $\mathbb{C}^{10}$ itself, and on the fields by pullback. The fermions S_{10+} act by supersymmetries; we choose $\varepsilon\in S_{10+}$ and consider the infinitesimal symmetry coming from ε, $(A,\Psi)\mapsto(A+\delta A,\Psi+\delta\Psi)$. We let

$$\begin{aligned}\delta A &= \Gamma(\Psi,\varepsilon)\\ \delta\Psi &= \rho^2(F_A\otimes\varepsilon)\end{aligned}$$

where Γ is the usual pairing $S_{10+}\otimes S_{10+}\to\mathbb{C}^{10}$, fiberwise (and again using the metric to identify vector fields and 1-forms), and where ρ^2 denotes the composite map

$$\Omega^2_{\mathbb{C}}(\mathbb{R}^{10})\otimes S_{10+}\to\Omega^1_{\mathbb{C}}(\mathbb{R}^{10})^{\otimes 2}\otimes S_{10+}\to\Omega^1_{\mathbb{C}}(\mathbb{R}^{10})\otimes S_{10-}\to S_{10+}$$

where the first map is the natural inclusion, and the latter maps are Clifford multiplication. That this gives a well-defined action of the supersymmetry algebra, at least on-shell, and that the Lagrangian is supersymmetric are proven in [**ABD⁺13**].

Remark 9.2.1. The on-shell condition here will require some care to treat rigorously. Rather than giving a well-defined Lie algebra action on the space of fields, the supersymmetry relations only hold up to terms that vanish after imposing the equations of motion. A priori this should give a well-defined homotopy action on the derived space of solutions to the equations of motion. A careful analysis of this action is beyond the scope of this paper.

Now, by the calculations above, considering the subspace of fields constant along the leaves of a foliation by six-dimensional affine subspaces produces a four-dimensional theory with $N = 4$ supersymmetry. This theory is called (pure) $N = 4$ *super Yang-Mills* in four dimensions. One can explicitly describe the fields and the action functional [**BSS77**] in this dimensionally reduced theory. The gauge field A breaks into a four-dimensional gauge field (which we'll also call A) and six scalar fields $\phi_1, \ldots, \phi_6$. The Weyl spinor Φ breaks into four four-dimensional Dirac spinors $\chi_1, \ldots \chi_4$. When we construct an $N = 4$ from the twistor space perspective we'll observe that the field content is the same (one can also define an action on super twistor space which recovers the dimensionally reduced action functional here. This was done by Boels, Mason and Skinner [**BMS07**]).

9.3. Twistor Space Formalism

Twistor space is a complex manifold whose geometry is closely related to that of (compactified) Minkowski space. At its root, twistor space $\mathbb{PT}$ is just the complex manifold $\mathbb{CP}^3$, but we can describe it in a way that explains why it might be related to the geometry of $\mathbb{R}^{1,3}$. Write $\mathbb{T}$ for the Dirac spinor representation $S = S_- \oplus S_+$ in signature $(1, 3)$, a 4-complex-dimensional vector space. This new notation is chosen for compatibility with the twistor literature. The *twistor space* $\mathbb{PT}$ is then the space of complex lines in $\mathbb{T}$.

Remark 9.3.1. Elsewhere when discussing four-dimensional spinors we've used Euclidean signature, and indeed since we're only discussing complex spinor representations here our classical field theories don't depend on a choice of signature. We've used the language of Lorentzian signature in the above construction of twistor space because of certain other aspects of twistor theory that appear in the literature, for instance the existence of the Penrose correspondence between the space of null twistors and complexified Minkowski space, that suggest that twistors are really most naturally related to Lorentzian geometry.

Fix a Hermitian inner product on the space S_+ of Weyl spinors. The space $\mathbb{T} = S_- \oplus S_+$ therefore admits a pseudo-Hermitian structure by

$$((\alpha_1, \beta_1), (\alpha_2, \beta_2)) \mapsto \langle \alpha_1, \overline{\beta}_2 \rangle + \langle \overline{\beta}_1, \alpha_2 \rangle$$

using the canonical isomorphism $S_- \cong \overline{S}_+$, which we observe has signature $(2, 2)$. This is called the *twistor norm*. The space of twistors with vanishing twistor norm is denoted $\mathbb{N} \subseteq \mathbb{T}$ and forms a seven-real-dimensional submanifold. Looking at complex lines contained in $\mathbb{N}$ defines $\mathbb{PN} \subseteq \mathbb{PT}$, a five-real-dimensional compact submanifold. Removing this submanifold splits $\mathbb{PT}$ into two components, $\mathbb{PT}_+$ and $\mathbb{PT}_-$ corresponding to twistors with positive and negative twistor norm respectively.

There are two natural maps associated to twistor space which we should describe. First define the *Penrose map* associated to an identification $S_+ \cong \mathbb{H}$ with the quaternions to be the map

$$p\colon\ \mathbb{PT} \cong \mathbb{CP}^3 \to \mathbb{HP}^1 \cong S^4$$

with fibers isomorphic to $\mathbb{CP}^1$ (the *twistor lines*). The space of null twistors $\mathbb{PN}$ maps to an equator $S^3 \subseteq S^4$. We choose a point in $p(\mathbb{PN})$ as a "point at infinity". The preimage $\mathbb{PT} \setminus \mathbb{CP}^1$ of the complement is isomorphic to $\mathbb{CP}^1 \times \mathbb{R}^4$ as a smooth manifold.

For concreteness, choose homogeneous coordinates Z_0, Z_1, Z_2, Z_3 on $\mathbb{T}$. The Penrose map is then given by

$$(Z_0 : Z_1 : Z_2 : Z_3) \mapsto (Z_0 + jZ_1 : Z_2 + jZ_3).$$

Say the point at infinity is $(1:0) \in \mathbb{HP}^1$. The complement of the twistor line at infinity is the set $\{(Z_0 : Z_1 : Z_2 : Z_3) \mid Z_2 \text{ and } Z_3 \text{ are not both } 0\}$. This allows us to define a *holomorphic* map

$$\pi \colon \mathbb{PT} \setminus \mathbb{CP}^1 \to \mathbb{CP}^1$$

$$(Z_0 : Z_1 : Z_2 : Z_3) \mapsto (Z_2 : Z_3).$$

In more coordinate-free language we can identify $\mathbb{PT} \setminus \mathbb{CP}^1$ with the total space of the rank 2 holomorphic vector bundle $\mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathbb{P}(S_+)$. The map π is the bundle map.

Remark 9.3.2. This is an instance of a more general construction due to Atiyah, Hitchin and Segal [**AHS78**] that makes sense starting from any pseudo-Riemannian 4-manifold X satisfying a certain curvature condition. In short, one can take the total space of the projectivized negative Weyl spinor bundle $\mathbb{P}(S_+)$ over X, and produce a canonical almost complex structure on this total space using the Clifford multiplication. This almost complex structure is integrable if one imposes the appropriate curvature condition. In the case where $X = \mathbb{R}^{1,3}$ is Minkowski space we obtain the total space of the trivial $\mathbb{P}(S_+)$ bundle, and the complex structure one defines is precisely the complex structure on $\mathbb{PT} \setminus \mathbb{CP}^1$ defined above.

The twistor space itself admits a supersymmetric extension.

Definition 9.3.3. *The* super twistor space *associated to a complex vector space W is the total space of the odd vector bundle*

$$\mathbb{PT}^W = \Pi(\mathcal{O}(1) \otimes W) \to \mathbb{PT}\,.$$

If we restrict to the preimage of $\mathbb{R}^4$ under the Penrose map p, we find a superspace which admits a natural action of the supersymmetry algebra $\mathcal{A}^W$ (where the R-symmetries act trivially). We'll construct supersymmetric field theories on $\mathbb{R}^4$ by compactification from theories on twistor space admitting manifest supersymmetry actions.

9.4. Holomorphic Chern-Simons Theory on Super Twistor Space

The power of the twistor space formalism lies in its ability to relate theories involving the *holomorphic* or *algebraic* geometry of (super) twistor spaces, and the metric geometry of 4-manifolds. We'll recall two types of theory modelling the theory of holomorphic principal bundles. First, let $X = (\Pi E \to X_{\text{even}})$ be a split algebraic supermanifold of complex dimension n, let G be a complex reductive group, and let P be a principal G-bundle on X_{even}.

The following theory of BG valued holomorphic maps was discussed in [**Cos11a**, section 11.2] (as an instance of a more general theory of holomorphic maps into a complex target stack). It will be an *analytic* perturbative field theory, i.e. a sheaf of complexes over a complex manifold X with respect to its analytic topology. Lacking a good theory of derived analytic geometry we won't be able to literally promote this to a non-perturbative field theory, we'll only be able to describe an analogous theory using algebraic bundles and the Zariski topology.

Definition 9.4.1. *The* curved $\beta\gamma$ system *on X (with target BG) near a holomorphic G-bundle P is the cotangent theory, as in definition 8.2.10, whose base is the elliptic L_∞ algebra*

$$\Omega^{0,\bullet}(X_{\text{even}}; \mathcal{O}_{\Pi E} \otimes \mathfrak{g}_P), \overline{\partial}).$$

Hence the underlying elliptic complex is $(\Omega^{0,\bullet}(X_{\text{even}}; \mathcal{O}_{\Pi E} \otimes \mathfrak{g}_P) \oplus \Omega^{n,\bullet}(X_{\text{even}}; \mathcal{O}_{\Pi E^\vee} \otimes \mathfrak{g}_P^*[n-3]), (\overline{\partial}, \overline{\partial}))$*, and the invariant pairing is given by the canonical pairings between $\mathfrak{g}$ and $\mathfrak{g}^*$ and between E and $E^\vee$, and the wedge pairing on forms.*

This perturbative description ought to arise as a description of the cotangent theory to the moduli space of holomorphic G-bundles on X, because the Dolbeault complex with coefficients in $\mathfrak{g}_P$ controls deformations of holomorphic G-bundles on X. This suggests an analogous *algebraic, non-perturbative* version of the classical field theory.

Definition 9.4.2. *The (algebraic)* curved $\beta\gamma$ system *on X (with target BG) is the cotangent theory whose local sections on $U \subseteq X$ are given by the derived stack*

$$T^*[-1]\,\mathrm{Bun}_G(U).$$

If X is smooth and proper – so $\mathrm{Bun}_G(U)$ is finitely presented – the global sections admit a natural shifted symplectic structure.

Remark 9.4.3. Since $\mathrm{Bun}_G(U)$ is not locally of finite presentation for general U, its cotangent complex is generally not perfect and hence one cannot define the (shifted) cotangent bundle as in the conventions section. On the other hand, one can always define the total space of a given quasi-coherent sheaf $\mathcal{F}$ on X in terms of the moduli problem whose R-points consists of maps $f\colon \mathrm{Spec}\, R \to X$ together with sections $\Gamma(\mathrm{Spec}\, R, f^*\mathcal{F})$. We won't make this technical definition precise here; we're most interested in describing the global sections of classical field theories on smooth projective varieties X. This remark should also be applied for later appearances of a cotangent space of a derived stack which is not locally of finite presentation.

In either the analytic or the algebraic setting we could instead consider a more general theory of holomorphic or algebraic maps into any target – this would define a more general curved $\beta\gamma$-system.

Starting from $N = 1$ and $N = 2$ super twistor space, one constructs supersymmetric gauge theories by taking the curved $\beta\gamma$ system on the complement of a twistor line in the super twistor spaces $\mathbb{PT}^{N=1}$ or $\mathbb{PT}^{N=2}$. For $N = 4$ super Yang-Mills however we'll do something different: we observe that the complex $\Omega^{0,\bullet}(X; \mathfrak{g}_P)$ where X is the complement of the line in $N = 4$ super twistor space (i.e. the restriction of the odd vector bundle defining super twistor space to $\mathbb{PT} \setminus \mathbb{CP}^1 \subseteq \mathbb{PT}$) *already* admits a degree -3 invariant pairing, and so defines a field theory. This is an instance of a more general family of theories.

Example 9.4.4. Let X be a compact super Calabi-Yau variety of complex dimension $n|m$, as in definition 8.1.19. Then the complex $\Omega^{0,\bullet}(X; \mathfrak{g}_P)$ admits a degree $-n$ invariant pairing by the invariant pairing on $\mathfrak{g}$ and the wedge pairing on forms. This pairing naturally lands in the Berezinian, which yields a density by applying the Calabi-Yau structure, an isomorphism of vector bundles $\mathrm{Ber}(X) \to \underline{\mathbb{C}}$. If $n = 3$, this defines a perturbative field theory on X which we call *holomorphic Chern-Simons theory*. This perturbative theory admits an algebraic non-perturbative analogue, as above. One can consider the non-perturbative algebraic classical field theory $\mathrm{EOM}(U) = \mathrm{Bun}_G(U)$, with (-1)-shifted symplectic structure arising via the derived AKSZ formalism [**PTVV13**, Theorem 2.5] from the 2-shifted symplectic structure on BG and the Calabi-Yau structure on X.

Remark 9.4.5. There's a certain amount of ambiguity in the terminology for these classical field theories. The theory we call the curved $\beta\gamma$ system with target BG is itself called holomorphic Chern-Simons theory in [**Cos10**]. In the case where X is a super Calabi-Yau 3-fold then the two theories are closely related: the holomorphic Chern-Simons theory (in our terminology) has the curved $\beta\gamma$ system as its cotangent theory, as in the book of Costello and Gwilliam [**CG15**].

Now, let $X = \mathbb{PT}^{N=4} \setminus \mathbb{CP}^1$, the complement of a line in $N = 4$ super twistor space. One observes (as noted by Witten [**Wit04b**]) that this space is super Calabi-Yau by computing the Berezinian. More generally, the Berezinian of the super projective space $\mathbb{CP}^{n|m}$ is computed to be

$$\begin{aligned}\mathrm{Ber}_{\mathbb{CP}^{n|m}} &\cong K_{\mathbb{CP}^n} \otimes_{\mathcal{O}} \wedge^m(\mathcal{O}(1) \otimes \mathbb{C}^m) \\ &\cong \mathcal{O}(-n-1) \otimes \mathcal{O}(m) \cong \mathcal{O}(m-n-1)\end{aligned}$$

(using a choice of trivialization of $\wedge^m \mathbb{C}^m$) which is trivial if and only if $m = n + 1$, for instance in the case $n = 3, m = 4$.

Remark 9.4.6. We should note that while $\mathbb{CP}^{3|4} \setminus \mathbb{CP}^1$ is super Calabi-Yau, it is not *compact* super Calabi-Yau. While holomorphic Chern-Simons on $\mathbb{PT}^{N=4}$ is a genuine classical field theory as in definition 8.2.6 with shifted symplectic structure on the space $\mathrm{Bun}_G(\mathbb{PT}^{N=4})$ of global solutions to the equations of motion given by the derived AKSZ formalism, the shifted symplectic form fails to be well-defined on the complement of a line. We expect at least a shifted Poisson structure to survive here, but since we won't need this shifted symplectic structure for the untwisted $N = 4$ moduli space in what follows – we'll construct the twisted theories of interest on $\mathbb{R}^4$, then generalize to arbitrary smooth algebraic surfaces by analogy – we'll ignore this subtlety in the present work.

Let's try to understand the theory we get when we perform compactification along the map $p\colon \mathbb{PT}^{N=4} \setminus \mathbb{CP}^1 \to \mathbb{R}^4$. Specifically let's verify that the field content agrees with the fields we described at the end of section 9.2. Our argument will follow the argument for the ordinary Penrose-Ward correspondence given by Movshev [**Mov08**], and cohomology calculations given in section 7.2 of the book of Ward and Wells [**WW91**]. We'll use the phrase *linearized* holomorphic Chern-Simons and $N = 4$ super Yang-Mills to mean the perturbative field theories obtained by forgetting the brackets in the L_∞ structure, leaving only a cochain complex. We'll do this calculation for the analytic, perturbative theory.

Remark 9.4.7. Note that we needed to trivialize $\wedge^4\mathbb{C}^4$ in order to define the super Calabi-Yau structure. This choice breaks the full $\mathfrak{gl}(4;\mathbb{C})$ of R-symmetries to $\mathfrak{sl}(4;\mathbb{C})$, as we remarked in section 8.1.1.

Proposition 9.4.8. *The compactification of linearized holomorphic Chern-Simons theory along the Penrose map p is equivalent to the linearized anti-self-dual $N = 4$ super Yang-Mills theory.*

Proof. To show this, we need to pushforward the sheaf of solutions to the classical equations of motion in the holomorphic Chern-Simons theory along p. This sheaf is just the complex $\Omega^{0,\bullet}(X;\mathfrak{g}_P)$

where X is the complement of the line in $N = 4$ super twistor space. That is, the complex

$$\begin{aligned}\bigoplus_{i\geq 0}\left(\Omega^{0,\bullet}(\mathbb{PT} \setminus \mathbb{CP}^1; \mathrm{Sym}^i(\Pi\mathcal{O}(-1)^4)\otimes_{\mathcal{O}}\mathfrak{g}_P)\right) &\cong \bigoplus_{i\geq 0}\left(\Omega^{0,\bullet}(\mathbb{PT} \setminus \mathbb{CP}^1; \wedge^i(\mathcal{O}(-1)^4)\otimes_{\mathcal{O}}\mathfrak{g}_P)\right) \\ &\cong \Omega^{0,\bullet}(\mathbb{PT} \setminus \mathbb{CP}^1; (\mathcal{O}\oplus\mathcal{O}(-4))\otimes_{\mathcal{O}}\mathfrak{g}_P) \\ &\quad \oplus\ \Omega^{0,\bullet}(\mathbb{PT} \setminus \mathbb{CP}^1; (\mathcal{O}(-1)\oplus\mathcal{O}(-3))\otimes_{\mathcal{O}}\mathfrak{g}_P)^4 \\ &\quad \oplus\ \Omega^{0,\bullet}(\mathbb{PT} \setminus \mathbb{CP}^1; \mathcal{O}(-2)\otimes_{\mathcal{O}}\mathfrak{g}_P)^6.\end{aligned}$$

We've grouped the terms here judiciously – they'll yield the gauge field, four spinor fields and six scalar fields we saw in section 9.2 respectively (with their corresponding antifields). To check this, we must compute the hypercohomology of these terms, complete with their actions of the algebra $\mathfrak{so}(4;\mathbb{C})$. This becomes a little simpler after identifying $\mathbb{PT} \setminus \mathbb{CP}^1$ with the total space of the rank two holomorphic vector bundle $\mathcal{O}(1)\otimes S_- \to \mathbb{P}(S_+)$. What's more, the pullback of the bundle $\mathcal{O}(k)$ on $\mathbb{P}(S_+)$ under the map π is precisely the vector bundle $\mathcal{O}(k)$ given by restriction from $\mathbb{PT} = \mathbb{CP}^3$. From this point of view we can identify

$$\Omega^{0,\bullet}(\mathbb{PT} \setminus \mathbb{CP}^1; \mathcal{O}(k)\otimes\mathfrak{g}_P) \cong \pi^*\left(\bigoplus_{i+j=\bullet}\Omega^{0,i}(\mathbb{P}(S_+); \mathcal{O}(k)\otimes\mathfrak{g}_P\otimes\wedge^j(\mathcal{O}(1)\otimes S_-))\right),$$

so $p_*(\Omega^{0,\bullet}(\mathbb{PT} \setminus \mathbb{CP}^1; \mathcal{O}(k)\otimes\mathfrak{g}_P)) \cong \Omega^0(\mathbb{R}^4)\otimes\left(\bigoplus_{i+j=\bullet}\Omega^{0,i}(\mathbb{P}(S_+); \mathcal{O}(k)\otimes\mathfrak{g}_P\otimes\wedge^j(\mathcal{O}(1)\otimes S_-))\right)$

as a sheaf on $\mathbb{R}^4$. We then compute the hypercohomology of the right hand side, which is just the cohomology of the coefficient coherent sheaf with an additional differential. Indeed, we can think of the complex as bigraded by the i and j gradings, and the cohomology of the coefficient coherent sheaf is precisely the E_1 page of the spectral sequence of the double complex. This page has form

$$\begin{pmatrix} H^0(\mathbb{P}(S_+);\mathcal{O}(k)\otimes\mathfrak{g}_P) \longrightarrow H^0(\mathbb{P}(S_+);\mathcal{O}(k+1)\otimes S_-\otimes\mathfrak{g}_P) \longrightarrow H^0(\mathbb{P}(S_+);\mathcal{O}(k+2)\otimes\mathfrak{g}_P) \\ \\ H^1(\mathbb{P}(S_+);\mathcal{O}(k)\otimes\mathfrak{g}_P) \longrightarrow H^1(\mathbb{P}(S_+);\mathcal{O}(k+1)\otimes S_-\otimes\mathfrak{g}_P) \longrightarrow H^1(\mathbb{P}(S_+);\mathcal{O}(k+2)\otimes\mathfrak{g}_P). \end{pmatrix}$$

$\otimes \quad C^\infty(\mathbb{R}^4)$

The page is concentrated in a single row and therefore the spectral sequence converges at the E_2 page unless $k=-2$, in which case there's one additional differential (from $(i,j)=(1,0)$ to $(0,2)$) and the complex converges at the E_3 page.

We begin with the first line (the term of interest in the ordinary, non-supersymmetric Penrose-Ward correspondence, and the term considered by Movshev [**Mov08**]). The coefficient sheaf is isomorphic to $((\mathcal{O}\oplus(\mathcal{O}(1)\otimes S_-)[-1]\oplus\mathcal{O}(2)[-2])\oplus(\mathcal{O}(-4)\oplus(\mathcal{O}(-3)\otimes S_-)[-1]\oplus\mathcal{O}(-2)[-2]))\otimes\mathfrak{g}_P$ whose cohomology is $\mathfrak{g}_P\oplus\mathfrak{g}_P[-1]\otimes(S_-\otimes S_+\oplus\operatorname{Sym}^2 S_+)\oplus\mathfrak{g}_P[-2]\otimes(\operatorname{Sym}^2 S_+\oplus S_-\otimes S_+)\oplus\mathfrak{g}_P[-3]$. Thus the corresponding term in the pushforward sheaf is

$$\Omega^0(\mathbb{R}^4;\mathfrak{g}_P\otimes(\mathbb{C}\oplus(V\oplus\operatorname{Sym}^2 S_+)[-1]\oplus(V\oplus\operatorname{Sym}^2 S_+)[-2]\oplus\mathbb{C}[-3]))$$

where $V\cong S_+\otimes S_-$ is the vector representation of $\mathfrak{so}(4;\mathbb{C})$. To compute the differential, we start with the first summand in the pushforward sheaf, $\Omega^0(\mathbb{R}^4)\otimes H^0(\mathcal{O}\oplus(\mathcal{O}(1)\otimes S_-)[-1]\oplus\mathcal{O}(2)[-2])$. This is the E_1 page of the spectral sequence of the double complex described above, and the differential is the image of the $\overline{\partial}$ operator. Concretely, in coordinates this operator has form $\partial_i e^i$, where x^i is a basis for $\mathbb{R}^4$, $\partial_i=\frac{\partial}{\partial x^i}$, and e^i is a degree 1 operator on $H^0(\mathcal{O}\oplus(\mathcal{O}(1)\otimes S_-)[-1]\oplus\mathcal{O}(2)[-2])$ associated to x^i. This operator arises by canonically identifying $H^0(\mathcal{O}(1)\otimes S_-)$ with $V=\mathbb{R}^4\otimes_{\mathbb{R}}\mathbb{C}$ so that every global section of $\mathcal{O}(1)\otimes S_-$ yields a degree 1 operator on $H^0(\wedge^\bullet(\mathcal{O}(1)\otimes S_-))$ via the natural map $\wedge^\bullet(H^0(\mathcal{O}(1)\otimes S_-))\to H^0(\wedge^\bullet(\mathcal{O}(1)\otimes S_-))$. Unpacking this calculation, we find exactly the differential in the Atiyah-Singer-Donaldson complex

$$\Omega^0(\mathbb{R}^4) \xrightarrow{d} \Omega^1(\mathbb{R}^4) \xrightarrow{d_+} \Omega^2_+(\mathbb{R}^4).$$

controlling an anti-self-dual connection. The remaining summand is Serre dual to the first summand, so the overall complex is the complex controlling an anti-self-dual Yang-Mills field as required.

Similarly, we analyse the second line. Now, the coefficient sheaf is isomorphic to $((\mathcal{O}(-1) \oplus (\mathcal{O} \otimes S_-)[-1] \oplus \mathcal{O}(1)[-2]) \oplus (\mathcal{O}(-3) \oplus (\mathcal{O}(-2) \otimes S_-)[-1] \oplus \mathcal{O}(-1)[-2])) \otimes \mathfrak{g}_P$, whose cohomology is $\mathfrak{g}_P[-1] \otimes (S_- \oplus S_+) \oplus \mathfrak{g}_P[-2] \otimes (S_- \oplus S_+)$ with the $\mathfrak{so}(4;\mathbb{C})$ action indicated by the notation. Thus the corresponding term in the pushforward sheaf is

$$(\Omega^0(\mathbb{R}^4; \mathfrak{g}_P \otimes (S[-1] \oplus S[-2])))^4$$

where $S = S_+ \oplus S_-$. We analyse the differential in a similar way to the above, focusing on the first summand $\Omega^0(\mathbb{R}^4) \otimes H^0(S_-[-1] \oplus \mathcal{O}(1)[-2])$ (the other term is Serre dual to this one). Again, in a specified basis, the differential is of the form $\partial_i e^i$, where now the e^i act according to the action of $x^i \in H^0(\mathcal{O}(1) \otimes S_-)$ on the complex $H^0(\mathrm{Sym}^\bullet(\mathcal{O}(1) \otimes S_-) \otimes \mathcal{O}(-1))$. Unpacking, this action map (from Sym^1 to Sym^2) is given by the composite

$$S_- \xrightarrow{x^i \otimes 1} V \otimes S_- \cong S_+ \otimes S_- \otimes S_- \twoheadrightarrow S_+ \otimes \wedge^2 S_- \cong S_+.$$

This composite is exactly the Clifford multiplication $\rho(x^i)$ by the vector x^i, so our overall differential is $\partial_i \rho(x^i)$. This is the Dirac operator $\not{d}$, so combining this term with its Serre dual we obtain the complex

$$\left(\Omega^0(\mathbb{R}^4; S) \xrightarrow{\not{d}} \Omega^0(\mathbb{R}^4; S)\right)^4$$

in degrees one and two, which is the linearized BV complex controlling four Dirac spinors.

Finally, we analyse the last line, which is the simplest algebraically, but whose differential is a little more subtle than the others. The coefficient sheaf is isomorphic to $(\mathcal{O}(-2) \oplus (\mathcal{O}(-1) \otimes S_-)[-1] \oplus \mathcal{O}[-2]) \otimes \mathfrak{g}_P$, whose cohomology is $\mathfrak{g}_P[-1] \oplus \mathfrak{g}_P[-2]$ with the trivial $\mathfrak{so}(4;\mathbb{C})$ action. Thus the corresponding term in the pushforward sheaf is

$$(\Omega^0(\mathbb{R}^4; \mathfrak{g}_P[-1] \oplus \mathfrak{g}_P[-2]))^6.$$

To compute the differential we have to do a little more than we did for the earlier terms, because now the E_1 and E_2 pages of the spectral sequence coincide, but there's a differential on the E_2 page increasing the j degree by two. This differential is of the form $D = \partial_i \partial_j (e^i \overline{\partial}^{-1} e^j)$, where the operator $e^i \overline{\partial}^{-1} e^j$ is obtained from the composite

$$\begin{tikzcd}
 & H^0(S_-(1)) \otimes \Gamma(\Omega^{0,0}_{\mathbb{CP}^1}(-1) \otimes S_-) \arrow[r] & \Gamma(\Omega^{0,0} \otimes \wedge^2 S_-) \cong H^0(\Omega^{0,0}_{\mathbb{CP}^1}) \\
H^0(S_-(1))^{\otimes 2} \otimes \Gamma(\Omega^{0,1}_{\mathbb{CP}^1}(-2)) \arrow[r] & H^0(S_-(1)) \otimes \Gamma(\Omega^{0,1}_{\mathbb{CP}^1}(-1) \otimes S_-) \arrow[u, "1 \otimes \overline{\partial}^{-1}"'] &
\end{tikzcd}$$

(where we've used Γ for the global sections of the infinite-type vector bundles $\Omega^{i,j}$ to emphasise that we're considering all forms, not just the Dolbeault cohomology, and where we've written $S_-(1)$ for $S_- \otimes \mathcal{O}(1)$) applied to $x^i \otimes x^j \in H^0(S_-(1))^{\otimes 2}$ and a representative for a cohomology class in $H^{0,1}(\mathbb{CP}^1; \mathcal{O}(-2))$. Here we use the fact that the operator $\overline{\partial}\colon \Omega^{0,0}_{\mathbb{CP}^1}(-1) \to \Omega^{0,1}_{\mathbb{CP}^1}(-1)$ induces an isomorphism on H^0. To compute the operator $e^i \overline{\partial}^{-1} e^j$ we follow the method of [**WW91**, Theorem 7.2.5]. There is a map of complexes

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}(-2) \arrow[r, "e^i"] \arrow[d, equal] & \mathcal{O}(-1) \otimes S_- \arrow[r, "e^i"] \arrow[d, equal] & \mathcal{O} \otimes \wedge^2 S_- \arrow[r] \arrow[d, "\delta^{ij}"] & 0 \\
 & \mathcal{O}(-2) \arrow[r, "e^i"] & \mathcal{O}(-1) \otimes S_- \arrow[r, "e^j"] & \mathcal{O} \otimes \wedge^2 S_- &
\end{tikzcd}$$

where the top row is exact, which yields a map between the spectral sequences computing the hypercohomology of the two rows. On the E_2 page of these spectral sequences, this map just yields a commutative square

$$\begin{tikzcd} H^0(\Omega^{0,1}_{\mathbb{CP}^1}(-2)) \arrow[r] \arrow[d, "\mathrm{id}"] & H^0(\Omega^{0,0} \wedge^2 S_-) \arrow[d, "\delta^{ij}"] \\ H^0(\Omega^{0,1}_{\mathbb{CP}^1}(-2)) \arrow[r, "e^i\overline{\partial}^{-1}e^j"] & H^0(\Omega^{0,0} \wedge^2 S_-), \end{tikzcd}$$

and the top arrow is an isomorphism because the corresponding sequence of complexes was exact, so the operator $e^i\overline{\partial}^{-1}e^j$ is obtained from δ^{ij} by a change of coordinates, and the second order operator D is conjugate to the Laplacian, as required. □

Remark 9.4.9. In the above calculation we've computed the BV complex for a perturbative classical field theory on $\mathbb{R}^4$ as a cochain complex with a pairing only. We *haven't* described the pushforward of the L_∞ structure. In other words we've shown that we obtain the expected quadratic terms in the action for an anti-self-dual $N = 4$ gauge theory, but we haven't checked that the correct interaction terms appear. In what follows we take the compactification of holomorphic Chern-Simons on twistor space as the *definition* of untwisted $N = 4$ anti-self-dual super Yang-Mills.

We won't investigate the action in detail, but the holomorphic Chern-Simons action functional yields an *anti-self-dual* super Yang-Mills theory after compactifying the twistor lines. There's an extra term that we can introduce into the action, of form

$$S_2(A) = \int_{\mathbb{R}^{4|8}} d\mu \log\det(\overline{\partial}|_{p^{-1}(\mu)}).$$

Boels, Mason and Skinner [**BMS07**] prove that the holomorphic Chern-Simons theory on $N = 4$ super twistor space with this additional term incorporated into the action recovers $N = 4$ super Yang-Mills after compactifying along the twistor lines.

Remark 9.4.10. We run into trouble when we try to define untwisted $N = 4$ super Yang-Mills theory non-perturbatively via compactification along the twistor fibers, because the Penrose map p is *not* holomorphic for any complex structure on $\mathbb{R}^4$. As such, a Zariski open set $U \subseteq \mathbb{C}^2$ does not lift to a Zariski set $p^{-1}(U) \subseteq \mathbb{PT} \setminus \mathbb{CP}^1$. This is not a problem in the analytic setting; any open set in a complex manifold admits a canonical complex structure, but generally not an *algebraic* structure. It is not particularly surprising that we encounter such problems: there's no reason that a metric-dependent theory like untwisted $N = 4$ gauge theory should admit a description purely in terms of algebraic geometry.

CHAPTER 10

Equations of Motion in the Twisted Theories

We'll now investigate the form of the classical field theories obtained from applying our holomorphic and topological twists to this $N = 4$ theory. The holomorphic twist will be the simplest, conceptually: the holomorphic twisting data is compatible with the structure of $\mathrm{Bun}_G(\mathbb{PT}^{N=4})$ as a fiberwise formal algebraic gauge theory over $\mathbb{PT} \setminus \mathbb{CP}^1$, so a canonical holomorphic twist exists by corollary 8.2.36, which can naturally be thought of as a fiberwise formal algebraic gauge theory over $\mathbb{C}^2$, and which generalizes to describe a fiberwise formal algebraic gauge theory over a *compact* complex algebraic surface X whose global sections are given by

$$\mathrm{EOM}_{\mathrm{hol}}(X) \cong T_{\mathrm{form}}[1]\underline{\mathrm{Map}}(\Pi TX, BG).$$

The A and B topological twists are more subtle, because they each break structures that survive the holomorphic twist: the B-twist breaks the section $\mathrm{Bun}_G(U) \to \mathrm{EOM}_{\mathrm{hol}}(U)$, while the A-twist breaks the projection map $\mathrm{EOM}_{\mathrm{hol}}(U) \to \mathrm{Bun}_G(U)$. However, we'll construct natural twists using example 8.2.28: the A-twist deforms the outer shifted tangent bundle to the de Rham prestack, while the B-twist deforms the source of the mapping stack to X_{dR}, yielding the cotangent theory to the moduli of G-local systems.

10.1. The Holomorphic Twist

First, recall that according to the superspace formalism, to define the holomorphically twisted theory we need to specify a complex structure on a 4-manifold. The perturbative piece of this calculation is contained in Costello's 2011 paper [**Cos11a**], but is included here for the reader's convenience. Recall that a *G-Higgs bundle* on a complex variety X is an algebraic G-bundle P

equipped with a section $\phi \in H^0(X, T_X^* \otimes \mathfrak{g}_P)$ such that $[\phi, \phi] = 0$. We'll write $\mathrm{Higgs}_G(X)$ for the moduli stack of G-Higgs bundles, and $\mathrm{Higgs}_G^{\mathrm{fer}}(X)$ for the moduli stack of G-Higgs bundles where the Higgs field is placed in *fermionic degree* (so the underlying bosonic piece is just $\mathrm{Bun}_G(X)$). This moduli space is described by the mapping stack $\underline{\mathrm{Map}}(\Pi TX, BG)$.

The Penrose-Ward correspondence tells us that $N = 4$ anti-self-dual super Yang-Mills corresponds to the compactification of holomorphic Chern-Simons on super twistor space along the Penrose map p, where the bundles are constrained to be trivializable along the twistor lines. As we remarked in 9.4.10 this is problematic when working algebraically, because the map p is not holomorphic, so the compactification is not well-defined. We'll motivate a definition of holomorphically twisted $N = 4$ theory by computing the twist of the holomorphic Chern-Simons theory (since, by lemma 9.1.5 the compactification of this twist is the desired twist of $N = 4$ theory).

We use the following trick: find a closed embedding $\iota\colon Z \subseteq \mathbb{PT} \setminus \mathbb{CP}^1$ such that the Penrose map p maps Z diffeomorphically onto $\mathbb{R}^4$. We define the compactification of an algebraic gauge theory along p to be the restriction of the theory to Z.

First, we'll check that the twisting data we've been discussing preserves the fibers of the maps σ from Bun_G and π to Bun_G as in corollary 8.2.36, so the twist remains fiberwise formal.

Proposition 10.1.1. *The twisting data associated to the holomorphic twist preserve the fibers of the zero section map $\sigma\colon \mathrm{Bun}_G(U) \to \mathrm{EOM}(U)$ and the projection map $\pi\colon \mathrm{EOM}(U) \to \mathrm{Bun}_G(U)$ for an open set $U \subseteq \mathbb{C}^2$, as in definition 8.2.30.*

Proof. We can check the holomorphic twist preserves the fibers at the super twistor space level. For holomorphic Chern-Simons theory on super-twistor space the relevant map

$$\pi\colon \mathrm{Bun}_G(\mathbb{PT}^{N=4} \setminus \mathbb{CP}^1) \to \mathrm{Bun}_G(\mathbb{PT} \setminus \mathbb{CP}^1)$$

is given by pulling back under the zero section of the super vector bundle $\Pi\mathcal{O}(-1)^4$. The twisting data acts by pairing with a section of $\mathcal{O}(1) \hookrightarrow (\mathcal{O}(-1)^4)^*$, the dual to the first factor, which acts on the fibers by multiplication by that section in the coefficient $\mathrm{Sym}(\Pi(\mathcal{O}(1)^4))$. In particular, the fibers are preserved, so the twisting data acts trivially on the image of $d\pi$. Also pairing with such a section preserves the zero-section of the bundle over $\mathbb{PT}$, thus the image of the section σ and therefore the twisting data acts trivially on the image of $d\sigma$. □

As such, we can compute the holomorphic twist by computing the restricted relative shifted tangent complex as a sheaf over Bun_G, twisting the fibers, and applying Gaitsgory-Rozenblyum's theorem as in corollary 8.2.36.

Theorem 10.1.2. *The solutions to the equations of motion in the holomorphically twisted $N = 4$ SYM theory on $\mathbb{C}^2$ near an open set U are given by*

$$\mathrm{EOM}_{\mathrm{hol}}(U) \cong T^*_{\mathrm{form}}[-1]\,\mathrm{Higgs}^{\mathrm{fer}}_G(U).$$

Note that remark 9.4.3 applies for this theorem for general open sets U. The choice of holomorphic supercharge we made corresponds to a choice of complex structure on the base space $\mathbb{R}^4$ of the Penrose map. For concreteness, let us note that for a holomorphic G-bundle P on $U \subset \mathbb{C}^2$, thought of as a Higgs bundle with trivial Higgs field, one has

$$\mathbb{T}_P[-1]\,\mathrm{Higgs}^{\mathrm{fer}}_G(U) \cong \mathcal{O}(U;\mathfrak{g}_P) \oplus \Omega^{\geq 1}_{\mathrm{alg}}(U;\mathfrak{g}_P) \cong \Omega^{\natural}_{\mathrm{alg}}(U;\mathfrak{g}_P),$$

with zero differential, where Ω^p_{alg} is naturally in fermionic degree $p \mod 2$ and cohomological degree 0. Here the first summand of the complex describes deformations of the holomorphic bundle P and the second summand describes deformations of the Higgs field $0 \in \Pi\Omega^1_{\mathrm{alg}}(U;\mathfrak{g}_P)$. We will see in the proof that the homomorphically twisted theory is the cotangent theory with the base $\mathrm{Higgs}^{\mathrm{fer}}_G(U)$, namely,

$$\mathbb{T}_P[-1]\,\mathrm{EOM}_{\mathrm{hol}}(U) = \Omega^{\natural}_{\mathrm{alg}}(U;\mathfrak{g}_P) \oplus \Omega^{\natural}_{\mathrm{alg}}(U;\mathfrak{g}_P)^{\vee}[-3]$$

with the Lie algebra structure being the base acting on the shifted cotangent fiber in a canonical way.

Remark 10.1.3. A priori, the twists of the full $N = 4$ super Yang-Mills theory and its anti-self-dual piece might differ. However, this is actually not the case. In the appendix of Costello's paper on supersymmetric field theories [**Cos11a**] it is shown that the Q_{hol} twist of perturbative $N = 4$ anti-self-dual Yang-Mills doesn't admit any deformations as a perturbative field theory. If the twist of the full theory differed from the twist of the anti-self-dual theory, then there would be a path of twisted theories deforming one into the other (by sending the additional term in the action for the full theory to zero), thus a non-trivial deformation of the perturbative theory. Hence we can compute our twist using twistor space without worrying about the additional Boels-Mason-Skinner term in the action: this is guaranteed to be Q_{hol}-exact.

Proof. We'll begin with a summary of the global structure of the proof. First, in view of lemma 9.1.5 we'll compute the twist of holomorphic Chern-Simons theory on super twistor space. This amounts to computing the shifted tangent complex and performing the twisting construction to get a new family over $\mathbb{PT} \setminus \mathbb{CP}^1$, with the structure of a family of pointed formal moduli problems over Bun_G. In order to obtain the compactified theory on $\mathbb{C}^2$, we will use the trick described above: we'll find a closed embedding $\iota\colon Z \subseteq \mathbb{PT} \setminus \mathbb{CP}^1$ such that the Penrose map p induces a diffeomorphism $Z \cong \mathbb{R}^4$ (and hence defines a complex structure on $\mathbb{R}^4$) and define the compactification to be the restriction of the family from $\mathbb{PT} \setminus \mathbb{CP}^1$ to Z. Since the result is a family over $Z \cong \mathbb{C}^2$ of pointed formal moduli problems over Bun_G, the above computation determines the moduli space of solutions in the twisted, compactified theory, using theorem 8.2.25.

We will compute the twisted theory at the level of twistor space. Choose an open set $U \subseteq \mathbb{PT} \setminus \mathbb{CP}^1$, an affine derived scheme V, and a smooth map $f\colon V \to \mathrm{Bun}_G(U)$. The shifted tangent complex at the map f to the $N = 4$ super twistor space theory was canonically quasi-isomorphic to

$$\Gamma(p^{-1}(U) \times V; f^*\mathfrak{g}) \cong \Gamma(U \times V; \pi_1^* \operatorname{Sym}(\Pi\mathcal{O}(-1)|_U^4) \otimes f^*\mathfrak{g}),$$

where we write $f^*\mathfrak{g}$ to denote the sheaf of Lie algebras on $U \times V$ obtained by pulling back $\mathfrak{g} = \mathbb{T}[-1]BG$ under a closed point f of $\underline{\mathrm{Map}}(V, \mathrm{Bun}_G(U)) \cong \underline{\mathrm{Map}}(U \times V, BG)$, and where $\pi_1 \colon U \times V \to U$ is the projection. From now on we'll just write $\mathcal{O}(k)$ for the restriction $\mathcal{O}(k)|_U$ when our arguments are independent of U. Recall that when we twist we modify the sections of our theory over U by adding a $\mathbb{C}^\times$ weight to the cohomological grading then introducing a new differential coming from the supercharge. We'll choose a $\mathbb{C}^\times$-action such that the first copy of $\mathcal{O}(-1)$ (corresponding to $e_1^* \in W^*$) has weight -1, the third copy of $\mathcal{O}(-1)$ (corresponding to $f_1^* \in W^*$) has weight 1, and the remaining two copies (corresponding to e_2^* and $f_2^* \in W^*$) have weight 0.

The holomorphic supercharge $Q_{\mathrm{hol}} = \alpha_1 \otimes e_1$ can be thought of as a section of $\Pi\mathcal{O}(1)$ which pairs non-trivially with the first factor of $\mathcal{O}(-1)^4$ (generated by $e_1^* \in W^*$) to define a map $\mathcal{O}(-1)^4 \to \mathcal{O}$, which extends to a sym-degree -1 derivation of $\mathrm{Sym}(\Pi\mathcal{O}(-1)^4)$. The section of $\Pi\mathcal{O}(1)$ in question, corresponding to $\alpha_1 \in S_+$, is given on the open set U by the homogeneous polynomial Z_2 in twistor coordinates, so the differential given by Q_{hol} is generated by the map that multiplies a section of $\mathcal{O}(-1)$ on the set U by Z_2. This preserves the cohomological grading, but increases the weight by 1, since it reduces the number of e_1^* factors by 1. The map "multiply by Z_2" from $\mathcal{O}(k)$ to $\mathcal{O}(k+1)$ is injective, and has cokernel isomorphic to $\mathcal{O}_Z(k+1) = \iota_*\mathcal{O}_{\{Z_2=0\}}(k+1)$ where $\mathcal{O}_Z$ is the structure sheaf of the zero locus of Z_2. Thus we compute the Q_{hol}-twisted shifted tangent complex to be the space of global sections of the sheaf

$$\pi_1^*\left(\mathcal{O}_Z \oplus \mathcal{O}_Z(-2) \oplus \Pi\mathcal{O}_Z(-1)^2\right) \otimes f^*\mathfrak{g} \oplus \pi_1^*\left(\mathcal{O}_Z(-3) \oplus \mathcal{O}_Z(-1) \oplus \Pi\mathcal{O}_Z(-2)^2\right) \otimes f^*\mathfrak{g}[-1],$$

arising from the cohomology of the operator

$$\begin{tikzcd}
-1 & & \mathcal{O}(-1)_1 \arrow[dl] & \Pi\mathcal{O}(-2)^2_{12,14} \arrow[dl] & \mathcal{O}(-3)_{124} \arrow[dl] & \\
0 & \mathcal{O} & \Pi\mathcal{O}(-1)^2_{2,4} & \mathcal{O}(-2)_{13} \oplus \mathcal{O}(-2)_{24} \arrow[dl] & \Pi\mathcal{O}(-3)^2_{123,134} \arrow[dl] & \mathcal{O}(-4)_{1234} \arrow[dl] \\
1 & & \mathcal{O}(-1)_3 & \Pi\mathcal{O}(-2)^2_{23,34} & \mathcal{O}(-3)_{234} &
\end{tikzcd}$$

where in the diagram cohomological degree runs vertically, and the subscripts represent symmetric products of the four factors of $\Pi\mathcal{O}(-1)^4$. This result actually defines the BV complex of a cotangent theory whose base is the first factor – $\pi_1^*\left(\mathcal{O}_Z \oplus \mathcal{O}_Z(-2) \oplus \Pi\mathcal{O}_Z(-1)^2\right) \otimes f^*\mathfrak{g}$ – alone, since there is a canonical quasi-isomorphism of complexes

$$\left(\mathcal{O}(k-1)[1] \to \mathcal{O}(k)\right)^! \cong \left(\mathcal{O}(-k-4) \to \mathcal{O}(-k-3)[-1]\right)[3]$$

for each k by identifying the sheaf of densities with $\mathcal{O}(-4)[3]$ (the canonical sheaf shifted so that its cohomology is concentrated in degree zero) – where the morphisms are given by pairing with the section α_1 of $\mathcal{O}(1)$ – and therefore an invariant pairing on $\mathfrak{g}$ provides an isomorphism of coherent sheaves

$$\pi_1^*\left(\mathcal{O}_Z(-3) \oplus \mathcal{O}_Z(-1) \oplus \Pi\mathcal{O}_Z(-2)^2\right) \otimes f^*\mathfrak{g} \cong \pi_1^*\left(\left(\mathcal{O}_Z \oplus \mathcal{O}_Z(-2) \oplus \Pi\mathcal{O}_Z(-1)^2\right) \otimes f^*\mathfrak{g}\right)^![-2].$$

Since the original Lie algebra structure comes from the tensor product of sheaves and the Lie algebra structure on $f^*\mathfrak{g}$ (in the diagram, this pairs objects with their reflection through the center, with complementary subscripts), the induced Lie structure is that of a cotangent theory, using the nondegenerate invariant pairing.

After identifying $\mathcal{O}_Z(-1)^2 \cong \Omega^1_{Z,\mathrm{alg}}$ by choosing a trivialization, we obtain an isomorphism of coherent sheaves of graded Lie algebras

$$\pi_1^*\left(\mathcal{O}_Z \oplus \mathcal{O}_Z(-2) \oplus \Pi\mathcal{O}_Z(-1)^2\right) \otimes f^*\mathfrak{g} \cong (\pi_1')^*\Omega^\natural_{Z,\mathrm{alg}} \otimes f^*\mathfrak{g}$$

over Z, where Ω^1 is fermionic but in cohomological degree 0 and π_1' is the projection $Z \times V \to Z$. With this, the holomorphically twisted shifted tangent complex becomes

$$\Gamma(U \times V; (\pi_1')^*\Omega^\natural_{Z,\mathrm{alg}} \otimes f^*\mathfrak{g}) = \Omega^\natural_{\mathrm{alg}}((U \cap Z) \times V; f^*\mathfrak{g})$$

where we abuse notation to write $f^*\mathfrak{g}$ both for the sheaf on $U \times V$ and for its restriction to $(U \cap Z) \times V$.

Now, we have to compactify the twisted complex along the Penrose map. We might worry that this is undefined since p is not holomorphic, but we note that p maps $\{Z_2 = 0\}$ diffeomorphically onto $\mathbb{R}^4$ and henceforth identify Z as $\mathbb{R}^4$ (thus in particular defining a complex structure on $\mathbb{R}^4$). Then for $U \subset \mathbb{C}^2$, and a smooth map $f\colon V \to \mathrm{Bun}_G(U)$, one obtains the shifted tangent complex of the cotangent theory whose base is perturbatively given by $\Omega^\natural_{\mathrm{alg}}(U \times V; f^*\mathfrak{g})$ with zero differential, and where Ω^i is placed in fermionic degree $i \mod 2$.

It remains to globalize our computation using theorem 8.2.25. By definition of the tangent complex as a quasi-coherent sheaf, it is enough to check that for any affine derived scheme V over $\mathrm{Bun}_G(U)$, the local sections on V of the restricted shifted tangent complexes to $\mathrm{EOM}_{\mathrm{hol}}(U)$ and $T^*[-1]\,\mathrm{Higgs}^{\mathrm{fer}}_G(U)$ are equivalent as dg Lie algebras. This is exactly what we checked above: the local sections on V of the restricted tangent complex to $\mathrm{Higgs}^{\mathrm{fer}}_G(U)$ are precisely given by $\Omega^\natural_{\mathrm{alg}}(U \times V; f^*\mathfrak{g})$ with zero differential, and with Ω^i in fermionic degree $i \mod 2$, so the calculation above of the restricted shifted tangent complex to the holomorphically twist moduli space provides the desired dg Lie algebra equivalence for each f. Thus we obtain an equivalence

$$\mathbb{T}_{\mathrm{EOM}_{\mathrm{hol}}(U)}[-1] \cong \mathbb{T}_{T^*_{\mathrm{form}}[-1]\,\mathrm{Higgs}^{\mathrm{fer}}_G(U)}[-1]$$

of sheaves of dg Lie algebras, and therefore by theorem 8.2.25 an equivalence of derived stacks as required. $\square$

Remark 10.1.4. If we were working in an analytic framework, we could do this calculation by literally compactifying along the twistor lines. If $U \subseteq \mathbb{C}^2$ is an *analytic* open set then its pullback $p^{-1}U$ to twistor space admits a canonical complex structure despite p not being holomorphic.

Given the above calculation, we can *define* the holomorphic twist of $N = 4$ theory on any complex proper algebraic surface X using the superspace formalism of section 8.1.2.

Definition 10.1.5. *The* holomorphically twisted $N = 4$ theory *on a complex proper algebraic surface X is the assignment of derived stacks with*

$$\mathrm{EOM}_{\mathrm{hol}}(U) = T^*_{\mathrm{form}}[-1]\,\mathrm{Higgs}^{\mathrm{fer}}_G(U)$$

where $U \subseteq X$ is a Zariski open set, with the canonical -1-shifted symplectic structure on the global sections.

10.2. The B-twist

We'll now proceed to compute the B-twist of $N = 4$ super Yang-Mills on a complex proper algebraic surface X. This will again be a cotangent theory, but now to the moduli space $\mathrm{Loc}_G(X)$ of G-bundles with *flat connection.* As before, we'll compute the B-twist on flat space first – computing the twist of the holomorphically twisted theory on $\mathbb{C}^2$ with respect to the further B supercharge – then note that the superspace formalism allows us to extend the theory to one on general complex (proper) algebraic surfaces.

Unlike the example of the holomorphic twist in the previous section, the B supercharge will preserve the fibers of the projection map $\pi \colon \mathrm{EOM}_{\mathrm{hol}}(U) \to \mathrm{Bun}_G(U)$, but *not* of the section $\sigma \colon \mathrm{Bun}_G(U) \to \mathrm{EOM}_{\mathrm{hol}}(U)$. As such we will not be able to directly apply theorem 8.2.26 to describe a canonical twist. Instead, we'll observe that the moduli space $\mathrm{EOM}_{\mathrm{hol}}(U)$ has the structure of a mapping space, and the twisting data acts on the source of the mapping space alone, which *does* admit a natural deformation describable by theorem 8.2.26, yielding a natural B-twist.

We begin by describing $\mathrm{EOM}_{\mathrm{hol}}(U)$ in a slightly different way. Using the language of the Hodge prestack, as in example 8.2.28, we can rewrite the moduli space of solutions to the equations of motion in the holomorphic twist in a way natural for constructing our further A- and B-twists. There is a $\mathbb{C}^\times$ action α on $\mathrm{EOM}_{\mathrm{hol}}(U)$, which acts on the base space $\mathrm{Higgs}_G^{\mathrm{fer}}(U)$ of the shifted cotangent bundle in a way that on the fibers of the projection $\mathrm{Higgs}_G^{\mathrm{fer}}(U) \to \mathrm{Bun}_G(U)$ it does with weight minus one by rescaling the Higgs field.

Definition 10.2.1. *We'll write* $\mathrm{Higgs}_G^{\mathrm{bos}}(U)$ *for the formal completion*

$$\mathrm{Higgs}_G^{\mathrm{bos}}(U) = \mathrm{Higgs}_G(U)^{\wedge}_{\mathrm{Bun}_G(U)} = \underline{\mathrm{Map}}(T[1]U, BG)^{\wedge}_{\underline{\mathrm{Map}}(U,BG)}.$$

The superscript "bos" (for bosonic) is intended to contrast with the fermionic Higgs moduli space of the previous section, and to remind the reader that this formal Higgs moduli space differs slightly from the definition that more normally appears in the literature.

Lemma 10.2.2. *The regrading of the moduli space* $\mathrm{EOM}_{\mathrm{hol}}(U)$ *for a smooth surface* U *with respect to this* $\mathbb{C}^\times$*-action* α *is equivalent to the mapping stack*

$$T^*_{\mathrm{form}}[-1]\underline{\mathrm{Map}}(U_{\mathrm{Dol}}, BG)^{\wedge}_{\mathrm{Bun}_G(U)} \cong T^*_{\mathrm{form}}[-1]\,\mathrm{Higgs}_G^{\mathrm{bos}}(U).$$

Proof. We saw in theorem 10.1.2 for the surface $\mathbb{C}^2$, which we used as a definition for more general surfaces, that

$$\begin{aligned}
\mathrm{EOM}_{\mathrm{hol}}(U) &\cong T^*_{\mathrm{form}}[-1]\,\mathrm{Higgs}_G^{\mathrm{fer}}(U) \\
&\cong T^*_{\mathrm{form}}[-1]\underline{\mathrm{Map}}(\Pi TU, BG) \\
&\cong T^*_{\mathrm{form}}[-1](\underline{\mathrm{Map}}(\Pi TU, BG)^{\wedge}_{\mathrm{Bun}_G(U)}).
\end{aligned}$$

The $\mathbb{C}^\times$-action we've described acts on the fiber of ΠTU with weight one, so the regraded space is equivalent to

$$\mathrm{EOM}^{\alpha}_{\mathrm{hol}}(U) \cong T^*_{\mathrm{form}}[-1](\underline{\mathrm{Map}}(T[1]U, BG)^{\wedge}_{\mathrm{Bun}_G(U)}).$$

In turn, the shifted tangent bundle $T[1]U$ is equivalent to U_{Dol} (because U is a smooth scheme, so $T[1]U \cong T_{\mathrm{form}}[1]U$), so $\mathrm{EOM}^{\alpha}_{\mathrm{hol}}(U) \cong T^*_{\mathrm{form}}[-1]\,\mathrm{Higgs}^{\mathrm{bos}}_G(U)$ as required. □

Remark 10.2.3. The formal completion at $\mathrm{Bun}_G(U)$ is necessary for the bosonic but not the fermionic Higgs moduli space because, while the fibers of the map $\mathrm{Higgs}^{\mathrm{fer}}_G(U) \to \mathrm{Bun}_G(U)$ are purely fermionic, and therefore formal, the map $\mathrm{Higgs}_G(U) \to \mathrm{Bun}_G(U)$ has non-formal fibers, so the map is not a nil-isomorphism. Taking the completion while we regrade is necessary for the regraded theory to still be a formal algebraic gauge theory.

Now, let's describe a twist of the holomorphic theory with respect to the B-supercharge. The idea is that, viewing $\mathrm{EOM}_{\mathrm{hol}}(U)$ as a mapping space as in lemma 10.2.2 we can canonically deform the *source* from U_{Dol} to U_{dR}, for instance by applying theorem 8.2.26 to the symmetry generated by a non-vanishing degree one vector field on $T[1]U$. This will contrast with the A-twist in the next section, where we'll deform the global shifted cotangent bundle construction in a similar way.

Theorem 10.2.4. *The algebraic classical field theory* EOM_B *which assigns to a complex algebraic surface* U *the derived stack*

$$\mathrm{EOM}_B(U) = T^*_{\mathrm{form}}[-1]\,\mathrm{Loc}_G(U)$$

arises as a natural deformation of $\mathrm{EOM}_{\mathrm{hol}}(U)$ *which, if* $U = \mathbb{C}^2$*, defines a twist of* $N = 4$ *super Yang-Mills theory with respect to the topological supercharge* Q_B*.*

Remark 10.2.5. As we noted in remark 9.4.6, this theory is only a true algebraic classical field theory according to definition 8.2.6 if U is proper, ensuring that $\mathrm{Loc}_G(U)$ is finitely presented, so has a perfect tangent complex. In general the theory exists as an assignment of (possibly infinite type) derived stacks, but the presymplectic form on the shifted cotangent complex may be degenerate.

Proof. We'll build a canonical twist as discussed in remark 8.2.35. More specifically, we'll describe a deformation of the regrading $\mathrm{EOM}^{\alpha}_{\mathrm{hol}}(U)$ for a general surface U, then observe that if U is a Zariski open subset of $\mathbb{C}^2$ then it satisfies the conditions of definition 8.2.21.

For a fixed complex algebraic surface U, we consider the derived stack

$$\mathcal{M}'(U) = T^*_{\mathrm{form}}[-1]\underline{\mathrm{Map}}_{\mathbb{A}^1}(U_{\mathrm{Hod}}, BG \times \mathbb{A}^1),$$

the formal shifted cotangent to the mapping stack relative to $\mathbb{A}^1$. This admits a flat map to $\mathbb{A}^1$ whose fiber over t is canonically equivalent to $T^*_{\mathrm{form}}[-1]\underline{\mathrm{Map}}(U_{t\text{-dR}}, BG)$ – as in example 8.2.28 – so the general fiber is equivalent to $\mathcal{M}^{Q_B}(U) = T^*_{\mathrm{form}}[-1]\,\mathrm{Loc}_G(U)$, and whose fiber over zero is equivalent to $T^*_{\mathrm{form}}[-1]\,\mathrm{Higgs}_G(U)$. We've therefore defined a deformation of the regrading $\mathcal{M}^{\alpha}(U) = \mathrm{EOM}^{\alpha}_{\mathrm{hol}}(U)$, via the embedding $\mathrm{EOM}^{\alpha}_{\mathrm{hol}}(U) \to \mathrm{Higgs}_G(U)$, whose general fiber is the desired twisted moduli space.

Now, we must check the hypotheses of definition 8.2.21; that is, that for every closed point $P \in \mathrm{Bun}_G(U)$ we can find a section s such that $s(0) = \sigma_\alpha(P)$ and such that the relative shifted tangent complex agrees with the twist of the zero fiber as a perturbative field theory. For every closed point of $\mathcal{M}^{Q_B}(U)$ – just a closed point $A = (P, \nabla)$ of the base space $\mathrm{Loc}_G(U)$ – there's a natural section $s \colon \mathbb{A}^1 \to \mathcal{M}'$ given by rescaling the connection, such that the shifted tangent complex restricted to s is equivalent to the $\mathbb{C}[t]$-module

$$s^*\mathbb{T}_{\mathcal{M}'}[-1] = \big((\Omega^\bullet_{\mathrm{alg}}(U;\mathfrak{g}_P) \oplus \Omega^\bullet_{\mathrm{alg}}(U;\mathfrak{g}_P)^\vee[-3]) \otimes \mathbb{C}[t]), (td_A, td_A)\big)$$

where d_A is the algebraic covariant derivative associated to the flat connection ∇ on U. This defines a twist of the perturbative field theory $\mathbb{T}_P[-1]\,\mathrm{EOM}_{\mathrm{hol}}(U) = \Omega^\natural_{\mathrm{alg}}(U;\mathfrak{g}_P) \oplus \Omega^\natural_{\mathrm{alg}}(U;\mathfrak{g}_P)^\vee[-3]$ by the B-twisting data. □

It is immediate to identify compactification of the twisted theory along an algebraic curve.

Corollary 10.2.6. *For a product $\Sigma_1 \times \Sigma_2$ of algebraic curves, the B-twist of $N = 4$ super Yang-Mills theory satisfies*

$$\mathrm{EOM}_B(\Sigma_1 \times \Sigma_2) = T^*_{\mathrm{form}}[-1]\underline{\mathrm{Map}}((\Sigma_1)_{\mathrm{dR}}, \mathrm{Loc}_G(\Sigma_2)).$$

Proof. This follows from the definition $\mathrm{Loc}_G(X) = \underline{\mathrm{Map}}(X_{\mathrm{dR}}, BG)$ and the adjunction

$$\underline{\mathrm{Map}}(X \times Y, Z) = \underline{\mathrm{Map}}(X, \underline{\mathrm{Map}}(Y, Z)).$$

□

Remark 10.2.7. One can read this corollary as saying that the B-twisted theory compactifies to the B-model with target $\mathrm{Loc}_G(\Sigma_2)$. A completely perturbative description was given by Costello [**Cos11a**], which was not enough to identify $\mathrm{Loc}_G(\Sigma_2)$ as an algebraic stack. One should note that here we identify the target as the moduli stack of de Rham local systems, as opposed to Betti local systems, which is more aligned with the usual formulation of the geometric Langlands correspondence. This result is somewhat surprising, because it has been widely believed that the Kapustin-Witten story can only capture the topological aspects of the correspondence.

One might worry that one shouldn't expect the twist by a *topological* supercharge to depend on a choice of complex structure on spacetime, which our examples clearly do. Because this theory on X didn't necessarily arise from twisting a theory with respect to global topological twisting data, there's no reason that the moduli space $\mathrm{EOM}_B(U)$ shouldn't depend on a complex algebraic structure on U, and in general it *does* depend on this choice.

A more familiar example of this phenomenon is provided by Donaldson-Witten theory as a topological twist of $N = 2$ super Yang-Mills. While the theory on a flat space is truly topological, if one uses the superspace formalism to extend this theory to a general 4-manifold one finds that the moduli space of solutions to the equations of motion is built from the moduli space of instantons, which – if $b_2^+ = 1$ – may depend on the metric of the underlying 4-manifold, not just its diffeomorphism type. A discussion in the physics literature can be found in the 1998 paper of Moore and Witten [**MW98**].

From the point of view of the current work, this subtlety is necessary if we intend to recover a statement as the geometric Langlands conjecture, which is dependent on changes in the algebraic/holomorphic structure on a curve from a topologically twisted theory.

Remark 10.2.8. In theories like the B-twist, we would like to be able to talk about the germs of solutions to the equations of motion near some (smooth) submanifold of positive real codimension, especially codimension 1 submanifolds of form $\Sigma \times S^1$, where Σ is an algebraic curve: these germs of solutions correspond to the classical phase space in the 2d theory obtained by compactification along Σ. With the ideal, analytic definition 8.2.3 of a classical field theory this would be possible: one could define the space of *germs of solutions to the equations of motion* along a submanifold $Y \subseteq X$ to be the inverse image $\iota^{-1}\mathcal{M}$, where $\iota\colon Y \hookrightarrow X$ was the inclusion map. As we'll see, this would give very natural examples for an analytic version of the B-twisted classical field theory, but using our algebraic definition we'll need to use a slightly different construction.

Suppose we indeed had an algebraic model for the holomorphically twisted $N = 4$ theory with open sections on an *analytic* open set U given by $T^*[-1]\,\mathrm{Higgs}_G^{\mathrm{fer}}(U)$, interpreted in some natural way. Then we could make a claim of the following sort.

Claim. *If $Y \subseteq X$ is a compact oriented codimension k submanifold, then the germs of solutions to the equations of motion near Y in a B-twisted $N = 4$ theory are given by*

$$\mathrm{EOM}_B(Y) = T^*_{\mathrm{form}}[k-1]\,\mathrm{Loc}_G(Y)$$

where $\mathrm{Loc}_G(Y)$ is the space of germs of flat connections near $Y \subseteq X$.

Proof. To identify the moduli space of germs along Y we choose a tubular neighborhood U of Y in X, and use Poincaré duality to identify the compactly supported sections of the shifted tangent complex on Y with the compactly supported sections of the complex $\Omega^\bullet(U; \mathfrak{g}_P)[1]$ of $\mathrm{Loc}_G(U)$ plus a shift of its dual. Indeed, global sections of the inverse image $\iota^{-1}\,\mathrm{EOM}_B(Y)$ are just compactly supported sections of EOM_B on a tubular neighborhood U of Y. We have quasi-isomorphisms

$$\begin{aligned}(\Omega^\bullet_c(U; \mathfrak{g}_P)[1])^\vee &\cong (\Omega^\bullet(Y; \mathfrak{g}_P)[1])^\vee \\ &\cong \Omega^\bullet(Y; \mathfrak{g}_P^*)[\dim Y - 1] \\ &\cong \Omega^\bullet(Y; \mathfrak{g}_P^*)[3-k] \\ &\cong (\Omega^\bullet(Y; \mathfrak{g}_P^*[1])[1])\,[1-k]\end{aligned}$$

which gives the total compactly supported tangent complex $\Omega^\bullet_c(U; \mathfrak{g}_P \oplus \mathfrak{g}_P^*[1])[1]$ a $(k-1)$-symplectic structure which splits globally as the sum of a sheaf of complexes and a shift of its dual. Thus, after an application of a version of theorem 8.2.25 in analytic derived geometry we identify the moduli space of solutions with the appropriate shifted cotangent bundle. $\square$

We'll give an algebraic version of this claim for manifolds of form $\Sigma \times U$ for $U = S^1$ or $U = \mathrm{pt}$ below.

As discussed in the remark, we would like to make sense of what a theory assigns to a submanifold of nonzero codimension. Because our framework uses an algebraic structure of a submanifold in an essential way – we defined the B-twist by twisting theories only naturally defined for algebraic varieties – we'll need to extend our formalism. One observes that the base of the cotangent sheaf defining the B-twist can be described by $U \mapsto \mathrm{Loc}_G(U) = \mathrm{Map}(U_{\mathrm{dR}}, BG)$ for $U \subset X$ and that this assignment makes sense for a more general class of derived stacks than just algebraic varieties.

Specifically, let's consider compact connected manifolds U so that $U \times \Sigma$ has dimension less than four (formally, we're considering spaces of positive codimension for the 2-dimensional theory obtained by compactification along Σ): the only possibilities are the circle and the point. These are modelled by derived stacks S^1_B and pt, so we will simply consider $\Sigma \times U \mapsto \underline{\mathrm{Map}}((\Sigma \times U)_{\mathrm{dR}}, BG)$ for $U = S^1_B$ or $U = \mathrm{pt}$.

While it is natural to consider the assignment $V \mapsto \mathrm{Loc}_G(V)$ to such extended objects, the (-1)-shifted cotangent bundle is not: the degree of the shift must change depending on the dimension of V. In order to understand what this means, let us view $\mathrm{EOM}_B(X) = T^*_{\mathrm{form}}[-1]\mathrm{Loc}_G(X)$, where X is a smooth and proper algebraic surface, as arising by applying theorem 8.2.25 to a sheaf of dg Lie algebras over $\mathrm{Loc}_G(X)$ given by the dg Lie algebra equivalence

$$\begin{aligned}
\mathbb{T}_{T^*_{\mathrm{form}}[-1]\mathrm{Loc}_G(X)}[-1] &= \mathbb{T}_{\mathrm{Loc}_G(X)}[-1] \oplus (\mathbb{T}_{\mathrm{Loc}_G(X)}[-1])^\vee[-3] \\
&= \mathbb{T}_{\mathrm{Loc}_G(X)}[-1] \oplus \mathbb{L}_{\mathrm{Loc}_G(X)}[-2] \\
&= \mathbb{T}_{\mathrm{Loc}_G(X)}[-1] \oplus \mathbb{T}_{\mathrm{Loc}_G(X)},
\end{aligned}$$

where we use the (-2)-shifted symplectic structure of $\mathrm{Loc}_G(X) = \underline{\mathrm{Map}}(X_{\mathrm{dR}}, BG)$ obtained from the AKSZ construction using the 4-orientation on X_{dR} to identify the -2-shifted cotangent complex with the tangent complex [**PTVV13**, Theorem 2.5]. This is an equivalence of dg Lie algebras, where the second summand is treated as a module for the first summand. We'll extend this description of the moduli space of solutions to the equations of motion, to *define* the moduli space for spaces of form $\Sigma \times U$.

Definition 10.2.9. *For $U = S^1_B$ or $U = \mathrm{pt}$, we define $\mathrm{EOM}_B(\Sigma \times U)$ on X to be the derived stack obtained by applying the theorem 8.2.25 to the sheaf $\mathbb{T}_{\mathrm{Loc}_G(\Sigma\times U)}[-1] \oplus \mathbb{T}_{\mathrm{Loc}_G(\Sigma\times U)}$ of Lie algebras over $\mathrm{Loc}_G(\Sigma \times U)$.*

Corollary 10.2.10. *There is an equivalence of derived stacks*

$$\mathrm{EOM}_B(\Sigma \times S^1_B) = T^*_{\mathrm{form}}(\mathcal{L}\,\mathrm{Loc}_G(\Sigma)).$$

Proof. By definition, it is enough to compare the shifted tangent complexes of $\mathrm{EOM}_B(\Sigma\times S^1_B)$ and $T^*_{\mathrm{form}}(\mathcal{L}\,\mathrm{Loc}_G(\Sigma))$ as sheaves of Lie algebras over $\mathcal{L}\,\mathrm{Loc}_G(\Sigma)$. There are Lie algebra equivalences

$$\begin{aligned}
\mathbb{T}_{\mathrm{EOM}_B(\Sigma\times S^1_B)}[-1] &= \mathbb{T}_{\mathrm{Loc}_G(\Sigma\times S^1_B)}[-1]\oplus\mathbb{T}_{\mathrm{Loc}_G(\Sigma\times S^1_B)} \\
&= \mathbb{T}_{\mathrm{Loc}_G(\Sigma\times S^1_B)}[-1]\oplus\mathbb{L}_{\mathrm{Loc}_G(\Sigma\times S^1_B)}[-1] \\
&= \mathbb{T}_{\mathcal{L}\,\mathrm{Loc}_G(\Sigma)}[-1]\oplus(\mathbb{T}_{\mathcal{L}\,\mathrm{Loc}_G(\Sigma))}[-1])^\vee[-2] \\
&= \mathbb{T}_{T^*_{\mathrm{form}}(\mathcal{L}\,\mathrm{Loc}_G(\Sigma))}[-1]
\end{aligned}$$

where we use the (-1)-shifted symplectic structure of $\mathrm{Loc}_G(\Sigma\times S^1_B)=\underline{\mathrm{Map}}((\Sigma\times S^1_B)_{\mathrm{dR}},BG)\cong\underline{\mathrm{Map}}(\Sigma_{\mathrm{dR}}\times S^1_B,BG)$ provided by the AKSZ construction, using the 2-orientation on Σ_{dR} and the 1-orientation on S^1_B. $\square$

Note that the result is a 0-shifted symplectic derived stack. This is an expected property of a *phase space* in a classical field theory, i.e. the space the theory assigns to a proper codimension 1 submanifold. According the Kapustin-Witten program, this space should – under geometric quantization – yield the Hochschild homology of the category the relevant *extended* 2d topological quantum field theory assigns to the point, expected to be the category on the B-side of the geometric Langlands correspondence. We intend to address this in the sequel to this work.

Finally, we can similarly understand what the B-twisted theory assigns to spaces of codimension 2.

Corollary 10.2.11. *For a smooth projective curve Σ, the moduli space of germs of solutions to the equations of motion on $\Sigma\times\mathrm{pt}$ is given by*

$$\mathrm{EOM}_B(\Sigma\times\mathrm{pt})\cong T^*_{\mathrm{form}}[1]\,\mathrm{Loc}_G(\Sigma).$$

Proof. The argument here is very similar to the computation of the phase space in corollary 10.2.10. We apply theorem 8.2.25 to the sheaf

$$\mathbb{T}_{\mathrm{Loc}_G(\Sigma)}[-1] \oplus \mathbb{T}_{\mathrm{Loc}_G(\Sigma)}$$

on $\mathrm{Loc}_G(\Sigma)$. There are dg Lie algebra equivalences

$$\begin{aligned}\mathbb{T}_{\mathrm{Loc}_G(\Sigma)}[-1] \oplus \mathbb{T}_{\mathrm{Loc}_G(\Sigma)} &\cong \mathbb{T}_{\mathrm{Loc}_G(\Sigma)}[-1] \oplus \mathbb{L}_{\mathrm{Loc}_G(\Sigma)} \\ &\cong \mathbb{T}_{T^*_{\mathrm{form}}[1]\,\mathrm{Loc}_G(\Sigma)}\end{aligned}$$

using the 0-shifted symplectic structure on $\mathrm{Loc}_G(\Sigma)$. Again, applying theorem 8.2.25 completes the proof. □

Remark 10.2.12. In order to perform this calculation, we were forced to extend a natural calculation of EOM_B for algebraic varieties to spaces of form $\Sigma \times U_B$ by hand. In order to obtain a theory compatible with geometric Langlands, as proposed by Kapustin and Witten, we are forced to perform this procedure, where we replace a theory which is "de Rham" in all four directions with a theory that is de Rham in two directions and Betti (purely topological) in the remaining two. It is worth noting that these theories are very different: the purely de Rham theory is determined entirely by its local operators, whereas the de Rham-Betti theory admits non-trivial line operators (indeed, these are critical for the geometric Langlands program). Having made this modification, one can go further to investigate a theory in which all four directions are topological; an understanding of such a theory should lead to a physical description of the "Betti Langlands correspondence" of Ben-Zvi and Nadler, as discussed by Ben-Zvi–Brochier–Jordan [**BZBJ15**, 6.2] and Ben-Zvi–Nadler [**BZN16**].

10.3. The A-twist as a Limit of Holomorphic-Topological Twists

Understanding the A-twisted theory will be slightly different to our calculation for the B-twist, because the A-twisted theory is no longer a cotangent theory. However, it will be a cotangent theory upon a certain compactification. In fact, we will realize that the A-twist arose as a limit of holomorphic-topological twists, all of which yield cotangent theories upon such a compactification.

We'll begin by calculating the solutions to the equations of motion in the holomorphic-topological twists by an analogous procedure to the one we used for the B-twist. A crucial difference from the previous twists is that the relevant twisting data fails to preserve the fibers of the morphism $\pi\colon \mathrm{EOM}^{\alpha}_{\mathrm{hol}}(X) \to \mathrm{Bun}_G(X)$ defining the fiberwise formal algebraic gauge theory. However, for the A-twist, the fibers of the morphism $\sigma\colon \mathrm{Bun}_G(X) \to \mathrm{EOM}^{\alpha}_{\mathrm{hol}}(X)$ *are* preserved from the twisting data, so it's possible to define a canonical twist by applying the general construction 8.2.34 based on the general Gaitsgory-Rozenblyum correspondence in theorem 8.2.26.

Let $Q_\lambda = Q_{\mathrm{hol}} + \lambda(\alpha_2^\vee \otimes f_2^*) + (\alpha_2 \otimes e_2)$ be a holomorphic-topological supercharge as described at the end of section 8.1.1 (so $Q_\lambda \to Q_A$ as $\lambda \to 0$). We'll first consider a twisted theory with respect to these supercharges where $\lambda \in \mathbb{C}^\times$ on a space of form $X = \Sigma_1 \times \Sigma_2$, where Σ_i are smooth algebraic curves. We'll have to be careful: if $\lambda \neq 0$ then the twisting data is equivariant *neither* for the projection π, *nor* for the section σ, so there is no chance of constructing the twist canonically from formal, linear algebraic data. We will however describe a natural deformation of the holomorphically twisted theory, for each λ, including $\lambda = 0$ that yields a twist as defined in section 8.2, guided by the superspace description of the supersymmetry action.

Recall that the holomorphic-topological twist Q_λ for $\lambda \in \mathbb{C}^\times$ corresponds – in the superspace formalism – to the vector field $\overline{\partial}_{\Sigma_1} + d_{\Sigma_2} + \frac{\partial}{\partial \varepsilon}$ on $\Sigma_1 \times \Sigma_2$. The Q_λ-twisted theory admits a description in terms of moduli space of λ-connections, as in definition 8.2.27; let's describe this. Let U_1 and U_2 be smooth complex curves; we'll describe $\mathrm{EOM}_\lambda(U_1 \times U_2)$, where the supercharge Q_λ acts holomorphically in the first complex direction and topologically in the second direction. Since the twisting procedure for a supercharge Q that splits as $Q' + Q''$ with Q' purely of positive helicity and Q'' purely of negative helicity can be performed in steps without changing the result, as in remark 8.1.9, or more concretely by performing two deformations, then obtaining a composite deformation by restricting to the diagonal $\mathbb{A}^1 \subseteq \mathbb{A}^1 \times \mathbb{A}^1$, we first consider the twist by the vector field $\overline{\partial}_{\Sigma_1} + d_{\Sigma_2}$ and then by $\frac{\partial}{\partial \varepsilon}$.

When we twist with respect to the supercharge $\overline{\partial}_{\Sigma_1} + d_{\Sigma_2}$, it is clear from a similar line of reasoning to the one employed in theorem 10.2.4 that there is a natural twisted moduli space of solutions to the equations of motion on $U_1 \times U_2$ given by the (-1)-shifted formal cotangent space to the moduli stack of principal G-bundles on $U_1 \times U_2$ together with a formal Higgs field on U_1 and a flat λ-connection on Σ_2, that is, the mapping space

$$T^*_{\text{form}}[-1]\left(\underline{\text{Map}}\left((U_1)_{\text{Dol}} \times (U_2)_{\lambda\text{-dR}}, BG\right)^{\wedge}_{\underline{\text{Map}}(U_1 \times (U_2)_{\lambda\text{-dR}}, BG)}\right).$$

More precisely, there is a deformation of the holomorphically twisted moduli space given by the relative mapping space

$$T^*_{\text{form}}[-1]\left(\underline{\text{Map}}_{\mathbb{A}^1}\left((U_1)_{\text{Dol}} \times (U_2)_{\text{Hod}}, BG \times \mathbb{A}^1\right)^{\wedge}_{\underline{\text{Map}}_{\mathbb{A}^1}(U_1 \times (U_2)_{\text{Hod}}, BG \times \mathbb{A}^1)}\right),$$

whose fiber over λ is given by the mapping space above, and when U_1 and U_2 are both Zariski open subsets of $\mathbb{C}$ this defines a twist in the sense of definition 8.2.21.

As for the second summand, $\frac{\partial}{\partial \varepsilon}$, this supercharge has a very natural description when $U = X$ is proper, in which case it becomes the non-vanishing vector field of degree 1, because

$$T^*_{\text{form}}[-1]\underline{\text{Map}}(X_{\text{Dol}}, BG) = T_{\text{form}}[1]\underline{\text{Map}}(X_{\text{Dol}}, BG)$$

using the (-2)-shifted symplectic structure of the mapping stack from the AKSZ construction [**PTVV13**, Theorem 2.5].

The following proposition describes what happens when we perform the two supercharges successively.

Proposition 10.3.1. *If Σ_1 and Σ_2 are proper smooth curves, the moduli space of solutions to the equations of motion in the Q_λ twist of $N = 4$ gauge theory is equivalent to the de Rham prestack*

$$\mathrm{EOM}_\lambda(\Sigma_1 \times \Sigma_2) \cong \left(\underline{\mathrm{Map}}\left((\Sigma_1)_{\mathrm{Dol}} \times (\Sigma_2)_{\lambda\text{-}dR}, BG\right)^\wedge_{\underline{\mathrm{Map}}(\Sigma_1 \times (\Sigma_2)_{\lambda\text{-}dR}, BG)}\right)_{\mathrm{dR}}.$$

Proof. Since Σ_1 and Σ_2 are proper, the mapping space $\mathcal{X} = \underline{\mathrm{Map}}\left((\Sigma_1)_{\mathrm{Dol}} \times (\Sigma_2)_{\lambda\text{-dR}}, BG\right)$ and its formal completion are -2-shifted symplectic by the AKSZ construction. Indeed, BG is naturally 2-shifted symplectic and $(\Sigma_1)_{\mathrm{Dol}}$ and $(\Sigma_2)_{\lambda\text{-dR}}$ are both $\mathcal{O}$-compact and $\mathcal{O}$-2-oriented by their fundamental classes. Using this shifted symplectic form, we can identify $T^*[-1]\mathcal{X}$ with $T[1]\mathcal{X}$. The result then follows by example 8.2.28. □

This Q_λ-twisted moduli space has another description, which realizes the compactified theory as a cotangent field theory on Σ_1. For a convenient future reference, we first note the following lemma on some useful canonical equivalences of derived stacks.

Lemma 10.3.2. (1) *For a reduced scheme Y and any prestack $\mathcal{X}$, there is an equivalence*

$$\underline{\mathrm{Map}}(Y, \mathcal{X}_{\mathrm{dR}}) \cong \underline{\mathrm{Map}}(Y, \mathcal{X})_{\mathrm{dR}}.$$

(2) *For a smooth projective curve Σ and a k-shifted symplectic derived stack $\mathcal{X}$, there is an equivalence*

$$T^*_{\mathrm{form}}[k-2]\underline{\mathrm{Map}}(\Sigma, \mathcal{X}) \cong \underline{\mathrm{Map}}(T[1]\Sigma, \mathcal{X})^\wedge_{\underline{\mathrm{Map}}(\Sigma, \mathcal{X})}.$$

(3) *For a derived Artin stack $\mathcal{X}$ locally of finite presentation, there is an equivalence*

$$T^*_{\mathrm{form}}[k]T_{\mathrm{form}}[\ell]\mathcal{X} \cong T_{\mathrm{form}}[\ell]T^*_{\mathrm{form}}[k-\ell]\mathcal{X}$$

for all integers k and ℓ.

Proof. (1) We analyse the S-points for an arbitrary cdga S. There are equivalences

$$\begin{aligned}\underline{\mathrm{Map}}(Y,\mathcal{X})_{\mathrm{dR}}(S) &\cong \underline{\mathrm{Map}}(Y,\mathcal{X})(S^{\mathrm{red}}) \\ &\cong \mathrm{Map}(Y \times \operatorname{Spec} S^{\mathrm{red}}, \mathcal{X}) \\ &\cong \mathrm{Map}(Y^{\mathrm{red}} \times \operatorname{Spec} S^{\mathrm{red}}, \mathcal{X}) \\ &\cong \mathrm{Map}(Y \times \operatorname{Spec} S, \mathcal{X}_{\mathrm{dR}}) \\ &\cong \underline{\mathrm{Map}}(Y, \mathcal{X}_{\mathrm{dR}})(S).\end{aligned}$$

(2) Note that both the left-hand and right-hand sides are pointed formal moduli problems over the mapping space $\underline{\mathrm{Map}}(\Sigma,\mathcal{X})$ theorem 8.2.25 it suffices to provide an equivalence of their shifted relative tangent bundles as sheaves of dg Lie algebras. We observe that

$$\begin{aligned}\mathbb{T}_{T^*[k-2]\underline{\mathrm{Map}}(\Sigma,\mathcal{X})/\underline{\mathrm{Map}}(\Sigma,\mathcal{X})}[-1] &\cong \mathbb{L}_{\underline{\mathrm{Map}}(\Sigma,\mathcal{X})}[k-2][-1] \\ \text{and } \mathbb{T}_{\underline{\mathrm{Map}}(T[1]\Sigma,\mathcal{X})/\underline{\mathrm{Map}}(\Sigma,\mathcal{X})}[-1] &\cong (\mathbb{T}_{\underline{\mathrm{Map}}(\Sigma,\mathcal{X})} \to \sigma^*\mathbb{T}_{\underline{\mathrm{Map}}(T[1]\Sigma,\mathcal{X})})[-1] \\ &\cong (\mathbb{T}_{\underline{\mathrm{Map}}(\Sigma,\mathcal{X})} \to \sigma^*\mathbb{L}_{\underline{\mathrm{Map}}(T[1]\Sigma,\mathcal{X})}[k-2])[-1]\end{aligned}$$

where σ is the morphism of mapping stacks obtained by precomposition with the projection $T[1]\Sigma \to \Sigma$, and where on the last line we used the $(k-2)$-shifted symplectic structure on $\underline{\mathrm{Map}}(T[1]\Sigma,\mathcal{X}) \cong \underline{\mathrm{Map}}(\Sigma_{\mathrm{Dol}},\mathcal{X})$ obtained by the AKSZ construction. Note that the Lie algebra structure is trivial on both sides. The two-step complexes on the right-hand side just spell out the definition of the relative tangent complex, as an object of the derived category of sheaves.

The map σ induces a map of sheaves

$$\mathbb{T}_{\underline{\mathrm{Map}}(\Sigma,\mathcal{X})}[-1] \to \sigma^*\mathbb{T}_{\underline{\mathrm{Map}}(T[1]\Sigma,\mathcal{X})}[-1]$$

or dually, with a shift, a map

$$\sigma^* \mathbb{L}_{\underline{\mathrm{Map}}(T[1]\Sigma, \mathcal{X})}[k-3] \to \mathbb{L}_{\underline{\mathrm{Map}}(\Sigma, \mathcal{X})}[k-3].$$

We'll show that the kernel of this map is equivalent to $\mathbb{T}_{\underline{\mathrm{Map}}(\Sigma, \mathcal{X})}$, and therefore the induced map between relative tangent complexes is an equivalence. It suffices to check this claim for the fiber at each map $f \colon \Sigma \to \mathcal{X}$. At such a fiber, the map of sheaves induced by σ is given by the projection

$$\Gamma(\Sigma; \mathbb{L}_{\mathcal{X}} \otimes (\mathcal{O}_\Sigma[2] \oplus K_\Sigma[1]))[k-3] \to \Gamma(\Sigma; \mathbb{L}_{\mathcal{X}} \otimes K_\Sigma)[k-2].$$

On the other hand, the inclusion of a fiber of $\mathbb{T}_{\underline{\mathrm{Map}}(\Sigma, \mathcal{X})}[-1]$ is given by the composite

$$\Gamma(\Sigma; \mathbb{T}_{\mathcal{X}})[-1] \to \Gamma(\Sigma; \mathbb{T}_{\mathcal{X}} \otimes (\mathcal{O}_\Sigma \oplus K_\Sigma[-1])[-1] \cong \Gamma(\Sigma; \mathbb{L}_{\mathcal{X}}[k] \otimes (\mathcal{O}_\Sigma \oplus K_\Sigma[-1])[-1],$$

whose image is precisely the kernel of the projection, as required. Therefore the relative tangent complexes to our two derived stacks are equivalent, so the derived stacks themselves are equivalent, as required.

(3) Since both $T^*_{\mathrm{form}}[k]T_{\mathrm{form}}[\ell]\mathcal{X}$ and $T_{\mathrm{form}}[\ell]T^*_{\mathrm{form}}[k-\ell]\mathcal{X}$ define pointed formal moduli problems over $\mathcal{X}$, it suffices by theorem 8.2.25 to prove an equivalence for the restricted shifted tangent complexes as sheaves of Lie algebras over $\mathcal{X}$. We realize such an equivalence as the composite

$$\begin{aligned}
\sigma^* \mathbb{T}_{T^*_{\mathrm{form}}[k]T_{\mathrm{form}}[\ell]\mathcal{X}}[-1] &\cong ((\mathbb{T}_{\mathcal{X}} \oplus \mathbb{T}_{\mathcal{X}}[\ell]) \oplus (\mathbb{L}_{\mathcal{X}} \oplus \mathbb{L}_{\mathcal{X}}[-\ell])[k])[-1] \\
&\cong (\mathbb{T}_{\mathcal{X}} \oplus \mathbb{T}_{\mathcal{X}}[\ell] \oplus \mathbb{L}_{\mathcal{X}}[k] \oplus \mathbb{L}_{\mathcal{X}}[k-\ell])[-1] \\
&\cong ((\mathbb{T}_{\mathcal{X}} \oplus \mathbb{L}_{\mathcal{X}}[k-\ell]) \oplus (\mathbb{T}_{\mathcal{X}} \oplus \mathbb{L}_{\mathcal{X}}[k-\ell])[\ell])[-1] \\
&\cong \sigma^* \mathbb{T}_{T_{\mathrm{form}}[\ell]T^*_{\mathrm{form}}[k-\ell]\mathcal{X}}[-1]
\end{aligned}$$

of dg Lie algebra equivalences, where the Lie structure on the second line is given by the bracket on the first factor, the action of the first factor on each of the others, and the pairing between the second and fourth factors, taking values in the third factor.

□

Remark 10.3.3. (1) The equivalence $\underline{\mathrm{Map}}(Y, \mathcal{X}_{\mathrm{dR}}) \cong \underline{\mathrm{Map}}(Y, \mathcal{X})_{\mathrm{dR}}$ arises as an equivalence of the full Hodge stack. For this, it is enough to observe that $\underline{\mathrm{Map}}(Y, T_{\mathrm{form}}[1]\mathcal{X}) \cong T_{\mathrm{form}}[1]\underline{\mathrm{Map}}(Y, \mathcal{X})$ has the same relative shifted tangent complex over $\underline{\mathrm{Map}}(Y, \mathcal{X})$, which is immediate.

(2) The third equivalence for $\ell = 1$ is also compatible with its de Rham deformation. More precisely, under the equivalence $T_{\mathrm{form}}[1]T^*_{\mathrm{form}}[-k]\mathcal{X} \cong T^*_{\mathrm{form}}[1-k]T_{\mathrm{form}}[1]\mathcal{X}$, we can transfer the natural deformation of the shifted tangent complex on the left-hand side corresponding to the family of sheaves $t \cdot \mathrm{id}\colon \pi^*\mathbb{T}_{T^*_{\mathrm{form}}[-k]\mathcal{X}} \to \pi^*\mathbb{T}_{T^*_{\mathrm{form}}[-k]\mathcal{X}}$ over $\mathbb{A}^1$, where π is the projection $T_{\mathrm{form}}[1]T^*_{\mathrm{form}}[-k]\mathcal{X} \to T^*_{\mathrm{form}}[-k]\mathcal{X}$, to the right-hand side. The result is the pullback under the map $T^*_{\mathrm{form}}[1-k]T_{\mathrm{form}}[1]\mathcal{X} \to T_{\mathrm{form}}[1]\mathcal{X}$ of the deformation $t \cdot \mathrm{id}\colon \pi'^*\mathbb{T}_{\mathcal{X}} \to \pi'^*\mathbb{T}_{\mathcal{X}}$, where now π' is the projection $T_{\mathrm{form}}[1]\mathcal{X} \to \mathcal{X}$. Now, we can consider the formal completions of both sides of our equivalence with respect to $T^*_{\mathrm{form}}[-k]\mathcal{X}$ to obtain a pair of equivalent pointed formal moduli problems over $T^*_{\mathrm{form}}[-k]\mathcal{X}$. By theorem 8.2.25 these are determined by (equivalent) sheaves of dg Lie algebras over $T^*_{\mathrm{form}}[-k]\mathcal{X}$, and we've described equivalent 1-parameter deformations of these sheaves, and therefore of the resulting formal moduli problems under $T^*_{\mathrm{form}}[-k]\mathcal{X}$. The fibers over 1 of these deformed moduli problems are given by

$$(T^*_{\mathrm{form}}[-k]\mathcal{X})_{\mathrm{dR}} \cong T^*_{\mathrm{form}}[1-k](\mathcal{X}_{\mathrm{dR}})$$

where the latter is a formal moduli problem under $T^*[-k]\mathcal{X}$ by the composite

$$T^*_{\mathrm{form}}[-k]\mathcal{X} \to (T^*_{\mathrm{form}}[-k]\mathcal{X})_{\mathrm{dR}} \to \mathcal{X}_{\mathrm{dR}} \cong T^*_{\mathrm{form}}[1-k](\mathcal{X}_{\mathrm{dR}}).$$

Theorem 10.3.4. *The moduli space of solutions to the equations of motion on the product $\Sigma_1 \times \Sigma_2$ of two smooth projective curves after applying the Q_λ-twist is equivalent to*

$$\mathrm{EOM}_\lambda(\Sigma_1 \times \Sigma_2) \cong T^*_{\mathrm{form}}[-1]\underline{\mathrm{Map}}\left(\Sigma_1, \mathrm{Loc}_G^\lambda(\Sigma_2)_{\mathrm{dR}}\right)$$

in a canonical way.

Remark 10.3.5. This statement is not contentless, despite the fact that it involves the cotangent bundle of a de Rham stack, which is necessarily trivial. Indeed, the equivalence is compatible with the deformation to the whole Hodge stack. All such statements appearing in the paper arise as specializations of equivalences of Hodge stacks.

Proof. We begin with the derived stack on the right-hand side. Since $\mathrm{Loc}_G^\lambda(\Sigma_2) = \underline{\mathrm{Map}}(\Sigma_{\lambda\text{-dR}}, BG)$ is 0-shifted symplectic by the AKSZ construction, there is an equivalence $T[1]\,\mathrm{Loc}_G^\lambda(\Sigma_2) \cong T^*[1]\,\mathrm{Loc}_G^\lambda(\Sigma_2)$, so in particular $\mathrm{Loc}_G^\lambda(\Sigma_2)_{\mathrm{Dol}} = T_{\mathrm{form}}[1]\,\mathrm{Loc}_G^\lambda(\Sigma_2)$ is 1-shifted symplectic. We have equivalences

$$\begin{aligned}
T^*_{\mathrm{form}}[-1]\underline{\mathrm{Map}}\left(\Sigma_1, \mathrm{Loc}_G^\lambda(\Sigma_2)_{\mathrm{dR}}\right) &\cong T^*_{\mathrm{form}}[-1]\left(\underline{\mathrm{Map}}(\Sigma_1, \mathrm{Loc}_G^\lambda(\Sigma_2))_{\mathrm{dR}}\right) \\
&\cong \left(T^*_{\mathrm{form}}[-2]\underline{\mathrm{Map}}(\Sigma_1, \mathrm{Loc}_G^\lambda(\Sigma_2))\right)_{\mathrm{dR}} \\
&\cong \left(\underline{\mathrm{Map}}(T[1]\Sigma_1, \mathrm{Loc}_G^\lambda(\Sigma_2))^\wedge_{\underline{\mathrm{Map}}(\Sigma_1, \mathrm{Loc}_G^\lambda(\Sigma_2))}\right)_{\mathrm{dR}} \\
&\cong \left(\left(\underline{\mathrm{Map}}\left((\Sigma_1)_{\mathrm{Dol}}, \mathrm{Loc}_G^\lambda(\Sigma_2)\right)\right)^\wedge_{\underline{\mathrm{Map}}(\Sigma_1, \mathrm{Loc}_G^\lambda(\Sigma_2))}\right)_{\mathrm{dR}} \\
&\cong \left(\left(\underline{\mathrm{Map}}\left((\Sigma_1)_{\mathrm{Dol}} \times (\Sigma_2)_{\lambda\text{-dR}}, BG\right)\right)^\wedge_{\underline{\mathrm{Map}}(\Sigma_1, \mathrm{Loc}_G^\lambda(\Sigma_2))}\right)_{\mathrm{dR}} \\
&= \mathrm{EOM}_\lambda(\Sigma_1 \times \Sigma_2),
\end{aligned}$$

where on the first line we used lemma 10.3.2 part 1, on the second line we used remark 10.3.3 part 2, and on the fifth line we used the adjunction

$$\underline{\mathrm{Map}}((\Sigma_1)_{\mathrm{Dol}} \times (\Sigma_2)_{\lambda\text{-dR}}, BG) = \underline{\mathrm{Map}}((\Sigma_1)_{\mathrm{Dol}}, \underline{\mathrm{Map}}((\Sigma_2)_{\lambda\text{-dR}}, BG)).$$

Now in view of remark 10.3.3, one can note that the whole equivalences work at the level of Hodge stacks. □

Remark 10.3.6. We have two apparently different-looking descriptions of our moduli space, but the point is that one can use either one. For the rest of the paper, we won't use this latter description. On the other hand, when $\lambda = 0$, this theorem amounts to identifying the compactification of the A-twisted theory along Σ_2 with the A-model with target $\mathrm{Higgs}_G(\Sigma_2)$, as expected from the physics literature. This can also be understood as an algebraization and globalization of Costello's perturbative description of the A-model in the smooth category [**Cos11a**].

Let's now discuss what this assigns to objects of nonzero codimension as we did in section 10.2:

$$\mathrm{EOM}_\lambda(\Sigma \times U) \cong \left(\underline{\mathrm{Map}}\left(\Sigma_{\mathrm{Dol}} \times U_{\lambda\text{-dR}}, BG\right)^{\wedge}_{\underline{\mathrm{Map}}(\Sigma \times U_{\lambda\text{-dR}}, BG)}\right)_{\mathrm{dR}}$$

as in proposition 10.3.1 the assignment naturally extends to $U = S^1_B$ or $U = \mathrm{pt}$.

We'll describe it in a way designed to illustrate the connection with geometric Langlands. However, the argument we gave for theorem 10.3.4 no longer applies. Instead of a (-1)-shifted cotangent space, we'll produce a 0-shifted cotangent space. In the A-twist, the degree of shifting comes naturally so we don't need any auxiliary step: de Rham stack can be regarded as k-shifted symplectic for any k, but that being realized as a Hodge stack over $\mathbb{A}^1$ determines the unique number k in such a way that ensures compatibility for any $t \in \mathbb{A}^1$.

Proposition 10.3.7. *The phase space* $\mathrm{EOM}_\lambda(\Sigma \times S^1_B)$ *in the* Q_λ*-twisted theory is equivalent to*

$$T^*_{\mathrm{form}}(\underline{\mathrm{Map}}(S^1_B, \mathrm{Bun}_G(\Sigma))_{\mathrm{dR}}).$$

In particular the result is independent of the value of λ*. The equivalence arises by taking the fiber at 1 of an equivalence of deformations, whose fiber at 0 is an equivalence*

$$\underline{\mathrm{Map}}\left(\Sigma_{\mathrm{Dol}} \times (S^1_B)_{\lambda\text{-}dR}, BG\right)_{\mathrm{Dol}} \cong T^*_{\mathrm{form}} T_{\mathrm{form}}[1]\underline{\mathrm{Map}}(S^1_B, \mathrm{Bun}_G(\Sigma)).$$

Proof. First, observe that $(S^1_B)_{\lambda\text{-dR}} \cong S^1_B$ for all $\lambda \in \mathbb{C}$. Indeed, any topological space Y viewed as a derived stack has trivial tangent complex, so $(Y_B)_{\mathrm{Hod}} \cong Y_B \times \mathbb{A}^1$. According to proposition 10.3.1 and lemma 10.3.2 part 2 we have

$$\begin{aligned}
\mathrm{EOM}_\lambda(\Sigma \times S^1_B) &\cong \left(\underline{\mathrm{Map}}\left(\Sigma_{\mathrm{Dol}} \times (S^1_B)_{\lambda\text{-dR}}, BG\right)^\wedge_{\underline{\mathrm{Map}}(\Sigma \times (S^1_B)_{\lambda\text{-dR}}, BG)}\right)_{\mathrm{dR}} \\
&\cong \left(\underline{\mathrm{Map}}(T[1]\Sigma, \mathrm{Loc}_G(S^1))^\wedge_{\underline{\mathrm{Map}}(\Sigma \times (S^1_B)_{\lambda\text{-dR}}, BG)}\right)_{\mathrm{dR}} \\
&\cong (T^*_{\mathrm{form}}[-1]\underline{\mathrm{Map}}(\Sigma, \mathrm{Loc}_G(S^1)))_{\mathrm{dR}}.
\end{aligned}$$

This falls into a family of equivalences, by replacing the de Rham prestack with the Hodge prestack, whose central fiber is given by the formal completion

$$\begin{aligned}
T_{\mathrm{form}}[1]\underline{\mathrm{Map}}\left(\Sigma_{\mathrm{Dol}} \times (S^1_B)_{\lambda\text{-dR}}, BG\right) &\cong T_{\mathrm{form}}[1]T^*_{\mathrm{form}}[-1]\underline{\mathrm{Map}}(\Sigma, \mathrm{Loc}_G(S^1)) \\
&\cong T^*_{\mathrm{form}}T_{\mathrm{form}}[1]\underline{\mathrm{Map}}(\Sigma, \mathrm{Loc}_G(S^1)),
\end{aligned}$$

by lemma 10.3.2 part 3. To conclude the proof we observe that the degree 1 symmetry of the tangent complex generating the de Rham deformation via example 8.2.28 corresponds – under the equivalence – to the symmetry on the right-hand side deforming $T^*_{\mathrm{form}}T_{\mathrm{form}}[1]\underline{\mathrm{Map}}(\Sigma, \mathrm{Loc}_G(S^1))$ to $T^*_{\mathrm{form}}(\underline{\mathrm{Map}}(\Sigma, \mathrm{Loc}_G(S^1))_{\mathrm{dR}})$ by remark 10.3.3 part 2. $\square$

Given that the A-twist is computed by an identical procedure to the more general λ-twist, one might ask what the point is of considering the λ family of twists at all. The claim, which I intend to address in more detail in future work, is that in order to see more refined structures in the geometric Langlands program, it is necessary to consider such twists. The following remark provides a hint of this structure.

Remark 10.3.8. The curve $\Sigma = \mathbb{CP}^1$ deserves a little more attention; we'll describe an infinitesimal version of the above calculation, explicitly using the family of theories obtained by varying λ. Instead of describing the solutions to the equations of motion on the derived stack $\mathbb{CP}^1 \times S^1_B$, we'll instead consider a different complex structure on a complex neighborhood of $S^2 \times S^1$. The following construction should be thought of as informal and motivational, since we'll use complex analytic constructions that don't make sense in derived algebraic geometry. Consider the complex manifold

$$(\mathbb{C} \times \mathbb{C}^\times) \setminus (\{0\} \times S^1).$$

Note that there are diffeomorphisms $\mathbb{C} \times \mathbb{C}^\times \cong \mathbb{C} \times (0, \infty) \times S^1 \simeq B^3 \times S^1$ for an open three-ball B^3 around 0. Removing $\{0\} \times S^1$ from $\mathbb{C} \times \mathbb{C}^\times$ corresponds to removing $\{0\} \times S^1$ from $B^3 \times S^1$ on the right-hand side, yielding a diffeomorphism $(B^3 \setminus \{0\}) \times S^1 \simeq (S^2 \times (-1, 1)) \times S^1$. Thus we can think of $(\mathbb{C} \times \mathbb{C}^\times) \setminus (\{0\} \times S^1)$ as a complex manifold thickening $S^2 \times S^1$.

From proposition 10.3.1, the space of solutions to the equations of motion is obtained by applying the de Rham space construction to the moduli space of G-bundles on $(\mathbb{C} \times \mathbb{C}^\times) \setminus (\{0\} \times S^1)$ with a Higgs field on $\mathbb{C}$ and a flat λ-connection on $\mathbb{C}^\times$. Let us denote the two connected components of $\mathbb{C}^\times \setminus S^1$ by A_{in} and A_{out}. Note that a G-bundle P on $(\mathbb{C} \times \mathbb{C}^\times) \setminus (\{0\} \times S^1)$ is equivalent to the data of a triple $(P', \phi_{\text{in}}, \phi_{\text{out}})$, where P' is the restriction of P to $\mathbb{C}^\times \times \mathbb{C}^\times$, ϕ_{in} is the extension of $P'|_{\mathbb{C}^\times \times A_{\text{in}}}$ to $\mathbb{C} \times A_{\text{in}}$, and ϕ_{out} is the extension of $P'|_{\mathbb{C}^\times \times A_{\text{out}}}$ to $\mathbb{C} \times A_{\text{out}}$. Note that ignoring the annular factor we would obtain a G-bundle on a "bubbled" plane $B := \mathbb{C} \amalg_{\mathbb{C}^\times} \mathbb{C}$ made by gluing the two planes along $\mathbb{C}^\times$.

Then we can describe the moduli space of solutions to the equations of motion on $(\mathbb{C} \times \mathbb{C}^\times) \setminus (\{0\} \times S^1)$ as a datum $(P', \phi_{\text{in}}, \phi_{\text{out}})$ of this form, together with a Higgs field and a flat λ-connection in the two complex directions. Since we have a flat λ-connection in the $\mathbb{C}^\times$-direction throughout, we can understand the space of germs of solutions to the equations of motion near $S^2 \times S^1$ as the de Rham stack of $\underline{\text{Map}}(S^1_B, \text{Higgs}^{\text{bos}}_G(B))$. It is essential here to have $\lambda \neq 0$: otherwise we cannot simply describe the moduli spaces in a way that depend only on the topology of $\mathbb{C}^\times$, and not its algebraic structure.

Finally, we can replace $\mathbb{C}$ by the formal disk $\mathbb{D}$. One then obtains as the space of solutions to the equations of motion

$$\text{EOM}(\mathbb{B} \times S^1_B) \cong T^*_{\text{form}}(\underline{\text{Map}}(S^1_B, \text{Bun}_G(\mathbb{B}))_{\text{dR}})$$

where $\mathbb{B}$ is the "formal bubble" $\mathbb{B} := \mathbb{D} \amalg_{\mathbb{D}^\times} \mathbb{D}$. The space of G-bundles on the formal bubble $\mathbb{B}$ is a familiar space in geometric representation theory: the quotient of the *affine Grassmannian* $\mathcal{G}\mathrm{r}_G$ by the arc group $G(\mathbb{C}[[t]])$. We'll investigate the action of a quantization of this moduli space $\mathrm{EOM}(\mathbb{B} \times S^1_B)$ on a quantization of $\mathrm{EOM}(\Sigma \times S^1)$ for general surfaces Σ, inherited from the geometric structure of the bases of these cotangent spaces in future work.

To conclude this chapter, we'd also like to understand germs of solutions to the equations of motion near manifolds of codimension 2.

Proposition 10.3.9. $\mathrm{EOM}_\lambda(\Sigma \times \mathbb{C}) \cong T^*_{\mathrm{form}}[1](\mathrm{Bun}_G(\Sigma)_{\mathrm{dR}})$. *The equivalence arises as the fiber at 1 of an equivalence of deformations, whose fiber over 0 is*

$$\underline{\mathrm{Map}}(\Sigma_{\mathrm{Dol}} \times \mathrm{pt}_{\lambda\text{-}dR}, BG)_{\mathrm{Dol}} \cong T^*_{\mathrm{form}}[1] T_{\mathrm{form}}[1] \,\mathrm{Bun}_G(\Sigma)^{\wedge}_{T^*_{\mathrm{form}}\,\mathrm{Bun}_G(\Sigma)}.$$

Proof. Lemma 10.3.2 provides an equivalence

$$\begin{aligned}
\mathrm{EOM}_\lambda(\Sigma) &\cong \left(\underline{\mathrm{Map}}(\Sigma_{\mathrm{Dol}} \times \mathrm{pt}_{\lambda\text{-dR}}, BG)^{\wedge}_{\underline{\mathrm{Map}}(\Sigma \times \mathrm{pt}_{\lambda\text{-dR}}, BG)}\right)_{\mathrm{dR}} \\
&\cong \left(\underline{\mathrm{Map}}(T[1]\Sigma, BG)^{\wedge}_{\underline{\mathrm{Map}}(\Sigma \times \mathrm{pt}_{\lambda\text{-dR}}, BG)}\right)_{\mathrm{dR}} \\
&\cong (T^*_{\mathrm{form}} \underline{\mathrm{Map}}(\Sigma, BG))_{\mathrm{dR}}
\end{aligned}$$

using the 2-shifted symplectic structure on BG. As in the proof of proposition 10.3.7, this equivalence arises as the generic fiber of a natural deformation, whose fiber over zero is

$$\begin{aligned}
T_{\mathrm{form}}[1]\underline{\mathrm{Map}}(\Sigma_{\mathrm{Dol}} \times \mathrm{pt}_{\lambda\text{-dR}}, BG) &\cong T_{\mathrm{form}}[1] T^*_{\mathrm{form}} \underline{\mathrm{Map}}(\Sigma, BG)) \\
&\cong T^*_{\mathrm{form}}[1] T_{\mathrm{form}}[1] \underline{\mathrm{Map}}(\Sigma, BG)) \\
&\cong T^*_{\mathrm{form}}[1] T_{\mathrm{form}}[1] \,\mathrm{Bun}_G(\Sigma).
\end{aligned}$$

Again, as in lemma 10.3.2 we observe by remark 10.3.3 part 2 that the degree 1 symmetry of the tangent complex generating the de Rham deformation corresponds to the symmetry on the right-hand side deforming the Dolbeault stack to the Hodge prestack, thus providing an equivalence of deformations, as required. □

Remark 10.3.10. As above, if $\Sigma = \mathbb{CP}^1$ we have the option to perform an infinitesimal construction. By the same reasoning as in remark 10.3.8 one can choose as a thickening $(\mathbb{C} \times \mathbb{C}) \setminus (\{0\} \times I)$, where I is the imaginary axis in the second factor: there is a diffeomorphism

$$S^2 \times (-1,1) \times I \simeq (B^3 \setminus \{0\}) \times I \simeq (\mathbb{C} \times \mathbb{C}) \setminus (\{0\} \times I)$$

which we again think of as a choice of complex thickening of S^2. Running through the same calculation as in remark 10.3.8 we end up with the moduli space of germs of solutions to the equations of motion $T^*_{\mathrm{form}}[1](\mathrm{Bun}_G(\mathbb{B})_{\mathrm{dR}})$. This will naturally appear in an interpretation of geometric Satake as arising from line operators.

Part 3

Outlook and Future Work

CHAPTER 11

Abelian Duality for $N = 4$ Theories

The first step to truly combining the two main parts of this thesis will be to extend the analysis of part 1 to twisted supersymmetric abelian (higher) gauge theories. In particular, a sufficiently rich construction of abelian duality for the Kapustin-Witten twisted abelian $N = 4$ gauge theories should recover the abelian version of the geometric Langlands correspondence, as realized independently by Laumon [**Lau96**] and Rothstein [**Rot96**] by a twisted Fourier-Mukai transformation.

This analysis will require two new pieces of input.

(1) It will be necessary to investigate exactly how abelian duality interacts with supersymmetry. Indeed, extending the analysis of part 1 to the untwisted abelian $N = 4$ theory is not difficult, since the fields in this abelian theory are not coupled to one another: one simply defines the dual on 2-form fields using the Hodge star as usual, and identifies scalar fields as dual to scalar fields and positive helicity spinor fields as dual to negative helicity spinor fields in a natural way. This provides a correspondence of factorization algebras, but how does it descend to the twisted theories? Ideally, the morphisms of this correspondence will be equivariant for appropriate actions of the $N = 4$ supersymmetry algebra, so that the action of the A-supercharge on one side corresponds to the action of the B-supercharge on the other. If this is the case then the duality descends to a correspondence relating the factorization algebras of dual twisted theories.

(2) Additionally, we need a procedure extending abelian duality from a correspondence on the level of the factorization algebras of observables to a correspondence of the categories of branes. A natural first step towards this understanding would be to apply theorems of Scheimbauer [**Sch14**], which associate to a locally constant factorization algebra an

extended topological field theory (in the sense of Lurie) whose target is an $\mathbb{E}_n$ Morita category. In particular, to a surface Σ, the theorem will associate an $\mathbb{E}_2$ algebra (with additional structure) whose module category we can compare to the expected categories from geometric class field theory. In order to apply these results, it would be necessary to extend the factorization algebra of observables to a constructible factorization algebra on a certain stratified space.

CHAPTER 12

Constructing Topological Field Theories

There's another important step which needs to be investigated to really connect the methods of Kapustin and Witten with geometric representation theory and the geometric Langlands correspondence: which aspects of the classical theories discussed in part 2 can be quantized? For instance, in chapter 10 we identified the space of germs of solutions to the equations of motion near (certain) 3-manifolds in the A- and B-twisted $N = 4$ theories as the total spaces of cotangent bundles. In particular, this description yields canonical polarizations on these moduli spaces, allowing us to define the Hilbert spaces associated to these theories by geometric quantization. Specifically, the Hilbert space associated of the A- and B-twisted theories after dimensional reduction along Σ have the forms $\mathcal{H}_A(\Sigma) = \Omega^\bullet(\mathcal{L}\,\mathrm{Bun}_G(\Sigma))$ and $\mathcal{H}_B(\Sigma) = \mathcal{O}(\mathcal{L}\,\mathrm{Loc}_G(\Sigma))$ [1]. Can we go further, and identify 2d topological quantum field theories (in the sense of Atiyah and Segal, or even extended theories in the sense of Lurie [**Lur09b**]) which quantise our 2d classical field theories, and which assign these Hilbert spaces to the circle?

This question makes sense not just for the 2d theories arising by dimensional reduction from the Kapustin-Witten theories, but for the algebraic A- and B-models with target *any* derived stack X. That is, the 2d classical field theories sending a smooth proper curve Σ to $T^*_{\mathrm{form}}[-1](\underline{\mathrm{Map}}(\Sigma, X))$ or $\underline{\mathrm{Map}}(\Sigma, X)_{\mathrm{dR}}$, which have Hilbert spaces of the forms described above if we let $X = \mathrm{Loc}_G(\Sigma)$ or $\mathrm{Bun}_G(\Sigma)$ respectively.

If X is sufficiently well-behaved, a theorem of Ben-Zvi and Nadler allows us to identify the Hilbert spaces of the A- and B-models with the Hochschild homologies of certain categories of sheaves.

[1] There's an implicit functional analytic choice here; I'm describing the Hilbert space using algebraic functions on the base of the cotangent bundle, not for instance L^2-functions with respect to some measure. We won't attempt to literally describe a complete innner product on these vector spaces. As we'll remark shortly, it may be more natural to consider algebraic *distributions* rather than algebraic functions.

From the point of view of the cobordism hypothesis, in a 2d extended topological field theory assigning the category $\mathcal{C}$ to the point, the vector space associated to the circle with its cylindrical framing is exactly the Hochschild homology $\mathrm{HH}_\bullet(\mathcal{C})$.

Definition 12.1. *The space $\omega(X)$ of* distributions *on a derived stack X is the vector space $p_* p^! \mathbb{C}$, where p is the unique map $X \to \mathrm{pt}$, and the pull-back and push-forward maps are the functors between categories of ind-coherent sheaves. Similarly, the space $\omega_{\mathrm{dR}}(X)$ of* de Rham distributions *on X is the vector space $p_{\mathrm{dR},*} p^!_{\mathrm{dR}} \mathbb{C}$, where we now pull back and push forward using the functors between categories of D-modules.*

Proposition 12.2. *Let X be a derived stack which is quasi-compact with affine diagonal ("QCA"). The vector space $\mathcal{H}_A = \omega_{\mathrm{dR}}(\mathcal{L}X)$ is canonically isomorphic to the Hochschild cohomology of the category $\mathrm{D}(X)$ of D-modules on X. The vector space $\mathcal{H}_B = \omega(\mathcal{L}X)$ is canonically isomorphic to the Hochschild homology of the category $\mathrm{IndCoh}(X)$.*

This result is an immediate consequence of a result of Ben-Zvi and Nadler [**BZN13**, Proposition 4.2].

Remark 12.3. This doesn't immediately apply to the Hilbert spaces associated to the A-twisted Kapustin-Witten theory, since $\mathrm{Bun}_G(\Sigma)$ is not quasi-compact. However, we expect that the result still holds for this stack. Our main piece of evidence for this is a result of Drinfeld and Gaitsgory [**DG11**], establishing that even though $\mathrm{Bun}_G(\Sigma)$ is not quasi-compact it still has a well-behaved theory of D-modules. In particular, its category of D-modules is compactly generated.

Remark 12.4. In order to obtain the Hochschild homology of the category of ind-coherent sheaves on X with singular support in $Y \subseteq \mathrm{Sing}(X)$, note that there are pull-back and pushforward functors that preserve specified singular support conditions (by composing $p^!$ with the appropriate colocalization functor Ψ_X^Y as defined by Arinkin and Gaitsgory), to which we can apply Ben-Zvi and Nadler's result.

Now, we'd like to extend our Hilbert spaces to 2d topological quantum field theories quantizing the classical A- and B-models. Heuristically, it should be possible to form such an extension using pull-push operations. Indeed, on the B-side, as a functor from the bordism category viewed as a 1-category, this can already be made precise; in work with Saul Glasman I'm currently extending this to a functor of quasi-categories. We view this as an algebraic version of *string topology operations* as defined by Godin [**God07**].

Let U be a 2-dimensional bordism, and consider the correspondence

$$\begin{tikzcd} & \mathrm{EOM}(U) \arrow[dl] \arrow[dr] & \\ \mathrm{EOM}(\partial_{\mathrm{in}} U) & & \mathrm{EOM}(\partial_{\mathrm{out}} U) \end{tikzcd}$$

where the morphisms are given by restriction. In the A- and B-models we can identify this as a Lagrangian correspondence between cotangent bundles, where $\mathrm{EOM}(U)$ is the conormal bundle to a space mapping to the product of the two base spaces (specifically, it will look like the conormal to the mapping space $\underline{\mathrm{Map}}(U, X)$ in $T^*\underline{\mathrm{Map}}(\partial_{\mathrm{in}} U \sqcup \partial_{\mathrm{out}} U, X)$. We would like to define an operation by pulling back and pushing forward distributions on the resulting correspondence between the bases. A priori we can't canonically pull back distributions, but we can define a pullback by constructing a relative Calabi-Yau structure for one of the maps. More specifically, if $f \colon X \to Y$ is a morphism of derived stacks, and we're given an equivalence of functors $\Omega \colon f^* \to f^![-d]$,there's a canonical commutative triangle of derived stacks

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "p"'] & Y \arrow[d, "q"] \\ & \mathrm{pt} \end{tikzcd}$$

and we can identify a morphism of cochain complexes

$$\begin{aligned}\omega(Y) &= q_* q^! \mathbb{C} \\ &\to q_* f_* f^* q^! \mathbb{C} \\ &\to q_* f_* f^! q^! \mathbb{C}[-d] \\ &= p_* p^! \mathbb{C}[-d] = \omega(X)[-d]\end{aligned}$$

where we used the unit for the adjunction between f^* and f_* on the second line and the Calabi-Yau structure Ω for f on the third line.

Claim. *If X is proper, we can construct a relative Calabi-Yau for the restriction maps above, and therefore define by pull-push a 2d topological field theory, as a functor from the 1-categorical bordism category to cochain complexes. If X is only separated, we can still construct a functor from the "non-compact" bordism category, in which all components of a bordism are required to have non-empty incoming boundary (see Lurie's article [**Lur09b**] for a discussion of such theories).*

In work in progress, Glasman and I are extending this functor to a map of quasi-categories, where the source bordism category is modelled by a quasi-category of *ribbon graphs*.

Remark 12.5. Unfortunately, the stack $\mathrm{Loc}_G(\Sigma)$ is far from being separated, so it doesn't appear that our methods will work to define a 2d TQFT modelling the dimensionally reduced B-twisted theory. To truly access this theory additional ideas will be needed.

CHAPTER 13

Vacua and Singular Support Conditions

In this chapter, I'll discuss ongoing joint work with Philsang Yoo.

In order to recover the geometric Langlands correspondence from gauge theory, it is still necessary to explain where the corrections to the best hope conjecture arise. For instance, it would be very desirable to have a physical origin for the singular support conditions introduced by Arinkin and Gaitsgory, in order to correct the incompatibility of the best hope conjecture with the geometric Eisenstein series functors (as we discussed in section 1.2.3 of the introduction).

We propose to explain the appearance of these singular support conditions by restricting to branes in S-dual theories which are compatible with a fixed choice of *vacuum*. In the 4 dimensional A- and B-twisted quantum field theories, there is an action of the algebra $Z(S^3)$ of local operators (using functorial notation for a TQFT suggestively) on the category $Z(\Sigma)$ associated to a curve Σ. This action yields a morphism of $\mathbb{E}_2$ algebras from $Z(S^3)$ to the Hochschild cohomology $\mathrm{HH}^\bullet(Z(\Sigma))$, which we'll describe in the examples of the Kapustin-Witten twisted theories.

Definition 13.1. *The* moduli space of vacua *in an n-dimensional topological field theory Z is the spectrum of the algebra of local operators* $\mathrm{Spec}(\mathrm{Obs}^{\mathrm{q}})$. *In general, this algebra is an $\mathbb{E}_n$-algebra, but we'll restrict attention to the case where it's actually genuinely commutative up to a degree shift (as in [**AG12**, section 3.6])* [1].

Remark 13.2. This is a special case of a definition of vacua in a general, not necessarily topological theory. See Costello-Gwilliam [**CG15**, section 4.9] for details.

[1]Thus avoiding interacting with any subtle notions from spectral algebraic geometry.

Given a category $\mathcal{L}$ of *line operators* in a topological field theory, one obtains an algebra of *local operators* as the endomorphism algebra $\mathrm{End}_{\mathcal{L}}(1_{\mathcal{L}})$ (using the state operator correspondence for a topological theory). The category of line operators acts on the category $\mathcal{C}$ of branes along some codimension 2 submanifold, so we note that the action

$$\begin{aligned} &\mathcal{L} \to \mathrm{Fun}(\mathcal{C}, \mathcal{C}) \\ \text{induces } &\mathrm{End}_{\mathcal{L}}(1_{\mathcal{L}}) \to \mathrm{End}_{\mathrm{Fun}(\mathcal{C},\mathcal{C})}(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}}) \\ &\qquad = \mathrm{HC}^\bullet(\mathcal{C}) \to \mathrm{End}_{\mathcal{C}}(\mathcal{F}). \end{aligned}$$

That is, the action of line operators on branes induces an action of the local operators on each brane $\mathcal{F} \in \mathcal{C}$.

In the Kapustin-Witten B-twisted theory, the category of line operators is $\mathrm{IndCoh}_{\mathcal{N}}(\mathrm{Hecke}^{G^\vee}_{\mathrm{spec}})$, i.e. the category on the B-side of the geometric Satake correspondence. The fact that these comprise *all* the line operators, not just the subcategory of Wilson operators, is a theorem of Kapustin, Setter and Vyas [**KSV10**] (although we've made a different functional analytic choice of sheaves, but this won't matter for the present argument – the algebra of endomorphisms of the unit will be the same).

One checks that the algebra $\mathrm{End}_{\mathcal{L}}(1_{\mathcal{L}})$ of local operators is equivalent to $\mathcal{O}(\mathfrak{g}[2]/G)$, and therefore to $\mathcal{O}(\mathfrak{h}[2]/W)$. The moduli space of vacua in this theory is therefore given by the affine dg-scheme $\mathfrak{h}[2]/W$. A naïve geometric quantization procedure suggests assigning to the curve Σ the category $Z_{\text{naïve}}(\Sigma) = \mathrm{IndCoh}(\mathrm{Loc}_{G^\vee}(\Sigma))$. The above argument gives us a morphism

$$v_{\mathcal{F}} \colon \mathcal{O}(\mathfrak{h}[2]/W) \to \mathrm{HH}^\bullet(\mathrm{IndCoh}(\mathrm{Loc}_{G^\vee}(\Sigma)) \to \mathrm{End}^\bullet(\mathcal{F}).$$

Choosing a $\mathbb{C}^\times$ orbit $[x] \subseteq \mathfrak{h}/W$ yields an ideal in $\mathcal{O}(\mathfrak{h}[2]/W)$. We think of this as a *choice of vacuum state*. We expect to recover Arinkin-Gaitsgory's nilpotent singular support condition by making a choice of vacuum, in the following way.

Conjecture 13.3. *For the vacuum state $x = 0$, the full subcategory generated by those objects $\mathcal{F}$ such that $v_{\mathcal{F}}(\mathfrak{m}_0) = 0$ is the category* $\mathrm{IndCoh}_{\mathcal{N}}(\mathrm{Loc}_{G^\vee}(\Sigma))$ *of objects with nilpotent singular support.*

On the A-side, the geometric Satake isomorphism gives us, by the same procedure as above, an action of the algebra of local operators on each A-brane. We intend to show that the vacuum condition associated to $x = 0$ is vacuous for the category $\mathrm{D}(\mathrm{Bun}_G(\Sigma))$, yielding the statement of the geometric Langlands correspondence as given by Arinkin and Gaitsgory, by restricting to the categories compatible with the vacuum 0 on each side.

What about more general choices of vacua? On the B-side, choosing a $\mathbb{C}^\times$ through the origin imposes a more complicated singular support condition on B-branes. For instance, if we choose a regular semisimple element x, the singular support condition includes, in particular, a *support* condition, saying that a compatible brane is set-theoretically supported only on the locus $\mathrm{Loc}_{H^\vee}(\Sigma) \subseteq \mathrm{Loc}_{G^\vee}(\Sigma)$ of completely reducible bundles. In general, the set-theoretic support will be broken to subsets associated to a union of Levi subgroups.

On the A-side, in order to identify the category of branes compatible with a particular vacuum condition, one needs to identify a category that should correspond to the full category $\mathrm{IndCoh}(\mathrm{Bun}_G(\Sigma))$ on the A-side, then identify singular supports of objects in this larger category. Guided by Arinkin and Gaitsgory's approach to the geometric Satake correspondence, we propose that this larger category should have an analogous form to the category Arinkin and Gaitsgory call $\mathrm{Sph}_G^{\mathrm{ren}}$: the "renormalized" spherical Hecke category.

There is a natural map of derived stacks $f_x \colon \mathrm{Gr}_G \to \mathrm{Bun}_G(\Sigma)$ associated to each closed point $x \in \Sigma$. Qualitatively, the Beilinson-Drinfeld description of the affine Grassmannian tells us that it can be represented as the moduli space of G-bundles on Σ which are trivialized away from x; from this point of view the map f_x merely forgets the trivialization. More concretely, we can identify

f_x as a map of mapping stacks induced from the natural map $\Sigma \setminus \{x\} \to \mathrm{pt}$ as follows.

$$\begin{aligned}\mathrm{Gr}_G &= \underline{\mathrm{Maps}}(\mathbb{D} \sqcup_{\mathbb{D}^\times} \mathrm{pt}, BG) \\ &\to \underline{\mathrm{Maps}}(\mathbb{D} \sqcup_{\mathbb{D}^\times} (\Sigma \setminus \{x\}), BG) \\ &= \underline{\mathrm{Maps}}(\Sigma, BG) \\ &= \mathrm{Bun}_G(\Sigma).\end{aligned}$$

Definition 13.4. *The* renormalized *category of A-branes on Σ is the category generated by objects M in $\mathrm{D}(\mathrm{Bun}_G(\Sigma))$ such that the pullback $f_x^! M$ under the map $f_x \colon \mathrm{Gr}_G \to \mathrm{Bun}_G(\Sigma)$ for each point $x \in \Sigma$ is compact, i.e. a bounded complex of D-modules with coherent cohomology.*

We intend to use the Satake action on this category to identify a full subcategory of objects with a particular singular support condition, i.e. objects compatible with a choice of vacuum.

CHAPTER 14

$N = 2$ Theories and the Future

This final section is perhaps the most speculative; I will propose a program (which I hope to follow jointly with Philsang Yoo) in which we might rigorously produce geometric Langlands like equivalences and structures from a family of S-dualities described by Gaiotto [**Gai12**]. These dualities come from the six-dimensional perspective on S-duality of $N = 4$ super Yang-Mills theory that we discussed in section 1.3.5: there are S-dualities between theories arising from dimensional reduction of theory X along a curve Σ associated to each element of the mapping class group of Σ.

The theories arising by dimensional reduction of theory X to 4-dimensions are called theories of "class S" [**Wit97**]. They all have $N = 2$ supersymmetry, but like theory X itself, they generically are not expected to come from quantization of a classical Lagrangian field theory. Also like theory X itself, there generally isn't a mathematical construction of most aspects of these theories. There are however a few special points that do admit Lagrangian descriptions, and therefore can be investigated using the methods of this thesis. $N = 4$ super Yang-Mills is one example, as are – for instance – SU(2) quiver gauge theories (as discussed in the first section of Gaiotto's paper [**Gai12**]). Finally, there's an SU(3) $N = 2$ gauge theory which was conjectured by Argyres and Seiberg [**AS07**] to be S-dual to an SU(2) theory coupled to a theory with E_6 flavour symmetry. Gaiotto argues that the duality of Argyres and Seiberg is also an example of his generalized S-duality.

As a starting point, it's possible to perform the analysis of part 2 of this thesis, but for $N = 2$ super Yang-Mills theory with arbitrary matter multiplets. Such theories admit a twistor space description (starting from holomorphic Chern-Simons theory on $N = 2$ twistor space with an additional field: a holomorphic section of the associated bundle to the holomorphic G-bundle for some representation). It's also a straightforward algebraic problem to classify the possible twists, which in particular include a unique topological twist and a holomorphic-topological twist first described by Kapustin [**Kap06**] which are both fixed by S-duality. By repeating the analysis of chapter 10 for these theories, we can obtain a description of the phase spaces in these twisted theories associated to the product of a Riemann surface with a circle, we can describe the geometric quantization and the moduli spaces of vacua to obtain conjectural descriptions of the categories of branes in these $N = 2$ theories.

Having done this, Gaiotto duality provides conjectural equivalences between these categories, generalizing the geometric Langlands correspondence. For instance, we should begin by investigating SU(2) quiver gauge theories. Here Gaiotto's duality acts as the mapping class group on a Teichmüller space viewed as a moduli space of coupling constants, so we obtain a collection of dual theories, for instance, one associated to each cusp in the moduli space. This will hopefully serve as an access point in to the far more complicated family of examples associated to higher rank type A gauge groups.

References

[ABD+13] Alexandros Anastasiou, Leron Borsten, Michael Duff, Leo Hughes, and Silvia Nagy. Super Yang-Mills, division algebras and triality. *arXiv preprint arXiv:1309.0546*, 2013.

[AS07] Philip C Argyres and Nathan Seiberg. S-duality in $N = 2$ supersymmetric gauge theories. *Journal of High Energy Physics*, 2007(12):088, 2007.

[AG12] Dima Arinkin and Dennis Gaitsgory. Singular support of coherent sheaves, and the geometric Langlands conjecture. *arXiv preprint arXiv:1201.6343*, 2012.

[AHS78] Michael Atiyah, Nigel Hitchin, and Isadore Singer. Self-duality in four-dimensional Riemannian geometry. *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, 362(1711):425–461, 1978.

[BD95] John Baez and James Dolan. Higher-dimensional algebra and topological quantum field theory. *Journal of Mathematical Physics*, 36(11):6073–6105, 1995.

[BGH03] Dongsu Bak, Michael Gutperle, and Shinji Hirano. A dilatonic deformation of AdS_5 and its field theory dual. *Journal of High Energy Physics*, 2003(05):072, 2003.

[Bar95] José Barbón. Generalized abelian S-duality and coset constructions. *Nuclear Physics B*, 452(1):313–330, 1995.

[Bat79] Marjorie Batchelor. The structure of supermanifolds. *Transactions of the American Mathematical Society*, 253:329–338, 1979.

[Bea14a] Chris Beasley. Abelian duality at higher genus. *Journal of High Energy Physics*, 7(157), 2014.

[Bea14b] Chris Beasley. Global aspects of abelian duality in dimension three. *Journal of High Energy Physics*, 8(146), 2014.

[BD04] Alexander Beilinson and Vladimir Drinfeld. *Chiral algebras*, volume 51 of *Colloquium Publications*. American Mathematical Society Providence, RI, 2004.

[BBK13] Oren Ben-Bassat and Kobi Kremnizer. Non-archimedean analytic geometry as relative algebraic geometry. *arXiv preprint arXiv:1312.0338*, 2013.

[BZBJ15] David Ben-Zvi, Adrien Brochier, and David Jordan. Integrating quantum groups over surfaces: quantum character varieties and topological field theory. *arXiv preprint arXiv:1501.04652*, 2015.

[BZN13] David Ben-Zvi and David Nadler. Nonlinear traces. *arXiv preprint arXiv:1305.7175*, 2013.

[BZN16] David Ben-Zvi and David Nadler. Betti spectral gluing. *arXiv preprint arXiv:1602.07379*, 2016.

[BIK08] Dave Benson, Srikanth Iyengar, and Henning Krause. Local cohomology and support for triangulated categories. In *Annales Scientifiques de l'ÉNS*, volume 41, pages 575–621, 2008.

[BF08] Roman Bezrukavnikov and Michael Finkelberg. Equivariant Satake category and Kostant–Whittaker reduction. *Mosc. Math. J*, 8(1):39–72, 2008.

[BMS07] Rutger Boels, Lionel Mason, and David Skinner. Supersymmetric gauge theories in twistor space. *Journal of High Energy Physics*, 2007(02):014, 2007.

[BSS77] Lars Brink, John Schwarz, and Joel Scherk. Supersymmetric Yang-Mills theories. *Nuclear Physics B*, 121(1):77–92, 1977.

[BD04] Bogusław Broda and Grzegorz Duniec. Abelian duality in three dimensions. *Physical Review D*, 70(10):107702, 2004.

[Bry93] Jean-Luc Brylinski. *Loop Spaces, Characteristic Classes and Geometric Quantization*, volume 107 of *Progress in Mathematics*. Birkhäuser, 1993.

[BG14] Kevin Buzzard and Toby Gee. The conjectural connections between automorphic representations and Galois representations. In *Automorphic Forms and Galois Representations*, volume 1, pages 135–187. Cambridge University Press, 2014.

[Cal13] Damien Calaque. Lagrangian structures on mapping stacks and semi-classical TFTs. *arXiv preprint arXiv:1306.3235*, 2013.

[CPT+15] Damien Calaque, Tony Pantev, Bertrand Toën, Michel Vaquié, and Gabriele Vezzosi. Shifted Poisson structures and deformation quantization. *arXiv preprint arXiv:1506.03699*, 2015.

[CR12] David Carchedi and Dmitry Roytenberg. Homological algebra for superalgebras of differentiable functions. *arXiv preprint arXiv:1212.3745*, 2012.

[Clo90] Laurent Clozel. Motifs et formes automorphes: applications du principe de fonctorialité. In *Automorphic forms, Shimura varieties, and L-functions*, volume 1, pages 77–159. 1990.

[Cos07] Kevin Costello. Topological conformal field theories and Calabi–Yau categories. *Advances in Mathematics*, 210(1):165–214, 2007.

[Cos10] Kevin Costello. A geometric construction of the Witten genus, I. *Proceedings of the International Congress of Mathematicians, (Hyderabad, 2010)*, 2010.

[Cos11a] Kevin Costello. Notes on supersymmetric and holomorphic field theories in dimensions 2 and 4. *arXiv preprint arXiv:1111.4234*, 2011.

[Cos11b] Kevin Costello. *Renormalization and Effective Field Theory*, volume 170. AMS, 2011.

[CG15] Kevin Costello and Owen Gwilliam. *Factorization Algebras in Quantum Field Theory.* 2015. book in progress, available at `http://people.mpim-bonn.mpg.de/gwilliam/Volume%201%20%2816%20December%202015%29.pdf`.

[CS76] Eugène Cremmer and Joël Scherk. Spontaneous compactification of space in an Einstein-Yang-Mills-Higgs model. *Nuclear Physics B*, 108(3):409–416, 1976.

[dHDV78] Alessandro d'Adda, Roger Horsley, and Paolo Di Vecchia. Supersymmetric magnetic monopoles and dyons. *Physics Letters B*, 76(3):298–302, 1978.

[Del99] Pierre Deligne. *Quantum Fields and Strings; A Course for Mathematicians: Notes on Spinors*, volume 1. AMS, 1999.

[DF99] Pierre Deligne and Daniel Freed. *Quantum Fields and Strings; A Course for Mathematicians: Classical Field Theory*, volume 1. AMS, 1999.

[Dir31] Paul Dirac. Quantised singularities in the electromagnetic field. In *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, volume 133, pages 60–72. The Royal Society, 1931.

[DP12] Ron Donagi and Tony Pantev. Langlands duality for Hitchin systems. *Inventiones mathematicae*, 189(3):653–735, 2012.

[Dou01] Michael Douglas. D-branes, categories and $N = 1$ supersymmetry. *Journal of Mathematical Physics*, 42(7):2818–2843, 2001.

[Dri83] Vladimir Drinfeld. Two-dimensional ℓ-adic representations of the fundamental group of a curve over a finite field and automorphic forms on GL(2). *American Journal of Mathematics*, 105(1):85–114, 1983.

[DG11] Vladimir Drinfeld and Dennis Gaitsgory. Compact generation of the category of D-modules on the stack of G-bundles on a curve. *arXiv preprint arXiv:1112.2402*, 2011.

[EY90] Tohru Eguchi and Sung-Kil Yang. $N = 2$ superconformal models as topological field theories. *Modern Physics Letters A*, 5(21):1693–1701, 1990.

[EY15] Chris Elliott and Philsang Yoo. Geometric Langlands twists of $N = 4$ gauge theory from derived algebraic geometry. *arXiv preprint arXiv:1507.03048*, 2015.

[FO98] José Figueroa-O'Farril. Electromagnetic duality for children. Available at author's website: `http://www.maths.ed.ac.uk/~jmf/Teaching/Lectures/EDC.pdf`, 1998.

[Fre00] Dan Freed. Dirac charge quantization and generalized differential cohomology. *Surveys in Differential Geometry*, VII:129194, 2000.

[Fre93] Daniel Freed. Locality and integration in topological field theory. In M.A del Olmo, M. Santander, and J.M. Guilarte, editors, *Group Theoretical Methods in Physics*, volume 2, pages 35–54. 1993.

[FQ93] Daniel Freed and Frank Quinn. Chern-Simons theory with finite gauge group. *Communications in Mathematical Physics*, 156(3):435–472, 1993.

[Fre07] Edward Frenkel. Lectures on the Langlands program and conformal field theory. In *Frontiers in number theory, physics, and geometry II*, pages 387–533. Springer, 2007.

[Fre10] Edward Frenkel. Gauge theory and Langlands duality. *Astérisque*, 332, 2010.

[FGV02] Edward Frenkel, Dennis Gaitsgory, and Kari Vilonen. On the geometric Langlands conjecture. *Journal of the American Mathematical Society*, 15(2):367–417, 2002.

[Gai12] Davide Gaiotto. $N = 2$ dualities. *Journal of High Energy Physics*, 2012(08):59, 2012.

[GW09a] Davide Gaiotto and Edward Witten. S-duality of boundary conditions in $N = 4$ super Yang-Mills theory. *Advances in Theoretical and Mathematical Physics*, 13(3):721–896, 2009.

[GW09b] Davide Gaiotto and Edward Witten. Supersymmetric boundary conditions in $N = 4$ super Yang-Mills theory. *Journal of Statistical Physics*, 135(5-6):789–855, 2009.

[Gai01] Dennis Gaitsgory. Construction of central elements in the affine Hecke algebra via nearby cycles. *Inventiones mathematicae*, 144(2):253–280, 2001.

[Gai04] Dennis Gaitsgory. On a vanishing conjecture appearing in the geometric Langlands correspondence. *Annals of Mathematics*, 160(2):617–682, 2004.

[Gai11a] Dennis Gaitsgory. Ind-coherent sheaves. *arXiv preprint arXiv:1105.4857*, 2011.

[Gai11b] Dennis Gaitsgory. Notes on geometric Langlands: Quasi-coherent sheaves on stacks. Available at author's website `http://www.math.harvard.edu/~gaitsgde/GL`, 2011.

[Gai11c] Dennis Gaitsgory. Notes on geometric Langlands: Stacks. Available at author's website `http://www.math.harvard.edu/~gaitsgde/GL`, 2011.

[Gai13] Dennis Gaitsgory. Outline of the proof of the geometric Langlands conjecture for $GL(2)$. *arXiv preprint arXiv:1302.2506*, 2013.

[GRa] Dennis Gaitsgory and Nick Rozenblyum. A study in derived algebraic geometry, part III.1: Deformation theory. Available at author's website `http://www.math.harvard.edu/~gaitsgde/GL`, accessed October 10th 2015.

[GRb] Dennis Gaitsgory and Nick Rozenblyum. A study in derived algebraic geometry, part III.2: (ind) inf-schemes. Available at author's website `http://www.math.harvard.edu/~gaitsgde/GL`, accessed October 10th 2015.

[GRc] Dennis Gaitsgory and Nick Rozenblyum. A study in derived algebraic geometry, part IV.1: Formal moduli. Available at author's website `http://www.math.harvard.edu/~gaitsgde/GL`, accessed October 10th 2015.

[GRd] Dennis Gaitsgory and Nick Rozenblyum. A study in derived algebraic geometry, part IV.3: Formal groups and Lie algebras. Available at author's website `http://www.math.harvard.edu/~gaitsgde/GL`, accessed October 10th 2015.

[GRe] Dennis Gaitsgory and Nick Rozenblyum. A study in derived algebraic geometry, part IV.4: Lie algebroids. Available at author's website `http://www.math.harvard.edu/~gaitsgde/GL`, accessed October 10th 2015.

[GRf] Dennis Gaitsgory and Nick Rozenblyum. A study in derived algebraic geometry, part IV.5: Infinitesimal differential geometry. Available at author's website `http://www.math.harvard.edu/~gaitsgde/GL`, accessed October 10th 2015.

[Get09] Ezra Getzler. Lie theory for nilpotent L_∞-algebras. *Annals of mathematics*, 170(1):271–301, 2009.

[God07] Véronique Godin. Higher string topology operations. *arXiv preprint arXiv:0711.4859*, 2007.

[Gör10] Ulrich Görtz. Affine Springer fibers and affine Deligne-Lusztig varieties. In Alexander Schmitt, editor, *Affine Flag Manifolds and Principal Bundles*, pages 1–50. Springer, 2010.

[GG14] Ryan Grady and Owen Gwilliam. L_∞-algebras and derived loop spaces. *arXiv preprint arXiv:1404.5426*, 2014.

[GW06] Sergei Gukov and Edward Witten. Gauge theory, ramification, and the geometric Langlands program. *arXiv preprint hep-th/0612073*, 2006.

[Gwi12] Owen Gwilliam. *Factorization Algebras and Free Field Theory*. PhD thesis, Northwestern University, 2012.

[GJF12] Owen Gwilliam and Theo Johnson-Freyd. How to derive Feynman diagrams for finite-dimensional integrals directly from the BV formalism. *arXiv preprint arXiv:1202.1554*, 2012.

[Har96] Jeffrey Harvey. Magnetic monopoles, duality, and supersymmetry. *arXiv preprint hep-th/9603086*, 1996.

[HT03] Tamás Hausel and Michael Thaddeus. Mirror symmetry, Langlands duality, and the Hitchin system. *Inventiones mathematicae*, 153(1):197–229, 2003.

[Hen15] Benjamin Hennion. *Formal loops, Tate objects and tangent Lie algebras*. PhD thesis, Université Montpellier 2, 2015. Preliminary version available as an arXiv preprint: arXiv:1412.0053.

[Hin01] Vladimir Hinich. DG coalgebras as formal stacks. *Journal of pure and applied algebra*, 162(2):209–250, 2001.

[HKK+03] Kentaro Hori, Sheldon Katz, Albrecht Klemm, Rahul Pandharipande, Richard Thomas, Cumrun Vafa, Ravi Vakil, and Eric Zaslow. *Mirror Symmetry*. Number 1 in Clay Mathematics Monographs. American Mathematical Society, Providence, RI, 2003.

[JT80] Arthur Jaffe and Clifford Taubes. *Vortices and monopoles: structure of static gauge theories*, volume 2 of *Progress in Physics*. Birkhäuser, 1980.

[Joy11] Dominic Joyce. An introduction to d-manifolds and derived differential geometry. *Moduli spaces, LMS Lecture Notes*, 411:230–281, 2011.

[Kap06] Anton Kapustin. Holomorphic reduction of $N = 2$ gauge theories, Wilson-'t Hooft operators, and S-duality. *arXiv preprint hep-th/0612119*, 2006.

[Kap08] Anton Kapustin. A note on quantum geometric Langlands duality, gauge theory, and quantization of the moduli space of flat connections. *arXiv preprint arXiv:0811.3264*, 2008.

[KSV10] Anton Kapustin, Kevin Setter, and Ketan Vyas. Surface operators in four-dimensional topological gauge theory and Langlands duality. *arXiv preprint arXiv:1002.0385*, 2010.

[KT09] Anton Kapustin and Mikhail Tikhonov. Abelian duality, walls and boundary conditions in diverse dimensions. *Journal of High Energy Physics*, 2009(11):006, 2009.

[KW06] Anton Kapustin and Edward Witten. Electric-magnetic duality and the geometric Langlands program. *arXiv preprint hep-th/0604151*, 2006.

[Kel09] Gerald Kelnhofer. Functional integration and gauge ambiguities in generalized abelian gauge theories. *J. Geom. Physics*, 59:1017–1035, 2009.

[Kna97] Anthony Knapp. Introduction to the Langlands program. In *Proceedings of symposia in pure mathematics*, volume 61, pages 245–302. American Mathematical Society, 1997.

[Kne99] Marco Kneipp. Introduction to dualities in gauge theories. *Proceedings of the X Jorge André Swieca Summer School*, pages 291–317, 1999.

[Laf02] Laurent Lafforgue. Chtoucas de Drinfeld et correspondance de Langlands. *Inventiones mathematicae*, 147(1):1–241, 2002.

[Laf09] Vincent Lafforgue. Quelques calculs reliés à la correspondance de Langlands géométrique pour $\mathbb{P}^1$ (version provisoire). Available at author's website: `http://www.math.jussieu.fr/~vlafforg`, 2009.

[Laf12] Vincent Lafforgue. Chtoucas pour les groupes réductifs et paramétrisation de Langlands globale. *arXiv preprint arXiv:1209.5352*, 2012.

[Lau87] Gérard Laumon. Correspondance de Langlands Géométrique pour les corps de fonctions. *Duke Mathematical Journal*, 54:309–359, 1987.

[Lau96] Gerard Laumon. Transformation de Fourier généralisée. *arXiv preprint alg-geom/9603004*, 1996.

[Law93] Ruth Lawrence. Triangulations, categories and extended topological field theories. In *Quantum topology*, volume 3, pages 191–208. World Scientific, 1993.

[Loz99] Carlos Lozano. *Duality in Topological Field Theories*. PhD thesis, University of Santiago de Compostela, 1999.

[Lur09a] Jacob Lurie. *Higher topos theory*. Number 170 in Annals of Mathematics Studies. Princeton University Press, 2009.

[Lur09b] Jacob Lurie. On the classification of topological field theories. *Current developments in mathematics*, 2008:129–280, 2009.

[Lur11] Jacob Lurie. Derived algebraic geometry X: Formal moduli problems. *available at author's website: http://www.math.harvard.edu/lurie*, 2011.

[MV07] Ivan Mirković and Kari Vilonen. Geometric Langlands duality and representations of algebraic groups over commutative rings. *The Annals of Mathematics*, 166(1):95–143, 2007.

[MW98] Gregory Moore and Edward Witten. Integration over the u-plane in Donaldson theory. *Advances in Theoretical and Mathematical Physics*, 1:298–387, 1998.

[Mov08] Mikhail Movshev. A note on self-dual Yang-Mills theory. *arXiv preprint arXiv:0812.0224*, 2008.

[Nad09] David Nadler. Microlocal branes are constructible sheaves. *Selecta Mathematica*, 15(4):563–619, 2009.

[Nad12] David Nadler. The geometric nature of the fundamental lemma. *Bulletin of the American Mathematical Society*, 49(1):1–50, 2012.

[NZ09] David Nadler and Eric Zaslow. Constructible sheaves and the Fukaya category. *Journal of the American Mathematical Society*, 22(1):233–286, 2009.

[Ngô10] Bao Châu Ngô. Le lemme fondamental pour les algebres de Lie. *Publications mathématiques de l'IHÉS*, 111(1):1–169, 2010.

[Oli96] David Olive. Exact electromagnetic duality. *Nuclear Physics B-Proceedings Supplements*, 46(1):1–15, 1996.

[OW78] David Olive and Edward Witten. Supersymmetry algebras that include topological charges. *Physics Letters B*, 78(1):97–101, 1978.

[Osb79] Hugh Osborn. Topological charges for $N = 4$ supersymmetric gauge theories and monopoles of spin 1. *Physics Letters B*, 83(3):321–326, 1979.

[PTVV13] Tony Pantev, Bertrand Toën, Michel Vaquié, and Gabriele Vezzosi. Shifted symplectic structures. *Publications mathématiques de l'IHÉS*, 117(1):271–328, 2013.

[Pol74] Alexander Polyakov. Particle spectrum in quantum field theory. *Soviet Journal of Experimental and Theoretical Physics Letters*, 20:194, 1974.

[Pri15] J.P. Pridham. Shifted Poisson and symplectic structures on derived N-stacks. *arXiv preprint arXiv:1504.01940*, 2015.

[PS00] Emil M Prodanov and Siddhartha Sen. Abelian duality. *Physical Review D*, 62(4):045009, 2000.

[Rot96] Mitchell Rothstein. Sheaves with connection on abelian varieties. *Duke Mathematical Journal*, 84(3):565–598, 1996.

[Sch14] Claudia Scheimbauer. *Factorization Homology as a Fully Extended Topological Field Theory*. PhD thesis, ETH Zürich, 2014.

[SS79] Joel Scherk and John H Schwarz. Spontaneous breaking of supersymmetry through dimensional reduction. *Physics Letters B*, 82(1):60–64, 1979.

[SS93] John Schwarz and Ashoke Sen. Duality symmetries of 4D heterotic strings. *Physics Letters B*, 312(1):105–114, 1993.

[Set13] Kevin Setter. *Topological Quantum Field Theory and the Geometric Langlands Correspondence*. PhD thesis, California Institute of Technology, 2013.

[Sim90] Carlos Simpson. Harmonic bundles on noncompact curves. *Journal of the American Mathematical Society*, 3(3):713–770, 1990.

[Sim97] Carlos Simpson. The Hodge filtration on nonabelian cohomology. In *Proceedings of Symposia in Pure Mathematics*, volume 62, pages 217–284. American Mathematical Society, 1997.

[Sim98] Carlos Simpson. Algebraic aspects of higher nonabelian Hodge theory. In *Motives, polylogarithms and Hodge theory*, volume 3 of *Int. Press Lect. Ser*, pages 417–604. 1998.

[Sim09] Carlos Simpson. Geometricity of the Hodge filtration on the ∞-stack of perfect complexes over X_{DR}. *Mosc. Math. J*, 9(209):665–721, 2009.

[Spr98] Tonny Albert Springer. *Linear Algebraic Groups*. Birkhäuser Basel, second edition, 1998.

[SYZ96] Andrew Strominger, Shing-Tung Yau, and Eric Zaslow. Mirror symmetry is T-duality. *Nuclear Physics B*, 479(1):243–259, 1996.

[tH74] Gerard ‘t Hooft. Magnetic monopoles in unified gauge theories. *Nuclear Physics B*, 79(2):276–284, 1974.

[Tay04] Richard Taylor. Galois representations. In *Annales de la Faculté des sciences de Toulouse: Mathématiques*, volume 13, pages 73–119, 2004.

[Tes16] Jörg Teschner. Exact results on $N = 2$ supersymmetric gauge theories. In *New Dualities of Supersymmetric Gauge Theories*, Mathematical Physics Studies. Springer, 2016.

[Toë05] Bertrand Toën. Higher and derived stacks: A global overview. In *Proc. Sympos. Pure Math*, volume 80, pages 435–487, 2005.

[Toë14] Bertrand Toën. Derived algebraic geometry. *arXiv preprint arXiv:1401.1044*, 2014.

[TV08] Bertrand Toën and Gabriele Vezzosi. *Homotopical algebraic geometry II: Geometric stacks and applications*, volume 193 of *Memoirs of the American Mathematical Society.* American Mathematical Society, 2008.

[VW94] Cumrun Vafa and Edward Witten. A strong coupling test of S-duality. *Nuclear Physics B*, 431(1):3–77, 1994.

[Var04] Veeravalli Varadarajan. *Supersymmetry for mathematicians: an introduction*, volume 11 of *Courant Lecture Notes in Mathematics.* American Mathematical Soc., 2004.

[Vec97] Paolo Di Vecchia. Duality in supersymmetric gauge theories. *Surveys in High Energy Physics*, 10(1-4):119–151, 1997.

[Ver95] Erik Verlinde. Global aspects of electric-magnetic duality. *Nuclear Physics B*, 455(1):211–225, 1995.

[WW91] Richard Samuel Ward and Raymond O'Neil Wells. *Twistor geometry and field theory.* Cambridge University Press, 1991.

[Wei79] André Weil. Œuvres scientifiques. collected papers. vol. i (1926–1951). 1979.

[Wei80] Erick Weinberg. Fundamental monopoles and multimonopole solutions for arbitrary simple gauge groups. *Nuclear Physics B*, 167(3):500–524, 1980.

[Wit88a] Edward Witten. Topological quantum field theory. *Communications in Mathematical Physics*, 117(3):353–386, 1988.

[Wit88b] Edward Witten. Topological sigma models. *Communications in Mathematical Physics*, 118(3):411–449, 1988.

[Wit90] Edward Witten. A note on the antibracket formalism. *Modern Physics Letters A*, 5(07):487–494, 1990.

[Wit95a] Edward Witten. On S-duality in abelian gauge theory. *Selecta Mathematica*, (2):383–410, 1995.

[Wit95b] Edward Witten. Some comments on string dynamics. *arXiv preprint hep-th/9507121*, 1995.

[Wit97] Edward Witten. Solutions of four-dimensional field theories via M-theory. *Nuclear Physics B*, 500(1):3–42, 1997.

[Wit99] Edward Witten. *Quantum Fields and Strings; A Course for Mathematicians*, volume 2, chapter : Dynamics of Quantum Field Theory. AMS, 1999.

[Wit04a] Edward Witten. Conformal field theory in four and six dimensions. In Ulrike Tillmann, editor, *Topology, Geometry and Quantum Field Theory*, number 308 in London Mathematical Society Lecture Note Series, pages 405–420. Cambridge University Press, 2004.

[Wit04b] Edward Witten. Perturbative gauge theory as a string theory in twistor space. *Communications in Mathematical Physics*, 252(1-3):189–258, 2004.

[Wit08] Edward Witten. Gauge theory and wild ramification. *Analysis and Applications*, 6(04):429–501, 2008.

[Wit09] Edward Witten. Geometric Langlands from six dimensions. *arXiv preprint arXiv:0905.2720*, 2009.

[Wit12] Edward Witten. Notes on supermanifolds and integration. *arXiv preprint arXiv:1209.2199*, 2012.

Appendices

APPENDIX A

Supersymmetry Algebras

We'll begin by setting up some general language for describing supersymmetry algebras before describing the particular cases we're interested in (supersymmetry in 2, 4 and 10 dimensions). The notion of *twisting* supersymmetry algebras and supersymmetric field theories makes sense in any dimension and signature. The material in this appendix is standard. Proofs can be found for instance in [**Del99**] or [**Var04**].

Let p and q be non-negative integers, and let $n = p + q$. We'll describe supersymmetry algebras in pseudo-Riemannian signature (p, q). The main pieces of data that we'll need to specify are a spin representation and a spin-invariant vector-valued pairing on this representation.

Definition A.1. *A (real or complex) representation of the Lie algebra* $\mathfrak{so}(p, q)$ *is* spinorial *if it extends to a module for the even (real or complex) Clifford algebra* $\mathrm{Cl}^+(p, q)$*.*

There is a complete classification of spinorial $\mathfrak{so}(p, q)$ representations.

Proposition A.2. *Over* $\mathbb{C}$*,* $\mathfrak{so}(p, q)$ *either has a* unique *non-trivial irreducible representation* S *of dimension* $2^{\frac{n-1}{2}}$ *if* $p+q$ *is odd, or has two distinct non-trivial irreducible representations* $S_\pm$ *each of dimension* $2^{\frac{n}{2}-1}$ *if* $p+q$ *is even. In the latter case we write* S *for* $S_+ \oplus S_-$*. We call* S *the space of* Dirac spinors *and* $S_\pm$ *the spaces of positive and negative helicity* Weyl spinors*.*

Over $\mathbb{R}$*, the representation* S *is the complexification of a real representation* $S_\mathbb{R}$ *when* $p - q \equiv 0, 1$ *or* $7 \mod 8$*. The representations* $S_\pm$ *are the complexifications of real representations* $S_{\mathbb{R}\pm}$ *when* $p - q \equiv 0 \mod 8$*. We call* $S_\mathbb{R}$ *the space of* Majorana spinors *and* $S_{\mathbb{R}\pm}$ *spaces of* Majorana-Weyl spinors. *When instead $p-q \equiv 2$ or $6 \mod 8$ the representation $S_+ \oplus S_+^*$* [1] *is the complexification of a real representation, which we also denote by $S_\mathbb{R}$ and refer to as the space of Majorana spinors.*

We write $V_\mathbb{R}$ for the n-dimensional vector representation $\mathbb{R}^{p,q}$ of $\mathfrak{so}(p,q)$, and $V_\mathbb{C}$ for its complexification. The second component necessary to define supersymmetry algebras is the following.

Definition A.3. *A* pairing *on a spin representation Σ is a symmetric $\mathfrak{so}(p,q)$-equivariant linear map*

$$\Gamma \colon \Sigma \otimes \Sigma \to V_k$$

where $k = \mathbb{R}$ or $\mathbb{C}$.

Again, we have a good control over the existence and uniqueness of such pairings. We can construct them using the Clifford multiplication, and duality properties of the spinors.

Proposition A.4. *Over $\mathbb{C}$ there exist unique pairings (up to rescaling)*

$$\Gamma \colon S \otimes S \to V_\mathbb{C} \quad \text{if } n \equiv 1, 3, 5, \text{ or } 7 \mod 8$$

$$\Gamma \colon S_\pm \otimes S_\pm \to V_\mathbb{C} \quad \text{if } n \equiv 2 \text{ or } 6 \mod 8$$

$$\Gamma \colon S_\pm \otimes S_\mp \to V_\mathbb{C} \quad \text{if } n \equiv 0 \text{ or } 4 \mod 8.$$

These pairings descend to give unique $V_\mathbb{R}$-valued pairings on the Majorana or Majorana-Weyl spinors whenever they exist.

We can use this to describe pairings on more general spinorial representations. There are pairings on the representation $S \otimes W$ – where W is a finite-dimensional vector space – for each element of $\mathfrak{gl}(W)$. If we also require our pairings to be non-degenerate then there is a unique pairing up to $\mathfrak{so}(p,q)$-equivariant isomorphism.

Now, we can define the supersymmetry algebra associated to this data.

[1] A real form for $S_- \oplus S_-^*$ would also work; the two agree up to complex conjugation.

Definition A.5. *The (real)* supertranslation algebra *associated to a spinorial representation* Σ *of* $\mathfrak{so}(p,q)$ *is the super Lie algebra*

$$T = V_{\mathbb{R}} \oplus \Pi(\Sigma)$$

where the only bracket is the pairing $\Gamma \colon \Sigma \otimes \Sigma \to V_{\mathbb{R}}$. *The (real)* super Poincaré algebra *is the super Lie algebra*

$$P = (\mathfrak{so}(p,q) \ltimes V_{\mathbb{R}}) \oplus \Pi(\Sigma)$$

where there are brackets given by Γ*, by the internal bracket on the even piece and by the action of* $\mathfrak{so}(p,q)$ *on* Σ*. We define* complex *supertranslation and super Poincaré algebras analogously, with* $V_{\mathbb{R}}$ *replaced by* $V_{\mathbb{C}}$*, and with* Σ *a complex spinorial representation.*

To complete the definition, we need one more piece of data, namely a subalgebra of the R-symmetry algebra.

Definition A.6. *The* R-symmetry algebra *associated to a supertranslation algebra is the algebra of outer automorphisms acting trivially on the bosonic piece. Given a subalgebra* $\mathfrak{g}_R$ *of the R-symmetry algebra, the (real)* supersymmetry algebra *is the super Lie algebra*

$$\mathcal{A} = (\mathfrak{so}(p,q) \ltimes V_{\mathbb{R}}) \oplus \mathfrak{g}_R \oplus \Pi(\Sigma)$$

with brackets as before, plus the action of $\mathfrak{g}_R$ *on* Σ*. The complexified supersymmetry algebra is defined analogously.*

When $\Sigma = S^N$, we say there are N *supersymmetries*. When $\Sigma = S_+^{N_1} \oplus S_-^{N_2}$ we say there are (N_1, N_2) *supersymmetries*. If we impose the condition that the pairing Γ is non-degenerate then we can only have $N_1 \neq N_2$ when $n \equiv 2$ or $6 \mod 8$ in the complex case, or when $n \equiv 2$ or $6 \mod 8$ *and* $p \equiv q \mod 8$ in the real case.

Definition A.7. *A* supersymmetric field theory *on $\mathbb{R}^{p,q}$ is a field theory on $\mathbb{R}^{p,q}$ equipped with an action of the complexified supersymmetry algebra extending the natural action of the complexified Poincaré algebra $\mathfrak{so}(p,q) \ltimes V_{\mathbb{C}}$.*

Example A.8 (Dimension 4)**.** The principal theories that we're interested in this paper are supersymmetric theories in dimension 4. In this and the subsequent examples we'll be most interested in the complexified supersymmetry algebra, so the choice of signature won't be too important. For specificity we'll work in Euclidean signature $(4,0)$. Recall that we have an isomorphism of groups, $\mathrm{Spin}(4) \cong \mathrm{SU}(2)_+ \times \mathrm{SU}(2)_-$. Let S_+ and S_- be the complex 2-dimensional defining representations of the two copies of $\mathrm{SU}(2)$, respectively. Let $V_{\mathbb{R}}$ be the real 4-dimensional vector representation of $\mathrm{Spin}(4)$. If we define $V_{\mathbb{C}} := V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$, then there is an isomorphism $\Gamma\colon S_+ \otimes S_- \xrightarrow{\cong} V_{\mathbb{C}}$ as complex $\mathrm{Spin}(4)$-representations.

Let W be a finite-dimensional complex vector space. There is a natural non-degenerate pairing on the spinorial representation $(S_+ \otimes W) \oplus (S_- \otimes W^*)$, given by the isomorphism Γ and the canonical pairing $W \otimes W^* \to \mathbb{C}$. The *super-translation algebra* associated to W is the super Lie algebra

$$T^W = V_{\mathbb{C}} \oplus \Pi\left(S_+ \otimes W \oplus S_- \otimes W^*\right),$$

with Lie bracket given by this pairing.

One can compute that the R-symmetry algebra for this representation and pairing is the algebra $\mathfrak{gl}(W)$ acting on W and W^* by the fundamental and anti-fundamental representations respectively. Given a subalgebra $\mathfrak{g}_R \subseteq \mathfrak{gl}(W)$, there is an associated *supersymmetry algebra*

$$\mathcal{A}^W = (\mathfrak{so}(4;\mathbb{C}) \oplus \mathfrak{g}_R) \ltimes T^W.$$

If $\dim W = k$, we also denote this algebra by $\mathcal{A}^{N=k}$. We'll be particularly interested in the case where $\dim W = 4$ and $\mathfrak{g}_R = \mathfrak{sl}(4)$. As we'll see, this is the supersymmetry algebra that will act on $N = 4$ supersymmetric gauge theories.

Example A.9 (Dimension 2)**.** Two-dimensional theories will arise for us as dimensional reductions of 4d theories along a Riemann surface. Again, since we're most interested in the complexified supersymmetry algebra we'll not be too concerned about the choice of signature, but it is worth remarking that the case of Lorentzian signature is special due to the existence of Majorana-Weyl spinors. We have an isomorphism $\mathrm{Spin}(2) \cong \mathrm{U}(1)$. Let $S_\pm$ be the complex 1-dimensional representations of the circle of weight ± 1. The vector representation of $\mathrm{Spin}(2)$ corresponds to the weight two representation of $\mathrm{U}(1)$, so there are natural pairings $\Gamma \colon S_\pm \otimes S_\pm \to V_\mathbb{C}$ (using a canonical isomorphism between $V_\mathbb{C}$ and its dual).

Let W_+ and W_- be finite-dimensional complex vector spaces, and choose inner products $W_\pm \otimes W_\pm \to \mathbb{C}$. Combining this with the pairing above yields a pairing Γ on the spinorial representation $(S_+ \otimes W_+) \oplus (S_- \otimes W_-)$, and thus a super Poincaré algebra

$$P_2^{(W_+,W_-)} = (\mathfrak{so}(2;\mathbb{C}) \ltimes V_\mathbb{C}) \oplus \Pi\left((S_+ \otimes W_+) \oplus (S_- \otimes W_-)\right).$$

The R-symmetry algebra associated to this super Poincaré algebra is $\mathfrak{gl}(W_+) \oplus \mathfrak{gl}(W_-)$, and associated to a subalgebra $\mathfrak{g}_R$ of this algebra we produce a *supersymmetry algebra*

$$\mathcal{A}_2^{(W_+,W_-)} = (\mathfrak{so}(2;\mathbb{C}) \ltimes V_\mathbb{C}) \oplus \mathfrak{g}_R \oplus \Pi\left((S_+ \otimes W_+) \oplus (S_- \otimes W_-)\right).$$

If $\dim W_+ = N_1$ and $\dim W_- = N_2$, we say we have (N_1, N_2) supersymmetries, and write $\mathcal{A}_2^{(N_1,N_2)}$.

Let's describe dimensional reduction from 4 to 2 dimensions (for the complexified algebra, though we could also investigate the real case in Riemannian or Lorentzian signature). That is, take $\mathbb{C}^2 \subseteq \mathbb{C}^4$, and consider the subalgebra of the complex infinitesimal isometries $\mathfrak{so}(4;\mathbb{C}) \ltimes \mathbb{C}^4$ mapping this subspace to itself, which has form $(\mathfrak{so}(2;\mathbb{C}) \ltimes \mathbb{C}^2) \oplus \mathfrak{so}(2;\mathbb{C})$. Let S_+ and S_- be the spaces of 4d Weyl spinors. As modules for this subalgebra, the first $\mathfrak{so}(2;\mathbb{C})$ acts with weights $(\pm 1, \mp 1)$ on $S_\pm$ respectively, and the second $\mathfrak{so}(2;\mathbb{C})$ acts with weight $(\pm 1, \pm 1)$ on $S_\pm$. Thus the $N = k$ super Poincaré algebra in dimension 4 naturally dimensionally reduces to the $N = (2k, 2k)$ supersymmetry algebra in dimension 2, with R-symmetry group $\mathfrak{so}(2;\mathbb{C}) \cong \mathfrak{gl}(1;\mathbb{C})$.

Example A.10 (Dimension 10)**.** There is a supersymmetric gauge theory in dimension 10 which is "universal" in the sense that a range of supersymmetric gauge theories that are studied in lower dimensions arise from it by a combination of dimensional reduction and restriction of scalars [**ABD$^+$13**]. We'll focus on the case of minimal supersymmetry, i.e. $N = (1,0)$, describe the Majorana-Weyl spinor representations in signature $(1,9)$, then describe the complexification.

Abstractly, the classification A.2 tells us to expect a pair of mutually dual irreducible spinorial representations of $\mathfrak{so}(1,9)$ over the real numbers, each of dimension 16. We can actually describe these representations very concretely; the details are described by Deligne in [**Del99**, section 6].

It suffices to construct a non-trivial 32-dimensional module for the algebra $\mathrm{Cl}(V,Q)$, where V is 10-dimensional, and Q is a quadratic form of signature (1,9). Concretely, we'll set $V = \mathbb{O} \oplus H$ with $\mathbb{O}$ 8-dimensional and $H = \langle e, f \rangle$ 2-dimensional, and we set

$$Q(\omega + ae + bf) = \omega \cdot \overline{\omega} - ab$$

where $\omega \cdot \overline{\omega}$ is the octonion norm-squared. Let $S_{10}^{\mathbb{R}} = (\mathbb{O}^2) \oplus (\mathbb{O}^2)$ be a 32-dimensional real vector space. We must describe a Clifford multiplication $\rho \colon V \otimes S_{10}^{\mathbb{R}} \to S_{10}^{\mathbb{R}}$ making $S_{10}^{\mathbb{R}}$ into a module for $\mathrm{Cl}(V,Q)$. This is concretely given by

$$\rho \colon \mathbb{O} \oplus H \to \mathrm{End}(S_{10}^{\mathbb{R}})$$

where

$$\rho(\omega) = \begin{pmatrix} \begin{pmatrix} 0 & m_\omega \\ m_\omega & 0 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 & m_\omega \\ m_\omega & 0 \end{pmatrix} \end{pmatrix} \text{ for } \omega \in \mathbb{O},\ m_\omega(\alpha) = \overline{\omega} \cdot \overline{\alpha}$$

$$\rho(e) = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \quad \text{and} \quad \rho(f) = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \end{pmatrix}.$$

One can check that this gives a well-defined Clifford multiplication, and thus defines a 32-dimensional real spin representation which splits as a sum of two 16-dimensional representations of the even part of the Clifford algebra: call them $S^{\mathbb{R}}_{10+}$, spanned by the first and third components of $\mathbb{O}^4$, and $S^{\mathbb{R}}_{10-}$ spanned by the second and forth. There is also the induced pairing $\Gamma\colon S^{\mathbb{R}}_{10\pm} \otimes S^{\mathbb{R}}_{10\pm} \to V$, which one checks is given on $S^{\mathbb{R}}_{10+}$ and $S^{\mathbb{R}}_{10-}$ respectively by

$$\Gamma((\alpha_1, \alpha_2), (\beta_1, \beta_2)) = \overline{\alpha}_1 \cdot \overline{\beta}_1 + \overline{\alpha}_2 \cdot \overline{\beta}_2 - \mathrm{Tr}(\alpha_1 \cdot \overline{\beta}_1 + \alpha_2 \cdot \overline{\beta}_2) f$$

$$\text{and } \Gamma((\alpha_1, \alpha_2), (\beta_1, \beta_2)) = \overline{\alpha}_1 \cdot \overline{\beta}_1 + \overline{\alpha}_2 \cdot \overline{\beta}_2 + \mathrm{Tr}(\alpha_1 \cdot \overline{\beta}_1 + \alpha_2 \cdot \overline{\beta}_2) e$$

where $\mathrm{Tr}(\alpha) = \alpha + \overline{\alpha}$ is the octonionic reduced trace, and where the calculation is done using the identity $\langle \Gamma(s,t), v \rangle = (\rho(v)s, t)$ for spinors s, t and vectors v. This now gives us a complete description of the supersymmetry algebra in 10-dimensions: it is given by

$$(\mathfrak{so}(1,9) \ltimes \mathbb{R}^{1,9}) \oplus \Pi(S^{\mathbb{R}}_{10+})$$

with brackets given by the internal bracket on $\mathfrak{so}(1,9)$, the action of $\mathfrak{so}(1,9)$ on the translations, the action of $\mathfrak{so}(1,9)$ on the supersymmetries, and the pairing $\Gamma\colon S^{\mathbb{R}}_{10+} \otimes S^{\mathbb{R}}_{10+} \to \mathbb{R}^{1,9}$.

Finally, we can complexify the supersymmetry algebra to obtain a superalgebra of form

$$(\mathfrak{so}(10;\mathbb{C}) \ltimes \mathbb{C}^{10}) \oplus \Pi(S_{10+}).$$

The complexification $S_{10+} = S^{\mathbb{R}}_{10+} \otimes \mathbb{C}$ is a 16-complex dimensional Weyl spinor representation of $\mathfrak{so}(10;\mathbb{C})$. Clifford theory says that the complexification $\mathfrak{so}(10;\mathbb{C})$ embeds in the (even part of the) Clifford algebra $\mathrm{Cl}^+_{10} \cong \mathrm{Mat}_{16}(\mathbb{C}) \oplus \mathrm{Mat}_{16}(\mathbb{C})$ as the elements of spinor norm one. The Weyl spinors are the fundamental representation of the first matrix algebra factor.

More concretely, we write S_{10+} as $\mathbb{O}^2 \oplus i\mathbb{O}^2$ where $\mathbb{O}$ is a 4-complex dimensional vector space. We write $\mathbb{C}^{10}$ as $\mathbb{O} \oplus i\mathbb{O} \oplus \mathbb{C}\langle e, f\rangle$. The Clifford multiplication is then given by

$$\rho(\omega) = \begin{pmatrix} \begin{pmatrix} 0 & m_\omega \\ m_\omega & 0 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 & m_\omega \\ m_\omega & 0 \end{pmatrix} \end{pmatrix}, \quad \rho(i\omega) = \begin{pmatrix} 0 & \begin{pmatrix} 0 & m_\omega \\ m_\omega & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & m_\omega \\ m_\omega & 0 \end{pmatrix} & 0 \end{pmatrix} \quad \text{for } \omega \in \mathbb{O}$$

$$\rho(e) = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \quad \text{and} \quad \rho(f) = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \end{pmatrix}.$$

This complexified algebra dimensionally reduces to recover the $N = 4$ supersymmetry algebra discussed above in four-dimensions. We choose an embedding $\mathbb{C}^4 \hookrightarrow \mathbb{C}^{10}$ and consider the subalgebra of the supersymmetry algebra fixing this subspace. The bosonic piece has the form $\mathfrak{so}(4;\mathbb{C}) \ltimes V_{\mathbb{C}} \oplus \mathfrak{sl}(4;\mathbb{C})$, where the $\mathfrak{sl}(4;\mathbb{C})$ fixes the subspace pointwise (and arises from complexification of $\mathfrak{so}(6) \cong \mathfrak{su}(4)$). We must check that the action of $\mathfrak{so}(6;\mathbb{C}) \oplus \mathfrak{sl}(4;\mathbb{C})$ on the 16-complex-dimensional space of spinors recovers the space $S_+ \otimes W \oplus S_- \otimes W^*$ that we expect. We can do this by looking at the actions of the two summands separately, using that the action is still spinorial, and the fact that it arose as complexification of a representation for the (Lorentzian) real form.

Firstly, $\mathfrak{sl}(4;\mathbb{C})$ has two Weyl spinor representations, the fundamental W and the anti-fundamental W^*, and we must have equal numbers of each (since the complexification of the Majorana spin representation is their sum). The modules S_{10+} has no $\mathfrak{so}(6;\mathbb{C})$-fixed points, so there are no trivial factors and $S_{10+} \otimes_{\mathbb{R}} \mathbb{C} \cong (W \oplus W^*) \otimes_{\mathbb{C}} \mathbb{C}^2$. Secondly, $\mathfrak{so}(4;\mathbb{C})$ has two Weyl spinor representations S_+ and S_-. By the same argument we have equal numbers of each and there are no trivial summands, so $S_{10+} \otimes_{\mathbb{R}} \mathbb{C} \cong (S_+ \oplus S_-) \otimes_{\mathbb{C}} \mathbb{C}^4$. Finally, to describe the relationship between these two actions we observe that the actions commute and complexify a real Lie algebra action.

APPENDIX B

Lie Algebras and Deformation Theory

For motivation and reference, we've included the fundamental definitions and results on sheaves of Lie algebras and deformation theory. None of this material is original, and most of the results in the smooth category context be found in [**Cos11a**], [**GG14**] and appendix A of [**CG15**]. The derived deformation theoretic results we reference are due to Hinich [**Hin01**] and Getzler [**Get09**], or in a more homotopical setting to Lurie [**Lur11**] and Hennion [**Hen15**].

As we work in the setting of ∞-categories and the two operads Lie and L_∞ are homotopy equivalent we are free to use the languages of Lie and L_∞-algebras interchangeably, mainly choosing our terminology in order to be more compatible with the literature for the appropriate context.

Definition B.1. *A* curved L_∞ algebra *over a cdga R with respect to an ideal I is a locally free graded $R^\natural$ module L equipped with a degree 1 differential*

$$d\colon \widehat{\mathrm{Sym}}_{R^\natural}(L^\vee[-1]) \to \widehat{\mathrm{Sym}}_{R^\natural}(L^\vee[-1])$$

making $\widehat{\mathrm{Sym}}_{R^\natural}(L^\vee[-1])$ into a dg-module over R, such that d vanishes on Sym^0 *modulo the ideal I. We denote $\widehat{\mathrm{Sym}}_{R^\natural}(L^\vee[-1])$ by $C^\bullet(L)$ and call it the* Chevalley-Eilenberg algebra *of L.*

By taking the Taylor coefficients of the differential d we obtain a sequence of degree 0 graded anti-symmetric operations $\ell_n\colon (\wedge^n L)[n-2] \to L$, dual to the composite

$$L^\vee[-1] \hookrightarrow C^\bullet(L) \xrightarrow{d} C^\bullet(L) \twoheadrightarrow \mathrm{Sym}^n(L^\vee[-1])$$

which satisfy higher analogues of the Jacobi identities, recovering a more classical definition of a (curved) L_∞ algebra. One way of thinking about our definition is that Lie algebras are Koszul dual to commutative algebras, so defining the Lie algebra structure on L is equivalent to defining a commutative dga structure on its Koszul dual $C^\bullet(L)$.

We'll want to study versions of L_∞ algebras varying over a topological space. This will be useful for perturbative field theory, where an L_∞ algebra describes the deformations of a particular solution to the equations of motion on an open set U in spacetime, in order to describe the relationship between these solutions on different open sets.

Definition B.2. *A* local L_∞ algebra *over a manifold M is a cochain complex of vector bundles L over M such that the sheaf of sections is given the structure of a sheaf of L_∞ algebras where the operations ℓ_n are polydifferential operators.*

If G is an algebraic supergroup, a G-action on a local L_∞ algebra L is a $C^\bullet(G)$-module structure on $L(U)$ for each open set $U \subseteq X$ making L into a sheaf of curved L_∞ algebra over $C^\bullet(G)$ relative to the ideal $C^{>0}(G)$. Here $C^\bullet(G)$ denotes the complex where $C^i(G) = \mathcal{O}(G^i)$, with the usual differential using the group structure. One similarly defines a $\mathfrak{g}$-action for a super Lie algebra $\mathfrak{g}$ to be a local module structure on each open set for the Chevalley-Eilenberg complex $C^\bullet(\mathfrak{g})$.

The perturbative definition of a classical field theory used by Costello in [**Cos11a**] builds on the following definition capturing local geometry of a given space. The idea is that in algebraic geometry, one is able to investigate formal neighborhoods of a point by only considering local Artinian algebras.

Definition B.3. *A* formal derived moduli problem *is a functor F from the category* $\mathrm{Art}_{\mathrm{dg}}^{\leq 0}$ *of differential graded Artinian algebras cohomologically in degrees ≤ 0 to the category* sSet *of simplicial sets satisfying the following conditions:*

- *the space $F(\mathbb{C})$ is contractible.*
- *If $A \to B$ and $A' \to B$ are morphisms in* $\mathrm{Art}_{\mathrm{dg}}^{\leq 0}$ *which are surjections on H^0, then the induced map $F(A \times_B A') \to F(A) \times_{F(B)} F(A')$ is a homotopy equivalence.*

Note that the second condition ensures the ability to glue $\operatorname{Spec} A$ and $\operatorname{Spec} A'$ along $\operatorname{Spec} B$ whenever we have closed embeddings at the classical level.

For example, given a point $p \in X = \operatorname{Spec} R$ for $R \in \mathrm{cdga}^{\leq 0}$, or a maximal ideal $\mathfrak{m} \subset R$, the functor $X_p \colon \mathrm{Art}_{\mathrm{dg}}^{\leq 0} \to \mathrm{sSet}$ defined by

$$(A, \mathfrak{m}_A) \mapsto (\{\phi \colon R \to A \otimes \Omega^\bullet(\Delta^n) \mid \phi(\mathfrak{m}) = \mathfrak{m}_A \otimes \Omega^\bullet(\Delta^n)\})_{n \in \Delta}$$

is a formal moduli problem. Geometrically $X_p(A)$ encodes the data of infinitesimal extension of p via A.

The most important tool we are going to take advantage of in order to understand formal moduli problems is the Maurer-Cartan functor.

Definition B.4. *Let L be an L_∞ algebra. The* Maurer-Cartan functor $\mathrm{MC}_L \colon \mathrm{Art}_{\mathrm{dg}}^{\leq 0} \to \mathrm{sSet}$ *is defined to be the functor given by $(R, \mathfrak{m}) \mapsto \mathrm{MC}_L(R)$, where the simplicial set $\mathrm{MC}_L(R)$ has as n-simplices elements $\alpha \in L \otimes \mathfrak{m} \otimes \Omega^\bullet(\Delta^n)$ of cohomological degree 1, which satisfy the Maurer-Cartan equation*

$$\sum_{n \geq 0} \frac{1}{n!} \ell_n(\alpha^{\otimes n}) = 0.$$

This is not manifestly a homotopy invariant notion, and thus not manifestly well-defined. However, there is an equivalent rephrasing of the Maurer-Cartan functor that *is* manifestly homotopy invariant.

Proposition B.5. $\mathrm{Hom}_{\mathrm{cdga}_*}(C^\bullet(L), R) = \mathrm{MC}_L(R)$ *for $R \in \mathrm{Art}_{\mathrm{dg}}^{\leq 0}$.*

A proof of this fact appears in section 2.3 of Lurie [**Lur11**]; as we've phrased it it's implied by his Theorem 2.3.1.

Theorem B.6 ([**Lur11**, 2.0.2])**.** *The Maurer-Cartan functor provides an equivalence of categories*

$$\mathrm{MC} \colon \{L_\infty \text{ algebras}\} \to \{\text{formal derived moduli problems}\}$$

with quasi-inverse given by taking the -1-shifted tangent complex equipped with a canonical L_∞ structure.

We sometimes write BL for the formal moduli problem MC_L. Then the theorem in particular says the following

- There is an equivalence $\mathbb{T}_0[-1]BL \cong L$.
- Every formal derived pointed moduli problem X can be realized as BL_X for some L_∞ algebra L_X, in the sense that the formal derived moduli problem describing maps into X is equivalent to the formal moduli problem MC_{L_X}.

The proposition allows one to think of $C^\bullet(L)$ as the structure sheaf of the formal moduli problem BL. Note that $C^\bullet(L)$ is in general not an object of the category $\mathrm{cdga}^{\leq 0}$, having stacky nature.

For our purpose, it is important to understand mapping stacks in terms of an L_∞-algebra.

Lemma B.7. *Let L be an L_∞-algebra and A be an object of $\mathrm{Art}_{\mathrm{dg}}^{\leq 0}$. Then $L \otimes A$ is the L_∞-algebra governing the deformations of the constant map $\operatorname{Spec} A \to BL$.*

We only sketch the proof for the 0-simplex to give an idea.

Proof sketch. If B is another Artinian algebra, then $\alpha \in MC_{L\otimes A}(B)[0]$ is an element $\alpha \in (L \otimes A \otimes \mathfrak{m}_B)^1$ satisfying Maurer-Cartan equation. Since the maximal ideal of $A \otimes B$ is $\mathfrak{m}_A \otimes B + A \otimes \mathfrak{m}_B$, from $MC_{L\otimes A}(B) \subset MC_L(A \otimes B)$, an element $\alpha \in MC_{L\otimes A}(B)$ can be characterized as an element of $MC_L(A \otimes B)$ which vanishes modulo $\mathfrak{m}_A$. Hence, geometrically, $MC_{L\otimes A}(B)$ represents families of maps $\operatorname{Spec} A \to BL$ parametrized by $\operatorname{Spec} B$ which are constant at the unique geometric point $\operatorname{Spec} \mathbb{C} \in \operatorname{Spec} A$. $\square$

In other words, for the mapping stack $\underline{\mathrm{Map}}(X, Y)$, its formal derived moduli problem at f is controlled by the L_∞-algebra $\Gamma(X, f^* L_Y)$.

The main construction we are using in the paper is in an algebraic setting.

Theorem B.8. *[**Hen15**, 4.2.0.1] If $\mathcal{X}$ is a derived Artin stack locally of finite presentation, then its shifted tangent complex $\mathbb{T}_{\mathcal{X}}[-1]$ is a Lie algebra object of $\mathrm{QCoh}(\mathcal{X})$.*