b. Décharge quantique des trous noirs de Kerr-Newman.

VOLUME 35, NUMBER 7 PHYSICAL REVIEW LETTERS 18 AUGUST 1975

Quantum Electrodynamical Effects in Kerr-Newmann Geometries

Thibaut Damour*
Joseph Henry Physical Laboratories, Princeton University, Princeton, New Jersey 08540

and

Remo Ruffini†
Institute for Advanced Study, Princeton, New Jersey 08540
(Received 13 January 1975)

Following the classical approach of Sauter, of Heisenberg and Euler and of Schwinger the process of vacuum polarization in the field of a "bare" Kerr-Newman geometry is studied. The value of the critical strength of the electromagnetic fields is given together with an analysis of the feedback of the discharge on the geometry. The relevance of this analysis for current astrophysical observations is mentioned.

Deruelle and Ruffini[1] have recently pointed out how the positive- and negative-root solutions[2] of test particles in the background field of a collapsed object have to be identified with the classical limit of the "positive" and "negative" energy solutions of a relativistic quantum field.[3] From this immediately follows that in the ergosphere[4] or in the effective ergosphere[5] of a black hole the conditions are encountered for the possibility of particle creation through the Klein process[6] as analyzed in the framework of a flat-space theory by Sauter,[7] Heisenberg and Euler,[8] and Schwinger.[9] In this Letter we present the results of the analysis of the vacuum polarization of a Kerr-Newman[10] geometry with particular emphasis on (a) the limitations imposed by vacuum polarization on the strength of the electromagnetic field of a black hole,[11] (b) the efficiency of extracting rotational[12] and Coulomb energy[13] from a black hole by pair creation, and (c) the possibility of having observational consequences of astrophysical interest. It is important to stress that the processes here analyzed differ in three major respects from the one recently analyzed by Hawking.[14] The process of particle creation in the present framework (1) does not require a time-varying background geometry, (2) can occur for masses of current astrophysical interest ($M \gtrsim 10^{33}$

g), and (3) always increases the irreducible mass² of the black hole, and therefore the area of the horizon ($S=16\pi M_{ir}^2$).

We study the creation of electrons (or positrons) of mass μ and charge ϵ in the field of a charged rotating black hole described by the Kerr-Newman geometry.[10] The main point here is to realize that at each point we can introduce the following orthogonal tetrad, as defined by Carter[15]:

$$\omega^{(0)}=(\Delta/\Sigma)^{1/2}(dt-a\sin^2\theta d\varphi), \quad (1a)$$

$$\omega^{(1)}=(\Sigma/\Delta)^{1/2}dr, \quad (1b)$$

$$\omega^{(2)}=\Sigma^{1/2}d\theta, \quad (1c)$$

$$\omega^{(3)}=\sin\theta\Sigma^{-1/2}[(r^2+a^2)d\varphi-a\,dt], \quad (1d)$$

where $\Delta=r^2-2Mr+a^2+e^2$ and $\Sigma=r^2+a^2\cos^2\theta$, M being the mass, e the charge, and a the angular momentum per unit mass of the black hole.

In the local Lorentz frame defined by this tetrad the electric and magnetic fields are parallel. Their invariantly defined values, derived from the potential

$$A=-er(\Sigma\Delta)^{-1/2}\omega^{(0)}, \quad (2)$$

are then given by

$$E_{(1)}=e\Sigma^{-2}(r^2-a^2\cos^2\theta) \quad (3a)$$

and

$$B_{(1)}=e\Sigma^{-2}2ar\cos\theta. \quad (3b)$$

The probability of pair creation, in the limit of very small creation rates, has been shown to be proportional to the transmission coefficient $T=\exp(-\zeta)$ of the relativistic electron-positron field between the positive- and negative-energy states. The function ζ is defined to be the opacity of the barrier against pair creation.

If $\mu M\gg1$ it is possible to consider the electric and magnetic fields defined by Eqs. (3) as constant in a neighborhood of a few wavelengths of the point r, θ, φ, t we are considering.

We can then *locally* apply[16] the classical results of Sauter,[7] and Heisenberg and Euler,[8] on uniform and parallel electric and magnetic fields, to evaluate the opacity ζ against the production of electron-positron pairs. We then obtain

$$\zeta=\pi\mu^2/\epsilon E_{(1)}+2\pi(n+\tfrac{1}{2}+\sigma_{(1)})B_{(1)}/E_{(1)}. \quad (4)$$

Here n is the quantum number associated with harmonic oscillation in the plane orthogonal to E and B and $\sigma_{(1)}$ is the spin of the electron in the common direction of E and B. The minimum opacity is then obtained for $n=0$ and $\sigma_{(1)}=-\frac{1}{2}$.

The condition $n=0$ is equivalent to requiring that the created pair have $u_{(2)}=u_{(3)}=0$ which in turn implies physically that *the particles move, in the frame defined by Eqs. (1), along the common direction of the electric and magnetic field.*

This same result can be reinterpreted and given a different physical meaning by going back to the coordinate frame. We then have that the condition $n=0$ is equivalent to requiring that the angular momentum of the particle, $m=\mu u_\varphi+\epsilon A_\varphi$, is linked to its energy, $\omega=-(\mu u_t+\epsilon A_t)$, by the relation

$$m=a\omega\sin^2\theta. \quad (5)$$

Moreover the further dependence of Eq. (4) on σ shows that the pairs created are polarized and the ratio of production of particles with spin $\sigma_{(1)}=-\frac{1}{2}$ and $\sigma_{(1)}=\frac{1}{2}$ is $\exp(2\pi B_{(1)}/E_{(1)})$ which can be very significant if the Kerr-Newman geometry is endowed with a large value of a/M (~0.1).

Finally, the energy of the particle of the pair created is given by

$$\omega=e\epsilon r/\Sigma \quad (6)$$

which implies $\omega\sim10^{20}M/M_\odot$ eV for $M\lesssim10^7M_\odot$ and $\omega\sim10^{27}$ eV for $M\gtrsim10^7M_\odot$.

Still capitalizing on the frame defined by Eqs. (1) we can proceed to compute the rate R of pair creation using Schwinger's approach.[9]* Generalizing the formula given in Ref. 9 to the case in which both an electric and a magnetic field are present[16] we find that the number of pairs created is *

$$N=\int 2\,\mathrm{Im}\mathcal{L}(|g|)^{1/2}d^4x, \quad (7)$$

where $(|g|)^{1/2}=\Sigma\sin\theta$ and

$$2\,\mathrm{Im}\mathcal{L}=(4\pi)^{-1}(E_{(1)}\epsilon/\pi)^2\sum_{n=1}^{\infty}n^{-2}(n\pi B_{(1)}/E_{(1)})\coth(n\pi B_{(1)}/E_{(1)})\exp\{-n\pi\mu^2/\epsilon E_{(1)}\}. \quad (8)$$

The total rate of particles created is simply given by $R=\dot N$.* Since for each pair created the particle (or antiparticle) with the same sign of charge as the background geometry is expelled at infinity with an energy $\omega=\epsilon e r/\Sigma$ and an angular momentum $m=a\omega\sin^2\theta$ and the antiparticle (or particle) with the opposite sign is absorbed by the col-

* pour un résultat plus correct voir l'éq. (72) de l'article précédent(DamourI977a).

lapsed object we can give an explicit estimate of the decrease of charge, mass, and angular momentum of the black hole associated with this process of particle creation. We have

$$-\dot{e} = R\epsilon, \tag{9a}$$

$$-\dot{M} = R\langle\omega\rangle, \tag{9b}$$

$$-\dot{J} = R\langle m\rangle, \tag{9c}$$

where $\langle\omega\rangle$ and $\langle m\rangle$ represent some suitable mean value for the energy and angular momentum carried by the pairs. We then obtain

$$-\dot{e}/e \simeq \pi^{-3}(\epsilon\mu)^2 M_{ir}(E/E_c)^2 \exp-(\pi E_c/E), \tag{10}$$

where $E_c = \mu^2/\epsilon$, $E \simeq e/4M_{ir}^2$, and $(\epsilon\mu)^2\pi^{-3}M_{ir} \simeq 7.0\times10^{32}M_{ir}/M_\odot\ \mathrm{sec}^{-1}$, where M_{ir} is the irreducible mass of the black hole.[2] We can then estimate, corresponding to a $\Delta e = -e$, the change in the total mass energy[2] as well as the changes in irreducible mass and angular momentum of the black hole. We have $-\Delta M/M \simeq e^2/4M_{ir}^2 \simeq 3\times10^{-14}(M_{ir}/M_\odot)^2$ for $M_{ir} \leq 10^7 M_\odot$ and from Eq. (5) follows $\Delta J/J \simeq \frac{2}{3}\Delta M/M$.

Finally, we expect the change of irreducible mass to be relatively small; we do find[16] $\Delta M_{ir} \leq 10^{-2}\Delta M$. It is also interesting to notice that for fields $E \ll E_c$ the pair creation, though much reduced, approaches more and more reversibility.[16]

We can then reach the following conclusions: (1) For any black hole smaller than $7.2\times10^6 M_\odot$ the vacuum polarization and particle-creation processes by the Klein[6] mechanism can drastically modify their electromagnetic structure; as a direct consequence we can never have a magnetic field larger than 2×10^{12} G (assuming the mass of the black hole $M \gtrsim 1.0 M_\odot$).[17] (2) If the electric field of a black hole reaches the critical strength $E \sim \mu^2/(24\epsilon)$ then an abrupt discharge by pair creation could occur releasing an energy $\Delta M \gtrsim 10^{41}$ erg (see Table I). (3) The process of particle creation is an irreversible transformation in the sense of Ref. 2 and carries away both the charge and part of the angular momentum from the black hole (compare and contrast the enormous difference from the case $e=0$, see Ref. 12). The entire treatment presented here applies to the case of a "bare" solution in vacuum. It is important to approach the more general problem of the electrodynamics of a collapsed object sur-

TABLE I: Critical values of the electromagnetic fields and of the energy extractable from a Kerr-Newman geometry as a function of the irreducible mass. B (bare) indicates the limits imposed by the geometrical condition $a^2+e^2 \leq M^2$, P (polarized) indicates the limits imposed by vacuum polarization. Vacuum-polarization effects are important in the absence of plasma only if $M_{ir} \lesssim 7.2\times10^6 M_\odot$.

$M_{ir}/M_\odot$	Maximum strength of electromagnetic field in Gauss (B)	Maximum strength of electromagnetic field in Gauss (P)	Maximum net Charge in Electron Charge (B)	Maximum net Charge in Electron Charge (P)	Maximum energy Extractable in erg (B)	Maximum energy Extractable in erg (P)
1	1.18×10^{19}	2.00×10^{12}	2.14×10^{39}	3.63×10^{32}	1.79×10^{54}	5.15×10^{40}
10^2	1.18×10^{17}	1.88×10^{12}	2.14×10^{41}	3.41×10^{36}	1.79×10^{56}	4.55×10^{46}
10^4	1.18×10^{15}	1.77×10^{12}	2.14×10^{43}	3.21×10^{40}	1.79×10^{58}	4.03×10^{52}
10^6	1.18×10^{13}	1.67×10^{12}	2.14×10^{45}	3.03×10^{44}	1.79×10^{60}	3.59×10^{58}
10^8	1.18×10^{11}	1.18×10^{11}	2.14×10^{47}	2.14×10^{47}	1.79×10^{62}	1.79×10^{62}

rounded by a plasma[13] for two different and opposite reasons: (1) The presence of a plasma can modify the conditions under which vacuum polarization occurs and allow it to reach higher values for the magnetic fields, and (2) one must examine the possibility of a suitable mechanism to build up the electromagnetic field needed for this process of vacuum polarization to occur.[18]

It is by now clear that processes of this kind can be of fundamental importance for the understanding of the physics of binary x-ray sources and possibly of galactic nuclei. In particular this work naturally leads to a most simple model for the explanation of the recently discovered γ-rays bursts.[19] It is desirable that possible coincidences between γ-ray bursts and changes in the spectrum and intensity of x-ray sources of the kind noticed in Cygnus X1 or Cygnus X3 be analyzed with great care.[20]

---

*Jane Eliza Procter Fellow.

†Alfred P. Sloan Fellow.

[1]N. Deruelle and R. Ruffini, Phys. Lett. 52B, 437 (1974).

[2]D. Christodoulou and R. Ruffini, Phys. Rev. D 4, 3552 (1971).

[3]This result has been made manifest by a suitable choice of coordinates by T. Damour, to be published.

[4]See, e.g., M. Rees, R. Ruffini, and J. A. Wheeler, *Black Holes, Gravitational Waves and Cosmology* (Gordon and Breach, New York, 1974).

[5]G. Denardo and R. Ruffini, Phys. Lett. 45B, 259 (1973); G. Denardo, L. Hively, and R. Ruffini, Phys. Lett. 50B, 270 (1974).

[6]O. Klein, Z. Phys. 53, 157 (1929).

[7]F. Sauter, Z. Phys. 69, 742 (1931).

[8]W. Heisenberg and H. Euler, Z. Phys. 98, 714 (1936).

[9]J. Schwinger, Phys. Rev. 82, 664 (1951).

[10]E. T. Newman *et al.*, J. Math. Phys. (N.Y.) 6, 918 (1965).

[11]The analysis of the classical electromagnetic field of a Kerr-Newman geometry has been given by D. Christodoulou and R. Ruffini, in *Black Holes*, edited by B. de Witt and C. de Witt (Gordon and Breach, New York, 1973).

[12]The occurrence of pair creation in a Kerr geometry has been predicted for radiation ($\mu = 0$) with a classical example by Ya. Zel'dovich, Zh. Eksp. Teor. Fiz. 62, 2076 (1972) [Sov. Phys. JETP 35, 1085 (1972)], and has been confirmed for massless fields by A. A. Starobinsky, Zh. Eksp. Teor. Fiz. 64, 48 (1973) [Sov. Phys. JETP 37, 28 (1973)]. The formulation of this problem in the framework of a second-quantized massless field (spin 0 and spin $\frac{1}{2}$) has been given by W. Unruh, University of California at Berkeley Report, 1974 (to be published). The detailed study of this process for pairs of mass $\mu \neq 0$ has been given by N. Deruelle and R. Ruffini (to be published). They have found the transmission coefficient, and, therefore, the rate of pair creation to be significant only if $\mu/\Omega < 1$ (here and in the following $G = c = \hbar = 1$), Ω being the angular velocity of the black hole defined in Ref. 11. The extraction of rotational energy in a Kerr geometry can only be significant then if the black hole is smaller than $\sim 2\times 10^{17}$ g. The energy of the created pair is found to be $\omega \sim \hbar\Omega$.

[13]The vacuum polarization of a Reissner-Nordstrøm geometry has been studied by W. T. Zaumen, Nature (London) 247, 530 (1974), and G. W. Gibbons, Cambridge University Report, 1974 (to be published). The transmission coefficient here introduced generalizes the Gibbons result by giving its explicit radial dependence. Details of this general approach as well as comparison with exact numerical computations have been given by G. Denardo, to be published.

[14]S. Hawking, Nature (London) 248, 30 (1974).

[15]B. Carter, Commun. Math. Phys. 10, 280 (1968).

[16]For details see T. Damour and R. Ruffini, to be published. It is there proved that the local study of ζ here used is justified by a direct analysis of the transmission coefficient in the usual coordinate frame.

[17]This is in agreement with the result obtained by B. Carter, Phys. Rev. Lett. 33, 558 (1974), by using an order-of-magnitude estimate and concluding that the contribution of the electromagnetic field to the geometry can be neglected if $M \lesssim 10^5 M_\odot$. We here point out, however, that the contribution of electromagnetic field to any physical process occurring in the magnetosphere can still be outstanding.

[18]See R. Ruffini and J. Wilson, to be published.

[19]J. B. Strong, R. W. Klebesadel, and R. A. Olson, Astrophys. Lett. 188, L1 (1974).

[20]R. Ruffini, in Proceedings of the Seventh Texas Symposium on Relativistic Astrophysics, Dallas, Texas, 16–20 December 1974 (to be published).

c. Evaporation quantique des trous noirs.

PHYSICAL REVIEW D VOLUME 14, NUMBER 2 15 JULY 1976

Black-hole evaporation in the Klein-Sauter-Heisenberg-Euler formalism

Thibaut Damour* and Remo Ruffini†
Joseph Henry Physical Laboratories, Princeton, New Jersey 08540
and Physics Department, University of Western Australia, Nedlands, Australia
(Received 3 November 1975)

A generalization of the classical approach of barrier penetration introduced by Klein, Sauter, Heisenberg, and Euler to curved spaces endowed with future horizons is given. This technique allows one to recover most directly results obtained by Hawking recently. The treatment here presented encompasses, as special cases, the works of Deruelle and Ruffini, of Damour and Ruffini, and of Nakamura and Sato.

One of the most important results obtained in recent years in black-hole physics has been the realization that the total mass-energy of a black hole can be separated into three components[1]: the irreducible mass, the Coulomb energy, and the rotational energy. That both rotational and Coulomb energy could be in principle extractable by a set of classical gedanken experiments has been known for some time.[2] It has been only recently, however, that the quantum analog of these processes occurring in the "effective ergosphere" have been analyzed.[3] The use of the Klein-Sauter-Heisenberg-Euler formalism has led to a most direct understanding of these processes of vacuum polarization[4] and to detailed analyses of possible astrophysical interest.

Hawking[5] has suggested, however, that, also, by vacuum polarization processes the irreducible mass of a black hole could be radiated away. In the present paper we show how a generalization of our previous treatment of barrier penetration[4] leads to a clear understanding of this phenomenon.

We consider (a) a Kerr-Newman geometry endowed with a vacuum future horizon, (b) a massive charged scalar field Φ fulfilling the covariant Klein-Gordon equation in that background geometry, and (c) we assume analyticity properties of the wave function Φ in the complexified manifold.

The result can be obtained mathematically thanks to the existence of explicit asymptotic expressions for the field Φ near the horizon and at spatial infinity. Physically, it comes from the existence, inside the horizon, of a spacelike Killing vector ξ_t, which allows a classical particle as "seen" from infinity to reach a negative-energy state. In the quantum description, this phenomenon allows an antiparticle to reach positive-energy states. These states, classically confined in the black hole, can be tunneled out by a wave function "over" the horizon which gives rise to the creation of a pair: one particle (positive energy) going out and one antiparticle (negative energy) falling back toward the singularity. Note that this approach only requires the existence of a future horizon and is totally independent of any dynamical details of the process leading to the formation of this horizon.

As usual we consider the Kerr-Newman metric

$$ds^2 = \Sigma(\Delta^{-1} dr^2 + d\theta^2) + \Sigma^{-1}\sin^2\theta[(r^2+a^2)d\phi - a\,dt]^2 - \Sigma^{-1}\Delta(dt - a\sin^2\theta\, d\phi)^2 , \qquad (1)$$

with $\Delta = r^2 - 2Mr + a^2 + e^2 = (r-r_+)(r-r_-)$, where $r_\pm = M \pm (M^2 - a^2 - e^2)^{1/2}$ and $\Sigma = r^2 + a^2\cos^2\theta$, M being the mass, e the charge, and a the specific angular momentum of the black hole. (Here and in the following we choose $G = c = \hbar = 1$.) We also indicate by H_+ the future horizon. Introducing the coordinate r_*,

$$dr_*/dr = (r^2+a^2)/\Delta , \qquad (2a)$$

we have when $r \to r_+$ $(r > r_+)$

$$r_* \sim \frac{1}{2\kappa}\ln(r - r_+),$$

with (2b)

$$\kappa = \frac{1}{2}\,\frac{r_+ - r_-}{r_+^2 + a^2} .$$

We shall first treat the case of a Schwarzschild metric ($a = e = 0$). The scalar function Φ fulfilling the covariant Klein-Gordon equation in this given metric can be separated as

$$\Phi_\omega = (2\pi|\omega| r^2)^{-1/2} E_\omega(r_*, t) Y_l^m(\theta,\phi) , \qquad (3)$$

the Y_l^m being the usual spherical harmonics and E_ω being monochromatic in time. In the following we take $\omega > 0$, that is, a flux of *particles* at infinity, the flux of *antiparticles* being treated as usual by charge conjugation. It is easy to show that just outside the horizon H_+ $(r > 2M)$ two linearly independent solutions exist:

$$E_\omega^{\rm in} = e^{-i\omega(t+r_*)} = e^{-i\omega v} \qquad (4a)$$

and

$$E_\omega^{out}=e^{-i\omega(t-r_*)}=e^{2i\omega r_*}e^{-i\omega v}=(r-2M)^{i4M\omega}e^{-i\omega v},\tag{4b}$$

where we use the usual advanced Eddington-Finkelstein coordinates, $t+r_*=v, r, \theta, \phi$ in which the metric is well behaved and, in fact, analytic over the whole coordinate range $0<r<\infty$, $-\infty<v<\infty$ including H_+ ($r=2M$, $-\infty<v<\infty$).

While Eq. (4a) corresponds to a wave purely ingoing on H_+ and can be extended inside $r<2M$, Eq. (4b) represents an outgoing wave and has an infinite number of oscillations as $r\to 2M$ and therefore cannot be straightforwardly extended to the region inside H_+.[6] We will in the following use and generalize to analytic curved spaces the well-known result of flat-space relativistic wave theories[7,8]: The wave function $\Phi(x)$ describing a particle state (positive frequencies) can be analytically continued to complex points of the form $z=x+iy$ if y lies in the past cone; similarly, for an antiparticle state (negative frequencies) y has to lie in the future cone.

Since in Finkelstein coordinates the vector $\partial/\partial r$ is everywhere null and past-directed, the prescription[9] $r\to r-i0$ will yield the unique continuation of Eq. (4a) describing an *antiparticle* state,

$$\overline{P}_\omega=\overline{N}_\omega\Phi_\omega^{out}(r-2M-i0),\tag{5a}$$

or introducing the Heaviside function Y,

$$\overline{P}_\omega=\overline{N}_\omega[Y(r-2M)\Phi_\omega^{out}(r-2M)+e^{4\pi M\omega}Y(2M-r)\Phi_\omega^{out}(2M-r)],\tag{5b}$$

where $\overline{N}_\omega$ is a normalization factor such that

$$\langle\overline{P}_{\omega_1},\overline{P}_{\omega_2}\rangle=-\delta(\omega_1-\omega_2)\delta_{l_1l_2}\delta_{m_1m_2}.\tag{5c}$$

As Φ_ω was already normalized[10] it is very simple to obtain $\overline{N}_\omega$

$$|\overline{N}_\omega|^2=(e^{8\pi M\omega}-1)^{-1}.\tag{6}$$

Now Eq. (5b) describes the splitting of $\overline{P}_\omega$ in a wave outgoing from the horizon and a wave falling on the singularity (see Fig. 1). The probability flux carried away by this outgoing wave is simply $|\overline{N}_\omega|^2/2\pi$ per unit of time [see Eq. (3)] and only a fraction Γ of this flux will be transmitted to infinity, where Γ is the transmission coefficient of the potential and centrifugal barrier (Fig. 1). Using Eq. (6) we get at infinity an outgoing flux of particles of

$$(\Gamma/2\pi)(e^{8\pi M\omega}-1)^{-1}\tag{7}$$

per unit of time and per unit range of frequency, which is Hawking's result.[5]

If we consider now a scalar field in a Kerr-Newman geometry, it can be shown,[11] using the analogs of the Eddington-Finkelstein coordinates[12] and a corresponding gauge transformation for the electromagnetic field, that the normalized ingoing wave Φ_ω^{in} is regular at H_+ but that Φ_ω^{out} contains a factor $(r-r_+)^{i(\omega-\omega_0)/\kappa}$, where κ is given by Eq. (2b) and where

$$\omega_0=m\Omega+\epsilon V,\tag{8}$$

m being the usual azimuthal quantum number of the particle, ϵ its charge, and Ω and V being

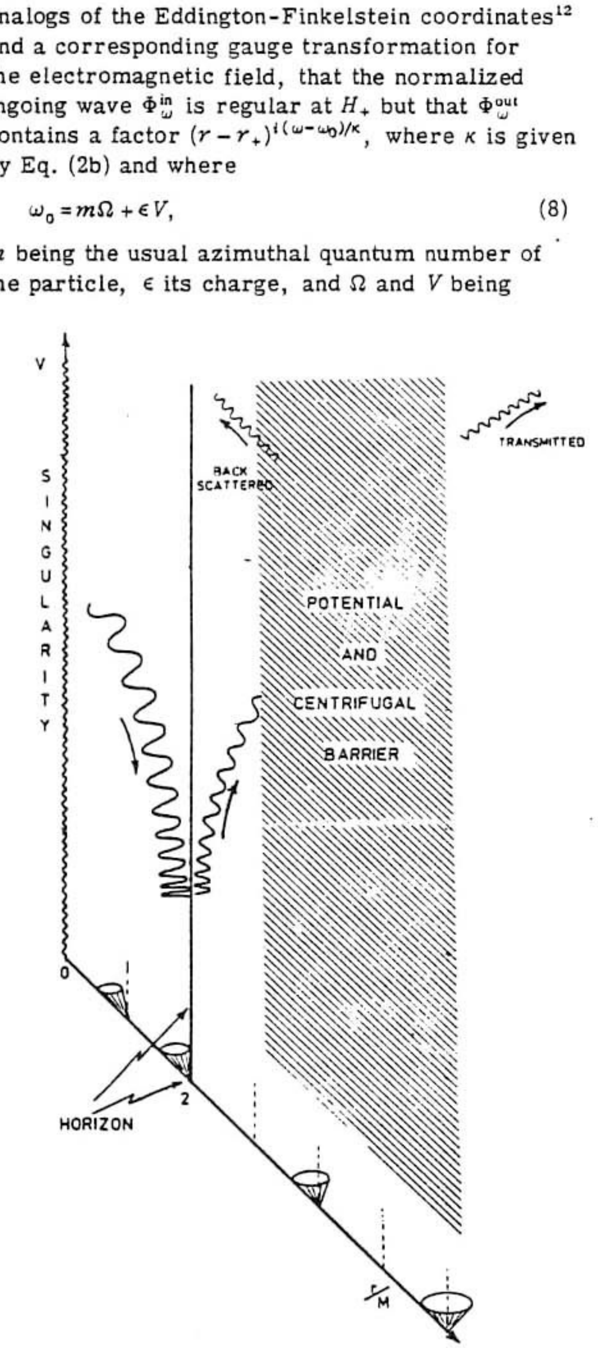

FIG. 1. In usual Eddington-Finkelstein coordinates, a qualitative representation is given of the splitting of the antiparticle state $\overline{P}_\omega$ [see Eq. (5b)] into two components: (a) a particle wave of strength $|\overline{N}_\omega|^2$ outgoing from the horizon and partially transmitted to infinity (if $\omega>\mu$) and partially backscattered into the hole, and (b) an antiparticle wave of strength $|\overline{N}_\omega|^2e^{8\pi M\omega}=1+|\overline{N}_\omega|^2$ with positive energy flux outgoing in the past from the singularity. This can always be interpreted as a negative-energy flux of antiparticles $|\overline{N}_\omega|^2$ ingoing in the future toward the singularity. $\overline{P}_\omega$ being normalized, $|\overline{N}_\omega|^2/2\pi$ yields the rate of pair creation per unit frequency from H_+.

respectively the angular velocity[13] and the electric potential[12] of the black hole. As in these coordinates the vector $\partial/\partial r$ is still null and past-directed we can use the same prescription as before, $r - r - i0$, to describe an antiparticle, which yields the splitting analogous to Eq. (5b):

$$\overline{P}_\omega = \overline{N}_\omega [Y(r - r_+)\Phi_\omega^{\rm out}(r - r_+) + e^{\pi(\omega-\omega_0)/\kappa} Y(r_+ - r)\Phi_\omega^{\rm out}(r_+ - r)] . \quad (9)$$

However, in the present situation of a Kerr-Newman geometry two drastically different situations occur for an energy of the wave $\omega > \omega_0$ or $\omega < \omega_0$.

For $\omega > \omega_0$ the norm of $\Phi_\omega^{\rm out}$ is positive and its flux is $+(2\pi)^{-1}$ so that one gets $|\overline{N}_\omega|^2 = (e^{2\pi(\omega-\omega_0)/\kappa} - 1)^{-1}$. And as usual only a positive fraction Γ of this flux is transmitted to infinity through the combined potential and centrifugal barriers.

For $\omega < \omega_0$ the norm of $\Phi_\omega^{\rm out}$ is negative as well as its flux $-(2\pi)^{-1}$ (antiparticles) so that one gets $|\overline{N}_\omega|^2 = (1 - e^{2\pi(\omega-\omega_0)/\kappa})^{-1}$. But we are precisely in the conditions of level crossing[4] *between* the horizon and spatial infinity so that a *negative* fraction Γ of this flux will be transmitted to infinity.

Therefore in both cases one observes a positive flux at infinity (particles) of

$$(\Gamma/2\pi)(e^{2\pi(\omega-\omega_0)/\kappa} - 1)^{-1} \quad (10)$$

per unit of time and per unit range of frequency.

The drastic difference between those two regimes appears clearly if we consider the limit where the effective temperature $\kappa/2\pi$ of the black hole tends to zero. In the Schwarzschild case the rate of particle creation goes then to zero. In the Kerr-Newman solution the rate goes also to zero if $\omega > \omega_0$, but in the range $\mu < \omega < \omega_0$ (where μ is the mass of the particles) the rate of particle creation tends to $-\Gamma/2\pi$ which is the phenomenon studied in Ref. 4.

If we now take into account that the effective temperature is given by $\kappa/2\pi \simeq 10^{-6} \times (M_\odot/M)$ °K we can conlcude that the only place where thermal contribution to the vacuum polarization of a black hole can exist is in the early stages of cosmology (black holes of $M < 10^{15}$ g). For macroscopic black holes, outcoming from stellar collapse, only the process of vacuum polarization described in Ref. 4 can possibly be observed.

---

*E. S. A. International fellow.

†Alfred P. Sloan fellow.

[1]D. Christodoulou, Phys. Rev. Lett. 25, 1596 (1970); D. Christodoulou and R. Ruffini, Phys. Rev. D 4, 3552 (1971). The result that the irreducible mass of a black hole can never decrease for classical transformations was independently shown by S. Hawking, Phys. Rev. Lett. 6, 1344 (1971).

[2]The extraction of rotational energy has been treated by R. Penrose and R. Floyd, Nature 229, 193 (1971). The extraction of Coulomb energy has been studied by G. Denardo and R. Ruffini, Phys. Lett. 45B, 259 (1973), and by G. Denardo, L. Hively, and R. Ruffini, Phys. Lett. 50B, 270 (1974). The region in which energy extraction processes can occur has been defined as the "effective ergosphere" in Ref. 1.

[3]The pioneering work in this field has been done by Ya. B. Zel'dovich, Zh. Eksp. Teor. Fiz. 62, 2076 (1972) [Sov. Phys.—JETP 35, 1085 (1972)], and A. A. Starobinsky, *ibid.* 64, 48 (1973) [*ibid.* 37, 28 (1973)], for the rotational energy and by G. W. Gibbons and S. Hawking, in *Gravitational Radiation and Gravitational Collapse*, edited by C. DeWitt (Reidel, New York, 1974), and W. T. Zaumen, Nature (London) 247, 530 (1974), for the Coulomb energy. Then the second-quantized formulations of these phenomena were given respectively by W. Unruh, Phys. Rev. D 10, 3194 (1974), and G. Gibbons, Commun. Math. Phys. 44, 245 (1975).

[4]The general outline of this approach is in the work by N. Deruelle and R. Ruffini, Phys. Lett. 52B, 437 (1974). The analysis of vacuum polarization processes were given in a Kerr-Newman geometry by T. Damour and R. Ruffini, Phys. Rev. Lett. 35, 463 (1975); in a Kerr geometry by N. Deruelle and R. Ruffini, Phys. Lett. 57B, 248 (1975); in a Reissner-Nordström geometry by T. Nakamura and H. Sato, Phys. Lett. 61B, 371 (1976).

[5]S. Hawking, Nature (London) 248, 30 (1974); Commun. Math. Phys. 43, 199 (1975).

[6]Some of the properties of this solution have been recently analyzed by D. G. Boulware, Phys. Rev. D 11, 1404 (1974). We disagree with the physical conclusions reached in that article and we explicitly show how our prescription of analytic continuation of the wave function leads to opposite physical conditions.

[7]See, e.g., R. Jost, in *The general theory of quantized fields* (American Mathematical Society, Providence, Rhode Island, 1965), p. 73.

[8]R. Penrose, Int. J. Theor. Phys. 1, 61 (1968), especially pp. 77 and 78.

[9]See, e.g., I. M. Guelfand and G. E. Chilov, *Les distributions* (Dunod, Paris, 1962), p. 43 ff.

[10]The norm is determined as usual by examining broad wave packets.

[11]The computation is similar to the one for the simpler case of a Reissner-Nordström geometry dealt with by G. Gibbons, Commun. Math. Phys. 44, 245 (1975); see p. 257.

[12]See, e.g., B. Carter, in *Black Holes*, edited by B. DeWitt and C. DeWitt (Gordon and Breach, New York, 1973).

[13]D. Christodoulou and R. Ruffini, in *Black Holes* (Ref. 12).

d. Résonances quantiques instables autour d'un trou noir.

LETTERE AL NUOVO CIMENTO VOL. 15, N. 8 21 Febbraio 1976

On Quantum Resonances in Stationary Geometries.

T. DAMOUR (*)

Joseph Henry Physical Laboratories - Princeton, N. J. 08540
Physics Department, University of Western Australia - Nedlands

N. DERUELLE (*)

Institute of Theoretical Astronomy - Cambridge, England

R. RUFFINI (**)

Institute for Advanced Study - Princeton, N. J. 08540
Physics Department. University of Western Australia - Nedlands

(ricevuto il 15 Ottobre 1975)

Detailed analyses have recently been made (1) of relativistic quantized fields in a classical background geometry described by Einstein field equations. Much emphasis has been directed toward the analysis of *a*) resonances of spin-0 and spin-$\frac{1}{2}$ fields in a Schwarzschild background geometry (2), *b*) scalar fields fulfilling the Klein-Gordon equation in a stationary geometry and their classical limits ($\hbar\to 0$, Hamilton-Jacobi equation) (3), *c*) pair creation processes occurring in stationary geometries endowed, as well, with electromagnetic fields (4).

The aim of this letter is to use some of the results presented in ref. (3,4) and show by an explicit example how in a stationary geometry or in the field of a collapsed object endowed with electromagnetic structure resonance states with $\Gamma < 0$ (growing with time) can be found.

The existence of these states is most clear if the effective-potential approach (3,4) is used and has its physical justification in the interplay of the two processes of pair creation (4) and resonance states (2,3) in the field of a collapsed object.

(*) E.S.A. International Fellow.
(**) Alfred P. Sloan Fellow.
(1) See, *e.g.*, *Proceedings of the Marcel Grossman Meeting*, edited by R. RUFFINI (Amsterdam, 1976).
(2) See, *e.g.*, J. A. WHEELER: *Transcending the law of conservation of leptons*, in *Quaderno No.* 157, Accademia Nazionale dei Lincei (Roma, 1971), p. 133.
(3) N. DERUELLE and R. RUFFINI: *Phys. Lett.*, **52** B, 437 (1974); in this paper the positive- and negative-root solutions introduced by D. CHRISTODOULOU and R. RUFFINI: *Phys. Rev. D*, **4**, 3552 (1971), are identified with the classical limits of the positive- and negative-energy states of a relativistic quantized field. See also T. DAMOUR: *Lett. Nuovo Cimento*, **12**, 315 (1975), where this correspondence is made manifest by a suitable choice of the co-ordinates.
(4) See, *e.g.*, T. DAMOUR and R. RUFFINI: *Phys. Rev. Lett.*, **35**, 463 (1975); N. DERUELLE and R. RUFFINI: *Phys. Lett.*, **53** B, (1975) and references mentioned there.

Let us consider a spin-0 boson field Φ of mass μ and charge ε in a Kerr-Newman geometry

$$(\nabla^\alpha - i\varepsilon A^\alpha)(\nabla_\alpha - i\varepsilon A_\alpha)\,\Phi = \mu^2\Phi \tag{1}$$

and

$$\mathrm{d}s^2 = \frac{\Sigma}{\Delta}\,\mathrm{d}r^2 + \Sigma\,\mathrm{d}\theta^2 + \frac{\sin^2\theta}{\Sigma}\,[(r^2+a^2)\,\mathrm{d}\varphi - a\,\mathrm{d}t]^2 - \frac{\Delta}{\Sigma}\,[\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\varphi]^2\,, \tag{2}$$

$$A = -\frac{er}{\Sigma}\,(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\varphi)\,, \tag{3}$$

where $\Delta = r^2 - 2Mr + e^2 + a^2$ and $\Sigma = r^2 + a^2\cos^2\theta$, with M the mass, e the charge and a the specific angular momentum of the background geometry (here and in the following we choose $G = c = \hbar = 1$).

The function Φ in eq. (1) is separable ([5]), we then have

$$\Phi = \psi(r)\,\chi(\theta)\exp\,[i(m\varphi - \omega t)]\,, \tag{4}$$

where $\chi(\theta)$ is expressible as a function of spheroidal harmonics ([6]) and m is the usual azimuthal quantum number.

Introducing a new radial co-ordinate x such that ([4])

$$\mathrm{d}x = [(r_+^2 + a^2)/r_+^2]\,r^2\,\mathrm{d}r/\Delta\,, \tag{5.1}$$

where

$$r_+ = M + (M^2 - a^2 - e^2)^{\frac{1}{2}}\,, \tag{5.2}$$

and such that $r = +\infty$ corresponds to $x = +\infty$ and $r = r_+$ to $x = -\infty$, we have for the radial dependence of the wave function

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = Wu \tag{6.1}$$

with $u = r\psi(r)$ and

$$W = \left(\frac{r_+^2}{r_+^2 + a^2}\right)^2\left\{\frac{\Delta}{r^2}\left[\mu^2 + \frac{K}{r^2} + \frac{2M}{r^3} - \frac{2(a^2+e^2)}{r^4}\right] - \frac{1}{r^4}\,[(r^2+a^2)\,\omega - am - e\varepsilon r]^2\right\}, \tag{6.2}$$

where K is given in the first approximation by ([6])

$$K = l(l+1) - 2ma\omega + a^2\omega^2 + \left[1 - \frac{(2m-1)(2m+1)}{(2l-1)(2l+3)}\right]\frac{\gamma^2}{2} + O(\gamma^4) \tag{6.3}$$

with $\gamma^2 = a^2(\mu^2 - \omega^2)$.

([5]) See, e.g., B. CARTER: *Comm. Math. Phys.*, **10**, 280 (1968); D. BRILL, P. M. CHRZANOWSKI, C. M. PEREIRA, E. D. FACKERELL and J. R. IPSER: *Phys. Rev. D*, **5**, 1913 (1972).

([6]) J. MEIXNER and F. W. SCHÄFKE: *Mathieusche Funktionen und Sphäroidfunktionen* (Berlin, 1954).

We are here interested in analyzing quantum states corresponding to classical circular and elliptical orbits. It is well known [1-3] that the corresponding quantum states are described by resonances of the quantum field of the kind first considered by GAMOW [7] in the classical problem of the α-decay from a nucleus [8].

In sharp contrast with the resonance states studied in ref. [2] we are here interested in resonances which present a level crossing between the positive- and the negative-energy states. We give an explicit example of this new kind of resonances in fig. 1.

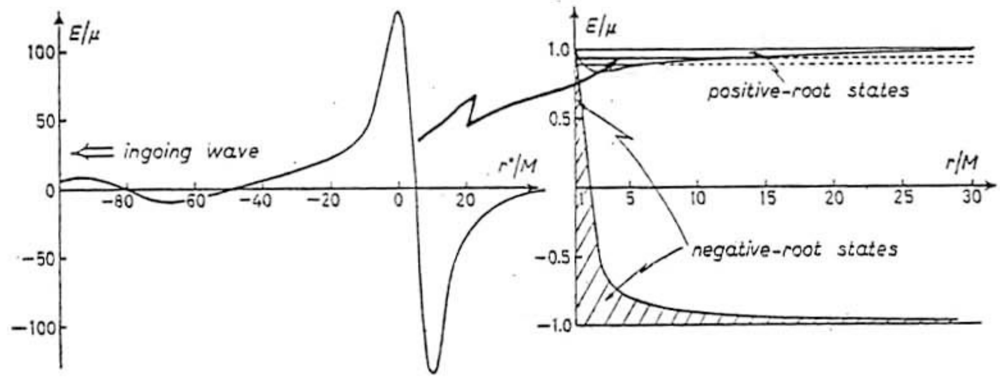

Fig. 1. – Example of resonance with level crossing between the positive- and the negative-energy states in the field of an extreme Kerr black hole with $a = M$. The scalar field is assumed to have $l = m = 2$ and a mass μ such that $\mu M = 1$. The energy of the resonance $E/\mu = 0.930$ corresponds to the first excited state $(1/\pi)\int_b^c (-W)^{\frac{1}{2}}\,\mathrm{d}x = 1.5$. The width of the resonance is $\Gamma/\mu = -4\cdot 10^{-4}$.

We approach the problem by the Gamow method using complex eigenvalues and the WKB approximation. Let us then first indicate by a, b and c the values of the x-co-ordinate corresponding to the zeros of the function $W(x, \omega)$, for a real value of ω, such that

$$-\infty < a < b < c < +\infty\,.$$

We then choose for $x > c$

$$u(x, \omega) = W^{-\frac{1}{4}} \exp\left[-\int_c^x W^{\frac{1}{2}}\,\mathrm{d}x\right] \tag{7}$$

for $x < a$, $u(x, \omega)$ is a mixture of equal-intensity ingoing and outgoing waves, the

---

([7]) G. GAMOW: *Zeits. Phys.*, **51**, 204 (1928). See also G. GAMOW: *Structure of Atomic Nuclei and Nuclear Transformations* (Oxford, 1937).

([8]) There is one important difference between the study of the resonances of a nucleus and the resonances of a quantized field around a collapsed object: the leakage in the case of a nucleus occurs always toward spatial infinity ($r \to \infty$) while in the case of a collapsed object occurs toward the horizon ($r \to r_+$ or $z \to -\infty$).

amplitude of which are complex conjugates:

(8.1) $$u(x,\omega) = \exp[-i\pi/4](-W)^{-\frac{1}{4}}\left[2\cos(I/2)\exp[\zeta/2] + \frac{i}{2}\sin(I/2)\exp[-\zeta/2]\right]\cdot \exp\left[-i\int_a^x(-W)^{\frac{1}{2}}\,\mathrm{d}x\right] + \exp[i\pi/4](-W)^{-\frac{1}{4}}\cdot \left[2\cos(I/2)\exp[\zeta/2] - \frac{i}{2}\sin(I/2)\exp[-\zeta/2]\right]\exp\left[+i\int_a^x(-W)^{\frac{1}{2}}\,\mathrm{d}x\right],$$

where

(8.2) $$I = 2\int_b^c(-W)^{\frac{1}{2}}\,\mathrm{d}x$$

and

(8.3) $$\zeta = 2\int_a^b(W)^{\frac{1}{2}}\,\mathrm{d}x\,.$$

The function $W(x)$ has the asymptotic behaviours

(9.1) $$\lim_{x\to+\infty} W(x) = \left(\frac{r_+^2}{r_+^2+a^2}\right)^2(\mu^2-\omega^2)$$

and

(9.2) $$\lim_{x\to-\infty} W(x) = -(\omega - m\Omega - \varepsilon V)^2\,,$$

where we have indicated by (9) $\Omega = a/(r_+^2+a^2)$ the angular velocity of the black hole and by (10) V its electric potential $V = er_+/(r_+^2+a^2)$.

Analytically continuing now the functions in eqs. (7) and (8) in the complex plane (11), we consider a complex eigenvalue

(10.1) $$\omega = \omega_0 - i\Gamma/2$$

with the following additional conditions

(10.2) $$\omega_0 \gg \Gamma\,,$$

(10.3) $$\omega_0 < \mu\,,$$

(10.4) $$\omega_0 < m\Omega + \varepsilon V\,.$$

(9) D. CHRISTODOULOU and R. RUFFINI: *On the electrodynamics of collapsed objects*, in *Black Holes*, edited by B. DE WITT and C. DE WITT (London, 1973).
(10) B. CARTER: in *Black Holes*, edited by B. DE WITT and C. DE WITT (London, 1973).
(11) See, e.g., G. BREIT and F. L. YOST: *Phys. Rev.*, 48, 203 (1935).

Equation (10.4) guarantees the condition of the level crossing between the positive- and the negative-energy solutions at resonance. This allows in the classical Gamow way (7) one to consider the entire spatial and time evolution of the resonance. However it is important to stress that the boundary conditions at the horizon are drastically influenced by the existence of the level crossing. We have to impose a *physically* ingoing wave at the horizon which corresponds here to a stream of antiparticles having a group velocity directed towards the hole. Therefore we have to keep in eq. (8.1) only the term in

$$\exp\left[+i\int_a^x(-W)^{\frac{1}{2}}\,\mathrm{d}x\right]\sim\exp\left[-i(\omega-m\Omega-\varepsilon V)x\right],$$

that is we must impose

$$2\cos(I/2)\exp[\zeta/2]+\frac{i}{2}\sin(I/2)\exp[-\zeta/2]=0\,. \tag{11}$$

But this stream of antiparticles carry a *negative flux* out of the potential well and therefore the conservation of flux implies that the function inside the potential well must be growing with time. This is easily checked by expanding eq. (11) which yields both the resonance condition

$$I(\omega_0)=(n+\tfrac{1}{2})\,2\pi\,, \tag{12.1}$$

where n is a positive or null integer, and

$$\Gamma=-\left[\exp[\zeta(\omega_0)]\,\frac{\mathrm{d}I(\omega_0)}{\mathrm{d}\omega_0}\right]^{-1}. \tag{12.2}$$

One can check straightforwardly that $\mathrm{d}I/\mathrm{d}\omega_0>0$.

In all the resonances having level crossing between the positive- and negative-energy states, we then have

$$\Gamma<0\,. \tag{13}$$

If following the Gamow approach we consider not only the time dependence of the resonance but also the space dependence implied by the presence of an imaginary component in the eigenvalue, we conclude that the eigenfunction is exponentially decreasing both for $x\to+\infty$ and $x\to-\infty$.

The radial dependence of the wave function at $x\to-\infty$ will, in fact, contain a factor $\exp[-i\omega x]=\exp[-i\omega_0x-(\Gamma/2)x]$. The absence of the usual divergence in the spatial dependence of the eigenfunction (7) should be interpreted as a direct consequence of the fact that the amplitude of the resonance goes to zero when $t\to-\infty$.

The exponential decrease of the function as $x\to-\infty$ also implies that the norm of the function Φ, defined by the conserved current, is convergent and is identically zero. Since $\Phi\to0$ as $t\to-\infty$, this implies that an equal number of particles and antiparticles have been created from the vacuum. Let us stress that, if eq. (10.4) is not fulfilled and if

$$\omega_0 > m\Omega + \varepsilon V \,, \tag{14.1}$$

then the resonant state will have its usual behaviour with a $\Gamma > 0$. If however

$$\omega_0 = m\Omega + \varepsilon V \,, \tag{14.2}$$

it is possible to have a resonance state with $\Gamma = 0$. This limiting case has as boundary condition at the horizon a wave function which tends to a constant value, which corresponds to a running wave.

It is possible, as usual (11), to build spatially bounded wave packets reproducing this wave till a cut-off in space which moves with the speed of light towards the horizon.

e- Correspondence entre les mouvements classiques et les états quantiques autour d'un trou noir.

LETTERE AL NUOVO CIMENTO VOL. 12, N. 9 1 Marzo 1975

On the Correspondence between Classical and Quantum Energy States in Stationary Geometries.

T. DAMOUR (*)

Joseph Henry Physical Laboratories, Princeton University - Princeton, N. J.

(ricevuto il 18 Dicembre 1974)

CHRISTODOULOU and RUFFINI (1) have pointed out the existence of positive- and negative-root solutions for a test particle moving in the field of a Kerr solution and their very important role with respect to reversible and irreversible transformations. Recently DERUELLE and RUFFINI (2) have pointed out that these positive- and negative-root states should be considered as the « classical » limit of the « positive » and « negative » solutions of the Klein-Gordon equation.

The aim of this letter is to point out that these results can be made manifestly clear by the consideration of a suitable choice of the co-ordinate system. This co-ordinate system however presents some disadvantages to carry out concrete computations.

The Hamiltonian-Jacobi equation for a particle of mass μ and charge ε is

$$-g^{\alpha\beta}(\partial_\alpha S-\varepsilon A_\alpha)(\partial_\beta S-\varepsilon A_\beta)-\mu^2=0\,. \tag{1}$$

In a Kerr-Newman field we have

$$\mathrm{d}s^2=\frac{\varrho^2}{\Delta}\mathrm{d}r^2+\varrho^2\mathrm{d}\theta^2+\frac{\sin^2\theta}{\varrho^2}[(r^2+a^2)\,\mathrm{d}\varphi-a\,\mathrm{d}t]^2-\frac{\Delta}{\varrho^2}[\mathrm{d}t-a\sin^2\theta\,\mathrm{d}\varphi]^2\,, \tag{2a}$$

$$A=-\frac{er}{\varrho^2}(\mathrm{d}t-a\sin^2\theta\,\mathrm{d}\varphi)\,, \tag{2b}$$

where $\Delta=r^2-2Mr+a^2+e^2$ and $\varrho^2=r^2+a^2\cos^2\theta$, M being the mass of the black hole, e its charge and a its angular momentum per unit mass.

Then one finds

$$-\Delta\left(\frac{\partial S}{\partial r}\right)^2+\frac{1}{\Delta}\left[(r^2+a^2)\frac{\partial S}{\partial t}+a\frac{\partial S}{\partial\varphi}+e\varepsilon r\right]^2-\mu^2r^2-\\-\left(\frac{\partial S}{\partial\theta}\right)^2-\left(\frac{1}{\sin\theta}\frac{\partial S}{\partial\varphi}+a\sin\theta\frac{\partial S}{\partial t}\right)^2-\mu^2a^2\cos^2\theta=0\,. \tag{3}$$

(*) Jane Eliza Procter Fellow.

(1) D. CHRISTODOULOU and R. RUFFINI: *Phys. Rev. D*, 4, 3552 (1971).

(2) N. DERUELLE and R. RUFFINI: *Phys. Lett.*, in press.

φ and t being kinosthenic, the separation is simple [3] and yields

$$S = R(r) + \Theta(\theta) + m\varphi - \omega t\,, \tag{4a}$$

$$\Delta\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)^2 - \frac{1}{\Delta}\left[(r^2 + a^2)\,\omega - am - e\varepsilon r\right]^2 + \mu^2 r^2 = -Q - (m - a\omega)^2\,, \tag{4b}$$

$$\left(\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right)^2 + \left(\frac{m}{\sin\theta} - a\omega\sin\theta\right)^2 + \mu^2 a^2\cos^2\theta = Q + (m - a\omega)^2\,, \tag{4c}$$

where Q is Carter's [3] constant, null for an equatorial motion.

One may write

$$\Delta^2\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)^2 = P(r;\,\omega;\,m;\,Q) \tag{5}$$

with

$$P(r;\,\omega;\,m;\,Q) \equiv \left[(r^2 + a^2)\,\omega - am - e\varepsilon r\right]^2 - \Delta\left[Q + (m - a\omega)^2 + \mu^2 r^2\right]. \tag{6}$$

The problem of the classical motion is thus reduced to the knowledge of the function $P(r;\,\omega;\,m;\,Q)$. In particular the accessible states are confined to the regions of the space $(r;\,\omega;\,m;\,Q)$ where $P \geq 0$, that is to say to the positive- and negative-root states of CHRISTODOULOU and RUFFINI [1]. The region where $P < 0$ is forbidden. Moreover the negative-root states are classically rejected.

For a quantum charged scalar particle the Klein-Gordon operator will be

$$(\nabla^\alpha - i\varepsilon A^\alpha)(\nabla_\alpha - i\varepsilon A_\alpha) - \mu^2\,, \tag{7}$$

which yields in the Kerr-Newman field

$$\partial_r \Delta \partial_r - \frac{1}{\Delta}\left[(r^2 + a^2)\,\partial_t + a\partial_\varphi + ie\varepsilon r\right]^2 - \mu^2 r^2 + \\ + \frac{1}{\sin\theta}\,\partial_\theta \sin\theta\,\partial_\theta + \left(\frac{1}{\sin\theta}\,\partial_\varphi + a\sin\theta\,\partial_t\right)^2 - \mu^2 a^2\cos^2\theta\,. \tag{8}$$

The separation is always possible [4]:

$$\varphi = \psi(r)\,\chi(\theta)\exp\left[i(m\varphi - \omega t)\right]; \tag{9}$$

it is sufficient to take for χ a spheroidal harmonic [5] $ps_l^m(\cos\theta;\,\gamma^2)$ solution of

$$\frac{1}{\sin\theta}\,\partial_\theta \sin\theta\,\partial_\theta y + \left[\lambda_l^m + \gamma^2\sin^2\theta - \frac{m^2}{\sin^2\theta}\right]y = 0\,. \tag{10a}$$

---

(3) B. CARTER: *Phys. Rev.*, **174**, 1559 (1968).
(4) B. CARTER: *Commun. Math. Phys.*, **10**, 280 (1968).
(5) J. MEIXNER and F. W. SCHÄFKE: *Mathieusche Funktionen und sphäroidfunktionen* (Berlin, 1954).

One must take

$$\gamma^2 = a^2(\mu^2 - \omega^2)\,, \tag{11}$$

then λ_l^m is a function of γ^2. If we define

$$Q_l^m(\gamma^2) = \lambda_l^m(\gamma^2) + \gamma^2 - m^2\,, \tag{12}$$

one has

$$Q_l^m(\gamma^2) = l(l+1) - m^2 + \frac{1}{2}\left[1 - \frac{(2m-1)(2m+1)}{(2l-1)(2l+3)}\right]\gamma^2 + O(\gamma^4)\,, \tag{13}$$

and the radial equation is

$$0 = \partial_r \Delta \partial_r \psi(r) + \left\{\frac{1}{\Delta}[(r^2 + a^2)\omega - am - e\varepsilon r]^2 - [Q_l^m(\gamma^2) + (a\omega - m)^2 + \mu^2 r^2]\right\}\psi(r)\,, \tag{10b}$$

or

$$0 = \Delta\partial_r \Delta \partial_r \psi(r) + P(r;\omega;m;Q_l^m(\gamma^2))\,\psi(r)\,, \tag{14}$$

P being the same function (6) as in the classical case. But eq. (14) has not the Schrödinger form. It is always possible to give it this form by a suitable change of ψ, but this alters the «effective potential» P when working with the variables r or r^* (6). Yet the new radial co-ordinate z defined by (7)

$$\mathrm{d}z = -\frac{\mathrm{d}r}{\Delta} \quad \text{or} \quad z = \int_r^\infty \frac{\mathrm{d}r}{\Delta} \tag{15}$$

yields immediately

$$\left\{\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^2 + P(r(z);\omega;m;Q_l^m(\gamma^2))\right\}\psi = 0 \tag{16a}$$

in the quantum case, and

$$\left(\frac{\mathrm{d}R}{\mathrm{d}z}\right)^2 = P(r(z);\omega;m;Q) \tag{16b}$$

in the classical case.

Now (16a) is a Schrödinger equation whose «effective potential» P is exactly the classical corresponding one (except for the dependence of Q on $\gamma^2 = a^2(\mu^2 - \omega^2)$). In fact (16b) is the WKB approximation of (16a). In order to study more closely this

---

(6) D. Brill, P. M. Chrzanowski, C. M. Pereira, E. D. Fackerell and J. R. Isper: *Phys. Rev. D*, **5**, 1913 (1972).

(7) One may also use a homographic function of z at the cost of a change of the function ψ. A similar transformation may be used too for θ. Let us notice that the z-transformation squeezes the infinity horizon to $z = 0$ and sends the black-hole horizon to $z = \infty$, contrarily to the r^*-transformation which sends the horizons to $r^* = +\infty$, and $r^* = -\infty$, which is generally better for the sake of numerical integration.

correspondence, let us introduce the new variables

$$(17)\qquad \begin{cases} \hat{r} = r/M\,, & \hat{a} = a/M\,, \quad \hat{e} = e/M\,, \quad \hat{z} = Mz\,, \\ \hat{\omega} = \omega/\mu\,, & \hat{\varepsilon} = \varepsilon/\mu\,, \\ \hat{m} = m/M\mu\,, & \hat{Q}_l^m = Q_l^m/(M\mu)^2\,. \end{cases}$$

Suppressing the carets one obtains

$$(18a)\qquad \left[\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^2 + (M\mu)^2 P(r(z);\,\omega;\,m;\,Q_l^m)\right]\psi(z) = 0\,,$$

$$(18b)\qquad \left(\frac{\mathrm{d}R}{\mathrm{d}z}\right)^2 = (M\mu)^2 P(r(z);\,\omega;\,m;\,Q)\,.$$

Now it is manifest that each resonance of (18*a*) corresponds necessarily to a classically bound state (with turning points $z = \alpha$ and $z = \beta$, where $P = 0$). But while the classical states form a continuum in the variable ω, the quantum states must satisfy a resonance condition which is in WKB approximation

$$(19)\qquad M\mu\int_\alpha^\beta \sqrt{P(r(z);\,\omega;\,m;\,Q_l^m)}\,\mathrm{d}z = (n + \tfrac{1}{2})\pi\,.$$

So that in the classical limit $M\mu \to \infty$ it is clear, accordingly with the results of DERUELLE and RUFFINI (²), that the separation of the energy levels of the resonances tends to zero (as $(M\mu)^{-1}$) and that the exponential in the forbidden region ($P < 0$) decreases very rapidly $\left(\text{as } \exp\left[-M\mu\int_\beta^\gamma \sqrt{-p}\,\mathrm{d}z\right]\right)$. Hence, in the limit $M\mu \to \infty$ the resonance states of the particle are in fact confined to the classical region of accessibility $P \geqslant 0$, but now it is possible to give a meaning to the negative-root states (²).

* * *

It is a pleasure to thank Dr. R. RUFFINI for interesting discussions.

T. DAMOUR
1 Marzo 1975
Lettere al Nuovo Cimento
Serie 2, Vol. **12**, pag. 315-318