Through a Glass Darkly:
The Structure of Cosmological Singularities

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Genericity of Cosmological Singularities?

Landau 1959: Is the big bang singularity of Friedmann universes a generic property of general relativistic cosmologies, or is it an artefact of the high degree of symmetry of these solutions?

Khalatnikov and Lifshitz 1963: look for generic inhomogeneous and anisotropic solution near a singularity

$$ds^2 = -dt^2 + (a^2 \ell_i \ell_j + b^2 m_i m_j + c^2 n_i n_j) dx^i dx^j$$

single homogeneous Friedmann scale factor $a(t) \to$ three inhomogeneous scale factors $a(t, \mathbf{x})$, $b(t, \mathbf{x})$, $c(t, \mathbf{x})$

KL63 did not succeed in finding the "general" solution of the complicated, coupled dynamics of a, b, c and tentatively concluded that a singularity is not generic.

Genericity of Cosmological Singularities?

local collapse: Penrose 1965; cosmology: Hawking 1966-7, Hawking-Penrose 1970: Theorems about genericity of cosmological "singularity".

They prove generic "incompleteness" of spacetime, without giving any information about the "singularity".

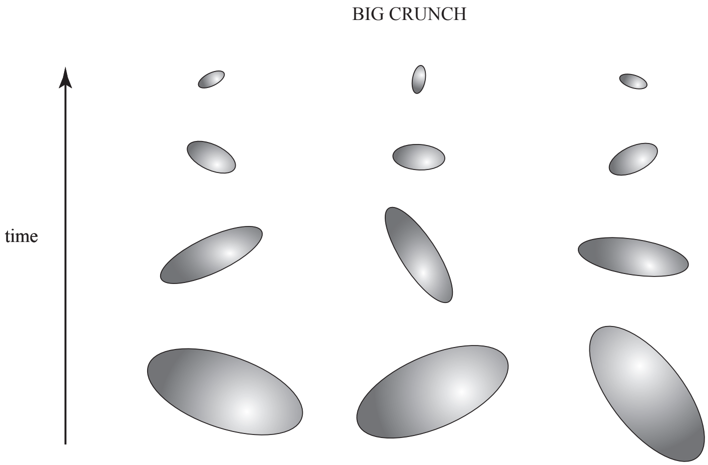
Belinsky, Khalatnikov, Lifshitz 1969:

• claim to construct the "general" solution near $abc \to 0$ of the coupled (inhomogeneous) dynamics of $a(t, \mathbf{x})$, $b(t, \mathbf{x})$, $c(t, \mathbf{x})$,

• find that, at each point of space $\mathbf{x}$, the dynamics of a, b, c is chaotic.

The BKL conjecture has been confirmed both by numerical simulations (Weaver-Isenberg-Berger 1998, Berger-Moncrief 1998, Berger et al 1998-2001; Garfinkle 2002-2007; Berger's Living Review) and by analytical studies (Damour-Henneaux-Nicolai 2003; Uggla et al 2003-2007; Damour-De Buyl 2008).

BKL chaos near a big bang or a big crunch



Dynamics of BKL a, b, c system

January 1968, here at the Institut Henri Poincaré, Isaak Khalatnikov gives a seminar in which he announces to the western world the results of BKL. He shows the system of equations for the three local scale factors a, b, c [with new time variable $d\tau = -dt/(abc)$]

$$2\frac{d^2 \ln a}{d\tau^2} = (b^2 - c^2)^2 - a^4$$

$$2\frac{d^2 \ln b}{d\tau^2} = (c^2 - a^2)^2 - b^4$$

$$2\frac{d^2 \ln c}{d\tau^2} = (a^2 - b^2)^2 - c^4$$

J.A. Wheeler was in the audience and immediately pointed out the possibility of a mechanical analogy for this model. He informed his former student Charles Misner (who was independently working on the Bianchi IX dynamics) of the BKL results. In 1969 Misner published a mechanical-like, Lagrangian analysis of the Bianchi IX (a, b, c) system under the catchy name of “mixmaster universe”.

Cosmological Billiards

(Misner 1969a [classical], 1969b [quantum], Chitre 1972 [unpublished], . . ., Damour-Henneaux-Nicolai 2003)

$$ds^2 = -dt^2 + (a^2\,\ell_i\,\ell_j + b^2\,m_i\,m_j + c^2\,n_i\,n_j)dx^i\,dx^j$$

exponential parametrisation: $a = e^{-\beta^1}$, $b = e^{-\beta^2}$, $c = e^{-\beta^3}$

Lagrangian ruling the dynamics of the β's at each spatial point

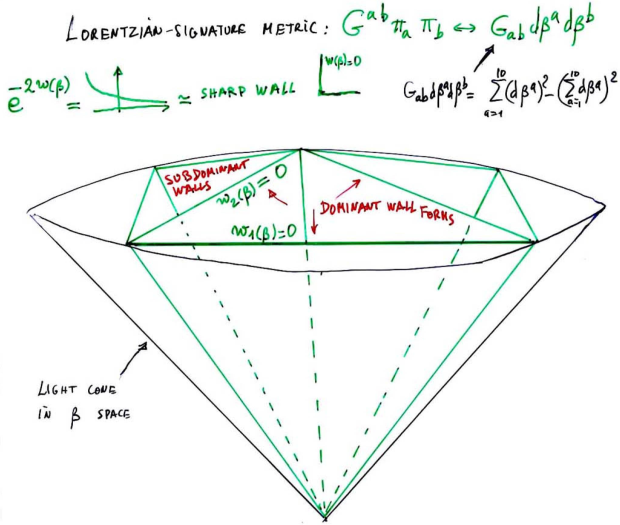
$$\mathcal{L} = \frac{1}{2}\,G_{ab}\,\dot\beta^a\,\dot\beta^b - V(\beta)$$

Kinetic metric $G_{ab}\,\dot\beta^a\,\dot\beta^b = \sum_a (\dot\beta^a)^2 - \left(\sum_a \dot\beta^a\right)^2$ (DeWitt metric)

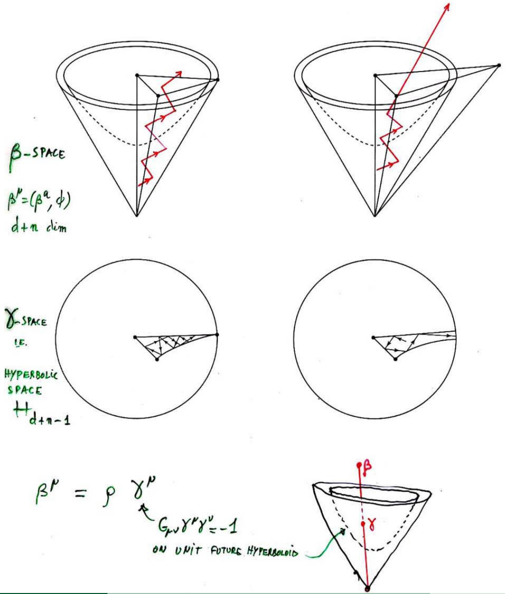
Potential $V(\beta) = \sum_a c_A(\ldots)\,e^{-2w_A(\beta)}$

Wall forms $w_A(\beta)$: e.g. gravitational walls: $w^{(g)}_{abc}(\beta) = \sum_e \beta^e + \beta^a - \beta^b - \beta^c$

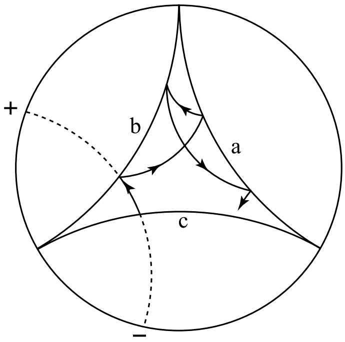
Billiard in β space

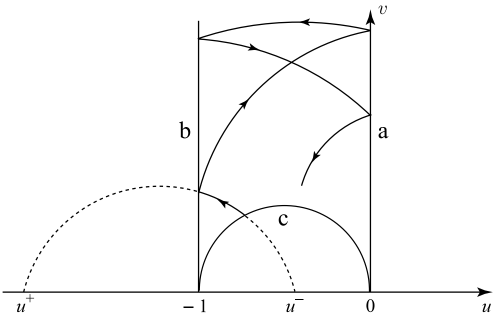


Einstein Billiards (chaotic versus non-chaotic)



Chaotic billiard for $D = 4$ gravity (BKL, Misner, Chitre)





Non-chaotic Billiards

Asymptotically Kasner-like; amenable to Fuchsian analysis

$D = 4$ gravity $+$ scalar field (Belinsky-Khalatnikov 73, Andersson-Rendall 01)

$D \geq 11$ pure gravity (Demaret et al 85, Damour-Henneaux-Rendall-Weaver 02)

Kac-Moody algebras

Generalization of the well-known "triangular" structure of $A_1 = so(3) = su(2) = sl(2)$: diagonalizable (Cartan) generator: J_z, and raising/lowering generators: $J_\pm = J_x \pm i\, J_y$ with $[J_z, J_+] = +J_+\,; \quad [J_z, J_-] = -J_-\,; \quad [J_+, J_-] = 2\, J_z$

Rank r: r mutually commuting Cartan generators h_i and r simple raising (e_i) and lowering (f_i) generators:

$$[h_i, h_j] = 0\,; [h_i, e_j] = A_{ij}\, e_j\,; \quad [h_i, f_j] = -A_{ij}\, f_j\,; \quad [e_i, f_j] = \delta_{ij}\, h_j$$

Serre relations: $ad_{e_i}^{1-A_{ij}}\, e_j = 0$; $ad_{f_i}^{1-A_{ij}}\, f_j = 0$

A_{ij} = Cartan matrix: $A_{ii} = +2$, $A_{ij} \in -\mathbb{N}$

Roots: α = linear form on Cartan: $h = \sum_i \beta^i h_i \to \alpha(h) = \alpha_i \beta^i$

$$E_\alpha \sim [e_{i_1}[e_{i_2}[e_{i_3}, \ldots]]] \qquad \alpha = n_1 \alpha^{(1)} + n_2 \alpha^{(2)} + \ldots + n_r \alpha^{(r)}$$

$$e_i = E_{\alpha^{(i)}} \text{ simple roots;} \quad \left[h, E_\alpha^{(s)}\right] = \alpha(h)\, E_\alpha^{(s)} \qquad A_{ij} = \frac{2(\alpha^{(i)}, \alpha^{(j)})}{(\alpha^{(i)}, \alpha^{(i)})}$$

Dynkin Diagrams (= Cartan Matrix) of E_{10} and AE_3

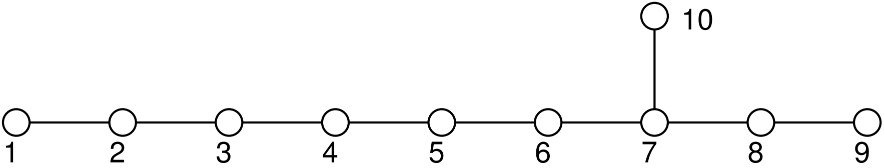
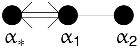


Figure: *Dynkin diagram of E_{10} with numbering of nodes.*

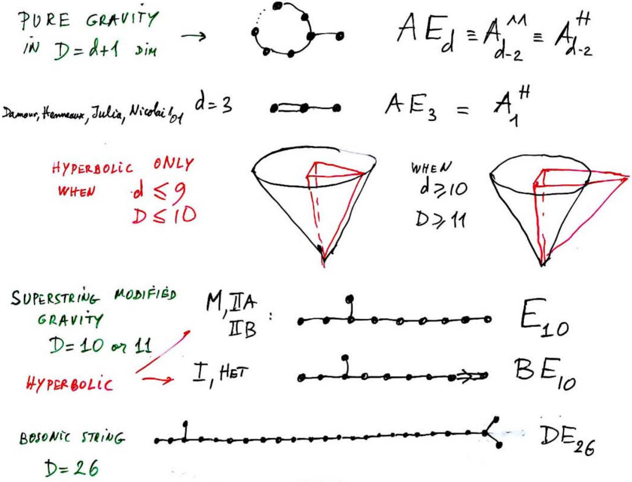
Cartan matrix of AE_3: $(A_{ij}) = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$



Dynkin diagram AE_3

Cosmological Singularities and Hyperbolic Kac-Moody Algebras: Billiard Walls = Kac-Moody Roots + much deeper gravity/coset correspondence

Damour, Henneaux 2001; Damour, Henneaux, Julia, Nicolai 2001; Damour, Henneaux, Nicolai 2002



Bosonic EOM of SUGRA$_{11}$

$D = 11$ spacetime, zero-shift slicing ($N^i = 0$) time-independent spatial coframe $\theta^a(x) \equiv E^a{}_i(x) dx^i$, $i = 1, \dots, 10$; $a = 1, \dots, 10$ choose time coordinate x^0 s.t. lapse $N = \sqrt{G}$ with $G := \det G_{ab}$

structure constants of frame: $d\theta^a = \frac{1}{2} C^a{}_{bc}\, \theta^b \wedge \theta^c$; frame derivative $\partial_a \equiv E^i{}_a(x)\partial_i$; 3-form $\mathcal{A}$; 4-form $\mathcal{F} = d\mathcal{A}$; $2G_{ad}\Gamma^d_{bc} = C_{abc} + C_{bca} - C_{cab} + \partial_b G_{ca} + \partial_c G_{ab} - \partial_a G_{bc}$

$$ds^2 = -N^2 (dx^0)^2 + G_{ab}\theta^a\theta^b$$

$$\mathcal{F} = \frac{1}{3!}\mathcal{F}_{0abc}\, dx^0 \wedge \theta^a \wedge \theta^b \wedge \theta^c + \frac{1}{4!}\mathcal{F}_{abcd}\,\theta^a \wedge \theta^b \wedge \theta^c \wedge \theta^d$$

$$\partial_0(G^{ac}\partial_0 G_{cb}) = \frac{1}{6} G\mathcal{F}^{a\beta\gamma\delta}\mathcal{F}_{b\beta\gamma\delta} - \frac{1}{72} G\mathcal{F}^{\alpha\beta\gamma\delta}\mathcal{F}_{\alpha\beta\gamma\delta}\delta^a_b - 2GR^a_b(\Gamma, C)$$

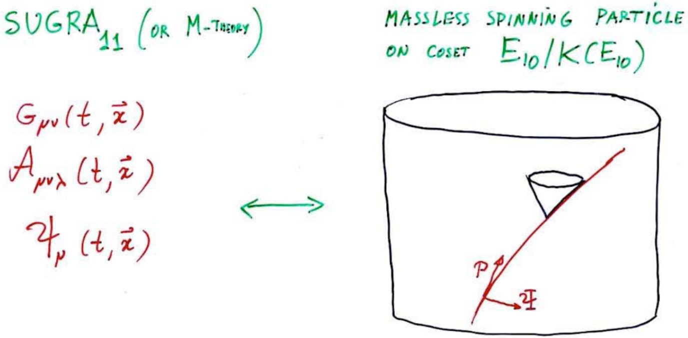
$$\begin{aligned} \partial_0(G\mathcal{F}^{0abc}) &= \frac{1}{144}\varepsilon^{abca_1a_2a_3b_1b_2b_3b_4}\mathcal{F}_{0a_1a_2a_3}\mathcal{F}_{b_1b_2b_3b_4} \\ &+ \frac{3}{2} G\mathcal{F}^{de[ab}C^{c]}{}_{de} - GC^e{}_{de}\mathcal{F}^{dabc} - \partial_d(G\mathcal{F}^{dabc}) \end{aligned}$$

$$\partial_0\mathcal{F}_{abcd} = 6\mathcal{F}_{0e[ab}C^e{}_{cd]} + 4\,\partial_{[a}\mathcal{F}_{0bcd]}$$

Gravity/Kac-Moody Coset Correspondence

Appearance of E_{10} in the “near cosmological singularity limit” (where a Belinski-Khalatnikov-Lifshitz chaotic behavior arises) suggests the existence of a supergravity/E_{10} coset correspondence (Damour, Henneaux, Nicolai ′02)

[related suggestions: E_{10}, Ganor ′99 ′04; E_{11} : West ′01]



Gravity/coset correspondence

(super)gravity $\leftrightarrow$ massless (spinning) particle on G/K

$g(t) \in G/K$; velocity $v \equiv \partial_t g\, g^{-1} \in \mathrm{Lie}(G)$ is decomposed into $v = \mathcal{P} + \mathcal{Q}$ where $\mathcal{Q} \in \mathrm{Lie}(K)$ and $\mathcal{P} = v^{\mathrm{sym}} = \frac{1}{2}(v + v^T) \in \mathrm{Lie}(G) - \mathrm{Lie}(K)$

Coset Action for massless particle:

$$S^{\mathrm{coset}}_{1_{\mathrm{BOS}}} = \int \frac{dt}{n(t)}\, \frac{1}{4}\, \langle \mathcal{P}(t), \mathcal{P}(t) \rangle$$

$n(t)$: coset lapse $\rightarrow$ constraint $\langle \mathcal{P}(t), \mathcal{P}(t) \rangle = 0$

For hyperbolic (or more generally Lorentzian) Kac-Moody algebras the coset G/K is an infinite dimensional Lorentzian space of signature $- + + + + + \ldots$

Evidence for gravity/coset correspondence

Damour, Henneaux, Nicolai 02; Damour, Kleinschmidt, Nicolai 06; de Buyl, Henneaux, Paulot 06; Kleinschmidt, Nicolai 06

Insert in $S_1^{\mathrm{COSET}} = \int dt \{ \frac{1}{4n(t)} \langle P(t), P(t) \rangle - \frac{i}{2} (\Psi(t) \mid \mathcal{D}^{\mathrm{vs}} \Psi(t))_{\mathrm{vs}} + \ldots \}$ the $GL(10)$ level expansion of the coset element

$$g(t) = \exp(h_b^a(t)\, K_a^b) \times$$

$$\times \exp\left[\frac{1}{3!} A_{abc}(t)\, E^{abc} + \frac{1}{6!} A_{a_1 \ldots a_6}(t) E^{a_1 \ldots a_6} + \frac{1}{9!} A_{a_0 | a_1 \ldots a_8}(t)\, E^{a_0 | a_1 \ldots a_8} + \ldots \right].$$

Agreement (up to height 29) of EOM of $g^{ab}(t) = (e^h)^a_c (e^h)^b_c$, $A_{abc}(t)$, $A_{a_1 \ldots a_6}(t)$, $A_{a_0 | a_1 \ldots a_8}(t)$, and $\Psi_a^{\mathrm{coset}}(t)$ with supergravity EOM (including lowest spatial gradients) for $G_{\mu\nu}(t, \mathbf{x}), \mathcal{A}_{\mu\nu\lambda}(t, \mathbf{x}), \psi_\mu(t, \mathbf{x})$ with dictionary:

$g^{ab}(t) = G^{ab}(t, \mathbf{x}_0), \qquad \dot{A}_{abc}(t) = \mathcal{F}_{0abc}(t, \mathbf{x}_0),$

$DA^{a_1 \ldots a_6}(t) = -\frac{1}{4!} \varepsilon^{a_1 \ldots a_6 b_1 \ldots b_4} \mathcal{F}_{b_1 \ldots b_4}(t, \mathbf{x}_0),$

$DA^{b | a_1 \ldots a_8}(t) = \frac{3}{2} \varepsilon^{a_1 \ldots a_8 b_1 b_2} C^b_{b_1 b_2}(t, \mathbf{x}_0)$

and $\Psi_a^{\mathrm{coset}}(t) = G^{1/4} \psi_a(t, \mathbf{x}_0)$

Moreover, $\exists$ roots in E_{10} formally associated with the infinite towers of higher spatial gradients of $G_{\mu\nu}(t, \mathbf{x}), \mathcal{A}_{\mu\nu\lambda}(t, \mathbf{x}), \psi_\mu(t, \mathbf{x})$

$K(E_{10})$ Structure of Gravitino Eq. of Motion

In the gauge $\psi_0^{(11)} = \Gamma_0 \Gamma^a \psi_a^{(11)}$, the equation of motion of the rescaled gravitino $\psi_a^{(10)} := g^{1/4} \psi_a^{(11)}$ (neglecting cubic terms) reads

$$
\begin{aligned}
\mathcal{E}_a \;=\;& \partial_t \psi_a^{(10)} + \omega_{t\,ab}^{(11)} \psi^{(10)b} + \frac{1}{4} \omega_{t\,cd}^{(11)} \Gamma^{cd} \psi_a^{(10)} \\
& - \frac{1}{12} F_{tbcd}^{(11)} \Gamma^{bcd} \psi_a^{(10)} - \frac{2}{3} F_{tabc}^{(11)} \Gamma^b \psi^{(10)c} + \frac{1}{6} F_{tbcd}^{(11)} \Gamma_a{}^{bc} \psi^{(10)d} \\
& + \frac{N}{144} F_{bcde}^{(11)} \Gamma^0 \Gamma^{bcde} \psi_a^{(10)} + \frac{N}{9} F_{abcd}^{(11)} \Gamma^0 \Gamma^{bcde} \psi_e^{(10)} - \frac{N}{72} F_{bcde}^{(11)} \Gamma^0 \Gamma_{abcdef} \psi^{(10)f} \\
& + N(\omega_{a\,bc}^{(11)} - \omega_{b\,ac}^{(11)}) \Gamma^0 \Gamma^b \psi^{(10)c} + \frac{N}{2} \omega_{a\,bc}^{(11)} \Gamma^0 \Gamma^{bcd} \psi_d^{(10)} - \frac{N}{4} \omega_{b\,cd}^{(11)} \Gamma^0 \Gamma^{bcd} \psi_a^{(10)} \\
& + N g^{1/4} \Gamma^0 \Gamma^b \Big(2 \partial_a \psi_b^{(11)} - \partial_b \psi_a^{(11)} - \frac{1}{2} \omega_{c\,cb}^{(11)} \psi_a^{(11)} - \omega_{0\,0a}^{(11)} \psi_b^{(11)} + \frac{1}{2} \omega_{0\,0b}^{(11)} \psi_a^{(11)} \Big).
\end{aligned}
$$

Apart from the last line, this is equivalent to the $K(E_{10})$-covariant equation

$$
0 = \overset{\text{vs}}{\mathcal{D}} \Psi(t) := \Big(\partial_t - \overset{\text{vs}}{\mathcal{Q}}(t) \Big) \Psi(t).
$$

expressing the parallel propagation of the $K(E_{10})$ vector-spinor $\Psi(t)$ along the $E_{10}/K(E_{10})$ worldline of the coset particle, with the $K(E_{10})$ connection $\mathcal{Q}(t) := \frac{1}{2}(v(t) - v^T(t)) \in \mathrm{Lie}(\mathrm{K}(\mathrm{E}_{10}))$, with $v(t) = \partial_t g g^{-1} \in \mathfrak{e}_{10} \equiv \mathrm{Lie}(E_{10})$.

Quantum Supersymmetric Bianchi IX (Damour-Spindel'13, '14)

Quantum version of Bianchi IX $N = 1$ supergravity: dof 6 $g_{ab}(t) +$ 12 $\psi^A_{\hat{\alpha}}(t)$ gravitino

Quantization:

$$\widehat{\pi}_a = -i \frac{\partial}{\partial \beta^a} \, ; \quad \widehat{p}_{\varphi_a} = -i \frac{\partial}{\partial \varphi_a}$$

$$\widehat{\Phi}^a_A \, \widehat{\Phi}^b_B + \widehat{\Phi}^b_B \, \widehat{\Phi}^a_A = G^{ab} \, \delta_{AB}$$

- The wave function of the universe $\Psi_\sigma(\beta^a, \varphi_a)$ is a 64-dimensional spinor of $\mathrm{Spin}\,(8,4)$ and the gravitino operators Φ^a_A are 64×64 "gamma matrices" acting on Ψ_σ, $\sigma = 1, \ldots, 64$

Supersymmetric action (first order form)

$$S = \int dt \left[\pi_a \, \dot{\beta}^a + p_{\varphi^a} \, \dot{\varphi}^a + \frac{i}{2} \, G_{ab} \, \Phi_A^a \, \dot{\Phi}_A^b + \bar{\Psi}'^A_{\hat{0}} \, \mathcal{S}_A - \tilde{N} H - N^a H_a \right]$$

G_{ab}: Lorentzian-signature quadratic form:

$$G_{ab} \, d\beta^a \, d\beta^b \equiv \sum_a (d\beta^a)^2 - \left(\sum_a d\beta^a \right)^2$$

G_{ab} defines the kinetic terms of the gravitino, as well as those of the β^a's:

$$\frac{1}{2} \, G_{ab} \, \dot{\beta}^a \, \dot{\beta}^b$$

Lagrange multipliers $\longrightarrow$ Constraints $\mathcal{S}_A \approx 0$, $H \approx 0$, $H_a \approx 0$

Kac-Moody Structures Hidden in the Quantum Hamiltonian

$$2\,\widehat{H} = G^{ab}(\widehat{\pi}_a + i\,A_a)(\widehat{\pi}_b + i\,A_b) + \widehat{\mu}^2 + W_g^{\rm bos}(\beta) + \widehat{W}_g^{\rm spin}(\beta) + \widehat{W}_{\rm sym}^{\rm spin}(\beta)\,.$$

$G_{ab} \leftrightarrow$ metric in Cartan subalgebra of AE_3

$$W_g^{\rm bos}(\beta) = \frac{1}{2}\,e^{-2\alpha_{11}^g(\beta)} - e^{-2\alpha_{23}^g(\beta)} + {\rm cyclic}_{123}$$

$$\widehat{W}_g^{\rm spin}(\beta, \widehat{\Phi}) = e^{-\alpha_{11}^g(\beta)}\widehat{J}_{11}(\widehat{\Phi}) + e^{-\alpha_{22}^g(\beta)}\widehat{J}_{22}(\widehat{\Phi}) + e^{-\alpha_{33}^g(\beta)}\widehat{J}_{33}(\widehat{\Phi})\,.$$

Linear forms $\alpha_{ab}^g(\beta) = \beta^a + \beta^b \Leftrightarrow$ six level-1 roots of AE_3

$$\widehat{W}_{\rm sym}^{\rm spin}(\beta) = \frac{1}{2}\,\frac{(\widehat{S}_{12}(\widehat{\Phi}))^2 - 1}{\sinh^2 \alpha_{12}^{\rm sym}(\beta)} + {\rm cyclic}_{123}\,,$$

Linear forms $\alpha_{12}^{\rm sym}(\beta) = \beta^1 - \beta^2$, $\alpha_{23}^{\rm sym}(\beta) = \beta^2 - \beta^3$, $\alpha_{31}^{\rm sym}(\beta) = \beta^3 - \beta^1$ $\Leftrightarrow$ three level-0 roots of AE_3

Spin dependent (Clifford) Operators coupled to AE_3 roots

$$
\begin{aligned}
\widehat{S}_{12}(\widehat{\Phi}) &= \frac{1}{2}[(\bar{\widehat{\Phi}}^3\,\gamma^{\hat{0}\hat{1}\hat{2}}(\widehat{\Phi}^1+\widehat{\Phi}^2)) + (\bar{\widehat{\Phi}}^1\,\gamma^{\hat{0}\hat{1}\hat{2}}\,\widehat{\Phi}^1) \\
&+ (\bar{\widehat{\Phi}}^2\,\gamma^{\hat{0}\hat{1}\hat{2}}\,\widehat{\Phi}^2) - (\bar{\widehat{\Phi}}^1\,\gamma^{\hat{0}\hat{1}\hat{2}}\,\widehat{\Phi}^2)]\,,
\end{aligned}
$$

$$
\widehat{J}_{11}(\widehat{\Phi}) = \frac{1}{2}[\;\bar{\widehat{\Phi}}^1\gamma^{\hat{1}\hat{2}\hat{3}}(4\widehat{\Phi}^1+\widehat{\Phi}^2+\widehat{\Phi}^3) + \;\bar{\widehat{\Phi}}^2\,\gamma^{\hat{1}\hat{2}\hat{3}}\,\widehat{\Phi}^3]\,.
$$

- $\widehat{S}_{12}$, $\widehat{S}_{23}$, $\widehat{S}_{31}$, $\widehat{J}_{11}$, $\widehat{J}_{22}$, $\widehat{J}_{33}$ generate (via commutators) a 64-dimensional representation of the (infinite-dimensional) "maximally compact" sub-algebra $K(AE_3) \subset AE_3$. [The fixed set of the (linear) Chevalley involution, $\omega(e_i) = -f_i$, $\omega(f_i) = -e_i$, $\omega(h_i) = -h_i$, which is generated by $x_i = e_i - f_i$.]

Solution space of quantum susy Bianchi IX: $N_F = 0$

Level $N_F = 0$: $\exists$ unique "ground state" $|f\rangle = C\, f_0(\beta)\, |0\rangle_-$ with

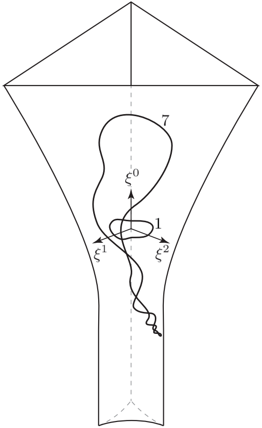
$$f_0(\beta) = abc\left[(b^2 - a^2)(c^2 - b^2)(c^2 - a^2)\right]^{3/8} e^{-\frac{1}{2}(a^2+b^2+c^2)}|\,0\,\rangle_-$$

This "ground state" (similar to the non susy ground state of Moncrief-Ryan 91) is localized in the middle of β space (or of a Weyl chamber) and decays in all directions in β space: small volume, large volume, large anisotropies. It describes a quantum universe which oscillates in shape and size, but stays of Planckian size

$\exists$ similar "discrete-spectrum" states at $N_F = 1, 2, 4, 5, 6$; however, it is only at levels $N_F = 0$ and 1 that these states decay in all directions and are square integrable at the symmetry walls.

Classical Bottle Effect

Classical confinement between $\mu^2 < 0$ for small volumes, and the usual closed-universe recollapse (Lin-Wald) for large volumes $\Rightarrow$ periodic, cyclically bouncing, solutions (Christiansen-Rugh-Rugh 95).

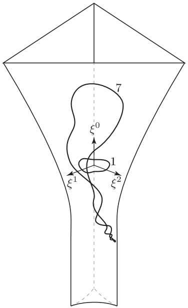


Quantum Bottle Effect ?

We conjecture the existence of a set of discrete quantum states (decaying in all directions in β space), corresponding (à la Selberg-Gutwiller) to the classical periodic solutions ? These would be excited avatars of the $N_F = 0$ "ground state"

$$\Psi_0 = (abc)\left[(b^2 - a^2)(c^2 - b^2)(c^2 - a^2)\right]^{3/8} e^{-\frac{1}{2}(a^2+b^2+c^2)} |\, 0\,\rangle_-$$

and define a kind of quantum storage ring of near-singularity states (ready for tunnelling, via inflation, toward large universes).



A Mathematically Precise Formulation of the BKL Conjecture

(Damour-Henneaux-Nicolai 2003, Damour-De Buyl 2008)

Technical tools (in spacetime dimension $D = d+1$; for simplicity for pure gravity $D \leq 10$)

- a quasi-Gaussian coordinate system (τ, x^i) with vanishing "shift" and a unit rescaled lapse $\tilde{N} = 1$ in $ds^2 = -(\tilde{N}\sqrt{g}d\tau)^2 + g_{ij}(\tau, x^k)\omega^i(x)\omega^j(x)$ where $\omega^i(x)$ is a time-independent coframe
- parametrize $d(d+1)/2 g_{ij}(\tau, x)$ by d "diagonal" dof $\beta^a(\tau, x)$ and $d(d-1)/2$ "off diagonal" dof $\mathcal{N}_i^a(\tau, x)$ (upper triangular matrix with $\mathcal{N}_i^i = 1$, $\mathcal{N}_i^a = 0$ if $i < a$) s.t.

$$g_{ij} = \sum_{a=1}^{d} e^{-2\beta^a} \mathcal{N}_i^a \mathcal{N}_j^a \quad \text{("Iwasawa decomposition")}$$

- use Arnowitt-Deser-Misner Hamiltonian formalism, i.e. first-order-in-time evolution system for

$$\beta^a(\tau, x), \pi_a(\tau, x), \mathcal{N}_i^a(\tau, x), \mathcal{P}_a^i(\tau, x) \quad \text{("conjugate momenta")}$$

More generally (with p-forms): $(\beta^a, \pi_a)(Q, P)$

Hamilton Evolution System in Iwasawa Variables

$$
\begin{aligned}
\mathcal{H}[\beta, Q; \pi, P] &= \mathcal{K} + \mathcal{V} \\
&= \frac{1}{4} G^{ab} \pi_a \pi_b + \sum_A c_A(Q, P, \partial_x \beta, \partial_x^2 \beta, \partial Q, \partial^2 Q) e^{-2 w_A(\beta)}
\end{aligned}
$$

$$
\begin{aligned}
\partial_\tau \beta^a &= \frac{1}{2} G^{ab} \pi_b \,, \\
\partial_\tau \pi_a &= \sum_A \left(2 c_A w_{A\,a} e^{-2 w_A(\beta)} + \partial_x \left(\frac{\partial c_A}{\partial \partial_x \beta^a} e^{-2 w_A(\beta)} \right) - \partial_x^2 \left(\frac{\partial c_A}{\partial \partial_x^2 \beta^a} e^{-2 w_A(\beta)} \right) \right) , \\
\partial_\tau Q &= \sum_A \frac{\partial c_A}{\partial P} e^{-2 w_A(\beta)} \,, \\
\partial_\tau P &= \sum_A \left(- \frac{\partial c_A}{\partial Q} e^{-2 w_A(\beta)} + \partial_x \left(\frac{\partial c_A}{\partial \partial_x Q} e^{-2 w_A(\beta)} \right) - \partial_x^2 \left(\frac{\partial c_A}{\partial \partial_x^2 Q} e^{-2 w_A(\beta)} \right) \right) ,
\end{aligned}
$$

Conjectured Behaviour of Iwasawa Variables

- All the "non-diagonal variables" (Q, P) [i.e. $\mathcal{N}, \mathcal{P}, \mathcal{A}_p, \pi_A = \mathcal{E}^p$] generically have limits on the singularity ($\tau \to +\infty$, with fixed spatial coordinates x^i)

$$Q_{(0)}(x) = \lim_{\tau\to+\infty} Q(\tau, x) \quad P_{(0)}(x) = \lim_{\tau\to+\infty} P(\tau, x)$$

- By contrast, the $2d$ "diagonal variables" $\beta^a(\tau, x), \pi_a(\tau x)$ have no limits (in chaotic case) but their asymptotic behaviour as $\tau \to +\infty$ can be described by a certain first-order-in-τ system of ODE's: the "asymptotic evolution system" (which is Toda-like)

$$\begin{aligned} \partial_\tau \beta_{(0)} &= \frac{1}{2}\pi_{(0)} \\ \partial_\tau \pi_{(0)} &= \sum_{\mathcal{A}} 2c_{\mathcal{A}}(Q_{(0)}, P_{(0)}, \partial_x Q_{(0)}) w_{\mathcal{A}} e^{-2w_{\mathcal{A}}(\beta_{(0)})} \\ \partial_\tau Q_{(0)} &= 0 \\ \partial_\tau P_{(0)} &= 0\,. \end{aligned}$$

"Chaotic analog" of the Asymptotically Velocity Term Dominated evolution (1/2)

For pure gravity

$$\begin{aligned}
\partial_\tau \beta^a_{(0)} &= \frac{1}{2} G^{ab} \pi_b^{(0)} , \\
\partial_\tau \pi_a^{(0)} &= -\frac{\partial}{\partial \beta^a_{(0)}} \left[\mathcal{V}_S^{\text{asymp}}(\beta_{(0)}; \mathcal{P}_{(0)}, \mathcal{N}_{(0)}) \right. \\
&\quad \left. + \mathcal{V}_G^{\text{asymp}}(\beta_{(0)}; \mathcal{P}_{(0)}, \mathcal{N}_{(0)}, \partial_x \mathcal{N}_{(0)}) \right] , \\
\partial_\tau \mathcal{N}^a_{(0)i} &= 0 , \\
\partial_\tau \mathcal{P}^i_{(0)a} &= 0 .
\end{aligned}$$

with

$$\mathcal{V}_S^{\text{asymp}} = \frac{1}{2} \sum_{a=1}^{d-1} e^{-2(\beta^{a+1} - \beta^a)} (\mathcal{P}^i_{(0)a} \mathcal{N}^{a+1}_{(0)i})^2 ,$$

and

$$\mathcal{V}_G^{\text{asymp}} = \frac{1}{2} e^{-2\alpha_{1d-1d}(\beta)} (C^1_{(0)d-1d})^2 .$$

where $\alpha_{abc}(\beta) = \beta^a + \sum_{e \neq b,c} \beta^e$

“Chaotic analog” of the Asymptotically Velocity Term Dominated evolution (2/2)

and where $C^a_{(0)bc} = -C^a_{(0)cb}$ denote the structure functions ($d\theta^a_{(0)} = -\frac{1}{2}\, C^a_{(0)bc}\, \theta^b_{(0)\wedge}\, \theta^c_{(0)}$) of the “asymptotic Iwasawa frame” $\theta^a_{(0)}(x) = \mathcal{N}^a_{(0)i}(x)\,\omega^i$.

This evolution system must be completed by the “asymptotic constraints”

$$\mathcal{H}^{\rm asymp}(\beta_{(0)}, \pi_{(0)}, \mathcal{N}_{(0)}, \partial_x \mathcal{N}_{(0)}, \mathcal{P}_{(0)}) = 0\,,$$
$$\mathcal{H}^{\rm asymp}_a(\mathcal{N}_{(0)}, \partial_x \mathcal{N}_{(0)}, \mathcal{P}_{(0)}) = 0\,,$$

The constraints are preserved by the asymptotic evolution system.

BKL conjecture in Iwasawa variables

Let, for $x \in U$, $(\beta_{(0)}(\tau,x), \pi_{(0)}(\tau,x), \mathcal{N}_{(0)}(\tau,x), \mathcal{P}_{(0)}(\tau,x))$ be a solution of the asymptotic evolution system , satisfying the asymptotic constraints, and such that the d x-dependent coefficients $\mathcal{P}_{(0)}$ $\mathcal{N}_{(0)}$ and $C_{(0)}$ (whose squares define the coefficients of the d exponential potential terms) do not vanish in the considered spatial domain (this avoid "spikes"). Then there exists a unique solution $(\beta(\tau,x), \pi(\tau,x), \mathcal{N}(\tau,x), \mathcal{P}(\tau,x))$ of the vacuum Einstein equations (including the constraints) such that the differences $\bar{\beta}(\tau,x) \equiv \beta(\tau,x) - \beta_{(0)}(\tau,x)$, $\bar{\pi}(\tau,x) \equiv \pi(\tau,x) - \pi_{(0)}(\tau,x)$, $\bar{\mathcal{N}}(\tau,x) \equiv \mathcal{N}(\tau,x) - \mathcal{N}_{(0)}(\tau,x)$, $\bar{\mathcal{P}}(\tau,x) \equiv \mathcal{P}(\tau,x) - \mathcal{P}_{(0)}(\tau,x)$ tend to zero as $x \in U$ is fixed and $\tau \to +\infty$.

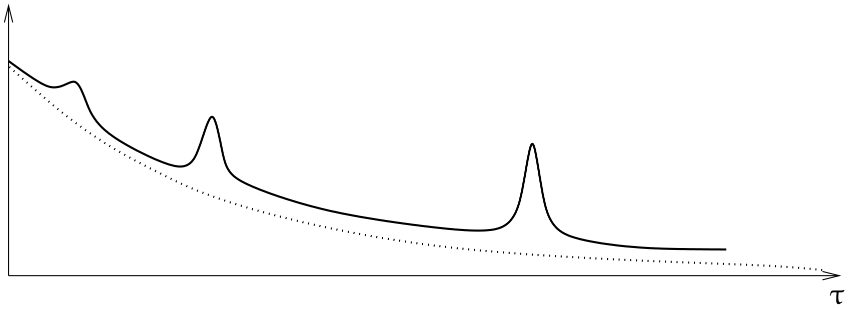
Physicist' proof: $\exists$ "generalized Fuchsian system" for the differenced variables

$$\bar{\beta}^a(\tau,x) \equiv \beta^a(\tau,x) - \beta^a_{(0)}(\tau,x), \bar{\pi}_a \equiv \pi_a - \pi_{(0)a}, \bar{Q} \equiv Q - Q_{(0)}, \bar{P} \equiv P - P_{(0)}$$

Generalized Fuchsian System for Differenced Variables

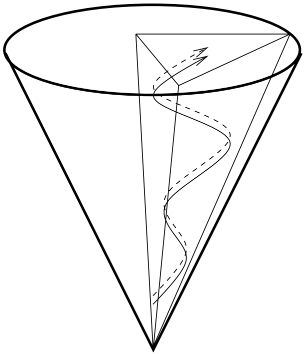
$$
\begin{aligned}
\partial_\tau \bar{\beta} - \frac{1}{2}\bar{\pi} &= 0 \\
\partial_\tau \bar{\pi} &= 2\sum_{\mathcal{A}} w_{\mathcal{A}} e^{-2w_{\mathcal{A}}(\beta_{[0]})} (c_{\mathcal{A}} e^{-2w_{\mathcal{A}}(\bar{\beta})} - c_{\mathcal{A}}(Q_{[0]}, P_{[0]}, \partial_x Q_{[0]})) \\
&+ 2\sum_{\mathcal{A}'} c_{\mathcal{A}'} w_{\mathcal{A}'} e^{-2w_{\mathcal{A}'}(\beta_{[0]})} e^{-2w_{\mathcal{A}'}(\bar{\beta})} \\
&+ \sum_{\mathcal{A}'} \partial_x \Big(\frac{\partial c_{\mathcal{A}'}}{\partial \partial_x \beta} e^{-2w_{\mathcal{A}'}(\beta_{[0]})} e^{-2w_{\mathcal{A}'}(\bar{\beta})} \Big) \\
&- \sum_{\mathcal{A}'} \partial_x^2 \Big(\frac{\partial c_{\mathcal{A}'}}{\partial \partial_x^2 \beta} e^{-2w_{\mathcal{A}'}(\beta_{[0]})} e^{-2w_{\mathcal{A}'}(\bar{\beta})} \Big)) \\
\partial_\tau \bar{Q} &= \sum_{\mathcal{A}} \frac{\partial c_{\mathcal{A}}}{\partial P} e^{-2w_{\mathcal{A}}(\beta_{[0]})} e^{-2w_{\mathcal{A}}(\bar{\beta})} \\
&+ \sum_{\mathcal{A}'} \frac{\partial c_{\mathcal{A}'}}{\partial P} e^{-2w_{\mathcal{A}'}(\beta_{[0]})} e^{-2w_{\mathcal{A}'}(\bar{\beta})} \\
\partial_\tau \bar{P} &= \sum_{\mathcal{A}} \left(-\frac{\partial c_{\mathcal{A}}}{\partial Q} e^{-2w_{\mathcal{A}}(\beta_{[0]})} e^{-2w_{\mathcal{A}}(\bar{\beta})} + \partial_x \Big(\frac{\partial c_{\mathcal{A}}}{\partial \partial_x Q} e^{-2w_{\mathcal{A}}(\beta_{[0]})} e^{-2w_{\mathcal{A}}(\bar{\beta})} \Big) \right) \\
&+ \sum_{\mathcal{A}'} \left(-\frac{\partial c_{\mathcal{A}'}}{\partial Q} e^{-2w_{\mathcal{A}'}(\beta_{[0]})} e^{-2w_{\mathcal{A}'}(\bar{\beta})} + \partial_x \Big(\frac{\partial c_{\mathcal{A}'}}{\partial \partial_x Q} e^{-2w_{\mathcal{A}'}(\beta_{[0]})} e^{-2w_{\mathcal{A}'}(\bar{\beta})} \Big) \right) \\
&- \sum_{\mathcal{A}'} \partial_x^2 \Big(\frac{\partial c_{\mathcal{A}'}}{\partial \partial_x^2 Q} e^{-2w_{\mathcal{A}'}(\beta_{[0]})} e^{-2w_{\mathcal{A}'}(\bar{\beta})} \Big) ,
\end{aligned}
$$

Schematic Behaviour of the Source Term of the Generalized Fuchsian System (replacing usual $e^{-\mu\tau}$)



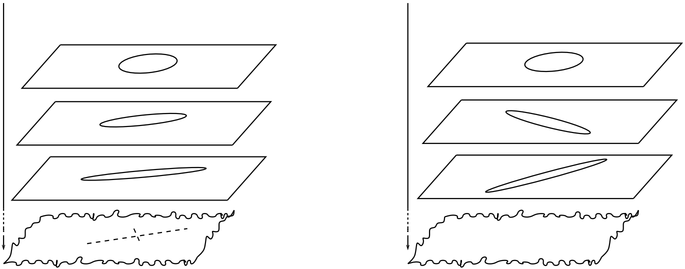
Asymptotic dynamics of diagonal variables at a given spatial point x

Idea: For each solution $\beta_{(0)}\pi_{(0)}Q_{(0)}P_{(0)}$ of the chaotic asymptotic evolution system there is a *unique* solution $\{\bar{\beta}, \bar{\pi}, \bar{Q}, \bar{P}\}$ of the generalized-Fuchsian differenced system that tends to *zero* as $\tau \to +\infty$



∃ ? well-defined asymptotic geometrical structure on the singularity?

non chaotic chaotic



Asymptotic Geometrical Structure in the Nonchaotic case

In the nonchaotic case, the solution is, at each spatial point, asymptotically Kasner-like

$$g_{ij}(t) = t^{2p_1} l_i l_j + t^{2p_2} m_i m_j + ... + t^{2p_d} r_i r_j \,,$$

The *Kasner coframes* (that diagonalize $k_{ij}(x)$ wrt $g_{ij}(x)$) $\omega^1_K = \ell_i dx^i$, $\omega^2_K = m_i dx^i, \ldots, \omega^d_K = r_i dx^i$ have finite limits at the singularity. They are defined up to (independent) rescalings and therefore provide a basis of preferred directions on the singular hypersurface, i.e. a *directional frame* (and coframe).

At each given spatial point x the geometrical structure defined by a directional frame (a set of directions) is invariant under the subgroup of diagonal matrices of $GL(d, R)$.

Chaotic case: Partially Framed Flag (Damour-De Buyl 08)

In the chaotic case, the existence of many variables having finite limits at the singularity, $\mathcal{N}_i^a(\tau, x) \to \mathcal{N}_{(0)\,i}^{a}(x)$, $\mathcal{P}_a^i(\tau, x) \to \mathcal{P}^i_{(0)\,a}(x)$ imply some asymptotic geometrical structure on the singular hypersurface.

But $\mathcal{N}_{(0)}, \mathcal{P}_{(0)}$ depend on the choice of coframe $\omega^i(x)$.

One can act, at each given spatial point x, on ω^i by $\Lambda \in GL(d, R)$.

Look for *canonical values* of $\mathcal{N}_{(0)}, \mathcal{P}_{(0)}$ that can be assigned by using $\Lambda(x)$

Generic answer: $\mathcal{N}^a_{(0)\,i}(x) = \delta^a_i$ and

$$\mathcal{P}_{(0)} = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ \mathcal{P}^2{}_1 & 0 & \dots & 0 & 0 \\ 0 & \mathcal{P}^3{}_2 & \cdots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & & 0 & 0 \\ 0 & 0 & \dots & \mathcal{P}^d{}_{d-1} & 0 \end{pmatrix} .$$

The *stabilizer* of this canonical structure is a proper subgroup of $GL(d, \mathbb{R})$. It defines an equivalence class of directional frames that can be called "partially framed flag".

Conclusions: Through a Glass Darkly

- We described a precise formulation of the BKL conjecture in Iwasawa variables. In these variables most field quantities have limits on the singularity, except for the diagonal (billiard) variables whose asymptotic behaviour is described by a chaotic Toda-like asymptotic evolution system.
- The deviations from the solutions of the chaotic asymptotic evolution system satisfy a generalized Fuchsian system (which should be amenable to a rigorous mathematical analysis).
- $\exists$ tantalizing evidence for the presence of hidden hyperbolic Kac-Moody structures in the near spacelike singularity regime.
- At zeroth order this is revealed in the fact that the BKL-Misner-type cosmological billiard dynamics is equivalent to billiard motion in the Weyl chamber of an hyperbolic Kac-Moody algebra.
- The evidence for Kac-Moody goes much beyond (both in bosonic and fermionic EOM and in classical/quantum effects). It suggests a gravity/coset correspondence: gravity dynamics $\leftrightarrow$ massless particle on infinite-dimensional (Lorentzian-signature) Kac-Moody coset G/K.