The Equivalence Principle and the Genesis of General Relativity

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Before Einstein

Newton : $\mathrm{m}a = \mathrm{F}_{\mathrm{tot}}$; $\mathrm{F}_{\mathrm{grav}} = \dfrac{G\mathrm{m}_A\mathrm{m}_B}{r^2}$

Gravitational potential (Laplace) : $\mathrm{F}_{\mathrm{grav}} = m\nabla\mathrm{U}$ weight ∝ quantity of matter

Poisson equation : $\Delta\mathrm{U} = -4\pi\,\mathrm{G}\,\rho$

Special Relativity and gravity : need to avoid instantaneous action at a distance.

Lorentz 1900, Poincaré 1906, Minkowski 1908, generalize $\mathrm{F}_{\mathrm{grav}}$

Scalar relativistic field theory : Einstein ≤ 1907, Abraham, 1912, Nordström 1912

$$\Box\Phi = 4\pi\,\sigma \quad ; \quad \sigma = ?$$

Einstein's Equivalence Principle 1907

« The happiest thought of my life »

Two facts : 1. Efface (locally) a real external gravitational field
2. Generate a (fictitious) gravitational field

« Hypothesis of a complete physical equivalence between a gravitational field and the corresponding acceleration of the reference system »

Together with $\mathrm{m} \; = \mathrm{E} \;/c^2$, it shows the necessity of going beyond SR

Observational consequences (drawn without field equations)

Gravitational redshift $\tau \simeq (1 - U/c^2)t$; spectral rays

$$c(x) \simeq c_0 \left(1 - \frac{1}{c^2} \mathrm{U}\ (\mathrm{x})\right) \Rightarrow$$

In 1911 $\delta\theta_{\mathrm{sun}} = \dfrac{2\mathrm{GM}}{c^2 R} \simeq 0.83''$ for grazing light

Formal consequences of EP

1912 « *The speed of light and the statics of the gravitational field* »

Considers a uniformly accelerated reference system (à la Born 1909) and derives (for the first time) the « Rindler metric »

$$ds^2 = -(c_0 + ax/c_0)^2\, dt^2 + dx^2 + dy^2 + dz^2$$

Suggests that $c(x) = c_0 + ax/c_0$ is exact for static, uniform gravitational field and that the spatial geometry stays Euclidean for this case.

1912 (same paper) Einstein mentions a uniformly rotating system and the fact that « π » is modified

$$\left[ds^2 = -(c^2 - \Omega^2(x^2+y^2))\, dt^2) + dx^2 + dy^2 + dz^2 - 2\Omega\, x\, dx\, dt + 2\Omega x\, dy\, dt\right]$$

Later in 1912 expects g_{oi} to be linked to motion of matter

Einstein's tools in building GR

All the forms of EP :

UFF → (wrongly) eliminates SR Scalar theory

inertia = gravitation → « Rindler metric » → rotating disk metric → $g_{\mu\nu}$

« Einstein's EP » → laws of non gravitational physics must be generally covariant in $g_{\mu\nu}$

$$S = -mc \int \sqrt{-g_{\mu\nu}\, d\,x^\mu dx^\nu}$$

« Strong EP » E=mc^2 → all forms of energy must contribute to mass, including gravitational energy

1912-1913 (Zurich notebook)

$$\begin{aligned} O = \sqrt{g}\ \nabla_\nu\ T^\nu_\mu &= \partial_\nu\ \left(\sqrt{g}T^\nu_\mu\right) - \frac{1}{2}\,\partial_\mu\, g_{\alpha\beta}\sqrt{g}\,T^{\alpha\beta} \\ &= \partial_\nu\ \left(\sqrt{g}\,T^\nu_\mu + t^\nu_\mu\right) \end{aligned}$$

Poisson eq + Maxwell eqs suggests $G_{\mu\nu}(g) = \kappa T_{\mu\nu}$

Tensor calculus, Riemannian geometry, Ricci tensor (with Marcel Grossmann)

Einstein’s Zurich Notebook

Vol 1 The Genesis of General Relativity edited by J. Renn

$$\begin{bmatrix}\mu\nu\\ l\end{bmatrix} = \frac{1}{2}\left(\frac{\partial g_{\mu l}}{\partial x_\nu} + \frac{\partial g_{\nu l}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_l}\right) \qquad \frac{\partial}{\partial x_k}\begin{bmatrix}i\,[illegible]\\ [illegible]\end{bmatrix} - \frac{\partial}{\partial x_i}\begin{bmatrix}k\,[illegible]\\ [illegible]\end{bmatrix}$$

$$(ik, lm) = \frac{1}{2}\left(\frac{\partial^2 g_{im}}{\partial x_k \partial x_l} + \frac{\partial^2 g_{kl}}{\partial x_i \partial x_m} - \frac{\partial^2 g_{il}}{\partial x_k \partial x_m} - \frac{\partial^2 g_{km}}{\partial x_i \partial x_l}\right) + \sum_{\rho\sigma} \gamma_{\rho\sigma}\left(\begin{bmatrix}im\\ \sigma\end{bmatrix}\begin{bmatrix}kl\\ \rho\end{bmatrix} - \begin{bmatrix}il\\ \sigma\end{bmatrix}\begin{bmatrix}km\\ \rho\end{bmatrix}\right)$$

Grossmann
Tensor vierter
Mannigfaltigkeit

$$\sum \gamma_{kl}(ik, lm)$$

$$\sum \gamma_{kl}\begin{bmatrix}kl\\ \rho\end{bmatrix} = \sum \gamma_{kl}\left[\frac{\partial g_{k\rho}}{\partial x_l} + \frac{\partial g_{l\rho}}{\partial x_k} - \frac{\partial g_{kl}}{\partial x_\rho}\right]$$

$$= \frac{\partial \lg G}{\partial x_\rho} + 2\sum_{kl} \gamma_{kl}\frac{\partial g_{k\rho}}{\partial x_l}$$

$$\frac{1}{4}\sum \gamma_{\rho\sigma}\left(\frac{\partial g_{i\sigma}}{\partial x_m} + \frac{\partial g_{m\sigma}}{\partial x_i} - \frac{\partial g_{im}}{\partial x_\sigma}\right)\left[-\frac{\partial \lg G}{\partial x_\rho} + 2\sum_{kl}\gamma_{kl}\frac{\partial g_{k\rho}}{\partial x_l}\right]$$

$$\sum \gamma_{kl}\gamma_{\rho\sigma}\left(\begin{bmatrix}im\\ \sigma\end{bmatrix}\begin{bmatrix}kl\\ \rho\end{bmatrix} - \begin{bmatrix}il\\ \sigma\end{bmatrix}\begin{bmatrix}km\\ \rho\end{bmatrix}\right)$$

$$= \sum_\rho -\left\{\begin{matrix}im\\ \rho\end{matrix}\right\}\cdot\frac{\partial \lg G}{\partial x_\rho} + 2\sum_{kl\rho}\left\{\begin{matrix}im\\ \rho\end{matrix}\right\}\cdot\gamma_{kl}\frac{\partial g_{k\rho}}{\partial x_l} - \sum_{\rho k l}\left\{\begin{matrix}il\\ \rho\end{matrix}\right\}\left(\frac{\partial g_{k\rho}}{\partial x_m}\right)\gamma_{kl}$$

$$+ \sum_{\rho l}\left\{\begin{matrix}il\\ \rho\end{matrix}\right\}\cdot\left\{\begin{matrix}\rho m\\ l\end{matrix}\right\}$$

$$\sum_k\left(\frac{\partial^2 g_{kk}}{\partial x_i \partial x_m} - \frac{\partial^2 g_{ik}}{\partial x_k \partial x_m} - \frac{\partial^2 g_{mk}}{\partial x_k \partial x_i}\right) = 0$$

Sollte verschwinden.

$$\varphi = \sum_{imkl}\gamma_{im}\gamma_{kl}\left(\frac{\partial^2 g_{im}}{\partial x_k \partial x_l} - \frac{\partial^2 g_{il}}{\partial x_k \partial x_m}\right)$$

$$+ \sum_{\rho\sigma imkl}\gamma_{\rho\sigma}\gamma_{im}\gamma_{kl}\left(\begin{bmatrix}im\\ \sigma\end{bmatrix}\begin{bmatrix}kl\\ \rho\end{bmatrix} - \begin{bmatrix}il\\ \sigma\end{bmatrix}\begin{bmatrix}km\\ \rho\end{bmatrix}\right)$$

$$\sum_{\rho\sigma imkl}\gamma_{\rho\sigma}\gamma_{im}\gamma_{kl}\left(\frac{\partial g_{i\sigma}}{\partial x_m} + \frac{\partial g_{m\sigma}}{\partial x_i} - \frac{\partial g_{im}}{\partial x_\sigma}\right)\left(\frac{\partial g_{k\rho}}{\partial x_l} + \frac{\partial g_{l\rho}}{\partial x_k} - \frac{\partial g_{kl}}{\partial x_\rho}\right)$$

$$-g_{i\sigma}\frac{\partial \gamma_{im}}{\partial x_m} \quad -g_{m\sigma}\frac{\partial \gamma_{im}}{\partial x_i} \quad -\gamma_{im}\frac{\partial g_{im}}{\partial x_\sigma} \quad -g_{k\rho}\frac{\partial \gamma_{kl}}{\partial x_l} \quad -g_{l\rho}\frac{\partial \gamma_{kl}}{\partial x_k} \quad -\gamma_{kl}\frac{\partial g_{kl}}{\partial x_\rho}$$

$$\sum \gamma_{\rho\sigma}\Bigl(\underbrace{g_{i\sigma}\frac{\partial \gamma_{\sigma m}}{\partial x_m} + g_{m\sigma}\frac{\partial \gamma_{im}}{\partial x_i}}_{\underbrace{\frac{\partial \gamma_{\rho m}}{\partial x_m} + \frac{\partial \gamma_{i\rho}}{\partial x_i}}_{2\frac{\partial \gamma_{\rho\alpha}}{\partial x_\alpha}}} + \underbrace{\gamma_{im}\frac{\partial g_{im}}{\partial x_\sigma}}_{\gamma_{\rho\sigma}\frac{\partial \lg G}{\partial x_\sigma}}\Bigr)\Bigl(g_{k\rho}\frac{\partial \gamma_{kl}}{\partial x_l} + g_{l\rho}\frac{\partial \gamma_{kl}}{\partial x_k} + \underbrace{\gamma_{kl}\frac{\partial g_{kl}}{\partial x_\rho}}_{\frac{\partial \lg G}{\partial x_\rho}}\Bigr)$$

$$\gamma_{\rho\sigma}\frac{\partial \lg G}{\partial \sigma}\frac{\partial \lg G}{\partial \rho} + \gamma_{\rho\sigma}\frac{\partial \lg G}{\partial x_\sigma}\left(g_{k\rho}\frac{\partial \gamma_{kl}}{\partial x_l} + g_{l\rho}\frac{\partial \gamma_{kl}}{\partial x_k}\right) + \frac{\partial \lg G}{\partial x_\rho}\cdot 2\cdot\frac{\partial \gamma_{\rho\alpha}}{\partial x_\alpha}$$

$$\frac{\partial \lg G}{\partial x_\sigma}\left(\frac{\partial \gamma_{\rho\alpha}}{\partial x_\alpha} + \frac{\partial \gamma_{\sigma\alpha}}{\partial x_\alpha}\right) \qquad 3\,\frac{\partial \lg G}{\partial x_\sigma}\frac{\partial \gamma_{\sigma\alpha}}{\partial x_\alpha} + \frac{\partial \lg G}{\partial x_\sigma}\frac{\partial \gamma_{\sigma\alpha}}{\partial x_\alpha}$$

$$+ 2\frac{\partial \gamma_{\rho\alpha}}{\partial x_\alpha}\left(g_{k\rho}\frac{\partial \gamma_{kl}}{\partial x_l} + g_{l\rho}\frac{\partial \gamma_{kl}}{\partial x_k}\right)$$

$$\cancel{4\frac{\partial \gamma_{\rho\alpha}}{}}\;\cancel{4 g_{k\rho}\frac{\partial \gamma_{k\alpha}}{\partial x_\alpha}\frac{\partial \gamma_{\rho\beta}}{\partial x_\beta}}$$

$l\rho \quad k\alpha$

$k\rho \quad \alpha\beta$

$$4\, g_{k\rho}\frac{\partial \gamma_{k\alpha}}{\partial x_\alpha}\frac{\partial \gamma_{\rho\beta}}{\partial x_\beta} + \cancel{2 g_{k\rho}\frac{\partial \gamma_{k\alpha}}{\partial x_\alpha}} \quad \Big/ \cdot\frac{1}{4}$$

war richtig.

Obstacles towards building GR

EP → «Rindler» $ds^2=-c^2(x)dt^2+dx^2+dy^2+dz^2$: strongly suggests that the newtonian limit involves

$$g_{00} \simeq -\left(1-\frac{U}{c^2}\right)^2 \quad , \quad g_{ij} \simeq \delta_{ij}$$

Mach's principle suggests that $T_{\mu\nu}$ (matter) uniquely determines $g_{\mu\nu}$ (inertia)

Conflict with general covariance (« hole argument »)

Looks for an intermediate (g-dependent) EP-GR covariance groupLinearized version of Ricci

tensor incompatible with expected Newtonian limit.

Already in 1912-1913 stumbles on $R^L_{\mu\nu} - \frac{1}{2}R^L\,\eta_{\mu\nu} \sim \Box\left(h_{\mu\nu} - \frac{1}{2}h\,\eta_{\mu\nu}\right) \sim T_{\mu\nu}$

Problem of the meaning of coordinates (especially after SR's breakthrough)

Unknown contracted Bianchi identitires

Unknown Noether-type theorems

Unknown concept of connection $\Gamma^\lambda_{\mu\nu}$

No definition of manifold (Weyl 1913 Concept of Riemann surface)

Unfashionable action principles (even Pauli in 1921)

No good notation, no summation convention

1913 Einstein's – Grossmann « outline theory »

Semingly impossible to satisfy all the wished physical and mathematical requirements with generally covevariant field equations.

Exclude field equations based on $G_{\mu\nu} \sim R_{\mu\nu}$

Looks for theory with restricted covariance but energy conservation and **strong** EP

$$\sum_{\alpha\beta\mu} \frac{\partial}{\partial x_\alpha}\left(\sqrt{-g}\,\gamma_{\alpha\beta}\,g_{\sigma\mu}\,\frac{\partial \gamma_{\mu\nu}}{\partial x_\beta}\right) = \varkappa(\mathfrak{T}_{\sigma\nu} + \mathfrak{t}_{\sigma\nu});$$

with $-2\varkappa \cdot \mathfrak{t}_{\sigma\nu} = \sqrt{-g}\left(\sum_{\beta\tau\varrho} \gamma_{\beta\nu}\,\frac{\partial g_{\tau\varrho}}{\partial x_\sigma}\,\frac{\partial \gamma_{\tau\varrho}}{\partial x_\beta} - \frac{1}{2}\sum_{\alpha\beta\tau\varrho} \delta_{\sigma\nu}\,\gamma_{\alpha\beta}\,\frac{\partial g_{\tau\varrho}}{\partial x_\alpha}\,\frac{\partial \gamma_{\tau\varrho}}{\partial x_\beta}\right);$

With $\sum_{\nu} \frac{\partial}{\partial x_\nu}(\mathfrak{T}_{\sigma\nu} + \mathfrak{t}_{\sigma\nu}) = 0.$

? What is the restricted covariance group ?

1913 – 1915 Progress preparing the 1915 breakthrough

Formulation of the « outline theory » by an action principle

Einstein realizes the non-uniquenes of outline theory

Einstein invents himself the technology for « Noether-like » theorems

Explores physical consequences of Outline theory :

Mercury's perihelion precession (with Besso) only 18''

Gravitomagnetism, Mach'type effects of induction of inertia

Problem with compatibility with naive EP (uniform acceleration, uniform rotation)

Discussions with colleagues : Abraham, Nordström

Einstein-Fokker reformulation of Nordström's scalar theory as

$$g_{\mu\nu} = \Phi^2 \eta\, g_{\mu\nu} \quad \text{and} \quad R = k\,T$$

The 1915 breakthrough

June –July 1915 Hilbert invites Einstein to give, during one week, six 2-hour lectures on General Relativity.

Hilbert gets henceforth interested in incorporating Einstein's ideas within his attempt at an all-encompassing axiomatic approach to physics (à la Gustav Mie).

4 November 1915 Restricts full covariance to unimodular diffeomorphisms Jacobian=1
Decomposes $R_{\mu\nu} = R^{\mathrm{I}}_{\mu\nu} + R^{\mathrm{II}}_{\mu\nu} \quad , \quad R^{\mathrm{I}}_{\mu\nu} = \partial_\sigma \Gamma^\sigma_{\mu\nu} - \Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\rho\nu}$

and shows that $R^{\mathrm{I}}_{\mu\nu} = \kappa T_{\mu\nu}$ derives from an action and admits a conservation law.

11 November 1915 shows that, under the assumption T = 0, one can write the fully covariant $R_{\mu\nu} = \kappa T_{\mu\nu}$ (which can be simplified to the 4 November form by imposing $\sqrt{g} = 1$)

18 November 1915 solves $R_{\mu\nu} = 0$, $\sqrt{g} = 1$ outside the Sun to second order in $1/c^2$ and finds an advance of Mercury' perihelion equal to 43''/cy and an unexpected Newtonian limit with $g_{ij} \neq \delta_{ij}$.

The 1915 breakthrough (2)

25 November 1915 He writes the general ($T \neq 0$), final generally covariant equations

$$R_{\mu\nu} = \kappa \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right)$$

and proves (when simplifying the calculation by using unimodular coordinates $\sqrt{g}=1$) that they imply the energy conservation

$$\sum_{\lambda} \frac{\partial}{\partial x_{\lambda}} \left(T_{\sigma}^{\lambda} + t_{\sigma}^{\lambda} \right) = 0$$

So that the strong EP is satisfied : the source of gravity is $T_{\mu}^{\nu} + t_{\mu}^{\nu}$

The Hilbert affair

On 20 November 1915 (after the first three communications of Einstein and before the final one of 25 November) Hilbert presents a communication to the Göttingen Academy. The published version of his communication includes both the Einstein-Hilbert action for gravity $\int d^4x\,\sqrt{g}\,R$ and the explicit form of Einstein's field equations $R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu} = \kappa T_{\mu\nu}$.

For a long time it was concluded that Hilbert had partially « scooped » Einstein in getting first the final field equations of GR. However, Corry, Renn, and Stachel (1997) found the first proofs of Hilbert's paper (December 1915).

These proofs (despite the theft of the a fraction of one page !) show that :

1. Hilbert has substantially amended / completed his paper between submission and publication
2. Hilbert initially postulated the necessity of breaking general covariance
3. Hilbert correctly postulated the action $\int d^4x\,\sqrt{g}\,R$ but very probably did not obtain the explicit form of Einstein's field equations until he saw Einstein's last paper.

CONCLUSIONS

Positive role of EP : led Einstein to discover key elements of GR, notably the crucial $g_{\mu\nu}(x)$ (via $S = -mc \int \sqrt{-g_{\mu\nu}\, d\,x^{\mu} dx^{\nu}}$, and Gauss theory of surfaces).

Negative role of EP : uniformly accelerated system (« Rindler metric ») blocked Einstein for years because it suggested only $g_{00}(x)$ excited.

Naive interpretation of EP (no dependence on horizontal velocity) led him to incorrectly reject Scalar theory.

Other principles used together with EP :

Mach principle (cf gas stove analogy), Energy Conservation Principle, Generalized Relativity Principle.

Fascinating to see the vision and stubborness of Einstein, and his readiness to abandon the so satisfying SR theory (cf Planck).