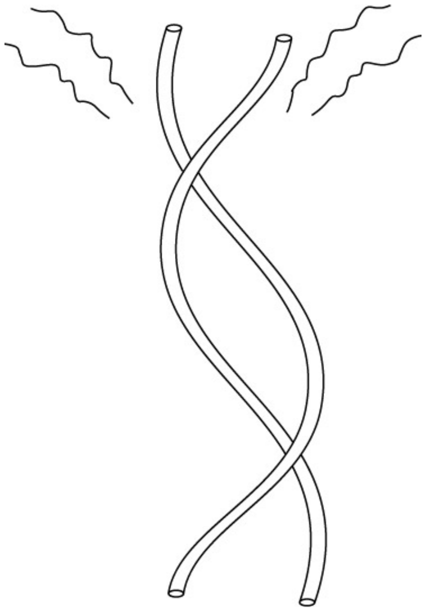
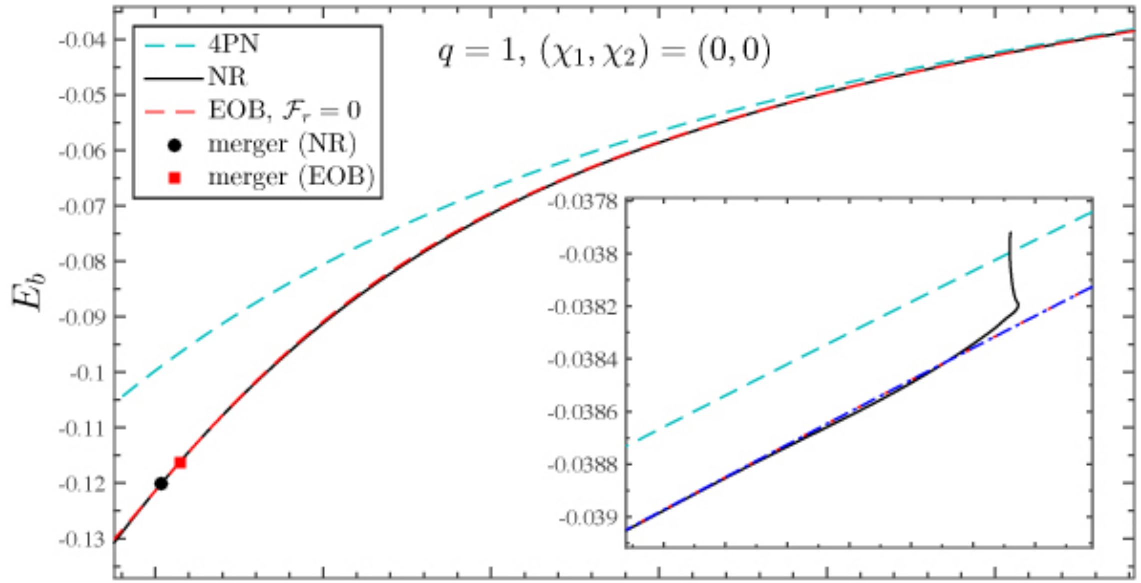
# The Problem of Motion in General Relativity


*Thibault Damour*

*Institut des Hautes Etudes Scientifiques (Bures-sur-Yvette, France)*



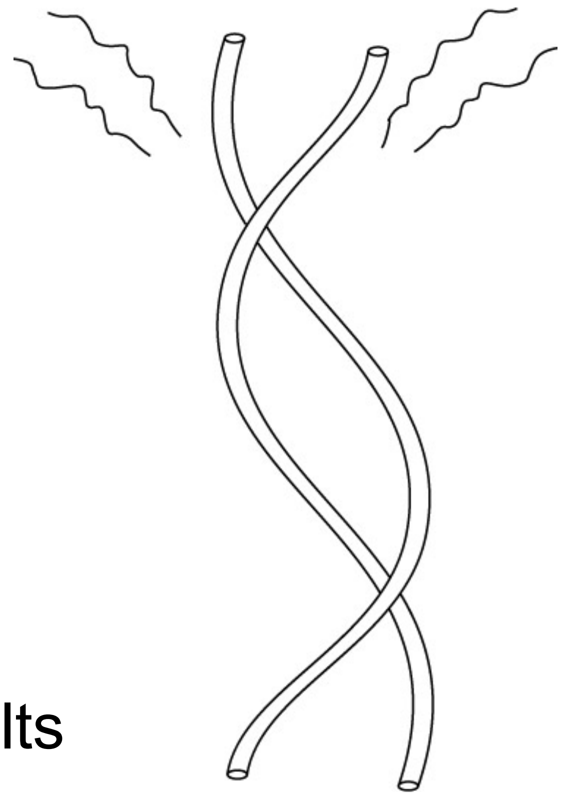

International Conference Relativity and Geometry
in Memory of André Lichnerowicz,
December 14-16, 2015, IHP, Paris

# The Problem of Motion in General Relativity

- Solve

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

or $R_{\mu\nu} = 0$



- Taking into account radiation damping effects

- Extract physical (gauge-invariant) results

# Early History of the Problem of Motion

Einstein 1912 : geodesic principle $-\int m\sqrt{-g_{\mu\nu}\, dx^\mu\, dx^\nu}$

Einstein 1913-1916 post-Minkowskian $g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)\ ,\ h_{\mu\nu} \ll 1$

Einstein, Droste : post-Newtonian $h_{00} \sim h_{ij} \sim \frac{v^2}{c^2}\ ,\ h_{0i} \sim \frac{v^3}{c^3}\ ,\ \partial_0 h \sim \frac{v}{c}\partial_i h$

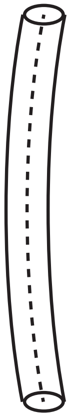
Mercury's perihelion : Einstein-Besso 1913, Droste 1914, Einstein 1915

N-body problem in PN theory: Droste 1916, De Sitter 1916, [Lorentz-Droste 1917], Chazy 1928,1930, Levi-Civita 1937

# Early History : Various Approaches

- Einstein-Grossmann ’13 : $\nabla_u\, u^\mu = 0 \,+\, T^{\mu\nu} = \rho\, u^\mu u^\nu \Rightarrow \nabla_\nu\, T^{\mu\nu} = 0$

- Post-Newtonian: 1916 in Leiden [with p = 0, “rigid spheres”]:
  Droste, Lorentz, Einstein (visiting Leiden), De Sitter ;
  Lorentz-Droste ‘17, Chazy ‘28, Levi-Civita ’37 ….,



- Later: $\nabla_\nu\, T^{\mu\nu} = 0 \quad ; \quad T^{\mu\nu} = \rho'\, u^\mu u^\nu + p\, g^{\mu\nu} \Rightarrow \nabla_u\, u^\mu = O(\nabla p)$
  Eddington’ 21, …, Lichnerowicz ‘39, Fock ‘39, Papapetrou ‘51,
  … Dixon ‘64, Bailey-Israël ‘75, Ehlers-Rudolph ‘77….

- Fitting a worldtube containing a singularity
  Weyl ’21, Einstein-Grommer ’27, Einstein-Infeld-Hoffman ‘38,…

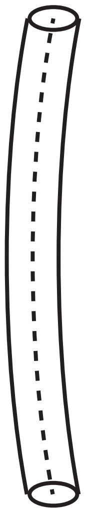
- δ-function source: Mathisson’31,…, Infeld ’54

- Regularization / Renormalization issues for point particles
  World-tubes : Dirac ’38, Bhabha-Harish-Chandra ‘44, …
  Analytic continuation: Riesz ‘39, Fremberg ‘46, Ma ‘47, …

# Lichnerowicz and the Problem of Motion

1939 " Problèmes Globaux" " conditions de raccordement "
Continuation: from the inside towards the outside, or from the outside to the inside
"la frontière S doit être dans ce cas [p = 0] engendrée par des géodésiques du $ds^2$ extérieur."

"Filling" ("meubler") a worldtube:
A fillable worldtube necessarily contains singularities of the exterior field
A static, filled 2-body spacetime necessarily contains singularities somewhere



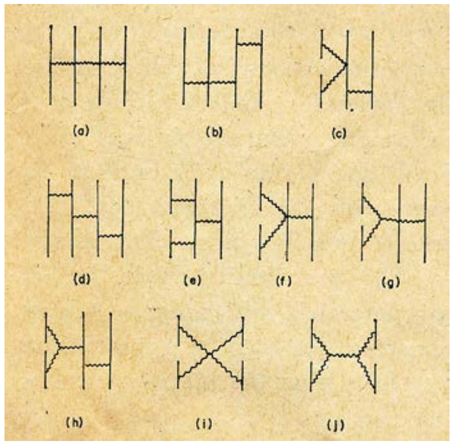
1943 "L'intégration des équations de la gravitation relativiste et le problème des n-corps"
" Beaucoup d'auteurs, entre autres de Sitter et Levi-Civita, se sont occupés du problème des n corps en relativité générale. Mais ils se sont toujours bornés à la recherche de solutions approchées et les approximations faites apparaissent parfois, notamment chez de Sitter, comme plus ou moins arbitraires. Je me propose ici de construire un exemple de données de Cauchy compatible avec le problème des n corps [$K = 0$, $g_{ij} = \varphi^4 \delta_{ij}$].
A de telles données correspondra une solution *rigoureuse* de ce problème, dont l'évolution dans le temps sera régie par les équations [d'évolution] et pourra être obtenue par une intégration numérique de ces équations."

Quite useful today in Numerical Relativity

# Later History : New Tools

- Post-Minkowskian: Bertotti ’56, Havas’57, Kerr’59, Bertotti-Plebanski’60, Havas-Goldberg’62, Schmutzer ‘66, …, Thorne-Kovacs ‘75, … , Rosenblum ’78, Westpfahl-Göller ‘79, Bel et al. ‘81, Damour ‘82, Ledvinka-Schäfer-Bicak ‘08
- ADM Hamiltonian formalism: Kimura ‘61 [1PN], Ohta-Okamura-Kimura-Hiida ‘74 [2PN], Schäfer ‘85 [2.5PN], Jaranowski-Schäfer ‘98 [3PN],….

- Reduced Action à la Fokker ‘29: Infeld ’57, Infeld-Plebanski ’60,




- Quantum S-matrix ➔ potential

one-graviton exchange : Corinaldesi ’56 ‘71, Barker-Gupta-Haracz 66, Barker-O’Connell 70, Hiida-Okamura 72

Nonlinear: Iwasaki 7 [1PN], Okamura-Ohta-Kimura-Hiida 73[2 PN]

# 1974-: Discovery of the Hulse-Taylor Binary Pulsar

Many entangled issues and a motivation for improving the methods

- need to reach the radiation-damping level 2.5 PN = (v/c)^5
- need an approach valid for strongly-self gravitating bodies
- need to tackle IR divergent integrals in PN schemes
- need to tackle UV divergent integrals when using δ-sources
- need to include spin effects
- need to derive gauge-invariant, observable effects
- need a relativistic timing formula

## 1979-1983: The “Quadrupole Controversy”

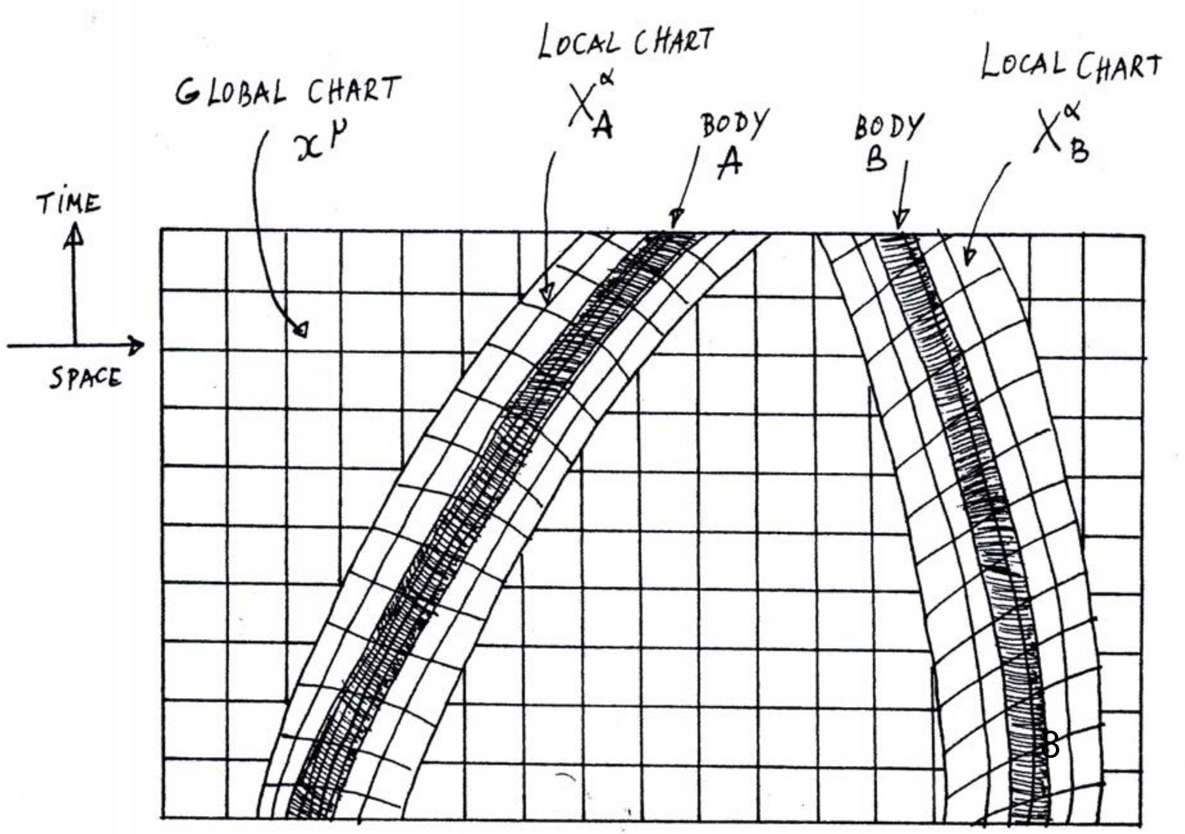
A flurry of (sometimes conflicting) results

- Old results: Eddington, Infeld, Peres, Carmeli, Havas, Hu, Burke, Chandrasekhar, ….
- New results: Walker-Will 80, Kerlick 80, Schutz 80, Breuer-Rudolph 81, Papapetrou-Linet 81, McCrea 81, Rosenblum 81, Damour-Deruelle 81-82, Futamase 83, Schäfer 85, Kopeikin 85, …..

## New Techniques in the Problem of Motion (1)

- Multi-chart approach and matched asymptotic expansions: necessary for strongly self-gravitating bodies (NS, BH) Manasse (Wheeler) ‘63, Demianski-Grishchuk ‘74, D’Eath ‘75, Kates ‘80, Damour ‘82

Useful even for weakly self-gravitating bodies, i.e.“relativistic celestial mechanics”, Brumberg-Kopeikin ’89, Damour-Soffel-Xu ‘91-94




Lack of rigorous mathemathical results

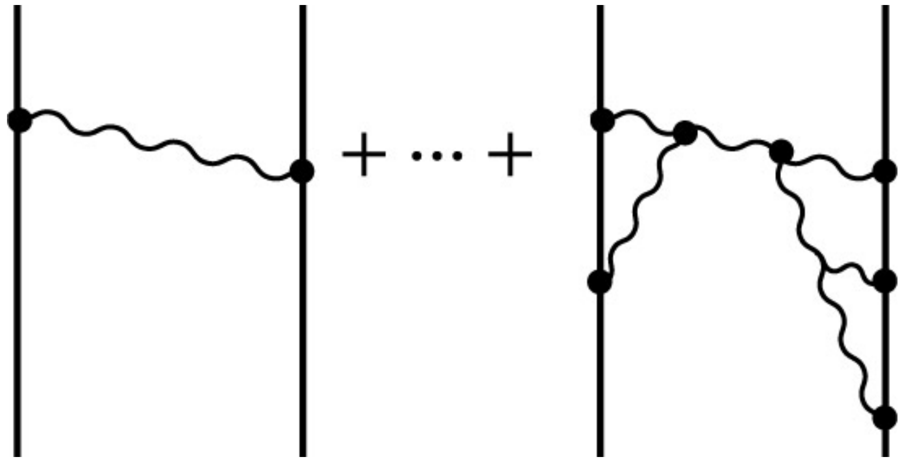
## New Techniques in the Problem of Motion (2)

Skeletonization : $T_{\mu\nu} \longrightarrow$ point-masses

delta-functions in GR : Infeld '54, Infeld-Plebanski '60,…, Eardley '75

QFT's dimensional regularization (Bollini-Giambiagi '72, t'Hooft-Veltman '72)

imported in GR (Damour '80, Damour-Jaranowski-Schäfer '01, …)



Feynman-like diagrams and

« Effective Field Theory » techniques

Bertotti-Plebanski '60,

Damour-Esposito-Farèse '96,

Goldberger-Rothstein '06, Porto '06, Gilmore-Ross' 08, Levi '10,

Foffa-Sturani '11 '13, Levi-Steinhoff '14, '15

# Post-Newtonian Equations of Motion [2-body, wo spins]

- 1PN (including $v^2/c^2$) Lorentz-Droste '17, EIH '38, Eddington-Clark '38
- 2PN (inc. $v^4/c^4$) Ohta-Okamura-Kimura-Hiida '74, Damour-Deruelle '81
  Damour '82, Schäfer '85, Kopeikin '85
- 2.5 PN (inc. $v^5/c^5$) Damour-Deruelle '81, Damour '82, Schäfer '85,
  Kopeikin '85
- 3 PN (inc. $v^6/c^6$) Jaranowski-Schäfer '98, Blanchet-Faye '00,
  Damour-Jaranowski-Schäfer '01, Itoh-Futamase '03,
  Blanchet-Damour-Esposito-Farèse' 04, Foffa-Sturani '11
- 3.5 PN (inc. $v^7/c^7$) Iyer-Will '93, Jaranowski-Schäfer '97, Pati-Will '02,
  Königsdörffer-Faye-Schäfer '03, Nissanke-Blanchet '05, Itoh '09
- 4PN (inc. $v^8/c^8$) Jaranowski-Schäfer '13, Foffa-Sturani '13,
  Bini-Damour '13, Damour-Jaranowski-Schäfer '14

New feature : non-locality in time

[Recent piecewize confirmations of 4PN: Bini-Damour-Geralico 15,
Hopper-Kavanagh-Ottewill 15, Bernard et al 15, Akcay-vandeMeent 15,…]

# 2-body Taylor-expanded N + 1PN + 2PN Hamiltonian

$$H_{\mathrm{N}}(\mathbf{x}_a, \mathbf{p}_a) = \frac{\mathbf{p}_1^2}{2m_1} - \frac{1}{2}\frac{Gm_1 m_2}{r_{12}} + (1 \leftrightarrow 2)$$

$$c^2 H_{1\mathrm{PN}}(\mathbf{x}_a, \mathbf{p}_a) = -\frac{1}{8}\frac{(\mathbf{p}_1^2)^2}{m_1^3} + \frac{1}{8}\frac{Gm_1 m_2}{r_{12}}\left(-12\frac{\mathbf{p}_1^2}{m_1^2} + 14\frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1 m_2} + 2\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1 m_2}\right)$$
$$+ \frac{1}{4}\frac{Gm_1 m_2}{r_{12}}\frac{G(m_1 + m_2)}{r_{12}} + (1 \leftrightarrow 2),$$

$$c^4 H_{2\mathrm{PN}}(\mathbf{x}_a, \mathbf{p}_a) = \frac{1}{16}\frac{(\mathbf{p}_1^2)^3}{m_1^5} + \frac{1}{8}\frac{Gm_1 m_2}{r_{12}}\left(5\frac{(\mathbf{p}_1^2)^2}{m_1^4} - \frac{11}{2}\frac{\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2 m_2^2} - \frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{m_1^2 m_2^2} + 5\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2 m_2^2}\right.$$
$$\left. - 6\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2 m_2^2} - \frac{3}{2}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2 m_2^2}\right)$$
$$+ \frac{1}{4}\frac{G^2 m_1 m_2}{r_{12}^2}\left(m_2\left(10\frac{\mathbf{p}_1^2}{m_1^2} + 19\frac{\mathbf{p}_2^2}{m_2^2}\right) - \frac{1}{2}(m_1 + m_2)\frac{27(\mathbf{p}_1\cdot\mathbf{p}_2) + 6(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1 m_2}\right)$$
$$- \frac{1}{8}\frac{Gm_1 m_2}{r_{12}}\frac{G^2(m_1^2 + 5m_1 m_2 + m_2^2)}{r_{12}^2} + (1 \leftrightarrow 2),$$

# 2-body Taylor-expanded 3PN Hamiltonian [JS 98, DJS 01]

$$
\begin{aligned}
c^6 H_{3\mathrm{PN}}(\mathbf{x}_a,\mathbf{p}_a) = &-\frac{5}{128}\frac{(\mathbf{p}_1^2)^4}{m_1^7} + \frac{1}{32}\frac{Gm_1m_2}{r_{12}}\Bigg(-14\frac{(\mathbf{p}_1^2)^3}{m_1^6} + 4\frac{((\mathbf{p}_1\cdot\mathbf{p}_2)^2+4\mathbf{p}_1^2\mathbf{p}_2^2)\mathbf{p}_1^2}{m_1^4m_2^2} + 6\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^4m_2^2}\\
&-10\frac{(\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2+\mathbf{p}_2^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2)\mathbf{p}_1^2}{m_1^4m_2^2} + 24\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^4m_2^2}\\
&+2\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3m_2^3} + \frac{(7\mathbf{p}_1^2\mathbf{p}_2^2-10(\mathbf{p}_1\cdot\mathbf{p}_2)^2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3m_2^3}\\
&+\frac{(\mathbf{p}_1^2\mathbf{p}_2^2-2(\mathbf{p}_1\cdot\mathbf{p}_2)^2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2^3} + 15\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3m_2^3}\\
&-18\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3m_2^3} + 5\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3m_2^3}\Bigg) + \frac{G^2m_1m_2}{r_{12}^2}\Bigg(\frac{1}{16}(m_1-27m_2)\frac{(\mathbf{p}_1^2)^2}{m_1^4}\\
&-\frac{115}{16}m_1\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2} + \frac{1}{48}m_2\frac{25(\mathbf{p}_1\cdot\mathbf{p}_2)^2+371\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2m_2^2} + \frac{17}{16}\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^3} + \frac{5}{12}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{m_1^3}\\
&-\frac{1}{8}m_1\frac{(15\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)+11(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1))(\mathbf{n}_{12}\cdot\mathbf{p}_1)}{m_1^3m_2} - \frac{3}{2}m_1\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3m_2}\\
&+\frac{125}{12}m_2\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2m_2^2} + \frac{10}{3}m_2\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\\
&-\frac{1}{48}(220m_1+193m_2)\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\Bigg) + \frac{G^3m_1m_2}{r_{12}^3}\Bigg(-\frac{1}{48}\left(425m_1^2+\left(473-\frac{3}{4}\pi^2\right)m_1m_2+150m_2^2\right)\frac{\mathbf{p}_1^2}{m_1^2}\\
&+\frac{1}{16}\left(77(m_1^2+m_2^2)+\left(143-\frac{1}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2} + \frac{1}{16}\left(20m_1^2-\left(43+\frac{3}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^2}\\
&+\frac{1}{16}\left(21(m_1^2+m_2^2)+\left(119+\frac{3}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\Bigg)\\
&+\frac{1}{8}\frac{G^4m_1m_2^3}{r_{12}^4}\left(\left(\frac{227}{3}-\frac{21}{4}\pi^2\right)m_1+m_2\right) + (1\leftrightarrow 2).
\end{aligned}
$$

# 2-body Taylor-expanded 4PN Hamiltonian [DJS, 2014]

$$
\begin{aligned}
c^8 H^{\rm local}_{\rm 4PN}(\mathbf{x}_a,\mathbf{p}_a) ={}& \frac{7(\mathbf{p}_1^2)^5}{256 m_1^9} + \frac{G m_1 m_2}{r_{12}} H_{48}(\mathbf{x}_a,\mathbf{p}_a) + \frac{G^2 m_1 m_2}{r_{12}^2} m_1 H_{46}(\mathbf{x}_a,\mathbf{p}_a) \\
&+ \frac{G^3 m_1 m_2}{r_{12}^3}\left(m_1^2 H_{441}(\mathbf{x}_a,\mathbf{p}_a) + m_1 m_2 H_{442}(\mathbf{x}_a,\mathbf{p}_a)\right) \\
&+ \frac{G^4 m_1 m_2}{r_{12}^4}\left(m_1^3 H_{421}(\mathbf{x}_a,\mathbf{p}_a) + m_1^2 m_2 H_{422}(\mathbf{x}_a,\mathbf{p}_a)\right) \\
&+ \frac{G^5 m_1 m_2}{r_{12}^5} H_{40}(\mathbf{x}_a,\mathbf{p}_a) + (1\leftrightarrow 2),
\end{aligned}
\tag{A3}
$$

$$
\begin{aligned}
H_{48}(\mathbf{x}_a,\mathbf{p}_a) ={}& \frac{45(\mathbf{p}_1^2)^4}{128m_1^8} - \frac{9(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2}{64m_1^6m_2^2} + \frac{15(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^3}{64m_1^6m_2^2} - \frac{9(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{16m_1^6m_2^2} \\
&- \frac{3(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{32m_1^6m_2^2} + \frac{15(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{64m_1^6m_2^2} - \frac{21(\mathbf{p}_1^2)^3\mathbf{p}_2^2}{64m_1^6m_2^2} - \frac{35(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{256m_1^5m_2^3} \\
&+ \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2}{128m_1^5m_2^3} + \frac{33(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3(\mathbf{p}_1^2)^2}{256m_1^5m_2^3} - \frac{85(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256m_1^5m_2^3} \\
&- \frac{45(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{128m_1^5m_2^3} - \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256m_1^5m_2^3} + \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{64m_1^5m_2^3} \\
&+ \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{64m_1^5m_2^3} - \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{64m_1^5m_2^3} + \frac{3\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{64m_1^5m_2^3} + \frac{55(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256m_1^5m_2^3} \\
&- \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{128m_1^5m_2^3} - \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{256m_1^5m_2^3} - \frac{23(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256m_1^5m_2^3} \\
&+ \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{128m_1^5m_2^3} - \frac{7(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256m_1^5m_2^3} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4\mathbf{p}_1^2}{64m_1^4m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4(\mathbf{p}_1^2)^2}{64m_1^4m_2^4} \\
&- \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{4m_1^4m_2^4} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{16m_1^4m_2^4} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{64m_1^4m_2^4} + \frac{21(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2\mathbf{p}_2^2}{64m_1^4m_2^4} \\
&- \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{32m_1^4m_2^4} - \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{4m_1^4m_2^4} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{16m_1^4m_2^4} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{16m_1^4m_2^4} \\
&- \frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{32m_1^4m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_2^2)^2}{64m_1^4m_2^4} - \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_2^2)^2}{32m_1^4m_2^4} - \frac{7(\mathbf{p}_1^2)^2(\mathbf{p}_2^2)^2}{128m_1^4m_2^4},
\end{aligned}
\tag{A4a}
$$

$$
\begin{aligned}
H_{46}(\mathbf{x}_a,\mathbf{p}_a) ={}& \frac{369(\mathbf{n}_{12}\cdot\mathbf{p}_1)^6}{160m_1^6} - \frac{889(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4\mathbf{p}_1^2}{192m_1^6} + \frac{49(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1^2)^2}{16m_1^6} - \frac{63(\mathbf{p}_1^2)^3}{64m_1^6} - \frac{549(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{128m_1^5m_2} \\
&+ \frac{67(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{16m_1^5m_2} - \frac{167(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2}{128m_1^5m_2} + \frac{1547(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_1\cdot\mathbf{p}_2)}{256m_1^5m_2} - \frac{851(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{128m_1^5m_2} \\
&+ \frac{1099(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256m_1^5m_2} + \frac{3263(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{1280m_1^4m_2^2} + \frac{1067(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{480m_1^4m_2^2} - \frac{4567(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2}{3840m_1^4m_2^2} \\
&- \frac{3571(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{320m_1^4m_2^2} + \frac{3073(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{480m_1^4m_2^2} + \frac{4349(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{1280m_1^4m_2^2} \\
&- \frac{3461\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{3840m_1^4m_2^2} + \frac{1673(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4\mathbf{p}_2^2}{1920m_1^4m_2^2} - \frac{1999(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2\mathbf{p}_2^2}{3840m_1^4m_2^2} + \frac{2081(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{3840m_1^4m_2^2} - \frac{13(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{8m_1^3m_2^3} \\
&+ \frac{191(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2}{192m_1^3m_2^3} - \frac{19(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384m_1^3m_2^3} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384m_1^3m_2^3} \\
&+ \frac{11(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{192m_1^3m_2^3} + \frac{77(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{96m_1^3m_2^3} + \frac{233(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_2^2}{96m_1^3m_2^3} - \frac{47(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{32m_1^3m_2^3} \\
&+ \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{384m_1^3m_2^3} - \frac{185\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{384m_1^3m_2^3} - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4}{4m_1^2m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4\mathbf{p}_1^2}{4m_1^2m_2^4} \\
&- \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3(\mathbf{p}_1\cdot\mathbf{p}_2)}{2m_1^2m_2^4} + \frac{21(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{16m_1^2m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{6m_1^2m_2^4} + \frac{49(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2\mathbf{p}_2^2}{48m_1^2m_2^4} \\
&- \frac{133(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{24m_1^2m_2^4} - \frac{77(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{96m_1^2m_2^4} + \frac{197(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_2^2)^2}{96m_1^2m_2^4} - \frac{173\mathbf{p}_1^2(\mathbf{p}_2^2)^2}{48m_1^2m_2^4} + \frac{13(\mathbf{p}_2^2)^3}{8m_2^6},
\end{aligned}
\tag{A4b}
$$

$$
\begin{aligned}
H_{441}(\mathbf{x}_a,\mathbf{p}_a) ={}& \frac{5027(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{384m_1^4} - \frac{22993(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2}{960m_1^4} - \frac{6695(\mathbf{p}_1^2)^2}{1152m_1^4} - \frac{3191(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{640m_1^3m_2} \\
&+ \frac{28561(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{1920m_1^3m_2} + \frac{8777(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384m_1^3m_2} + \frac{752969\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{28800m_1^3m_2} \\
&- \frac{16481(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{960m_1^2m_2^2} + \frac{94433(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{4800m_1^2m_2^2} - \frac{103957(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{2400m_1^2m_2^2} \\
&+ \frac{791(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{400m_1^2m_2^2} + \frac{26627(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_2^2}{1600m_1^2m_2^2} - \frac{118261\mathbf{p}_1^2\mathbf{p}_2^2}{4800m_1^2m_2^2} + \frac{105(\mathbf{p}_2^2)^2}{32m_2^4},
\end{aligned}
\tag{A4c}
$$

$$
\begin{aligned}
H_{442}(\mathbf{x}_a,\mathbf{p}_a) ={}& \left(\frac{2749\pi^2}{8192} - \frac{211189}{19200}\right)\frac{(\mathbf{p}_1^2)^2}{m_1^4} + \left(\frac{63347}{1600} - \frac{1059\pi^2}{1024}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2}{m_1^4} + \left(\frac{375\pi^2}{8192} - \frac{23533}{1280}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{m_1^4} \\
&+ \left(\frac{10631\pi^2}{8192} - \frac{1918349}{57600}\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{m_1^2m_2^2} + \left(\frac{13723\pi^2}{16384} - \frac{2492417}{57600}\right)\frac{\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2m_2^2} \\
&+ \left(\frac{1411429}{19200} - \frac{1059\pi^2}{512}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{m_1^2m_2^2} + \left(\frac{248991}{6400} - \frac{6153\pi^2}{2048}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^2m_2^2} \\
&- \left(\frac{30383}{960} + \frac{36405\pi^2}{16384}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2} + \left(\frac{1243717}{14400} - \frac{40483\pi^2}{16384}\right)\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2} \\
&+ \left(\frac{2369}{60} + \frac{35655\pi^2}{16384}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3m_2} + \left(\frac{43101\pi^2}{16384} - \frac{391711}{6400}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{m_1^3m_2} \\
&+ \left(\frac{56955\pi^2}{16384} - \frac{1646983}{19200}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2},
\end{aligned}
\tag{A4d}
$$

$$
H_{421}(\mathbf{x}_a,\mathbf{p}_a) = \frac{64861\mathbf{p}_1^2}{4800m_1^2} - \frac{91(\mathbf{p}_1\cdot\mathbf{p}_2)}{8m_1m_2} + \frac{105\mathbf{p}_2^2}{32m_2^2} - \frac{9841(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{1600m_1^2} - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{2m_1m_2},
\tag{A4e}
$$

$$
\begin{aligned}
H_{422}(\mathbf{x}_a,\mathbf{p}_a) ={}& \left(\frac{1937033}{57600} - \frac{199177\pi^2}{49152}\right)\frac{\mathbf{p}_1^2}{m_1^2} + \left(\frac{176033\pi^2}{24576} - \frac{2864917}{57600}\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2} + \left(\frac{282361}{19200} - \frac{21837\pi^2}{8192}\right)\frac{\mathbf{p}_2^2}{m_2^2} \\
&+ \left(\frac{698723}{19200} + \frac{21745\pi^2}{16384}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^2} + \left(\frac{63641\pi^2}{24576} - \frac{2712013}{19200}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2} \\
&+ \left(\frac{3200179}{57600} - \frac{28691\pi^2}{24576}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_2^2},
\end{aligned}
\tag{A4f}
$$

$$
H_{40}(\mathbf{x}_a,\mathbf{p}_a) = -\frac{m_1^4}{16} + \left(\frac{6237\pi^2}{1024} - \frac{169799}{2400}\right)m_1^3m_2 + \left(\frac{44825\pi^2}{6144} - \frac{609427}{7200}\right)m_1^2m_2^2.
\tag{A4g}
$$

$$
\begin{aligned}
H^{\rm nonloc}_{\rm 4PN}(t) ={}& -\frac{1}{5}\frac{G^2M}{c^8} I_{ij}^{(3)}(t) \\
&\times \mathrm{Pf}_{2r_{12}/c}\int_{-\infty}^{+\infty}\frac{\mathrm{d}v}{|v|} I_{ij}^{(3)}(t+v),
\end{aligned}
$$

# Gravitational Radiation

1916, 1918 Einstein : $h_+$, $h_x$ and quadrupole formula

Relativistic, multipolar extensions of LO quadrupole radiation :

Mathews '62,

Bonnor-Rotenberg '66,

Campbell-Morgan '71,

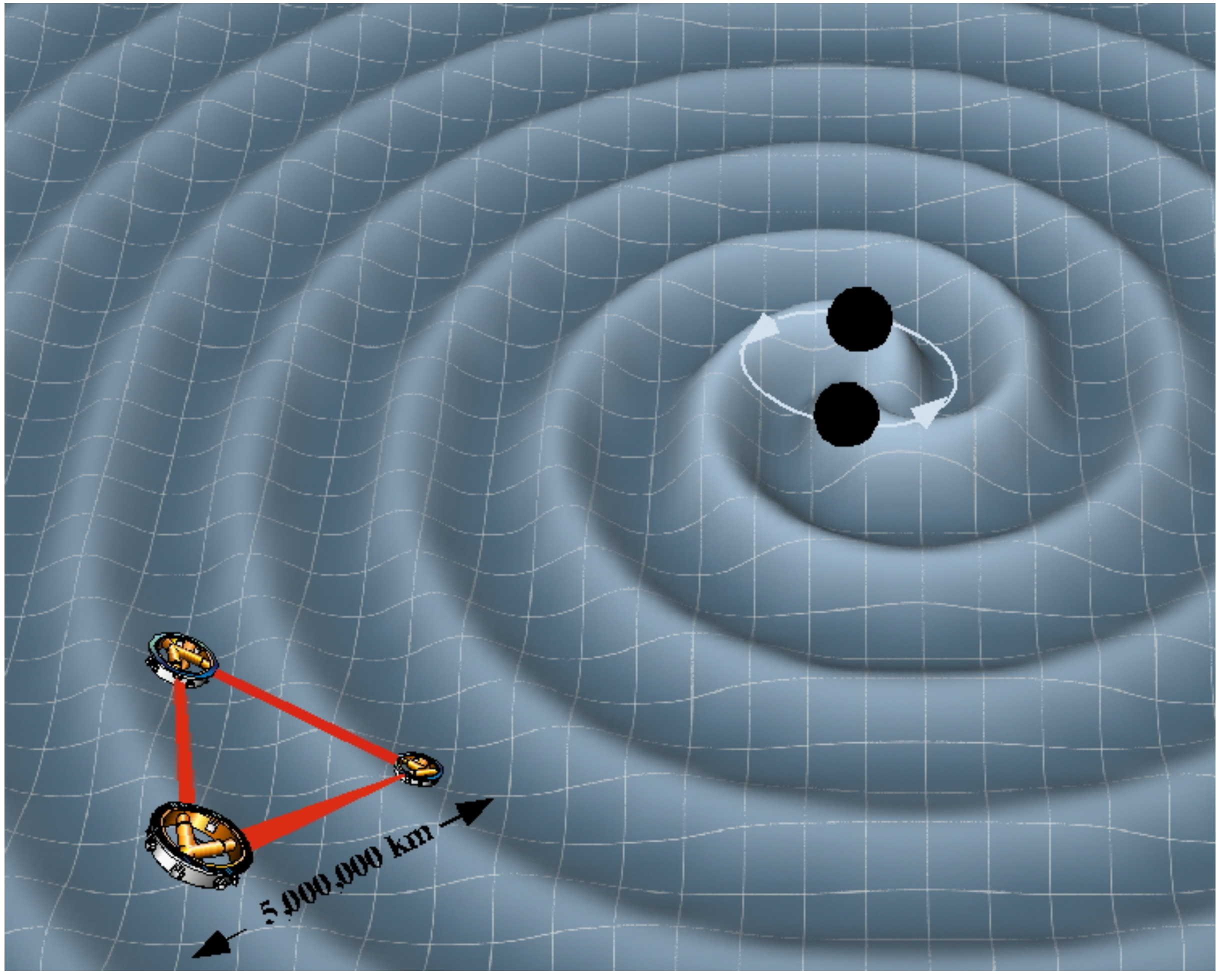
Campbell et al '75,

Thorne '80,

Blanchet-Damour '86,

Damour-Iyer '91,

Will et al '00



# Taylor-expanded 3.5PN waveform

Blanchet,Iyer, Joguet 02, Blanchet, Damour, Esposito-Farese, Iyer 04, Kidder 07, Blanchet et al. 08, Faye et al ’12, Bohé, Marsat, Blanchet, Buonanno 13-15

$$
\begin{aligned}
h^{22} = & -8\sqrt{\frac{\pi}{5}}\frac{G\nu m}{c^2 R}e^{-2i\phi}x\Big\{1 - x\Big(\frac{107}{42} - \frac{55}{42}\nu\Big) + x^{3/2}\Big[2\pi + 6i\ln\Big(\frac{x}{x_0}\Big)\Big] - x^2\Big(\frac{2173}{1512} + \frac{1069}{216}\nu - \frac{2047}{1512}\nu^2\Big) \\
& - x^{5/2}\Big[\Big(\frac{107}{21} - \frac{34}{21}\nu\Big)\pi + 24i\nu + \Big(\frac{107i}{7} - \frac{34i}{7}\nu\Big)\ln\Big(\frac{x}{x_0}\Big)\Big] \\
& + x^3\Big[\frac{27\,027\,409}{646\,800} - \frac{856}{105}\gamma_E + \frac{2}{3}\pi^2 - \frac{1712}{105}\ln 2 - \frac{428}{105}\ln x \\
& - 18\Big[\ln\Big(\frac{x}{x_0}\Big)\Big]^2 - \Big(\frac{278\,185}{33\,264} - \frac{41}{96}\pi^2\Big)\nu - \frac{20\,261}{2772}\nu^2 + \frac{114\,635}{99\,792}\nu^3 + \frac{428i}{105}\pi + 12i\pi\ln\Big(\frac{x}{x_0}\Big)\Big] + O(\epsilon^{7/2})\Big\},
\end{aligned}
$$

$$x = (M\Omega)^{2/3} \sim v^2/c^2$$

$$
\begin{aligned}
M &= m_1 + m_2 \\
\nu &= m_1 m_2/(m_1 + m_2)^2
\end{aligned}
$$

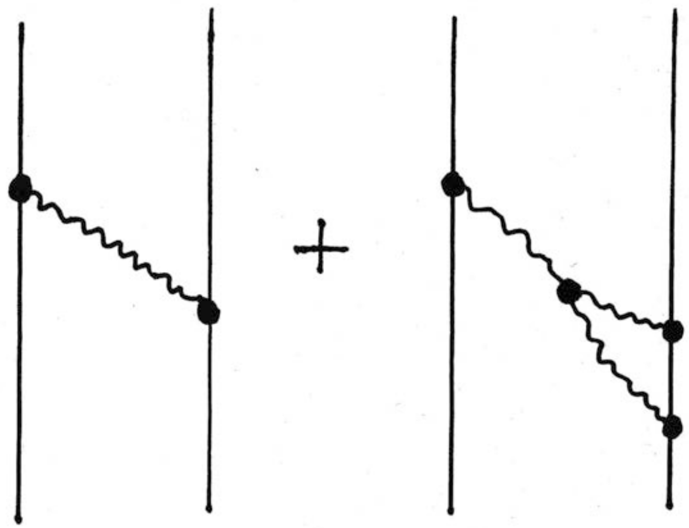
# Gravitational Radiation Damping : Two Views

Balance between the E and J of the system and GW losses at infinity

(Landau-Lifshitz, Fock, Peters-Mathews )

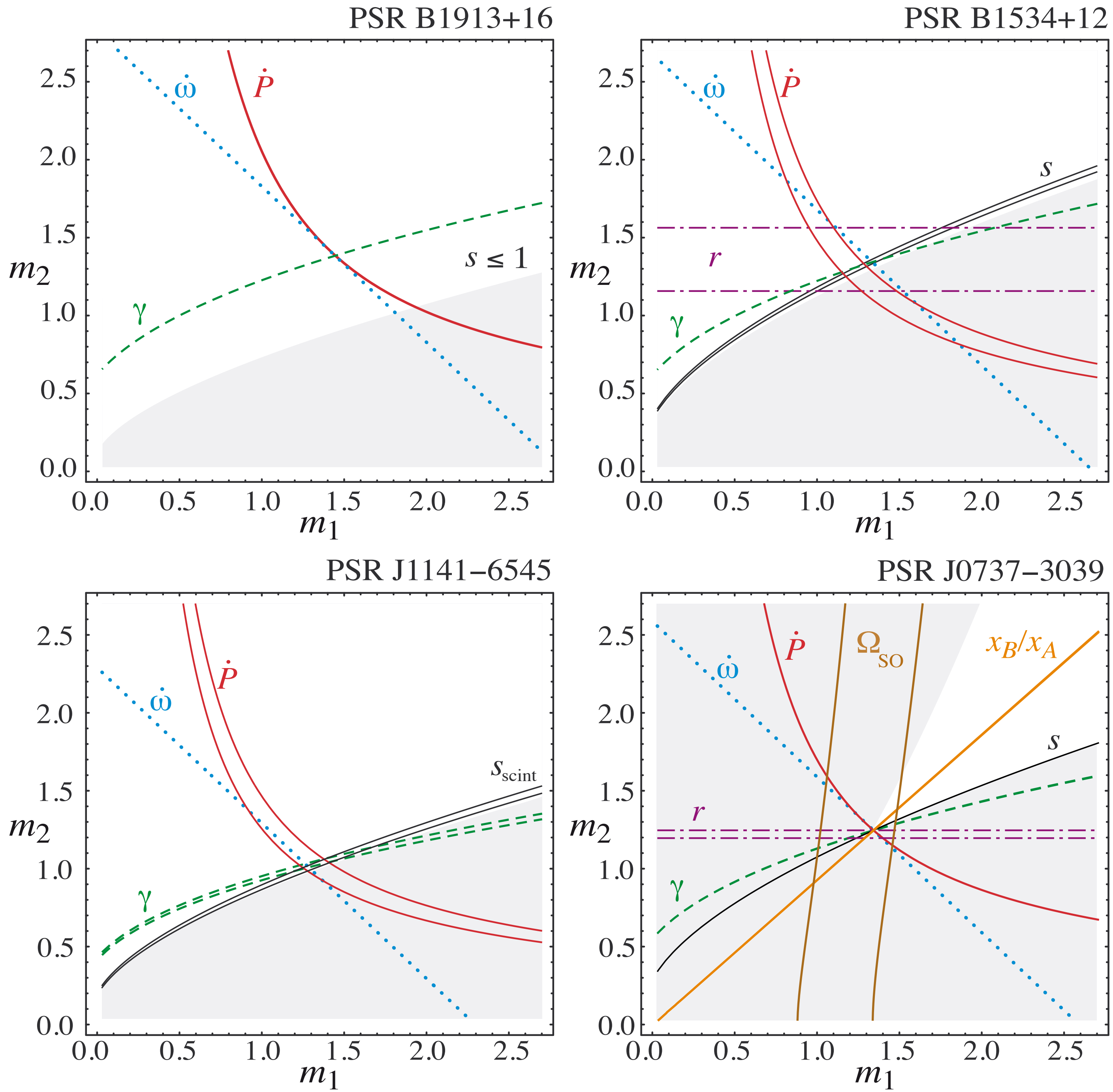
Radiation Reaction : « Laplace-Eddington effect »

Eddington 1924 : « The problem of the double star is more difficult ; we should have to take account of the effect of the gravitational field in disturbing the propagation of its own potentials and we cannot be sure that even the sign of (1) [quadrupole energy loss] is correct. »




1974 Hulse-Taylor binary pulsar : new motivation

2.5 PN equation of motion (T.D. '82) showed that binary pulsar data give a direct proof that the gravitational interaction propagates at velocity c à la GR


PSR B1913+16
PSR B1534+12
PSR J1141−6545
PSR J0737−3039
$\dot{\omega}$
$\dot{P}$
$\gamma$
$s \leq 1$
$s$
$r$
$s_{\rm scint}$
$\Omega_{\rm SO}$
$x_B/x_A$
$m_1$
$m_2$
0.0
0.5
1.0
1.5
2.0
2.5

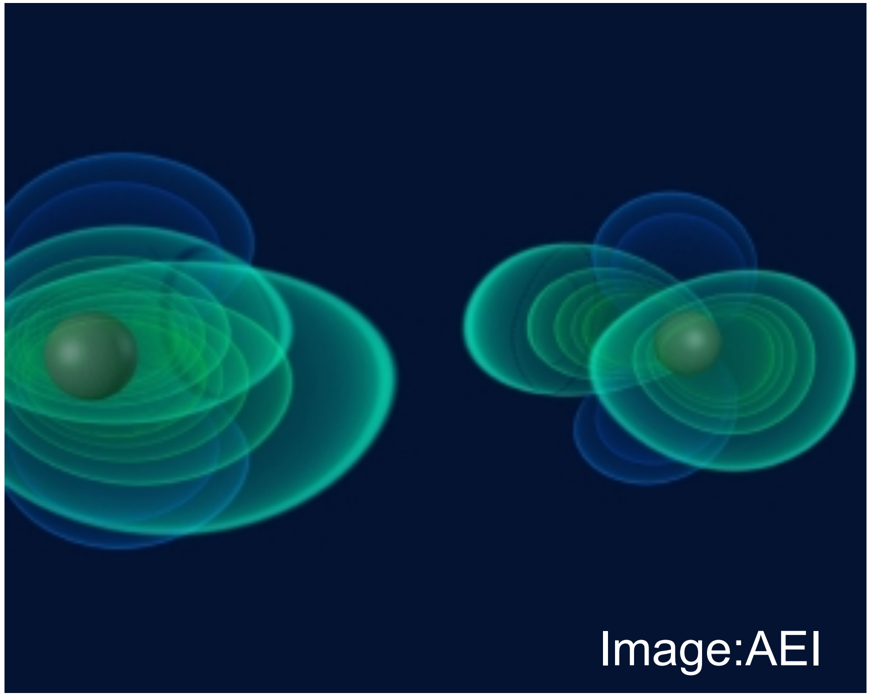
# Numerical Relativity

Uses solution of Initial-Value Constraints (à la Lichnerowicz, …) and a numerical

Integration of improved formulations of the Evolution Equations (à la Choquet-Bruhat, ….)

2005 breakthrough (Pretorius ’05, Campanelli et al ’06, Baker et al ’06, Diener et al ’06, Brügmann et al, Boyle et al,  Rezzolla, Hinder, Pollney, Reisswig, Mroué, Scheel,

Szilagyi, Pfeiffer, …) many impressive results for binary systems of BH or NS (Shibata, Hotokezaka, Rezzolla, Baiotti, Brügmann, Bernuzzi, Hilditch, ….)



# Caltech-Cornell-CITA catalog of gravitational waveforms

A catalog of 171 high-quality binary black-hole simulations for gravitational-wave astronomy [arXiv: 1304.6077]

Abdul H. Mroué,[1] Mark A. Scheel,[2] Béla Szilágyi,[2] Harald P. Pfeiffer,[1] Michael Boyle,[3] Daniel A. Hemberger,[3] Lawrence E. Kidder,[3] Geoffrey Lovelace,[4,2] Sergei Ossokine,[1,5] Nicholas W. Taylor,[2] Anıl Zenginoğlu,[2] Luisa T. Buchman,[2] Tony Chu,[1] Evan Foley,[4] Matthew Giesler,[4] Robert Owen,[6] and Saul A. Teukolsky[3]

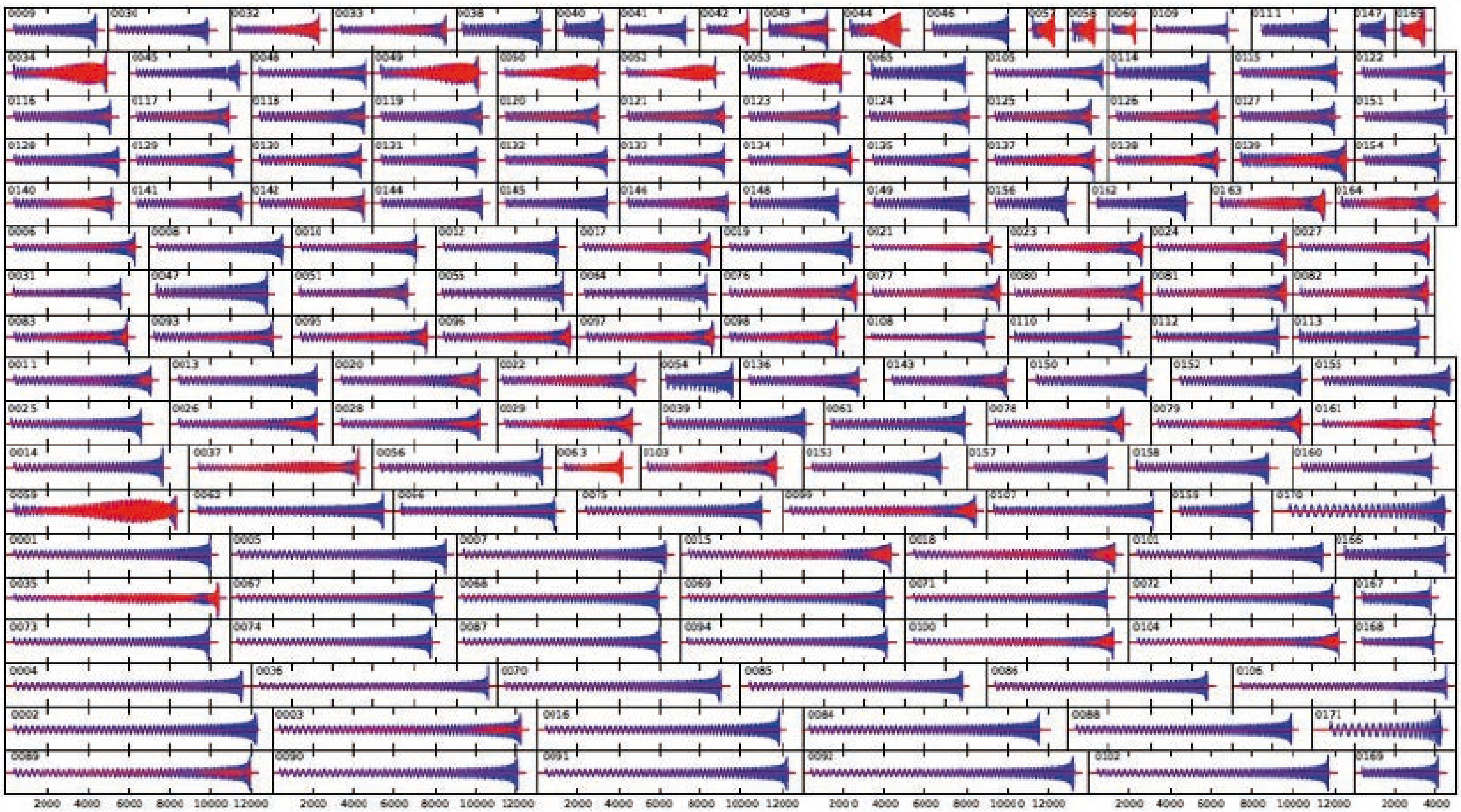


FIG. 3: Waveforms from all simulations in the catalog. Shown here are $h_+$ (blue) and $h_\times$ (red) in a sky direction parallel to the initial orbital plane of each simulation. All plots have the same horizontal scale, with each tick representing a time interval of $2000M$, where $M$ is the total mass.

# Gravitational Self-Force Theory : $m_1 << m_2$

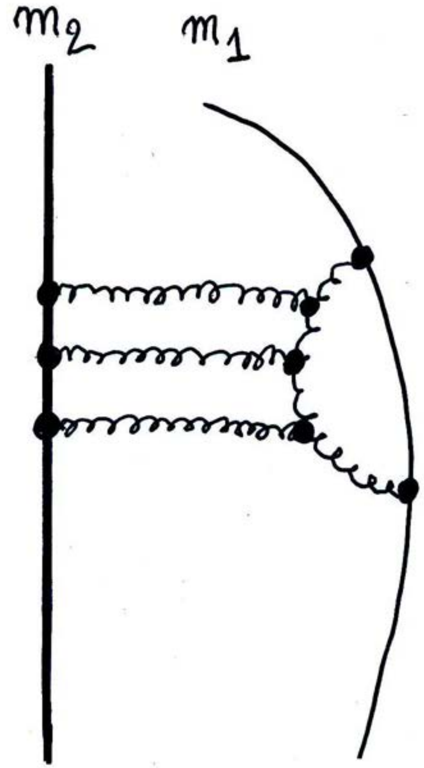
• First-order beyond the geodesic test-mass dynamics :

one-loop in BH background

DeWitt-Brehme ’60, Mino-Sasaki-Tanaka ’97,

Quinn-Wald ’97, …, Barack-Ori ’02,

Mino-Nakano-Sasaki ’03, Detweiler-Whiting ’03




• Uses BH perturbation theory : Regge-Wheeler ’57,

Zerilli ’70, Teukolsky ’72, Mano-Suzuki-Takasugi ’96

• Motivated by LISA-type astrophysics, but useful for getting the terms linear in $q = m_1/m_2$ either to (very) high PN orders, or numericallly.

# Gravitational Self-Force Theory : $m_1 << m_2$

- Analytical high-PN results : Blanchet-Detweiler-LeTiec-Whiting ’10, Damour ’10, Blanchet et al ’10, LeTiec et al ’12, Bini-Damour ’13-15, Kavanagh-Ottewill-Wardell ’15
- (gauge-invariant) Numerical results : Detweiler ’08, Barack-Sago ’09, Blanchet-Detweiler-LeTiec-Whiting ’10, Barack-Damour-Sago ’10, Shah-Friedman-Keidl ’12, Dolan et al ’14, Nolan et al ’15, …
- Analytical PN results from high-precision (hundreds to thousands of digits !) numerical results : Shah-Friedman-Whiting ’14, Johnson-McDaniel-Shah-Whiting ’15

# GSF: Analytical High PN results (1) (Bini-Damour '15)

$$
\begin{aligned}
a_{10}^{c} = {} & \frac{18605478842060273}{7079830758000}\ln(2) - \frac{1619008}{405}\zeta(3) - \frac{21339873214728097}{1011404394000}\gamma \\
& + \frac{27101981341}{100663296}\pi^6 - \frac{6236861670873}{125565440}\ln(3) + \frac{360126}{49}\ln(2)\ln(3) + \frac{180063}{49}\ln(3)^2 \\
& - \frac{121494974752}{9823275}\ln(2)^2 - \frac{242229836023352153}{549755813888}\pi^4 + \frac{1115369140625}{124540416}\ln(5) + \frac{96889010407}{277992000}\ln(7) \\
& + \frac{75437014370623318623299}{186907532011200000} - \frac{60648244288}{9823275}\ln(2)\gamma + \frac{200706848}{280665}\gamma^2 \\
& + \frac{11980569677139}{2306867200}\pi^2 + \frac{360126}{49}\gamma\ln(3), \\
a_{10}^{\ln} = {} & -\frac{21275143333512097}{2022808788000} + \frac{200706848}{280665}\gamma - \frac{30324122144}{9823275}\ln(2) + \frac{180063}{49}\ln(3), \\
a_{10}^{\ln^2} = {} & \frac{50176712}{280665}, \\
a_{10.5}^{c} = {} & -\frac{185665618769828101}{24473489040000}\pi + \frac{377443508}{77175}\ln(2)\pi + \frac{2414166668}{1157625}\pi\gamma - \frac{5846788}{11025}\pi^3 - \frac{246402}{343}\pi\ln(3), \\
a_{10.5}^{\ln} = {} & \frac{1207083334}{1157625}\pi.
\end{aligned}
$$

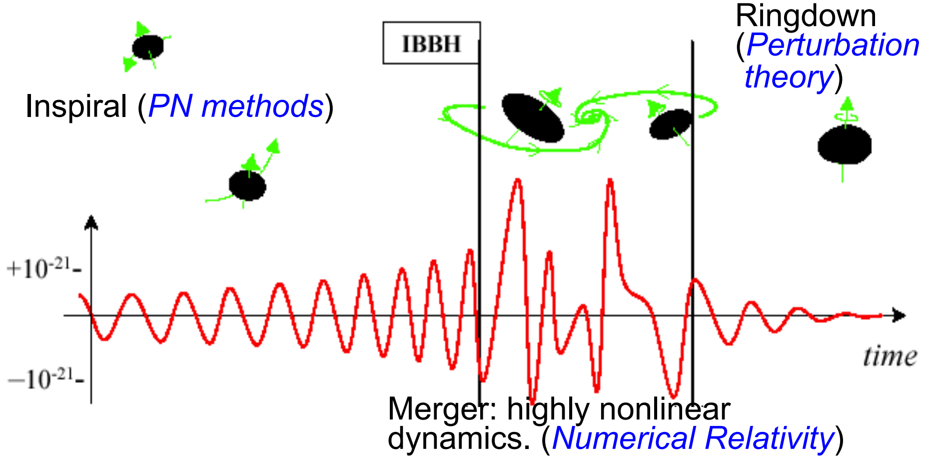
# GSF: Analytical High PN results (2)

(Kavanagh et al ’15)

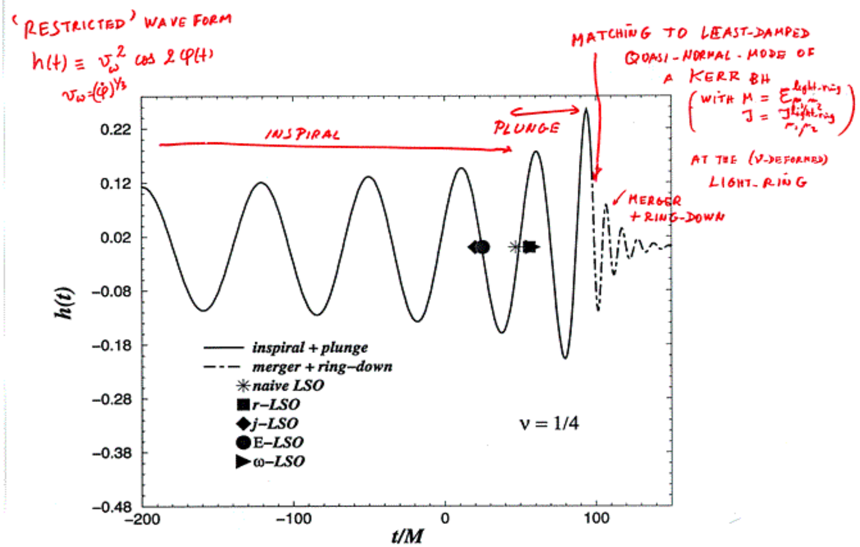
$$
\begin{aligned}
c_{15} = &-\frac{2069543450583769619340376724}{3254774420865060843 75}\zeta(3)+\frac{6519502629824500 7936}{22370298575625}\gamma\zeta(3)-\frac{5049442304}{25725}\gamma^2\zeta(3)+\frac{1262360576}{15435}\pi^2\zeta(3)\\
&+\frac{171722752}{441}\zeta(3)^2+\frac{1613866959570176}{496621125}\zeta(5)-\frac{343445504}{441}\gamma\zeta(5)-\frac{146997248}{105}\zeta(7)+\frac{56314978304}{385875}\zeta(3)\log^2(2)\\
&-\frac{106445664}{343}\zeta(3)\log^2(3)+\frac{1516709982448497976 96}{22370298575625}\zeta(3)\log(2)-\frac{190336581632}{1157625}\gamma\zeta(3)\log(2)\\
&+\frac{28863591064624341}{4909804900}\zeta(3)\log(3)-\frac{212891328}{343}\gamma\zeta(3)\log(3)-\frac{212891328}{343}\zeta(3)\log(2)\log(3)-\frac{77186767578125}{19876428}\zeta(3)\log(5)\\
&-\frac{2039263232}{3675}\zeta(5)\log(2)-\frac{49128768}{49}\zeta(5)\log(3)+\frac{298267427515018397019736592175289419501391539444290849}{6587612222544653226142468405031917319531250}\\
&-\frac{680766176845363776831328694806032908 7501419}{7043109481248037225626077295440625 00}\gamma+\frac{1598346944412603247831006289829388}{5261717150386770335918906 25}\gamma^2-\frac{10076471462159710276 4416}{335890033113009375}\gamma^3\\
&+\frac{461219496448}{72930375}\gamma^4-\frac{28338275082077591587855063450276303790065762907243197}{9997031558451434181157440457927557120000 00}\pi^2+\frac{2519117865539927569 1104}{67178006622601875}\gamma\pi^2\\
&-\frac{230609748224}{14586075}\gamma^2\pi^2+\frac{1054803233577572268947137877603911807762480362 41}{30424535483131602802509905532026880 0000}\pi^4+\frac{1262360576}{385875}\gamma\pi^4\\
&-\frac{620847283961296697269145713 1143}{266930151354100246118400}\pi^6+\frac{3573178781920929118281329}{151996487423754240}\pi^8-\frac{10136323685888}{72930375}\log^4(2)+\frac{38438712}{2401}\log^4(3)\\
&-\frac{1778960861264826796478 72}{549638236003106 25}\log^3(2)-\frac{89686013106176}{364651875}\gamma\log^3(2)+\frac{153754848}{2401}\log^3(2)\log(3)\\
&-\frac{131463845322790269123}{245735735245000}\log^3(3)+\frac{153754848}{2401}\gamma\log^3(3)+\frac{153754848}{2401}\log(2)\log^3(3)+\frac{11933074267578125}{51161925672}\log^3(5)\\
&+\frac{38782586741666289745954206352002 04}{1894218174139237320930806 25}\log^2(2)-\frac{344085637991460169215 1168}{1007670099339028125}\gamma\log^2(2)-\frac{16582891400192}{121550625}\gamma^2\log^2(2)\\
&+\frac{4145722850048}{72930375}\pi^2\log^2(2)-\frac{523697163373483905609}{245735735245000}\log^2(2)\log(3)+\frac{461264544}{2401}\gamma\log^2(2)\log(3)\\
&+\frac{454545357661890658883022992617 59}{656972822678988303488 0000}\log^2(3)-\frac{394391535968370807369}{245735735245000}\gamma\log^2(3)+\frac{230632272}{2401}\gamma^2\log^2(3)\\
&-\frac{96096780}{2401}\pi^2\log^2(3)-\frac{437493411770007517 3449}{245735735245000}\log(2)\log^2(3)+\frac{461264544}{2401}\gamma\log(2)\log^2(3)\\
&+\frac{230632272}{2401}\log^2(2)\log^2(3)+\frac{11933074267578125}{17053975224}\log^2(2)\log(5)-\frac{2505842696993145943705498046875}{402136320895332222431232}\log^2(5)\\
&+\frac{11933074267578125}{17053975224}\gamma\log^2(5)+\frac{11933074267578125}{17053975224}\log(2)\log^2(5)+\frac{4792950831647041514201025 1}{564646351702118400 00}\log^2(7)\\
&-\frac{181636067216895220421537747685253699734494659}{633879853312323350306346956589 6562500}\log(2)+\frac{7420366215521910854379953165301013 6}{473554543534809330232701 5625}\gamma\log(2)\\
&-\frac{14821693265224925154993 92}{1007670099339028125}\gamma^2\log(2)-\frac{4905667647488}{364651875}\gamma^3\log(2)+\frac{3712281154906676684511 68}{604602059603416875}\pi^2\log(2)\\
&+\frac{1226416911872}{72930375}\gamma\pi^2\log(2)+\frac{23792072704}{17364375}\pi^4\log(2)-\frac{41411583753971803023870951249358557477 27}{10826663159627448888019865600 0000}\log(3)\\
&+\frac{94593580011315754543320552762 39}{69155033966209295104 0000}\gamma\log(3)-\frac{394391535968370807369}{245735735245000}\gamma^2\log(3)+\frac{153754848}{2401}\gamma^3\log(3)\\
&+\frac{131463845322790269123}{196588588196000}\pi^2\log(3)-\frac{192193560}{2401}\gamma\pi^2\log(3)+\frac{8870472}{1715}\pi^4\log(3)\\
&+\frac{214411501060211389845962927148381}{131394564535797660697600 00}\log(2)\log(3)-\frac{437493411770007517 3449}{122867867622500}\gamma\log(2)\log(3)\\
&+\frac{461264544}{2401}\gamma^2\log(2)\log(3)-\frac{192193560}{2401}\pi^2\log(2)\log(3)+\frac{97861294850170985327709557611886523437 5}{179427491919561270211323849031 68}\log(5)\\
&-\frac{2505842696993145943705498046875}{201068160447666111215616}\gamma\log(5)+\frac{11933074267578125}{17053975224}\gamma^2\log(5)-\frac{59665371337890625}{204647702688}\pi^2\log(5)\\
&-\frac{2505842696993145943705498046875}{201068160447666111215616}\log(2)\log(5)+\frac{11933074267578125}{8526987612}\gamma\log(2)\log(5)\\
&-\frac{5858006173792308915665113013914648081}{32391919320751280297779200 0000}\log(7)+\frac{47929508316470415142010251}{28232317585105920000}\gamma\log(7)\\
&+\frac{47929508316470415142010251}{28232317585105920000}\log(2)\log(7)+\frac{7400249944258160101211}{656763448320000 00}\log(11),
\end{aligned}
$$

# Templates for GWs from BBH coalescence

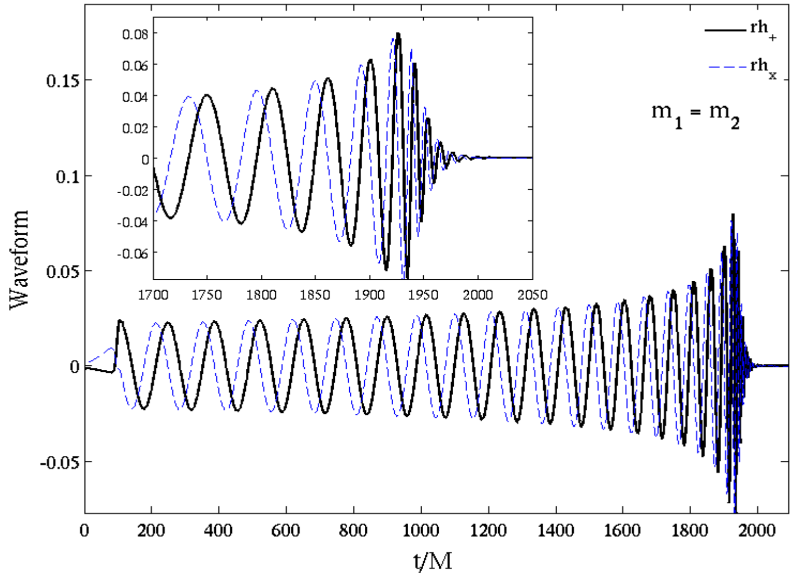
(Brady, Craighton, Thorne 1998)




(Buonanno & Damour 2000)




Numerical Relativity, the 2005 breakthrough:
Pretorius, Campanelli et al., Baker et al. …

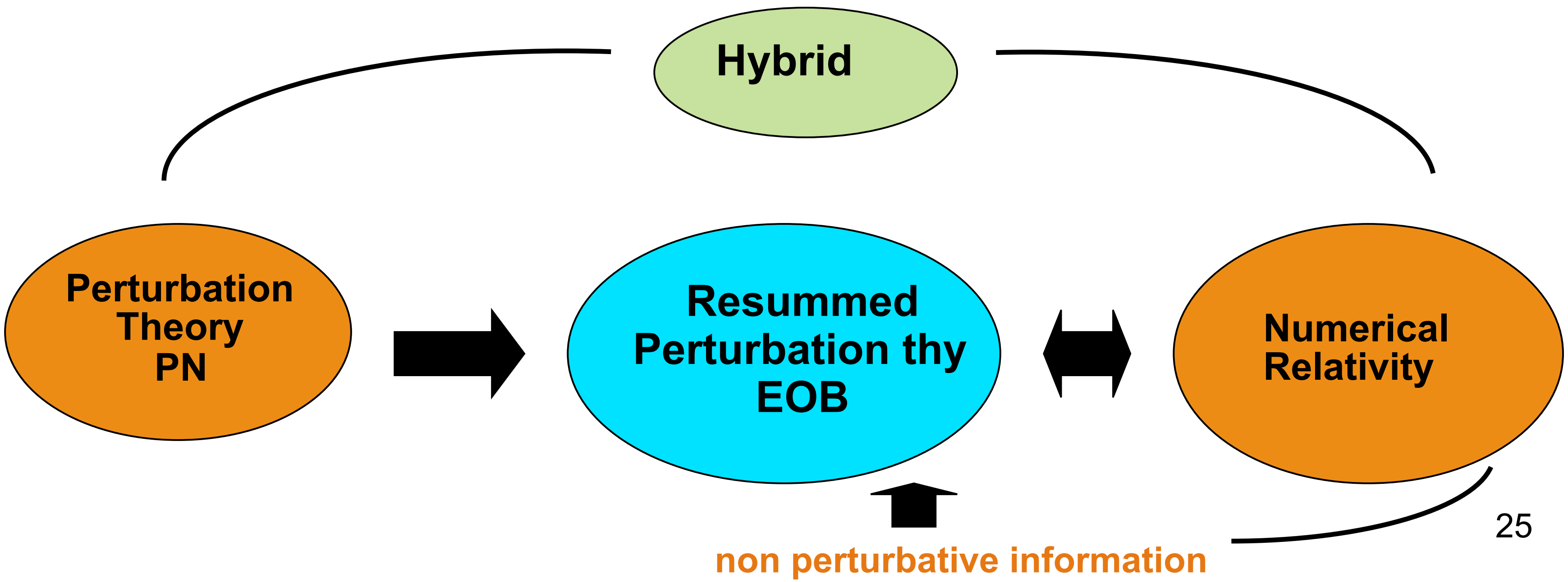


# Importance of an analytical formalism

- **Theoretical:** physical understanding of the coalescence process, especially in complicated situations (arbitrary spins)

- **Practical:** need many thousands of accurate GW templates for detection & data analysis;
  need some "analytical" representation of waveform templates as $f(m_1,m_2,\mathbf{S_1},\mathbf{S_2})$

- Solution: **synergy between analytical & numerical relativity**



## An improved analytical approach

# EFFECTIVE ONE BODY (EOB)

# approach to the two-body problem

| | |
|---|---|
| Buonanno,Damour 99 | (2 PN Hamiltonian) |
| Buonanno,Damour 00 | (Rad.Reac. full waveform) |
| Damour, Jaranowski,Schäfer 00 | (3 PN Hamiltonian) |
| Damour 01, Buonanno, Chen, Damour 05, Barausse, Buonanno, 10 … Damour,Nagar 14 | (spin) |
| Damour, Nagar 07, Damour, Iyer, Nagar 08 | (factorized waveform) |
| Buonanno, Cook, Pretorius 07, Buonanno, Pan, Taracchini,… | (comparison to NR) |
| Damour, Nagar 10 | (tidal effects) |
| Bini, Damour 13, Damour, Jaranowski, Schäfer 15 | (4 PN Hamiltonian) |

# EOB approach to the two-body problem

1. Replace the conservative PN dynamics of two bodies, $m_1$, $m_2$ by the dynamics of a particle of mass $\boxed{\mu = m_1 m_2/(m_1+m_2)}$ moving in some effective metric $\boxed{g_{\mu\nu}^{eff}}$ which is a deformation of a Schwarzschild (or Kerr) metric of mass $M = m_1 + m_2$ by the parameter :

$$\boxed{\nu = \mu/M = m_1 m_2/(m_1+m_2)^2}$$

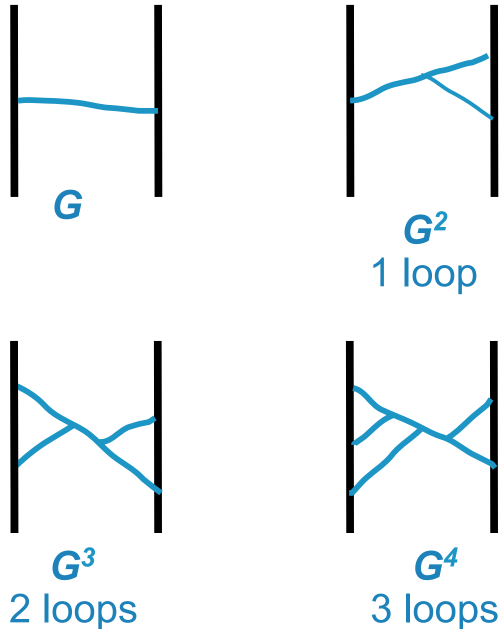
2. Resum the PN waveform $h^{resum}$

3. Add a radiation-reaction force $\boxed{\mathcal{F}[h^{resum}]}$

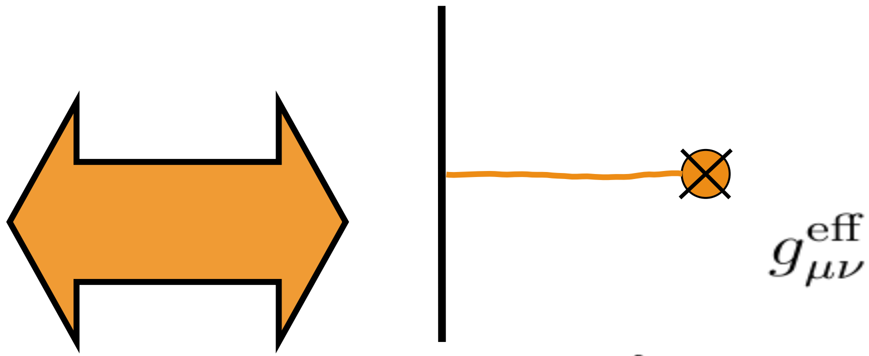
4. Complete the inspiral + plunge + merger gravitational waveform emitted by the EOB dynamics by attaching a QNM-constructed ringdown waveform (Vishveshwara 70, Davis-Ruffini-Press-Price 71, Davis-Ruffini-Tiomno 72)

# Real dynamics versus Effective dynamics

## Real dynamics




## Effective dynamics




$$S = -\int \mu ds + \ldots$$

$$H = H_0 + \left( G H_1 + \frac{G^2}{c^2} H_2 + \frac{G^3}{c^4} H_3 + \frac{G^4}{c^6} H_4 \right) \left( 1 + \frac{1}{c^2} + \ldots \right)$$

## Effective metric

$$\boxed{ds^2_{\text{eff}} = -A(r)dt^2 + B(r)dr^2 + r^2 \left( d\theta^2 + \sin^2\theta d\varphi^2 \right)}$$

# Two-body/EOB "correspondence": think quantum-mechanically (Wheeler)

Real 2-body system ($m_1$, $m_2$)
(in the c.o.m. frame)

an effective particle of mass $\mu$ in some effective metric $g_{\mu\nu}^{eff}(M)$

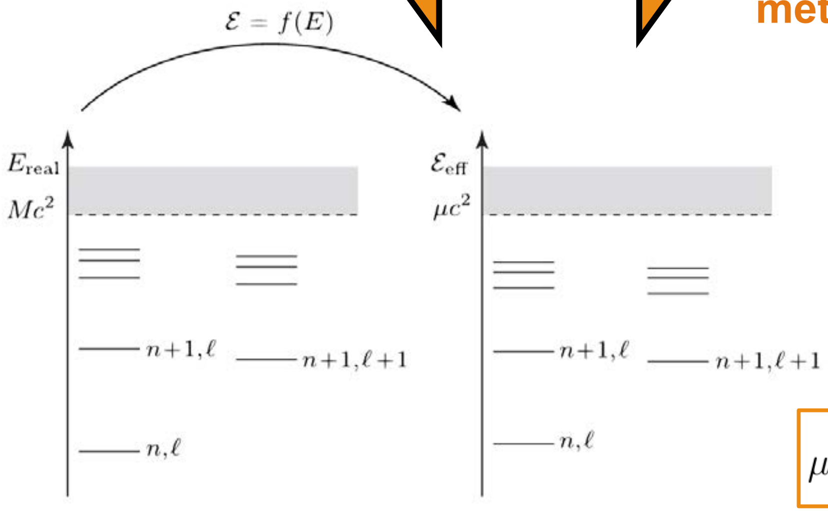



Figure 1: Sketch of the correspondence between the quantized energy levels of the real and effective conservative dynamics. $n$ denotes the 'principal quantum

$$\mu^2 + g_{\rm eff}^{\mu\nu} \frac{\partial S_{\rm eff}}{\partial x^\mu} \frac{\partial S_{\rm eff}}{\partial x^\nu} + \mathcal{O}(p^4) = 0$$

Sommerfeld "Old Quantum Mechanics":

$$J = \ell\hbar = \frac{1}{2\pi} \oint p_\varphi \, d\varphi$$
$$N = n\hbar = I_r + J$$
$$I_r = \frac{1}{2\pi} \oint p_r \, dr$$

$H^{classical}(q,p)$ → $H^{classical}(I_a)$ → $E^{\rm quantum}(I_a = n_a h) = f^{-1}\left[\mathcal{E}_{\rm eff}^{\rm quantum}\left(I_a^{\rm eff} = n_a h\right)\right]$

# The EOB energy map

Real 2-body system $(m_1, m_2)$
(in the c.o.m. frame)


an effective particle of Mass $\mu=m_1 m_2/(m_1+m_2)$ in some effective metric $g_{\mu\nu}^{eff}(M)$

Simple energy map

$$\mathcal{E}_{eff} = \frac{s - m_1^2 - m_2^2}{2M}$$

$$s = E_{\text{real}}^2$$

$$H_{\text{EOB}} = M\sqrt{1 + 2\nu\left(\hat{H}_{\text{eff}} - 1\right)}$$

$$\begin{aligned} M &= m_1 + m_2 \\ \nu &= m_1 m_2/(m_1 + m_2)^2 \end{aligned}$$

# Explicit form of the 3PN EOB effective Hamiltonian

**The effective metric $g_{\mu\nu}^{eff}(M)$** at 3PN

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

where the coefficients are a ν-dependent "deformation" of the Schwarzschild ones:

$$\boxed{A_{3PN}(R) = 1 - 2u + 2\nu u^3 + a_4 \nu u^4}$$

$$\boxed{a_4 = \frac{94}{3} - \frac{41}{32}\pi^2 \simeq 18.6879027}$$

$$(A(R)B(R))_{3PN} = 1 - 6\nu u^2 + 2(3\nu - 26)\nu u^3$$

***u = GM/(c²r)***

**Simple** effective Hamiltonian

$$\boxed{\hat{H}_{\text{eff}} \equiv \sqrt{p_{r_*}^2 + A\left(1 + \frac{p_\varphi^2}{r^2} + z_3 \frac{p_{r_*}^4}{r^2}\right)}.}$$

$$p_{r_*} = \left(\frac{A}{B}\right)^{1/2} p_r$$

**crucial EOB "radial potential" A(r)**

# 2-body Taylor-expanded 3PN Hamiltonian [JS98, DJS00,01]

$$H_{\mathrm{N}}(\mathbf{x}_a,\mathbf{p}_a) = \sum_a \frac{\mathbf{p}_a^2}{2\,m_a} - \frac{1}{2}\sum_a \sum_{b\neq a} \frac{G\,m_a\,m_b}{r_{ab}}\,.$$

$$H_{1\mathrm{PN}}(\mathbf{x}_a,\mathbf{p}_a) = -\frac{1}{8}\frac{(\mathbf{p}_1^2)^2}{m_1^3} + \frac{1}{8}\frac{Gm_1m_2}{r_{12}}\left[-12\frac{\mathbf{p}_1^2}{m_1^2} + 14\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2} + 2\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right] + \frac{1}{4}\frac{Gm_1m_2}{r_{12}}\frac{G(m_1+m_2)}{r_{12}} + (1\longleftrightarrow 2),$$

1PN

$$\begin{aligned}
H_{2\mathrm{PN}}(\mathbf{x}_a,\mathbf{p}_a) = {}& \frac{1}{16}\frac{(\mathbf{p}_1^2)^3}{m_1^5} + \frac{1}{8}\frac{Gm_1m_2}{r_{12}}\left[5\frac{(\mathbf{p}_1^2)^2}{m_1^4} - \frac{11}{2}\frac{\mathbf{p}_1^2\,\mathbf{p}_2^2}{m_1^2m_2^2} - \frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{m_1^2m_2^2} + 5\frac{\mathbf{p}_1^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right.\\
&\left. -6\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2m_2^2} - \frac{3}{2}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right]\\
&+\frac{1}{4}\frac{G^2m_1m_2}{r_{12}^2}\left[m_2\left(10\frac{\mathbf{p}_1^2}{m_1^2} + 19\frac{\mathbf{p}_2^2}{m_2^2}\right) - \frac{1}{2}(m_1+m_2)\frac{27\,(\mathbf{p}_1\cdot\mathbf{p}_2) + 6\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right]\\
&-\frac{1}{8}\frac{Gm_1m_2}{r_{12}}\frac{G^2(m_1^2+5m_1m_2+m_2^2)}{r_{12}^2} + (1\longleftrightarrow 2).
\end{aligned}$$

2PN

$$\begin{aligned}
H_{3\mathrm{PN}}^{\mathrm{reg}}(\mathbf{x}_a,\mathbf{p}_a) = {}& -\frac{5}{128}\frac{(\mathbf{p}_1^2)^4}{m_1^7} + \frac{1}{32}\frac{Gm_1m_2}{r_{12}}\left[-14\frac{(\mathbf{p}_1^2)^3}{m_1^6} + 4\frac{\big((\mathbf{p}_1\cdot\mathbf{p}_2)^2 + 4\,\mathbf{p}_1^2\,\mathbf{p}_2^2\big)\mathbf{p}_1^2}{m_1^4m_2^2} + \frac{\big(\mathbf{p}_1^2\,\mathbf{p}_2^2 - 2\,(\mathbf{p}_1\cdot\mathbf{p}_2)^2\big)(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2^3}\right.\\
&-10\frac{\big(\mathbf{p}_1^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2 + \mathbf{p}_2^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\big)\mathbf{p}_1^2}{m_1^4m_2^2} + 24\frac{\mathbf{p}_1^2\,(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^4m_2^2} + 2\frac{\mathbf{p}_1^2\,(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3m_2^3}\\
&+\frac{\big(7\,\mathbf{p}_1^2\,\mathbf{p}_2^2 - 10\,(\mathbf{p}_1\cdot\mathbf{p}_2)^2\big)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3m_2^3} + 6\frac{\mathbf{p}_1^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^4m_2^2}\\
&\left.+15\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3m_2^3} - 18\frac{\mathbf{p}_1^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3m_2^3} + 5\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3m_2^3}\right]\\
&+\frac{G^2m_1m_2}{r_{12}^2}\left[\frac{1}{16}(m_1-27m_2)\frac{(\mathbf{p}_1^2)^2}{m_1^4} - \frac{115}{16}m_1\frac{\mathbf{p}_1^2\,(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2} + \frac{1}{48}m_2\frac{25\,(\mathbf{p}_1\cdot\mathbf{p}_2)^2 + 371\,\mathbf{p}_1^2\,\mathbf{p}_2^2}{m_1^2m_2^2}\right.\\
&+\frac{17}{16}\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^3} - \frac{1}{8}m_1\frac{\big(15\,\mathbf{p}_1^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_2) + 11\,(\mathbf{p}_1\cdot\mathbf{p}_2)\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)\big)(\mathbf{n}_{12}\cdot\mathbf{p}_1)}{m_1^3m_2} + \frac{5}{12}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{m_1^3}\\
&-\frac{3}{2}m_1\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3m_2} + \frac{125}{12}m_2\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2m_2^2} + \frac{10}{3}m_2\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\\
&\left.-\frac{1}{48}(220m_1+193m_2)\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right] + \frac{G^3m_1m_2}{r_{12}^3}\left[-\frac{1}{48}\left(466\,m_1^2 + \left(473 - \frac{3}{4}\pi^2\right)m_1m_2 + 150\,m_2^2\right)\frac{\mathbf{p}_1^2}{m_1^2}\right.\\
&+\frac{1}{16}\left(77(m_1^2+m_2^2) + \left(143 - \frac{1}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2} + \frac{1}{16}\left(61\,m_1^2 - \left(43 + \frac{3}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^2}\\
&\left.+\frac{1}{16}\left(21(m_1^2+m_2^2) + \left(119 + \frac{3}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right]\\
&+\frac{1}{8}\frac{G^4m_1m_2^3}{r_{12}^4}\left[\left(\frac{227}{3} - \frac{21}{4}\pi^2\right)m_1 + m_2\right] + (1\longleftrightarrow 2).
\end{aligned} \tag{12}$$

3PN

# NR-completed resummed 5PN EOB radial A potential

4PN analytically complete + 5 PN logarithmic term in the A(u, nu) function,
With u = GM/R and nu = m1 m2 / (m1 + m2)^2
[Damour 09, Blanchet et al 10, Barack-Damour-Sago 10, Le Tiec et al 11,
Barausse et al 11, Akcay et al 12, Bini-Damour 13,
Damour-Jaranowski-Schäfer 14, Nagar-Damour-Reisswig-Pollney 15]

$$
\begin{aligned}
A(u;\nu,a_6^c) \quad &= \quad P_5^1\left[1-2u+2\nu\,u^3+\nu\left(\frac{94}{3}-\frac{41}{32}\pi^2\right)\,u^4\right.\\
&+ \quad \nu\left[-\frac{4237}{60}+\frac{2275}{512}\pi^2+\left(-\frac{221}{6}+\frac{41}{32}\pi^2\right)\nu+\frac{64}{5}\ln(16e^{2\gamma}u)\right]u^5\\
&+ \quad \left.\nu\left[a_6^c(\nu)-\left(\frac{7004}{105}+\frac{144}{5}\nu\right)\ln u\right]u^6\right]
\end{aligned}
$$

$$
a_6^{c\,\text{NR-tuned}}(\nu) = 81.38 - 1330.6\,\nu + 3097.3\,\nu^2
$$

# 2-body Taylor-expanded 4PN Hamiltonian [DJS, 2014]

$$
\begin{aligned}
c^8 H^{\text{local}}_{4\text{PN}}(\mathbf{x}_a,\mathbf{p}_a) &= \frac{7(\mathbf{p}_1^2)^5}{256m_1^9} + \frac{Gm_1m_2}{r_{12}}H_{48}(\mathbf{x}_a,\mathbf{p}_a) + \frac{G^2m_1m_2}{r_{12}^2}m_1H_{46}(\mathbf{x}_a,\mathbf{p}_a)\\
&+ \frac{G^3m_1m_2}{r_{12}^3}\left(m_1^2H_{441}(\mathbf{x}_a,\mathbf{p}_a) + m_1m_2H_{442}(\mathbf{x}_a,\mathbf{p}_a)\right)\\
&+ \frac{G^4m_1m_2}{r_{12}^4}\left(m_1^3H_{421}(\mathbf{x}_a,\mathbf{p}_a) + m_1^2m_2H_{422}(\mathbf{x}_a,\mathbf{p}_a)\right)\\
&+ \frac{G^5m_1m_2}{r_{12}^5}H_{40}(\mathbf{x}_a,\mathbf{p}_a) + (1\leftrightarrow 2),
\end{aligned}
\tag{A3}
$$

$$
\begin{aligned}
H_{48}(\mathbf{x}_a,\mathbf{p}_a) &= \frac{45(\mathbf{p}_1^2)^4}{128m_1^8} - \frac{9(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2}{64m_1^6m_2^2} + \frac{15(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^3}{64m_1^6m_2^2} - \frac{9(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{16m_1^6m_2^2}\\
&- \frac{3(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{32m_1^6m_2^2} + \frac{15(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{64m_1^6m_2^2} - \frac{21(\mathbf{p}_1^2)^3\mathbf{p}_2^2}{64m_1^6m_2^2} - \frac{35(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{256m_1^5m_2^3}\\
&+ \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2}{128m_1^5m_2^3} + \frac{33(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3(\mathbf{p}_1^2)^2}{256m_1^5m_2^3} - \frac{85(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256m_1^5m_2^3}\\
&- \frac{45(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{128m_1^5m_2^3} - \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256m_1^5m_2^3} + \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{64m_1^5m_2^3}\\
&+ \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{64m_1^5m_2^3} - \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{64m_1^5m_2^3} + \frac{3\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{64m_1^5m_2^3} + \frac{55(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256m_1^5m_2^3}\\
&- \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{128m_1^5m_2^3} - \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{256m_1^5m_2^3} - \frac{23(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256m_1^5m_2^3}\\
&+ \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{128m_1^5m_2^3} - \frac{7(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256m_1^5m_2^3} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4\mathbf{p}_1^2}{64m_1^4m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4(\mathbf{p}_1^2)^2}{64m_1^4m_2^4}\\
&- \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{4m_1^4m_2^4} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{16m_1^4m_2^4} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{64m_1^4m_2^4} + \frac{21(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2\mathbf{p}_2^2}{64m_1^4m_2^4}\\
&- \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{32m_1^4m_2^4} - \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{4m_1^4m_2^4} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{16m_1^4m_2^4} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{16m_1^4m_2^4}\\
&- \frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{32m_1^4m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_2^2)^2}{64m_1^4m_2^4} - \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_2^2)^2}{32m_1^4m_2^4} - \frac{7(\mathbf{p}_1^2)^2(\mathbf{p}_2^2)^2}{128m_1^4m_2^4},
\end{aligned}
\tag{A4a}
$$

$$
\begin{aligned}
H_{46}(\mathbf{x}_a,\mathbf{p}_a) &= \frac{369(\mathbf{n}_{12}\cdot\mathbf{p}_1)^6}{160m_1^6} - \frac{889(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4\mathbf{p}_1^2}{192m_1^6} + \frac{49(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1^2)^2}{16m_1^6} - \frac{63(\mathbf{p}_1^2)^3}{64m_1^6} - \frac{549(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{128m_1^5m_2}\\
&+ \frac{67(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{16m_1^5m_2} - \frac{167(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2}{128m_1^5m_2} + \frac{1547(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_1\cdot\mathbf{p}_2)}{256m_1^5m_2} - \frac{851(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{128m_1^5m_2}\\
&+ \frac{1099(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256m_1^5m_2} + \frac{3263(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{1280m_1^4m_2^2} + \frac{1067(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{480m_1^4m_2^2} - \frac{4567(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2}{3840m_1^4m_2^2}\\
&- \frac{3571(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{320m_1^4m_2^2} + \frac{3073(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{480m_1^4m_2^2} + \frac{4349(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{1280m_1^4m_2^2}\\
&- \frac{3461\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{3840m_1^4m_2^2} + \frac{1673(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4\mathbf{p}_2^2}{1920m_1^4m_2^2} - \frac{1999(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2\mathbf{p}_2^2}{3840m_1^4m_2^2} + \frac{2081(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{3840m_1^4m_2^2} - \frac{13(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{8m_1^3m_2^3}\\
&+ \frac{191(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2}{192m_1^3m_2^3} - \frac{19(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384m_1^3m_2^3} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384m_1^3m_2^3}\\
&+ \frac{11(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{192m_1^3m_2^3} + \frac{77(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{96m_1^3m_2^3} + \frac{233(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_2^2}{96m_1^3m_2^3} - \frac{47(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{32m_1^3m_2^3}\\
&+ \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{384m_1^3m_2^3} - \frac{185\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{384m_1^3m_2^3} - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4}{4m_1^2m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4\mathbf{p}_1^2}{4m_1^2m_2^4}\\
&- \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3(\mathbf{p}_1\cdot\mathbf{p}_2)}{2m_1^2m_2^4} + \frac{21(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{16m_1^2m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{6m_1^2m_2^4} + \frac{49(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2\mathbf{p}_2^2}{48m_1^2m_2^4}\\
&- \frac{133(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{24m_1^2m_2^4} - \frac{77(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{96m_1^2m_2^4} + \frac{197(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_2^2)^2}{96m_1^2m_2^4} - \frac{173\mathbf{p}_1^2(\mathbf{p}_2^2)^2}{48m_1^2m_2^4} + \frac{13(\mathbf{p}_2^2)^3}{8m_2^6},
\end{aligned}
\tag{A4b}
$$

$$
\begin{aligned}
H_{441}(\mathbf{x}_a,\mathbf{p}_a) &= \frac{5027(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{384m_1^4} - \frac{22993(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2}{960m_1^4} - \frac{6695(\mathbf{p}_1^2)^2}{1152m_1^4} - \frac{3191(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{640m_1^3m_2}\\
&+ \frac{28561(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{1920m_1^3m_2} + \frac{8777(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384m_1^3m_2} + \frac{752969\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{28800m_1^3m_2}\\
&- \frac{16481(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{960m_1^2m_2^2} + \frac{94433(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{4800m_1^2m_2^2} - \frac{103957(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{2400m_1^2m_2^2}\\
&+ \frac{791(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{400m_1^2m_2^2} + \frac{26627(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_2^2}{1600m_1^2m_2^2} - \frac{118261\mathbf{p}_1^2\mathbf{p}_2^2}{4800m_1^2m_2^2} + \frac{105(\mathbf{p}_2^2)^2}{32m_2^4},
\end{aligned}
\tag{A4c}
$$

$$
\begin{aligned}
H_{442}(\mathbf{x}_a,\mathbf{p}_a) &= \left(\frac{2749\pi^2}{8192} - \frac{211189}{19200}\right)\frac{(\mathbf{p}_1^2)^2}{m_1^4} + \left(\frac{63347}{1600} - \frac{1059\pi^2}{1024}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2}{m_1^4} + \left(\frac{375\pi^2}{8192} - \frac{23533}{1280}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{m_1^4}\\
&+ \left(\frac{10631\pi^2}{8192} - \frac{1918349}{57600}\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{m_1^2m_2^2} + \left(\frac{13723\pi^2}{16384} - \frac{2492417}{57600}\right)\frac{\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2m_2^2}\\
&+ \left(\frac{1411429}{19200} - \frac{1059\pi^2}{512}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{m_1^2m_2^2} + \left(\frac{248991}{6400} - \frac{6153\pi^2}{2048}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^2m_2^2}\\
&- \left(\frac{30383}{960} + \frac{36405\pi^2}{16384}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2} + \left(\frac{1243717}{14400} - \frac{40483\pi^2}{16384}\right)\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2}\\
&+ \left(\frac{2369}{60} + \frac{35655\pi^2}{16384}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3m_2} + \left(\frac{43101\pi^2}{16384} - \frac{391711}{6400}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{m_1^3m_2}\\
&+ \left(\frac{56955\pi^2}{16384} - \frac{1646983}{19200}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2},
\end{aligned}
\tag{A4d}
$$

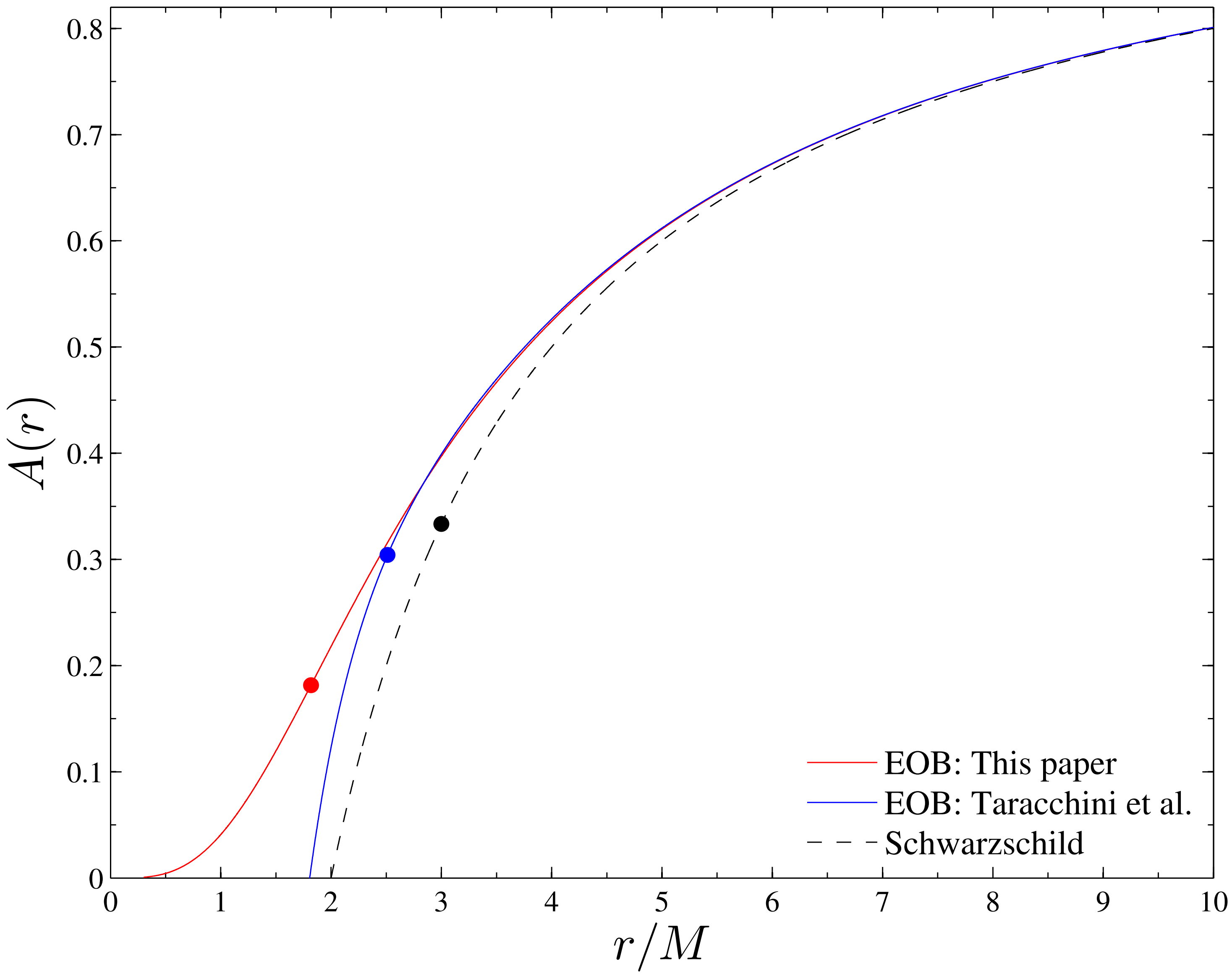
$$
H_{421}(\mathbf{x}_a,\mathbf{p}_a) = \frac{64861\mathbf{p}_1^2}{4800m_1^2} - \frac{91(\mathbf{p}_1\cdot\mathbf{p}_2)}{8m_1m_2} + \frac{105\mathbf{p}_2^2}{32m_2^2} - \frac{9841(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{1600m_1^2} - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{2m_1m_2},
\tag{A4e}
$$

$$
\begin{aligned}
H_{422}(\mathbf{x}_a,\mathbf{p}_a) &= \left(\frac{1937033}{57600} - \frac{199177\pi^2}{49152}\right)\frac{\mathbf{p}_1^2}{m_1^2} + \left(\frac{176033\pi^2}{24576} - \frac{2864917}{57600}\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2} + \left(\frac{282361}{19200} - \frac{21837\pi^2}{8192}\right)\frac{\mathbf{p}_2^2}{m_2^2}\\
&+ \left(\frac{698723}{19200} + \frac{21745\pi^2}{16384}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^2} + \left(\frac{63641\pi^2}{24576} - \frac{2712013}{19200}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\\
&+ \left(\frac{3200179}{57600} - \frac{28691\pi^2}{24576}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_2^2},
\end{aligned}
\tag{A4f}
$$

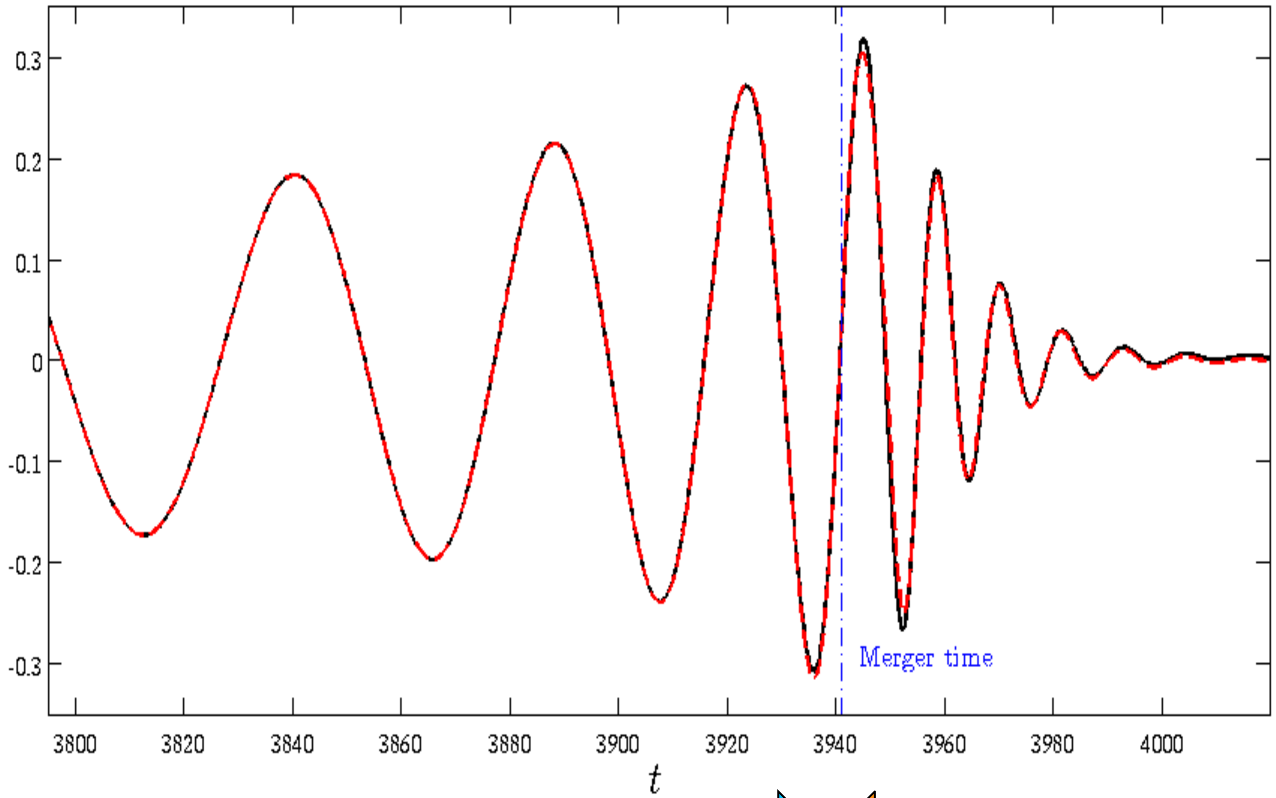
$$
H_{40}(\mathbf{x}_a,\mathbf{p}_a) = -\frac{m_1^4}{16} + \left(\frac{6237\pi^2}{1024} - \frac{169799}{2400}\right)m_1^3m_2 + \left(\frac{44825\pi^2}{6144} - \frac{609427}{7200}\right)m_1^2m_2^2.
\tag{A4g}
$$

$$
\begin{aligned}
H^{\text{nonloc}}_{4\text{PN}}(t) &= -\frac{1}{5}\frac{G^2M}{c^8}I^{(3)}_{ij}(t)\\
&\times \text{Pf}_{2r_{12}/c}\int_{-\infty}^{+\infty}\frac{\mathrm{d}v}{|v|}I^{(3)}_{ij}(t+v),
\end{aligned}
$$

# Comparing the EOB A potentials of Nagar et al 15 and Taracchini et al 14 to $A^{Schw} = 1 - 2M/r$

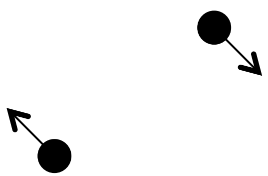


# Binary BH coalescence: Analytical Relativity vs NR[CC]



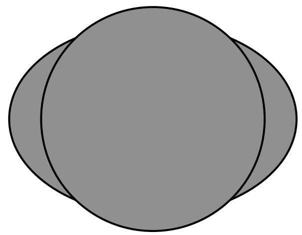

Inspiral + « plunge »

Ringdown

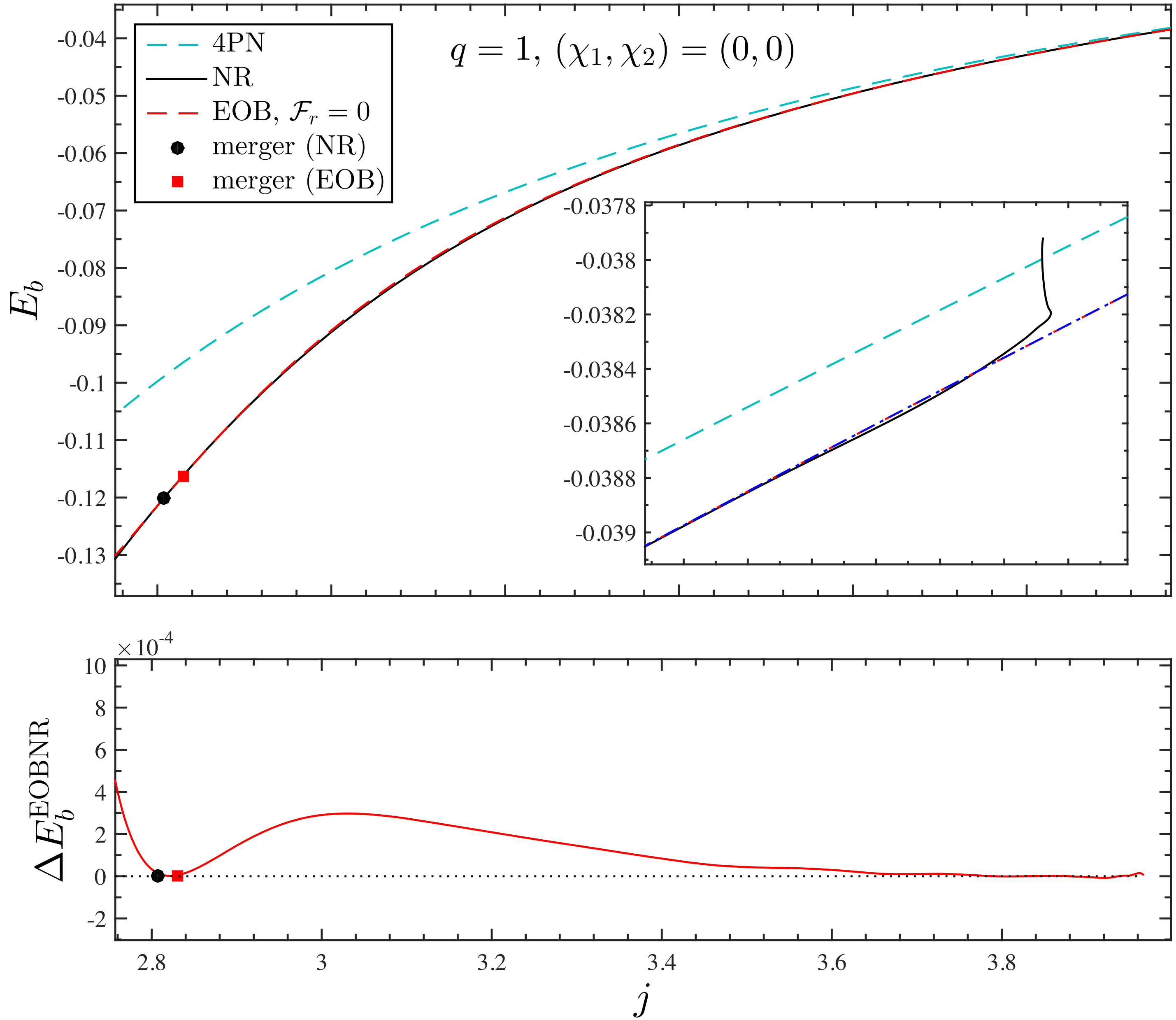


Two orbiting point-masses:
Resummed dynamics

Ringing BH



# Energy and Angular Momentum in Spinning and Nonspinning Black-Hole Binaries : E(J)




Damour,Nagar,

Pollney,Reisswig 12

Nagar,Damour,

Reisswig, Pollney 15

# ϱ(x) and periastron advance in (comparable-mass) Black Hole binaries (Le Tiec et al. 2011)

K(x)= Ω_ϕ/Ω_r;   x=(M Ω_ϕ)^(2/3)

$$K_{\rm EOB} = \sqrt{\frac{A'_p(u)}{D(u)\Delta(u)}},$$

where $A'_p = \mathrm{d}A_P/\mathrm{d}u$, and $\Delta = A_P A'_p + 2u(A'_p)^2 - uA_P A''_p$

$$K^{\nu}_{\rm GSF} = \frac{1}{\sqrt{1-6x}}\left[1 - \frac{\nu}{2}\frac{\rho(x)}{1-6x} + \mathcal{O}(\nu^2)\right],$$

3

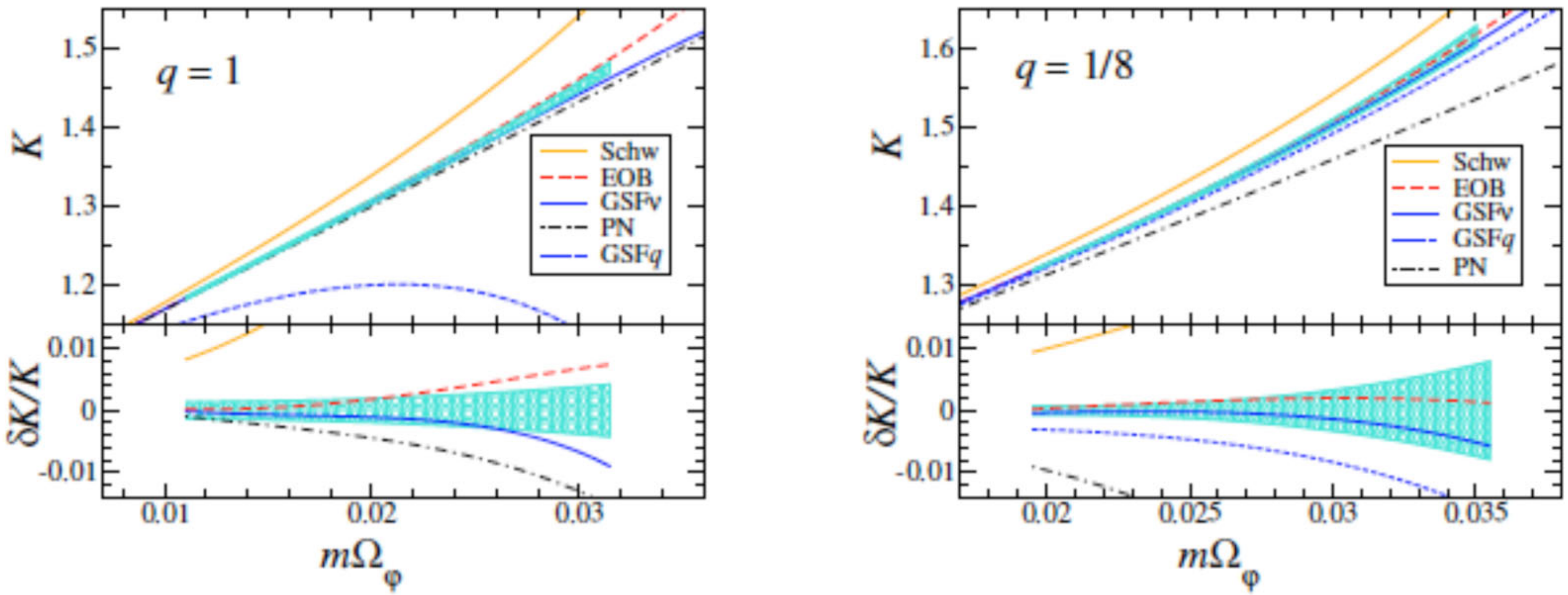


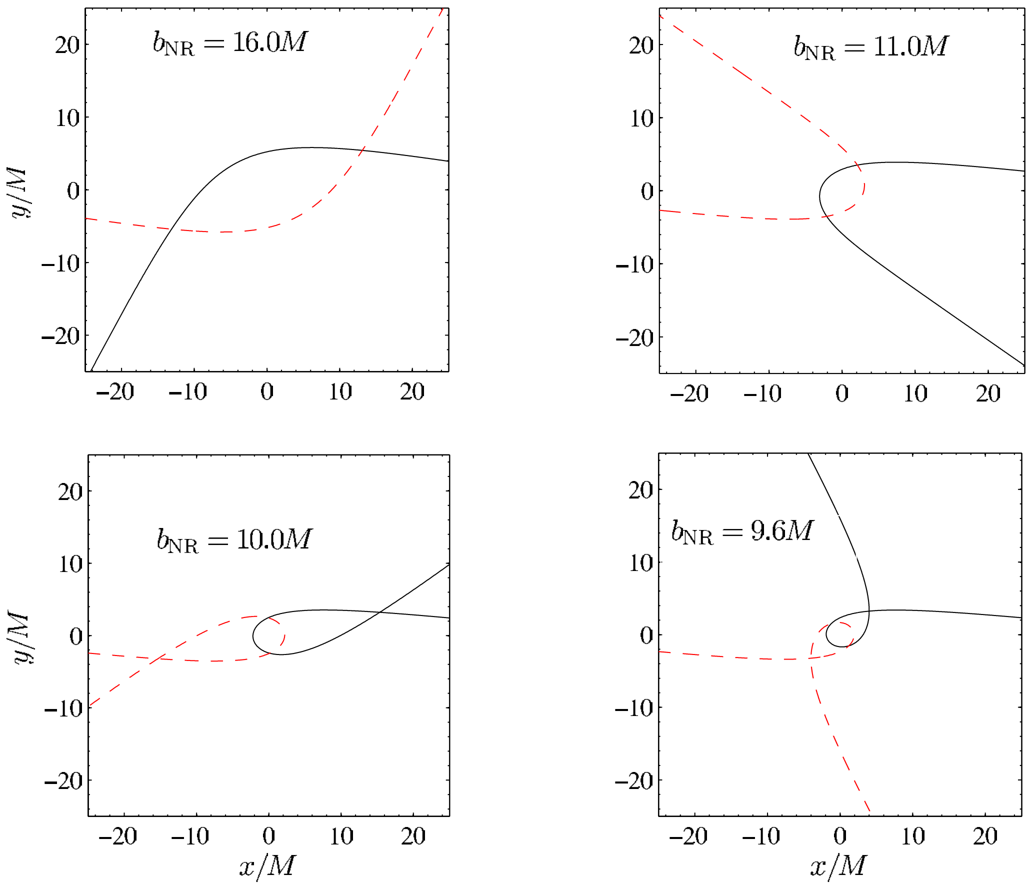

FIG. 1: The periastron advance $K$ of an equal mass black hole binary, in the limit of zero eccentricity, as a function of the orbital frequency $\Omega_\varphi$ of the circular motion. The NR results are indicated by the cyan-shaded region. The PN and EOB results are valid at 3PN order. The lower panel shows the relative difference $\delta K/K \equiv (K - K_{\rm NR})/K_{\rm NR}$.

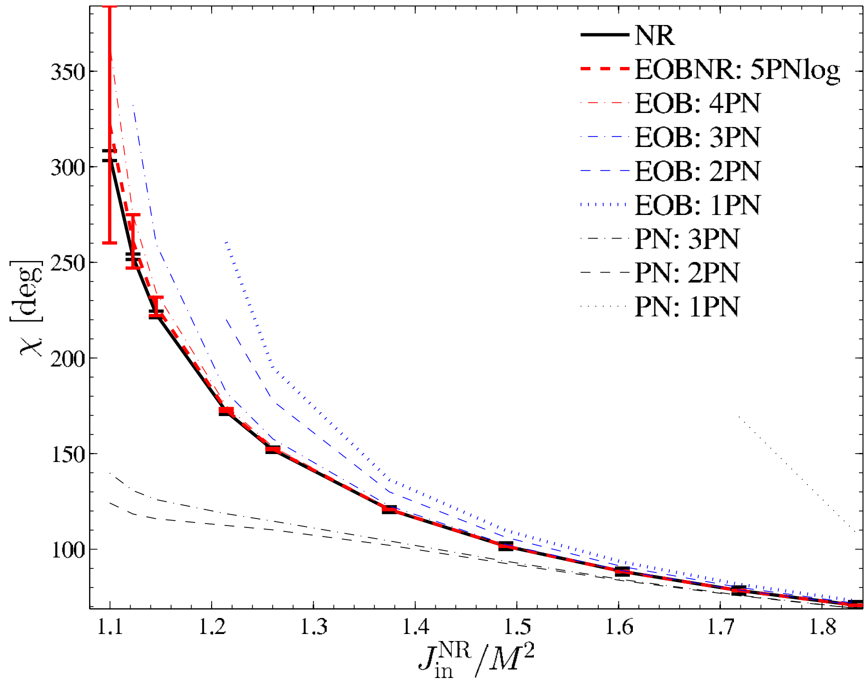
FIG. 2: Same as in Fig. 1, but for a mass ratio $q = 1/8$. Note that for an orbital frequency $m\Omega_\varphi \sim 0.03$, corresponding to a separation $r \sim 10m$, the periastron advance reaches half an orbit per radial period.

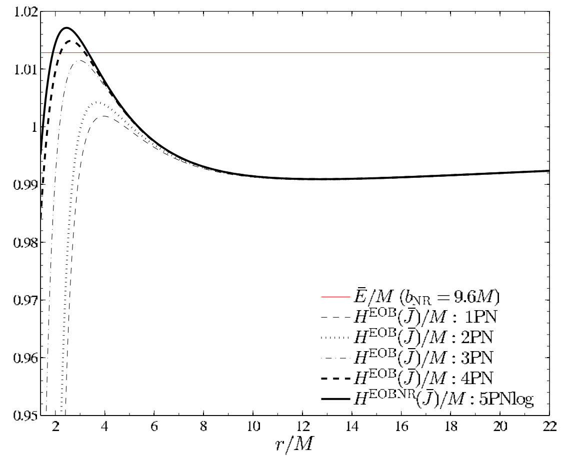
Since $\rho(x) > 0$ for all stable circular orbits, the $\mathcal{O}(q)$ GSF

# Strong-Field Scattering of Two Black Holes

(Damour, Guercilena, Hinder, Hopper, Nagar, Rezzolla 2014)

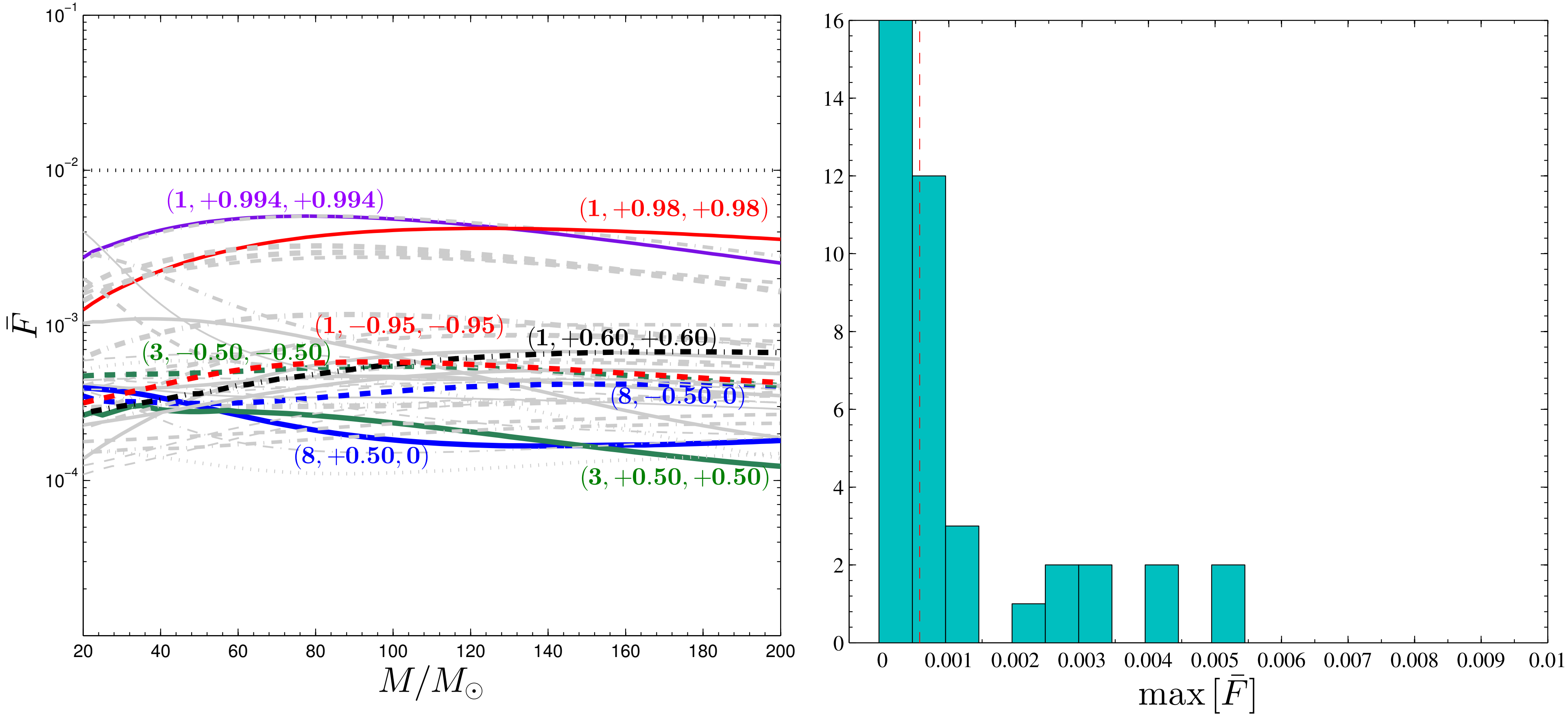


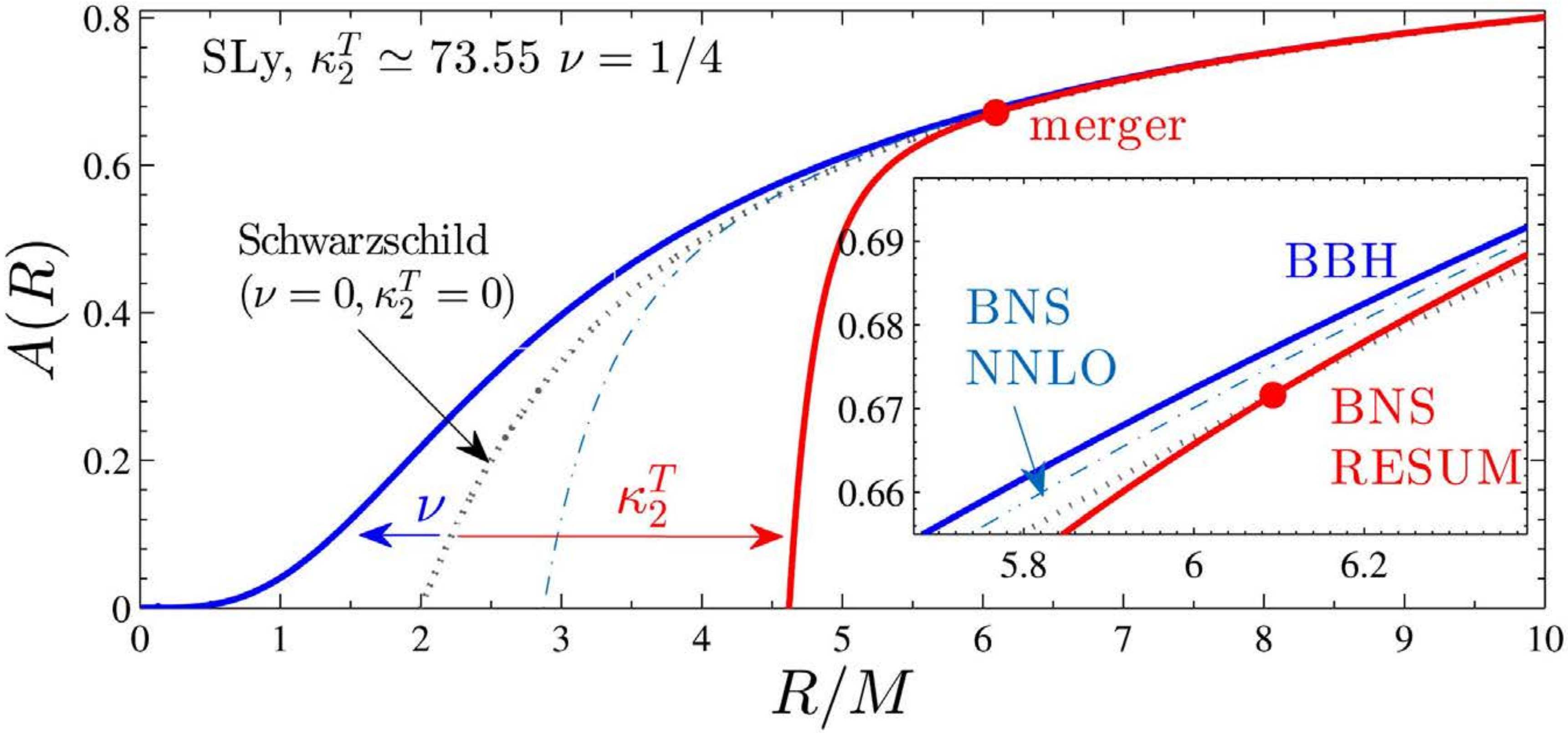
# EOB/NR unfaithfulness = 1 – F for spinning BBH

Nagar et al 2015 [ similar results in Taracchini et al 2014]



# Binary neutron stars: Tidal EOB [NR] potential

Bernuzzi, Nagar, Dietrich & Damour 2015 [based on Bini-Damour 2014]



# Conclusions on the "Problem of Motion"

Poincaré : "Il n'y a pas de problèmes résolus et d'autres qui ne le sont pas;
Il y a seulement des problèmes *plus ou moins* résolus."

Experimentally, gravitational wave astronomy is about to start :
LIGO/Virgo/GEO/KAGRA/LIGO-India/…
One of the prime GW sources that one expects to detect is the GW signal emitted by coalescing binary systems (BH, NS)

The study of the motion and radiation of binary systems recently made a lot of progress, in several different directions : PN, GSF, EOB, EFT, NR.
There is a useful synergy between the various methods (e.g. recent GSF confirmations of 4PN dynamics)

There exist a traditional synergy between Mathematical Relativity and Numerical Relativity (Darmois, Lichnerowicz, Choquet-Bruhat, ….)

Lack of mathematical results on n-body pb (Matched Asymptotic Expansions)

There exists a complementarity between Numerical Relativity and Analytical Relativity, especially when using the particular resummation of perturbative results defined by the Effective One Body formalism.
The NR- tuned EOB formalism is likely to be essential for computing the many thousands of accurate GW templates needed for LIGO/Virgo/...