CHAPITRE IV

EFFETS QUANTIQUES DANS LE CHAMP D'UN TROU NOIR

IV. EFFETS QUANTIQUES DANS LE CHAMP D'UN TROU NOIR.

Dans les chapitres précédents les trous noirs ont été étudiés comme des objets "classiques" au moyen des équations d'Einstein et d'un tenseur d'énergie impulsion décrivant la matière (par exemple sous forme d'un fluide) et le champ électromagnétique. Nous allons maintenant prendre en compte certains effets quantiques qui, non seulement sont importants d'un point de vue théorique (quantification en espace courbe, entropie quantique) mais ont aussi d'intéressantes conséquences pour l'Astrophysique (extraction d'énergie sous forme plus ou moins explosive).

Etant donné qu'à l'heure actuelle il est encore très problématique de quantifier des champs de gravitation forts (comme celui d'un trou noir) l'approche utilisée consiste à quantifier des champs d'épreuve dans une métrique de fond donnée. Bien sûr le champ d'épreuve considéré peut être un champ scalaire, neutrinique, électronique, photonique, ... et même gravifique (entendant par là une perturbation de la métrique de base).

Il reste alors le problème essentiel de la définition des particules correspondantes aux champs c'est-à-dire en d'autres termes de la séparation entre fréquences "positives" et "négatives", ou du choix d'un noyau G_1 (Lichnerowicz (1964)). Ce problème est loin d'être résolu en général, cependant dans les applications que nous avions en vue l'usage d'une approximation des hautes fréquences (W.K.B.) était suffisante pour obtenir le résultat souhaité. Soulignons d'ailleurs que cette méthode devient exacte quand elle est appliquée à la quantification des champs d'épreuve dans la métrique d'un trou noir à cause de la particularité qu'à ce dernier de causer des "décalages vers le rouge" infinis.

Nos résultats sont contenus dans les articles reproduits ci-dessous. Signalons simplement que ces articles contiennent :

a) Une revue de certaines des approches que l'on peut utiliser dans l'étude de champs quantiques plongés dans un champ classique extérieur stationnaire (Damour 1977). On considère surtout les cas où des particules réelles peuvent être crées, le paradigme de cette situation étant le paradoxe de Klein. A la suite de Sauter, Heisenberg et Euler nous approchons ce paradoxe en introduisant certains diagrammes d'énergie dont l'extension au cas des trous noirs s'est révélée être très utile. Les résultats nouveaux obtenus dans cet article sont :

- l'expression exacte du nombre de particules crées par un champ électromagnétique uniforme et constant (spin 0 et 1/2).

- la comparaison de cette expression avec le célèbre résultat de J. Schwinger. Il est montré en utilisant l'approche de E.C.G. Stueckelberg, que la formule de J. Schwinger ne décrit pas exactement la même chose, mais est parfaitement cohérente avec notre résultat.

- enfin les méthodes et résultats précédents sont appliqués au cas d'un trou noir tournant et chargé, ainsi qu'au phénomène d'évaporation des trous noirs découvert par Hawking (1975). Les détails techniques et les résultats précis de ces applications sont contenus dans les articles suivants.

b) Une étude de la décharge quantique explosive (analogue au claquage d'un condensateur) d'un trou noir de Kerr-Newman lorsque le champ électrique atteint sa valeur critique. (Damour et Ruffini 1975).

L'importance potentielle de cette analyse pour certaines observations astrophysiques ("γ-ray bursts") est soulignée. En effet cette décharge s'opère par l'émission d'une bouffée de particules chargées extrêmement énergétiques ($\sim 10^{20}$eV pour une masse solaire) qui emportent l'énergie électromagnétique du trou noir ($\sim 10^{41}$erg pour une masse solaire). Bien sûr la présence de plasma autour du trou noir convertirait l'essentiel de ce rayonnement particulaire en une violente bouffée de rayons gamma. Il serait donc très intéressant d'étudier les corrélations entre les " γ-ray bursts" et les rayons cosmiques de très hautes énergies.

c) Une démonstration directe et très courte du phénomène d'évaporation quantique des trous noirs découvert par S.W. Hawking. (Damour et Ruffini 1976).

Il est à remarquer que cette démonstration procède par analogie avec le paradoxe de Klein et utilise seulement l'état stationnaire "final" du trou noir sans se référer à l'effondrement gravitationnel. (voir aussi la fin de (Damour 1977)).

d) Une démonstration de l'existence d'états quantiques instables autour d'un trou noir tournant ou chargé. (Damour, Deruelle et Ruffini 1976).

Il s'agit d'états résonants à "largeur négative", c'est-à-dire croissant exponentiellement avec le temps, de particules massives en mouvement autour d'un trou noir.

L'interprétation de ces états se fait par la création de particules par le champ gravimagnétique ou électromagnétique du trou noir.

e) Enfin, en appendice, nous avons ajouté une note technique (Damour 1975b) qui analyse la correspondance entre les mouvements classiques et les états quantique dans le champ d'un trou noir. Cette note montre que la correspondance peut être resserrée par l'utilisation d'une coordonnée radiale nouvelle ($z = \int_r^\infty dr/\Delta$) qui se révélera aussi utile dans le chapitre VI.

En somme nous pouvons relier certains des résultats obtenus dans ce chapitre à l'approche utilisée dans les chapitres précédents en disant que dans la formule de masse de Christodoulou et Ruffini (1971) :

$$M^2 = \left(M_{ir} + \frac{e^2}{4M_{ir}}\right)^2 + \frac{J^2}{4M_{ir}^2} \quad (1)$$

- toute l'énergie coulombienne peut être extraite (de manière astrophysiquement réaliste) par des effets quantiques (cf.b) .

- la masse irréductible peut aussi s'évaporer en un rayonnement thermique d'origine quantique (cf. c)) contrairement au résultat "classique" (cf. chapitre III). Mais ce rayonnement est négligeable pour les trous noirs de tailles "astrophysiques". Cependant sur le plan des principes cette découverte de Hawking (1975) est très intéressante car elle confirme les conjectures thermodynamiques de Bekenstein (chapitre III) (Bekenstein 1975) tout en permettant de préciser le coefficient de conversion α que nous avions laissé indéterminé au chapitre III :

$$\alpha = \frac{\text{entropie du trou noir}}{\text{surface du trou noir}} = \frac{1}{4\hbar} \quad (2)$$

a. Paradoxe de Klein et polarisation du vide.

KLEIN PARADOX AND VACUUM POLARIZATION

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INTRODUCTION: This contribution reviews some of the methods which can be used when studying quantum fields in a given stationary classical external field. The attention is mainly directed towards cases where real pair creation can occur in such a stationary background. The paradigm of this situation is the Klein paradox. This paradox is best approached by the introduction of some energy diagrams (see the Figures) whose direct extension to black holes physics has proven to be very useful. Finally processes of real pair creation around a Kerr-Newman (charged and rotating) black hole and their feedback on the geometry are briefly discussed. It is also shown how the Hawking process can be recovered in this approach.

I KLEIN PARADOX IN FLAT SPACE

It is well known that every relativistic wave equation (e.g. Klein-Gordon's or Dirac's) admits symmetrically "positive frequency" as well as "negative frequency" solutions. Namely a relativistic wave of mass μ moving in free space and having a frequency ω and a wave vector $\underset{\sim}{k}$ must fullfill (here and in the following we choose $G = c = \hbar = 1$):

$$\omega^2 = \mu^2 + \underset{\sim}{k}^2 \tag{1a}$$

or

$$\omega = \pm (\mu^2 + \underset{\sim}{k}^2)^{\frac{1}{2}} \tag{1b}$$

This gives rise to the familiar spectrum of allowed states represented in Fig. I.

These states are the possible states of the quantum particle described by the wave. In the present situation (flat space, no external field) one can ignore the existence of these "negative" states because the "posirive" states are stable; that is, there exists no posibility of decay of a "positive" state into a "negative" state. But if, for instance, the particle bears an electric charge ε and is embedded in a constant electric potential V, ω will be shifted by the amount ε V. More precisely in the quasi-classical (W.K.B.) approximation the particle must fulfill:

$$(\omega - \varepsilon V)^2 = \mu^2 + \underset{\sim}{k}^2 \tag{2a}$$

or

$$\omega = \varepsilon V \pm (\mu^2 + \underset{\sim}{k}^2)^{\frac{1}{2}} \tag{2b}$$

In the case of an electric field $\underset{\sim}{E} = (o, o, E(z))$ uniform between z_1 and z_2 and null outside, the Fig. II represents the corresponding sketch of allowed states.

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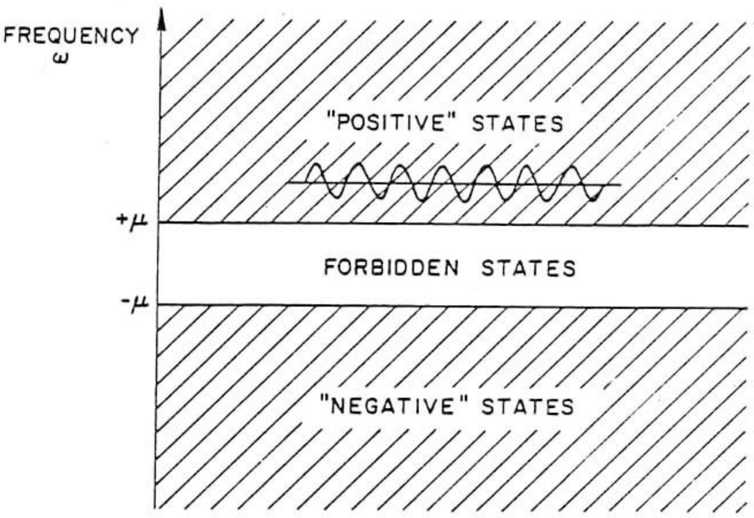

FIGURE I: The spectrum of allowed states for a free wave in flat space describing a quantum particle of mass μ is here represented in function of the frequency ω. In such a free field situation all the states are stable; that is, there is no possibility of a "positive" ("negative") wave decaying into a "negative" ("positive") one.

The key point now, which is the essence of the Klein paradox [1], is that the above mentioned stability of the "positive" states is lost for sufficiently strong electric fields. The same is true for "negative" states: for instance (see fig. II) a "negative" wave incident from the left will be both reflected back by the electric field and partly transmitted to the right as a "positive" wave. This transmission is nothing else but a Gamow tunnelling of the wave function through the classically forbidden states [2, 3]. Mathematically it is due to the fact that in Eq. (2a) k_z^2 will become negative in the barrier which means that the wave function will have an exponential behaviour ($\exp - \int |k_z| dz$ in W.K.B. approximation) instead of its usual oscillatory behaviour. The quantity of interest associated with this process will evidently be the transmission coefficient $|T|^2$ of the wave through the one-dimensional barrier between the "negative" and the "positive" states:

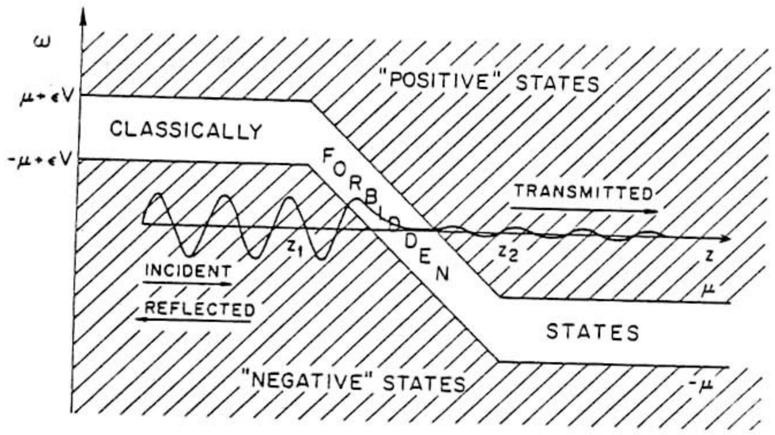

FIGURE II: In presence of a strong enough electric field the boundaries of the classically allowed states ("positive" or "negative") can be so tilted that a "negative frequency" is at the same level as a "positive frequency" (level crossing). Therefore a "negative" wave-packet incident from the left will be partially transmitted, after an exponential damping due to the tunnelling through the classically forbidden states, as a "positive" wave-packet outgoing to the right.

$$|T|^2 = |\text{transmitted flux}|/|\text{incident flux}| \qquad (3)$$

In W.K.B. approximation one has simply $|T|^2 = \exp -2 \int_{\text{barrier}} |k_z|\, dz$. There will also be a reflection coefficient $|R|^2$ for the wave whose link to $|T|^2$ depends on the spin of the particle.

For fermions, the probability density is always positive and the probability flux directed as the group velocity. Therefore, the reflected flux will be smaller than the incident one and more precisely.

$$|R|^2 = 1 - |T|^2 \qquad \text{(fermions)} \qquad (4a)$$

For bosons, "negative" waves have a negative density and therefore a flux oppositely directed to their group velocity. Consequently one sees immediately that the reflected flux will be larger than the incident one and that more precisely

$$|R|^2 = 1 + |T|^2 \qquad \text{(bosons)} \tag{4b}$$

But this last phenomenon: amplification of the wave by reflection that is "superradiance" though an interesting consequence of the spin-statistics connection, (see below) does not constitute the essence of the Klein paradox; what is important is the possibility of decay:

ingoing "positive" state → outgoing "negative" state

or

ingoing "negative" state → outgoing "positive" state

as exemplified by the scattering process described in Fig. II. And if such a decay process can take place, one can easily predict that there will be spontaneous real pair creation by the vacuum. For instance a heuristic reasoning using Dirac's Ocean of "negative" states (for electrons, but see II below for charged bosons) shows directly that the leakage of the "negative" states towards "positive" states will give rise to a continuous creation of electrons with a rate proportional to $|T|^2$ (see II below for details). Let us check this, using a second quantized approach [4].

II RATES OF PARTICLE CREATION

Along the lines of the scattering process described in Fig. II let us introduce [5-11] the ingoing and the outgoing states of the quantized field that we are considering. That is to say, if we want to investigate particle creation, by a given external field (electromagnetic and/or gravitational), in a given volume Ω of space-time, we have to introduce the solutions of the wave equation, written in that given background, from which one can build localized wave packets purely ingoing into Ω from the past. These states are the generalization of the ingoing stationary waves of a potential scattering process. In the following we will suppose that it has been possible to define meaning fully these ingoing states. We have to do the same thing for outgoing states, that is, states from which one can build localized wave packets purely outgoing from Ω into the future. Moreover it is essential that it has been possible to separate those ingoing (outgoing) states in "positive" and "negative" states. This choice is clear in the case of Fig. II where we have an effective potential with two asymptotic regions. See V below for the analogous situation in black holes physics.

Therefore we suppose that we have a complete basis of "positive" and "negative" ingoing modes $p_i^{in}(x)$ and $n_i^{in}(x)$ fullfilling the orthonormality relations

$$(p_i^{in}, p_k^{in}) = \delta_{ik} = \pm (n_i^{in}, n_k^{in}) \tag{5a}$$

$$(p_i^{in}, n_k^{in}) = 0 \tag{5b}$$

where (p, q) denotes the natural scalar product for our field, δ_{ik} is a general Kronecker symbol which may contain δ-functions in case of continuous normalization, and where, here and in the following, the upper sign is valid for fermions and the lower one for bosons.

Then we expand the quantized field as:

$$\phi(x) = \sum_i a_i^{in} p_i^{in}(x) + (b_i^{in})^+ n_i^{in}(x) \tag{6a}$$

with the (anti) commutation relations:

$$\left[a_i^{in}, (a_k^{in})^+\right]_\pm = \left[b_i^{in}, (b_k^{in})^+\right]_\pm = \delta_{ik} \tag{6b}$$

Finally the in-vacuum is defined by

$$a_i^{in} |{}^{in}_{vac}\rangle = b_i^{in} |{}^{in}_{vac}\rangle = 0 \tag{7}$$

that is, it is cancelled by the annihilation operators a^{in} and b^{in} for the in-particles and in-antiparticles of our field.

Evidently one can do the same thing with out-states and the phenomenon of particle creation will consist in that the in-vacuum contains out-states. More precisely, the mean number $\langle N_i \rangle = n_i$ of out-particles described by p_i^{out} that one will find in the in-vacuum is given by

$$n_i = \langle{}^{in}_{vac}| (a_i^{out})^+ a_i^{out} |{}^{in}_{vac}\rangle \tag{8}$$

With a corresponding expression for the mean number of out-antiparticles. But from

$$\phi = \sum_k a_k^{in} p_k^{in} + (b_k^{in})^+ n_k^{in} = \sum_i a_i^{out} p_i^{out} + (b_i^{out})^+ n_i^{out} \tag{9}$$

and the orthonormality relations (5) one gets

$$a_i^{out} = \sum_k \ (p_i^{out} , p_k^{in}) \ a_k^{in} + (p_i^{out} , n_k^{in}) \ (b_k^{in})^+ \qquad (10)$$

hence

$$n_i = \sum_k \ |(p_i^{out} , n_k^{in})|^2 \qquad (11)$$

So that, as predicted above, to each possible channel for a decay $n_k^{in} \to p_i^{out}$ with what we may call, a non vanishing transmission amplitude:

$$T_{ik} = (p_i^{out} , n_k^{in}) \qquad (12)$$

corresponds a mean number $|T_{ik}|^2$ of particles created in the mode p_i^{out}. The mean total number of particles created will be:

$$<N> = \sum_i \ n_i = \sum_{\substack{\text{channels}\\ i \ , \ k}} |T_{ik}|^2 \qquad (13)$$

Of special interest in the applications, is the case where it is possible to choose the in-basis and the out-basis in such a correspondence that a n_i^{in} decays only in a p_i^{out} with the same i, that is when there is only one channel possible. This case occurs for instance when enough symmetries are present for allowing the use of factorized partial waves.

In such a case one can write:

$$T_{ik} = T_i \ \delta_{ik} \qquad (14)$$

and therefore

$$n_i = |T_i|^2 \qquad (15)$$

Eq. (14) means that the ingoing state n_i^{in} contains the outgoing part $T_i \ p_i^{out}$. As we will consider in the following particle creation in a stationary background, we will be able to separate the time t in the modes by a factor $e^{-i\omega t}$. Therefore the modes will be defined by a continuous index ω as well as by some other indices α. Hence the transmission amplitude will read:

$$T_{ik} = T_{\omega_i \alpha_i} \ \delta(\omega_i - \omega_k) \ \delta_{\alpha_i \alpha_k} \qquad (16)$$

and we must use the familiar "golden rule" that [6]:

$$[\delta(\omega_i - \omega_k)]^2 = \delta(\omega_i - \omega_k)\frac{1}{2\pi}\int e^{i(\omega_i - \omega_k)t}\,dt = \delta(\omega_i - \omega_k)\frac{1}{2\pi}\int dt \tag{17}$$

which is easily justified by using nearly monochromatic wavepackets.

Therefore one gets a continuous rate of particle creation:

$$\frac{d}{dt} \langle N\rangle = \int d\omega \sum_{\alpha} \frac{1}{2\pi} |T_{\omega\alpha}|^2 \tag{18}$$

and in this formula $|T_{\omega\alpha}|^2$ is precisely the transmission coefficient previously introduced because the normalization in ω implies the normalization of the flux to $(2\pi)^{-1}$ which is just another phrasing of the preceding "golden rule". Finally it is of interest to compute the quantum fluctuations in the number of particles created. The number of particles in the state p_i^{out} is represented by the operator $N_i = (a_i^{out})^+ a_i^{out}$ which yields a mean number n_i of particles with a dispersion σ_i given by:

$$\sigma_i^2 = \langle {}^{in}_{vac} | N_i^2 | {}^{in}_{vac} \rangle - \langle {}^{in}_{vac} | N_i | {}^{in}_{vac} \rangle^2 \tag{19a}$$

which yields, using eq. (10), the usual formula for fermions or bosons:

$$\sigma_i^2 = n_i(1 \mp n_i) \tag{19b}$$

The total fluctuation is:

$$\sigma_N^2 = \sum_i \sigma_i^2 \tag{19c}$$

Therefore if each individual transmission coefficient is small that is if $n_i \ll 1$ (which does not mean necessarily that $\langle N\rangle$ is small due to the large number of possible channels) one gets the familiar $\langle N\rangle^{1/2}$ fluctuation.

It is interesting to interpret the preceding results on particle creation obtained by second quantization with the simple picture of the Dirac's ocean. In order to extend this picture to charged bosons let us begin with second quantization. We start from the expansion (6a) where the symbol Σ is understood as containing an integration over the frequency ω. The charge of the field ϕ is:

$$Q = \varepsilon(\phi, \phi) \tag{20a}$$

and its energy is

$$E = (\phi, i\partial_t\phi) \tag{20b}$$

where we use the scalar product in its usual order: $\phi^+ \times \phi$. Denoting by ω_i^+ the frequency of the "positive" state p_i^{in} and by ω_i^- the one of the "negative" state n_i^{in}, the sesquilinearity of the scalar product yields:

$$Q = \sum_i \varepsilon (a_i^{in})^+ a_i^{in} \pm \varepsilon b_i^{in} (b_i^{in})^+ \tag{21a}$$

$$E = \sum_i \omega_i^+ (a_i^{in})^+ a_i^{in} \pm \omega_i^- b_i^{in} (b_i^{in})^+ \tag{21b}$$

where, as before, the upper sign holds for fermions and the lower one for bosons. The use of the (anti)commutation relations (6a) yields:

$$Q = \sum_i \varepsilon (a_i^{in})^+ a_i^{in} - \varepsilon (b_i^{in})^+ b_i^{in} + \sum_{n_i^{in}} \pm \varepsilon \tag{22a}$$

$$E = \sum_i \omega_i^+ (a_i^{in})^+ a_i^{in} - \omega_i^- (b_i^{in})^+ b_i^{in} + \sum_{n_i^{in}} \pm \omega_i^- \tag{22b}$$

The last sums in (22a) and (22b) can be readily interpreted by saying that all the "negative" states n_i^{in} are filled by a wave normalized to unity which bears a charge $\pm\varepsilon$ and an energy $\pm\omega_i^-$. But in the case of Fig. II, or any similar situation, these waves will leak out of the "negative" sea and appear as an outgoing positive wave. More precisely in the case where only one channel is possible one can write:

$$n_i^{in} = R_i \, n_i^{out} + T_i \, p_i^{out} \tag{23a}$$

with

$$|R_i|^2 = 1 \mp |T_i|^2 \tag{23b}$$

The continuum normalization in ω implies that all the waves have a flux normalized to $(2\pi)^{-1}$ (an immediate corollary is that $|T_i|^2$ is identical with the transmission coefficient defined by Eq. (3)). Therefore in the case of fermions, the scattering process (23a) will be seen as an outgoing flux $|T_i|^2/2\pi$ of particles of charge $+\varepsilon$ and energy $+\omega_i^-$ associated to a defect of flux (hole) of $|T_i|^2/2\pi$ over the background sea which will appear as a $|T_i|^2/2\pi$ flux of antiparticles of charge $-(+\varepsilon)$ and energy $-(+\omega_i^-)$. For bosons the reflected part $R_i \, n_i^{out}$ will yield an excess of flux of $|T_i|^2/2\pi$ over the background sea and will appear directly as a $|T_i|^2/2\pi$ flux of antiparticles of charge $-\varepsilon$ and energy $-\omega_i^-$. (For bosons, antiparticles are directly "negative" waves and not holes). Therefore in both cases this simple picture yields the same quantitative predictions as the previous calculation (see Eq. (18)).

Let us now consider an application of what precedes to the study of pair creation by uniform electromagnetic fields in flat space.

III UNIFORM ELECTROMAGNETIC FIELDS

If we first consider following Sauter [2] a uniform electric field E in the z direction the electric potential will be (see Fig. II)

$$V = - E.z \tag{24}$$

We can use separated wave functions:

$$\phi = e^{i(k_x x + k_y y - \omega t)} \psi(z) \tag{25}$$

with continuous normalization in k_x, k_y and ω and discrete normalization for spin variables. In W.K.B. approximation, the equation for $\psi(z) \sim \exp i \int k_z \, dz$ is:

$$(\omega + Ez)^2 = \mu^2 + k_x^2 + k_y^2 + k_z^2 \tag{26}$$

independently of the spin.

Inside the barrier k_z^2 will be negative and the transmission coefficient will be

$$|T|^2 = e^{-\zeta} \tag{27a}$$

with the "opacity" ζ given by

$$\zeta = 2 \int_{\text{barrier}} |k_z| \, dz \tag{27b}$$

Therefore using Eq. 26 one gets

$$\zeta = \oint [\mu^2 + k_x{}^2 + k_y{}^2 - (\omega + \varepsilon Ez)^2]^{1/2} \, dz \tag{27c}$$

that is, trivially (with $\varepsilon E > o$):

$$\zeta = \pi \, (\mu^2 + k_x^2 + k_y^2)/\varepsilon E \tag{28}$$

Looking back to Eq. 25 we can write the transmission amplitude as:

$$T_{ik} = e^{i\alpha} e^{-\zeta/2} \, \delta(\omega_i - \omega_k) \, \delta(k_i^x - k_k^x) \, \delta(k_i^y - k_k^y) \, \delta_{\sigma_i \sigma_k} \tag{29}$$

where $e^{i\alpha}$ is a phase factor and $\delta_{\sigma_i \sigma_k}$ arises from the conservation of the projection σ of the spin on the z-direction.

The mean number of particles created is then obtained by using Eq. (13) and the "golden rule" (17) extended as usual to the other conjugated variables (x, k_x) and (y, k_y):

$$\langle N\rangle = \int \frac{dt\, d\omega}{2\pi} \frac{dx\, dk^x}{2\pi} \frac{dy\, dk^y}{2\pi} \sum_\sigma e^{-\zeta} \tag{30}$$

the integration over k_x and k_y is easily made by the substitution: $d k_x\, d k_y = \pi\, d k^2$ with $k^2 = k_x^2 + k_y^2$, the integration over ω is performed by noting that the frequency is linked to the position of the barrier by $d\omega = \varepsilon E\, dz$, with a result:

$$\langle N\rangle = \int d^4x \sum_\sigma \frac{1}{8\pi} \left(\frac{\varepsilon E}{\pi}\right)^2 e^{-\pi\mu^2/\varepsilon E} \tag{31a}$$

where $d^4x = dxdydz\, dt$ and where the numerical value of $\mu^2/\varepsilon = \mu^2c^3/\varepsilon\hbar$ ("critical electric field") is 4.414×10^{13} c.g.s. (for the production of electrons). In the case of spin 0 there are no supplementary channels due to the spin so that one gets a density of pair creation per unit space-time volume of:

$$\text{spin } 0\text{:}\quad (8\pi)^{-1}\, (\varepsilon E/\pi)^2\, e^{-\pi\mu^2/\varepsilon E} \tag{31b}$$

In the spin ½ case one has twice as many states due to $\sigma = \pm \frac{1}{2}$ which yields a spacetime density of pair creation:

$$\text{spin } \tfrac{1}{2}\text{:}\quad (4\pi)^{-1}\, (\varepsilon E/\pi)^2\, e^{-\pi\mu^2/\varepsilon E} \tag{31c}$$

These two results were obtained very simply by using a W.K.B. <u>approximation</u> but it happens surprisingly that they are <u>exact</u> because the opacity (28) is exact. This can be seen as follows: in the case of spin 0 the equation for $\psi(z)$ is:

$$d^2\psi(z)/dz^2 = \left[\mu^2 + k_x^2 + k_y^2 - (\omega + \varepsilon Ez)^2\right] \psi(z) \tag{32a}$$

introducing

$$\xi = (\varepsilon E)^{-\frac{1}{2}} (\omega + \varepsilon Ez) \tag{32b}$$

$$\lambda = (\mu^2 + k_x^2 + k_y^2)/\varepsilon E \tag{32c}$$

one gets
$$d^2\psi/d\xi^2 = (\lambda - \xi^2)\psi \tag{32d}$$

whose solution is the parabolic cylinder function [12, 13]

$$\psi = D_n(u) \tag{33a}$$

with
$$u = 2^{\frac{1}{2}}\, e^{-i\,\pi/4} \xi \tag{33b}$$

$$n = -\tfrac{1}{2} - i\,\lambda/2 \tag{33c}$$

when $\xi \to + \infty$ ψ is equivalent to $e^{-u^2/4}\, u^n$ which contains the phase factor $e^{i\xi^2/2}$, which implies a positive group velocity <u>both</u> for $\xi \to +\infty$ and $\xi \to -\infty$. Therefore Eq. (33a) describes a wave incident from the left and its transmission amplitude is obtained by rotating ξ of $e^{+i\pi}$ from $+\infty$ to $-\infty$ which transforms the transmitted wave into the incident one, hence: $T^{-1} = e^{in\pi} = e^{-i\frac{\pi}{2}} e^{\pi\lambda/2}$ which yields $|T|^2 = e^{-\pi\lambda}$ in perfect agreement with (28). The same conclusion is reached in the case of spin ½ simply because the iterated equation for $\psi(z)$ is obtained by the replacement $\lambda \to \lambda \pm i$ which does not alter the transmission coefficient.

Therefore the two expressions (31b) and (31c) yield exactly the rate of pair creation by a uniform electric field and yet they are different from Schwinger's expression [14]. This will be explained below, but let us first generalize these expressions to the situation where both an electric and a magnetic field (supposed uniforms) are present [3].

In such a case one can always use a frame where $\underset{\sim}{E}$ and $\underset{\sim}{B}$ are parallel (except when $|\underset{\sim}{E}| = |\underset{\sim}{B}|$ and $\underset{\sim}{E} \cdot \underset{\sim}{B} = 0$). In such a frame we can use separated wave functions:

$$\text{spin}\tfrac{1}{2}: \quad \phi = e^{i(k_x x - \omega t)}\, U_n\left[(\varepsilon B)^{\frac{1}{2}}(y + k_x/\varepsilon B)\right]\psi(z) \tag{34}$$

where U_n is a harmonic oscillator wave function of order n (n = 0,1,2,...) which describes the motion of the electron in the plane orthogonal to the common direction of E and B. We then get the same first order equations for $\psi(z)$ as before with, instead of Eq. (32c):

$$\lambda = u^2/\varepsilon E + (n + \tfrac{1}{2} + \sigma)\, 2B/E \tag{35}$$

where σ is the projection of the spin along the common direction of E and B. And as before the opacity is exactly given by

$$\zeta = \pi\lambda \tag{36}$$

The spin 0 case is obtained when $\sigma = 0$ instead of $\sigma = \pm\tfrac{1}{2}$. Then the pair creation rate is given by a sum over the channels:

$$\langle N\rangle = \int \frac{dx\, dk_x}{2\pi}\, \frac{d\omega\, dt}{2\pi} \sum_n \sum_\sigma e^{-\zeta} \tag{37}$$

which yields:

$$\text{spin 0:} \quad \langle N\rangle = \int d^4x\, \frac{1}{8\pi} \left(\frac{\varepsilon E}{\pi}\right)^2 \frac{\pi B/E}{sh(\pi B/E)}\, e^{-\pi u^2/\varepsilon E} \tag{38a}$$

$$\text{spin } \tfrac{1}{2}: \quad \langle N \rangle = \int d^4x \, \frac{1}{4\pi} \left(\frac{\varepsilon E}{\pi}\right)^2 \frac{\pi B/E}{\mathrm{th}(\pi B/E)} \, e^{-\pi\mu^2/\varepsilon E} \tag{38b}$$

IV COMPARISON WITH THE EFFECTIVE LAGRANGIAN APPROACH

This approach was pioneered by Heisenberg and Euler [3] and further developed by Schwinger [14]. Its main idea is that the polarization of the vacuum induced by (real or virtual) pair creation is equivalent to a non-linear modification of the field equations (Maxwell equations for electromagnetism) this modification being similar to the one used when working with dielectrics. We will not expound this method here but just quote the result: the effective Lagrangian for electromagnetism as modified by the creation of pairs of electronsand positrons in a uniform electromagnetic field is:

$$W = -\frac{F}{4\pi} - \frac{1}{8\pi^2} \int_0^\infty \frac{ds}{s^3} e^{-\mu^2 s} \left[(\varepsilon s)^2 \, G \, \frac{\mathrm{Re}\ \mathrm{ch}\ \varepsilon s X}{\mathrm{Im}\ \mathrm{ch}\ \varepsilon s X} - 1 - \frac{2}{3} (\varepsilon s)^2 \, F\right] \tag{39}$$

where $F = F_{\mu\nu} F^{\mu\nu}/4$, $G = F_{\mu\nu} {}^*F^{\mu\nu}/4$ and $X = [2(F + iG)]^{\frac{1}{2}}$

In presence of an electric field the integral in W has poles on the s-axis. If the integration path is considered to lie above this axis we obtain a positive imaginary contribution which is for parallel and uniform fields:

$$2 \ \mathrm{Im}\ W = \frac{1}{4\pi} \left(\frac{\varepsilon E}{\pi}\right)^2 \sum_{n=1}^{\infty} n^{-2} \frac{n\pi B/E}{\mathrm{th}(n\pi B/E)} \, e^{-n\pi\mu^2/\varepsilon E} \tag{40}$$

This is the probability per unit spacetime volume that at least one pair is created by the constant electromagnetic field or more precisely $\exp - \int d^4x \, 2 \ \mathrm{Im}\ W$ is the probability that no actual pair creation occurs during the history of the field [14, 11]. How is this probability related to the mean number of pairs created:

$$\langle N \rangle = \sum_i n_i = \int \frac{d^4x}{4\pi} \left(\frac{\varepsilon E}{\pi}\right)^2 \frac{\pi B/E}{\mathrm{th}(\pi B/E)} \, e^{-\pi\mu^2/\varepsilon E} \tag{41}$$

The answer can be found in the works of Nikishov [15], and, Narozhnyi and Nikishov [16] and uses the approach of Stueckelberg and Feynmann [17] [18]. We can expound it using the notation of II above and assuming for simplicity that T_{ik} is diagonal, therefore we have simply:

$$n_i^{in} = R_i \, n_i^{out} + T_i \, p_i^{out} \qquad \text{(no summation)} \tag{42}$$

where T_i is the transmission amplitude and R_i the reflexion amplitude. Let us denote:

$$|T_i|^2 \equiv n_i \tag{43a}$$

where n_i is now understood as a simple <u>notation</u> not yet related to the mean number of particles created.

We have therefore (see Eq. (4)):

$$|R_i|^2 = 1 \mp n_i \tag{43b}$$

where as before the upper sign is valid for fermions and the lower one for bosons. Now let us reinterpret the scattering process (42) "à la Stueckelberg" [17]:

$$n_i^{out} = R_i^{-1} n_i^{in} - (T_i/R_i) p_i^{out} \tag{44}$$

considering n_i^{out} as <u>incident from the future</u> and both <u>refracted in the past</u> and <u>reflected in the future</u> by the constant background field. The amplitude of "refraction" is R_i^{-1} and the one of "reflection" - T_i/R_i.

Therefore we can consider the new "reflection" coefficient.

$$|T_i/R_i|^2 = n_i/(1 \mp n_i) \tag{45}$$

as the relative probability for the creation of one pair: (n_i^{out}, p_i^{out}). The corresponding absolute probabilities are obtained by multiplying the relative ones by the probability $p_{i,o}$ that no pairs have appeared in the state i. Then the absolute probability for the creation of n pairs in the state i is:

$$p_{i,n} = p_{i,o} \, n_i^{\,n} \, (1 \mp n_i)^{-n} \tag{46}$$

For fermions n = 0,1 and for bosons n = 0,1,2, ... Writing that the total probability is unity, one obtains

$$p_{i,o} = (1 \mp n_i)^{\pm 1} \tag{47}$$

The associated generating functions are therefore:

$$p_i(\lambda) = \sum_n p_{i,n} \, \lambda^n = (1 \mp n_i \pm \lambda \, n_i)^{\pm 1} \tag{48}$$

with, correspondingly, for all the states:

$$P(\lambda) = \prod_i p_i(\lambda) = \sum_N P_N \lambda^N \tag{49}$$

Therefore one reaches simply the probabilities for the creation of N pairs throughout space-time.

For our purpose it will be sufficient to compute the mean number of pairs created in the state i by simply differentiating Eq. (48) with respect to λ and then setting $\lambda = 1$; this yields:

$$\langle n \rangle_i = n_i \tag{50}$$

in perfect agreement with Eq. (15) and (43a) above. And differentiating (49) yields the mean total number of pairs created:

$$\langle N \rangle = \sum_i n_i \tag{51}$$

which we can now compare to the absolute probability of the vacuum remaining a vacuum:

$$P_o = \prod_i p_{i,o} = \prod_i (1 \mp n_i)^{\pm 1} \tag{52a}$$

or $$P_o = \exp - 2 \operatorname{Im} W \tag{52b}$$

with $$2 \operatorname{Im} W = \sum_i \mp \ln(1 \mp n_i) = \sum_i \sum_{n=1}^{\infty} (\pm)^{n+1} n^{-1} n_i^{\,n} \tag{52c}$$

Equation (52c) which gives the imaginary part of the effective action provides us with the general connection we were looking for between the "transmission coefficient approach" and the "effective lagrangian" one.

When the individual transmission coefficients n_i are small, one will have $2 \operatorname{Im} W = \langle N \rangle$. In the case of uniform parallel electric and magnetic fields the transmission coefficients are exactly known (see Eqs. (35) and (36)) and a simple calculation shows that Eq. (52c) reduces to Eq. (40).

Finally the fluctuations (19) are easily obtained from (48).

V APPLICATIONS TO BLACK HOLES PHYSICS

A black hole provides us with a stationary external field (gravitational and electromagnetic) in which we can study the behaviour of various quantum particles. According to the approach used in I above pair creation will occur if a "negative" state has in fact a "positive" frequency as seen from infinity (where we detect the particles see Fig. 11). In W.K.B. approximation we have a well defined wave 4-vector k = d (phase) which fullfills:

$$(k - \varepsilon A)^2 = - \mu^2 \tag{53}$$

where A is the electric 4-potential and where the square is understood with respect to the background metric (signature +2).

Denoting by ξ_t the Killing vector expressing the stationarity of the spacetime (with $\xi_t^2 = -1$ at infinity) the frequency as seen from infinity is given by the scalar product:

$$\omega = - \xi_t \cdot k \tag{54}$$

Equation (53) is equivalent to:

$$k = \varepsilon A \pm \mu . u \tag{55}$$

where u is a unit future directed vector, (except when $\mu \to 0$ in which case $\mu . u$ is a null future directed vector). By comparison with Eq. (2) one sees that in the W.K.B. approximation the sign ± in Eq. (55) determines whether we are dealing with a "positive" or a "negative" state, therefore the frequency of a "negative" state as seen from infinity is:

$$\omega^- = \xi_t \cdot (\mu . u - \varepsilon A) \tag{56}$$

So that if there exists a region in space ("effective ergosphere" [19]) where μu can be chosen such that:

$$\omega^- > \mu \tag{57}$$

spontaneous pair creation will occur.

In the case of an uncharged black hole (A = 0) and taking $\mu \to o$ this condition is:

$$\xi_t . \ell > o \tag{58}$$

where ℓ is a future directed null vector. And this can be satisfied in a region where ξ_t ceases to be timelike for becoming space-like. Such a region, called ergosphere, exists outside the horizon for a rotating black hole. The crucial importance of the ergosphere for the energetics of blackholes was first noticed by Penrose and Floyd [20]. The corresponding pair creation process was predicted by Zel'dovich [21] and studied by Starobinsky [22] and Unruh [7] for massless fields. A detailed analysis of this process for massive particles has been given by Deruelle and Ruffini [23-25] using the generalization [19] to black holes physics of the one dimensional "effective potential" which was used in I and III above to compute the transmission coefficient from "negative" to "positive" states. Inequality (57) can be satisfied around a charged blackhole implying pair creation as predicted by Gibbons and Hawking [26] and by Zaumen [27] and studied by Zaumen [27] and Gibbons [10]. A detailed treatment of this phenomenon using the above quoted "effective potential" approach has been given by Nakamura and Sato [28]. The more general case of a charged rotating black hole has been studied by Damour and Ruffini [29] with special emphasis on the possible astrophysical consequences [29-31] of the pair creation process [32].

Finally even in the case of an uncharged non-rotating vacuum black hole the condition (58) will be satisfied inside the horizon where ξ_t becomes space-like. Therefore one foresees the possibility of spontaneous particle creation by a Schwarzschild black hole. Actually this effect has been discovered by Hawking [8, 9] (see also Boulware [33] and Gerlach [34]). But the proof makes an explicit use of the time-dependent phase (collapse) leading to the formation of the hole. It is therefore interesting to see how it is possible to generalize the purely static barrier penetration treatment to this case. See below and [40].

Let us now define more precisely the "effective potential" for a particle of mass μ and charge ε moving around a Kerr-Newman hole of mass M, specific angular momentum a and charge e. The background geometry is:

$$ds^2 = \Sigma[\Delta^{-1} dr^2 + d\theta^2] + \Sigma^{-1} \sin^2\theta \, [(r^2 + a^2)d\phi - a\, dt]^2 - \Sigma^{-1} \Delta \, [dt - a \sin^2\theta \, d\phi]^2 \tag{59a}$$

with $\Delta = r^2 - 2Mr + a^2 + e^2$ and $\Sigma = r^2 + a^2 \cos^2\theta$ and the background electromagnetic potential is

$$A = - er\Sigma^{-1} (dt - a \sin^2\theta \, d\phi) \tag{59b}$$

As was shown by Carter [35] the corresponding Hamilton-Jacobi equation, that is the W.K.B. approximation (53) with $k_\alpha = S_{,\alpha}$, is separable. Therefore choosing the action (that is, the phase) S as:

$$S = - \omega t + m\phi + \Theta(\theta) + R(z) \tag{60a}$$

where $dz = - dr/\Delta$ (60b)

one gets $(d\Theta/d\theta)^2 + (m/\sin\theta - a\,\omega \sin\theta)^2 + \mu^2 a^2 \cos^2\theta = K$ (60c)

$$(dR/dz)^2 = - Z \tag{60d}$$

where $Z = \Delta[\mu^2 r^2 + K] - [\omega(r^2 + a^2) - am - e\varepsilon r]^2$ (60e)

K being Carter's fourth constant (35), generalization of the total angular momentum.

According to Eq. (60d) the classical region of accessibility is defined by:

$$Z(r;\omega,m;\ K) \leq o \tag{61}$$

but Z is <u>quadratic</u> in ω therefore the condition (61) can be written as:

$$\omega \geq E_o^+\ (r;\ m;\ K) \tag{62a}$$

or

$$\omega \leq E_o^-\ (r;\ m;\ K) \tag{62b}$$

with

$$E_o^\pm = (r^2 + a^2)^{-1}\ [am + e\varepsilon r \pm [\Delta(\mu^2 r^2 + K)]^{1/2}] \tag{62c}$$

(62a) defines the positive-root states E^+ and (62b) the negative-root states E^- introduced by Christodoulou and Ruffini [19] for their important role with respect to reversible and irreversible transformations of a black hole. It is easy to see that they correspond with the ambiguity of sign of Eq. (55). Some instances of them, displaying the phenomenon of level crossing (57), are represented in Fig. III.

Now let us consider the corresponding quantum problem. For simplicity's sake we will consider a scalar particle of mass μ and charge ε. It must fulfill the Klein-Gordon equation in the external field (59):

$$(\nabla^\alpha - i\varepsilon A^\alpha)(\nabla_\alpha - i\varepsilon A_\alpha)\ \phi = \mu^2\ \phi \tag{63}$$

where the wave function ϕ can be separated [36] as:

$$\phi = e^{i(m\phi - \omega t)}\ \chi(\theta)\ \psi(z) \tag{64a}$$

with [37] $dz = -\ dr/\Delta$ and

$$[-(\sin\theta)^{-1}\ (d/d\theta)\ \sin\theta\ d/d\theta + (m/\sin\theta - a\omega\ \sin\theta)^2 + \mu^2 a^2\ \cos^2\theta]\chi = K\chi \tag{64b}$$

$$d^2\psi/dz^2 = Z\ \psi \tag{64c}$$

where Z is the same expression (60e) as before, K being now understood as an eigenvalue of the Eq. (64b) which is equivalent to the equation for spheroidal harmonics. This fact implies a close correspondence [37] between the quantum motion of the particle, described by Eq. (64c) and its classical limit: Eq. (60d) which is useful, for instance, in the search for resonances. Anyway in the case of level crossing as represented in Fig. III both the neighborhood of the horizon and spatial infinity are asymptotic regions where the local wave length is infinitely smaller than the local length scale and therefore where the W.K.B. ap-

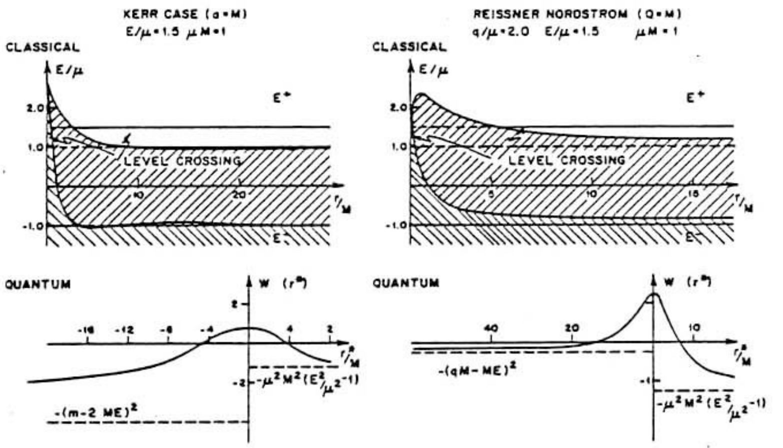

FIGURE III: In direct analogy with the flat-space situation depicted in Fig. II, the neighborhood of a black-hole can exhibit the phenomenon of level crossing between the "negative" and "positive" frequencies and therefore lead to spontaneous real pair creation. The upper part of the figure represents the quasi-classical diagrams of positive and negative-root states [19] in the two cases of a rotating (Kerr) or charged (Reissner-Nordstrom) black-hole. The lower part represents the explicit shape of the quantum potential which determines the behaviour of the wave function (using the coordinate $r_* = \int dr\, r^2/\Delta$). This shows the two asymptotic regions, the horizon ($r_* \to -\infty$) and spatial infinity ($r_* \to +\infty$), connected by a barrier where $W > o$.

proximation becomes exact. Hence, restricting ourselves to the study of pair creation processes occurring outside the horizon, we can apply the analysis of II above, where the choice of positive and negative states is provided by the asymptotic behaviour of the effective potential (see figure III and compare it to figure II). The rate of pair creation will be given by (see eq. (18)):

$$R \equiv \frac{d}{dt} \langle N \rangle = \int \frac{d\omega}{2\pi} \sum_{m,K} |T|^2_{\omega,m,K} \tag{65}$$

where $|T|^2$ is the transmission coefficient between "negative" and "positive" states computed by numerical integration of Eq. (64c) from the horizon to spatial infinity for all the modes (74a) fulfilling the condition for level crossing:

$$\mu < \omega < m\Omega + \epsilon V \tag{66}$$

where Ω is the angular velocity and V the electric potential of the black hole [38]. Inequality (66) is easily obtained by taking the limit on the horizon either of (56) or of (62b).

It is possible too to use Eq. (64c) for studying the quantum resonances around a black hole.

It was found [39] that when the phenomenon of resonance is combined with the one of level crossing one obtains unstable modes growing exponentially with time as a consequence of a continuous pair creation of bosons.

Finally let us consider vacuum polarization around a Kerr-Newmann hole [29] and its possible astrophysical consequences. The key point is, here, to realize that as we will be interested in black holes of current astrophysical interest, that is $M \gtrsim M_\odot$ and in the creation of pairs electrons-positrons, the compton wave length $1/\mu$ of the electron is much smaller than the characteristic dimension M of the hole. Therefore the only region where the behaviour of the particle is non-classical: that is during the tunnelling through the effective potential barrier, is very localized in the background geometry. There is no need for investigating the transmission coefficient by numerical means, a W.K.B. approximation is excellent:

$$\zeta = 2 \int_{\text{barrier}} Z^{\frac{1}{2}} \, dz \tag{67}$$

This can be done and agrees perfectly with what follows; but it is much simpler to exploit thoroughly the locality of the tunnelling by constructing in the neighborhood of any point (t, r, θ, ϕ) a local Lorentz frame. More precisely we can use the orthogonal tetrad defined by Carter [36]:

$$\omega^{(o)} = (\Delta/\Sigma)^{\frac{1}{2}} \, (dt - a \sin^2\theta \, d\phi) \tag{68a}$$

$$\omega^{(1)} = (\Sigma/\Delta)^{\frac{1}{2}} \, dr \tag{68b}$$

$$\omega^{(2)} = \Sigma^{\frac{1}{2}} \, d\theta \tag{68c}$$

$$\omega^{(3)} = \sin\theta \, \Sigma^{-\frac{1}{2}} [(r^2 + a^2) \, d\phi - a \, dt] \tag{68d}$$

In the inertial frame defined by (68) the background electric and the magnetic fields are parallel and directed along $\omega^{(1)}$. Their invariantly defined values are:

$$E_{(1)} = e\Sigma^{-2} \, (r^2 - a^2 \cos^2\theta) \tag{69a}$$

$$B_{(1)} = e\Sigma^{-2}\, 2\, ar \cos\theta \tag{69b}$$

Locally $\mu^{-2} R_{\alpha\beta\gamma\delta}$ and $\mu^{-1} F_{\alpha\beta;\gamma}$ are negligible therefore we can apply the flat space, uniform fields results of III above. This yields a transmission coefficient $|T|^2 = \exp -\zeta$ valid for spin ½, with

$$\zeta = \pi\, \mu^2/\varepsilon E_{(1)} + 2\pi(n + \tfrac{1}{2} + \sigma_{(1)})\, B_{(1)}/E_{(1)} \tag{70}$$

The transmission is then maximum when $n = o$ and $\sigma_{(1)} = -½$. The first condition means that the particles move along the common direction of E and B (in the frame (68)). Going back to the coordinate frame, this implies the following link between the energy and the angular momentum of the particle as seen from infinity:

$$m = a\omega \sin^2\theta \tag{71}$$

Moreover the second condition $\sigma_{(1)} = -½$ shows that the pairs created are polarized, the branching ratio between a spin $\sigma_{(1)} = -½$ and a spin $\sigma_{(1)} = +½$ being $\exp(2\pi\, B_{(1)}/E_{(1)})$.

The mean number of pairs created is given by Eq. (38) which yields:

$$\langle N\rangle = \int g^{½}\, d^4x\, \frac{1}{4\pi} \left(\frac{\varepsilon E_{(1)}}{\pi}\right)^2 \frac{\pi B_{(1)}/E_{(1)}}{th(\pi B_{(1)}/E_{(1)})}\; e^{-\pi\, \mu^2/\varepsilon E_{(1)}} \tag{72}$$

with $g^{½} = \Sigma \sin\theta$. Taking into account the charge ε, energy $\omega = e\varepsilon r/\Sigma$ and angular momentum $m = a\omega \sin^2\theta$ carried away by the particle expelled to infinity ($e\varepsilon > o$) we get explicit estimates of the decrease of charge, mass and angular momentum of the black hole:

$$-\dot{e} = R\varepsilon \tag{73a}$$

$$-\dot{M} = R \langle\omega\rangle \tag{73b}$$

$$-\dot{J} = R \langle m\rangle \tag{73c}$$

where $\langle\omega\rangle$ and $\langle m\rangle$ are suitable spatial mean values of ω and m. $R = d\langle N\rangle/dt$ is the total rate of pair creation.

Let us state some of the results obtained [29]: If the mass of the black hole is larger than $7.2 \times 10^6\, M_\odot$ particle creation processes are never important, but if $M < 7.2 \times 10^6\, M_\odot$ and if the electric field reaches a critical strength $\sim 2 \times 10^{12}$ c.g.s. then an abrupt discharge of the hole via pair creation could occur releasing an energy $\Delta M \sim 5 \times 10^{40}\, (M/M_\odot)^3$ erg with the emission of electrons (or positrons) of energy $\omega \sim 10^{20} (M/M_\odot)$ eV.

The possible astrophysical relevance of this event as well as the conditions for its occurence are discussed in references [29-31]. Let us point out that the process of particle creation studied here always increases the irreducible mass of the black hole [19], in contrast with the phenomenon studied by Hawking. However it has been shown how it is possible to extend the barrier penetration treatment to study the evaporation of the horizon itself, which occurs even for an uncharged, non-rotating black-hole. For technical details see reference [40].

It is first necessary to realize why the preceding treatment excluded from the start the study of the evaporation of the horizon of the black hole: this came from the fact that the modes were defined and studied only in the open region outside the horizon (see e.g. Fig. III which uses a "tortoise" coordinate r_* going to $-\infty$ at the horizon). This situation is easily cured by using a coordinate system regular on the future horizon (the only existing one for a blackhole formed by collapse). This allows to study the behaviour of the modes from the singularity to infinity. As we saw before, the states, locally of negative frequency, inside the horizon can have a positive frequency as seen from infinity, but these states are classically confined in the black hole due to the "one-way" character of the horizon. However, relativistic wave mechanics allows the "leakage" of these states through the light barrier (i.e. the horizon). Physically this comes from the fact, well known in flat space, that a wave packet built only with one half of the frequency spectrum can never have a spatially compact support; therefore the wave functions describing the negative states inside the blackhole will necessarily have a small tail outside the horizon. And this tail, after peeling off the horizon and crossing the potential and centrifugal barrier, will appear at infinity as a positive wave, giving rise as usual to a flux of particles at infinity. (See Fig. IV).

Mathematically the result is very simply obtained by noticing that in Eddington-Finkelstein coordinates $v=t+r_*$, r, θ, ϕ (for simplicity we consider the Schwarzschild case, see [40] for details in the general case) a wave describing an antiparticle inside the horizon with a positive frequency ω at infinity has a logarithmic singularity $(2M-r)^{i4M\omega}$ at the horizon. But, using a small spread $\Delta\omega$ around ω, one can build with these states nearly monochromatic wave packets sharply localized on the horizon (because locally these states have infinitesimal wavelengths). Introducing new coordinates tangent to the old ones but locally geodesic at one point of the horizon we get a localized wave packet with a behaviour $\sim (2M-\hat{r})^{i4M\omega}$ where $\hat{r}$ is the locally geodesic coordinate associated to r. Then one can work as in flat space, Fourier-analyze this packet and apply the well known flat space result that a negative frequency wave is analytically continuable to the complex points of the forward tube (points $x + iy$ where y lies in the forward light cone). As the vector $\partial/\partial r$ is null and past directed in Finkelstein coordinates so is $\partial/\partial\hat{r}$, this implies that our negative wave must be

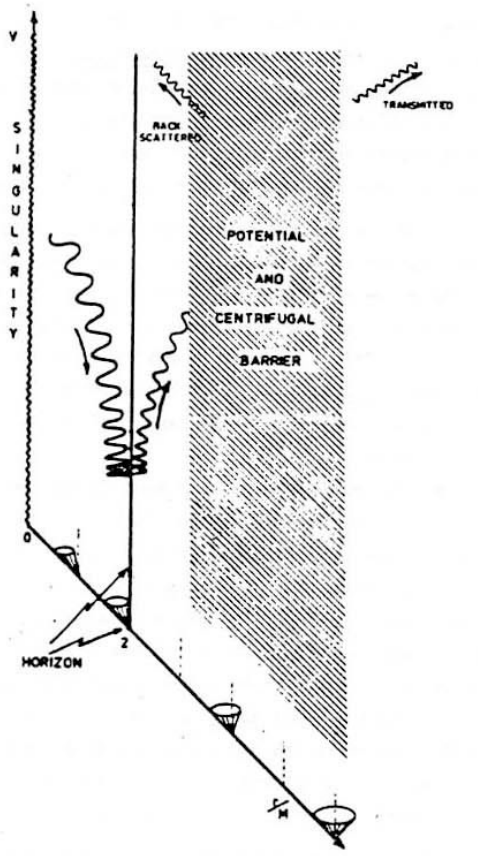

Figure IV: In usual Eddington-Finkelstein coordinates a qualitative representation is given of the splitting of a negative state into its inner component with negative energy flux falling on the singularity and its outer component (tail) peeling off the horizon and partially transmitted to infinity (if$\omega>\mu$) and partially back scattered into the hole (taken from ref. [40]).

analytic in the lower complex $\hat{r}$ plane. This fact tells us how to continue the logarithmic singularity from inside the horizon by an $e^{+i\pi}$ rotation in 2M-r yielding an outside tail term of

$$e^{-4\pi M\omega}(r-2M)^{i4M\omega}$$

necessarily associated to a state inside the horizon. After normalization of this negative state together with its associated tail and taking into account the transmission coefficient Γ of the wave through the combined potential and centrifugal barrier, one gets from formula (18) a continuous flux of particles at infinity. This flux is $(\Gamma/2\pi)\ (e^{8\pi M\omega}-1)^{-1}$ per unit of time and per unit range of frequency (which is Hawking's result [8]).

The same method can be straight forwardly applied to the general case of a Kerr-Newman black hole [40].

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