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# ON THE DIVERIO-TRAPANI CONJECTURE

BY YA DENG

ABSTRACT. – In this paper we establish effective lower bounds on the degrees of the Debarre and Kobayashi conjectures. Then we study a more general conjecture proposed by Diverio-Trapani on the ampleness of jet bundles of general complete intersections in complex projective spaces.

RÉSUMÉ. – Dans cet article, nous établissons des bornes inférieures effectives sur les degrés liés aux conjectures de Debarre et Kobayashi. Ensuite, nous étudions une conjecture plus générale proposée par Diverio-Trapani sur l'amplitude des fibrés de jets des intersections complètes générales dans les espaces projectifs complexes.

## 0. Introduction

A compact complex manifold $X$ is said to be Kobayashi (Brody) hyperbolic if there exists no non-constant holomorphic map $f : \mathbb{C} \to X$. As is well-known, a sufficient criteria for Kobayashi hyperbolicity is the ampleness of the cotangent bundle. Although the complex manifolds with ample cotangent bundles are expected to be reasonably abundant, there are few concrete constructions before the work of Debarre. In [8], Debarre proved that the complete intersection of sufficiently ample general hypersurfaces in a *complex abelian variety*, whose codimension is at least as large as its dimension, has ample cotangent bundle. He further conjectured that this result should also hold for intersection varieties of general hypersurfaces in *complex projective spaces* (the so-called *Debarre conjecture*). This conjecture was recently proved by Brotbek-Darondeau [4] and independently by Xie [26, 25], based on the ideas and explicit methods in [2].

THEOREM 0.1 (Brotbek-Darondeau, Xie). – *Let $X$ be an $n$-dimensional projective manifold equipped with a very ample line bundle $\mathscr{A}$. Then there exists $d_{\mathrm{Deb},n} \in \mathbb{N}$ depending only on the dimension $n$, such that for all $d \geqslant d_{\mathrm{Deb},n}$, the complete intersection of $c$-general hypersurfaces $H_1, \ldots, H_c \in |\mathscr{A}^d|$ has ample cotangent bundle, provided that $\frac{n}{2} \leqslant c \leqslant n$.*

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In [25], Xie was able to obtain an effective lower bound $d_{\mathrm{Deb},n} = n^{n^2}$ by working with (much more elaborated) explicit expressions of some symmetric differential forms. The result in [4] is "almost" effective on $d_{\mathrm{Deb},n}$, because it depends on some constant involved in some noetherianity argument, arising in their reduction to Nakamaye's theorem [22] for *families of zero-dimensional subschemes*.

One goal of the present paper is to provide an effective estimate for such a Nakamaye's theorem (see Theorem 2.10). In particular, as a complement of [4, Theorem 1.1], we can improve Xie's effective lower bound $d_{\mathrm{Deb},n}$.

THEOREM A. – *In the same setting as Theorem 0.1, one can take*

$$d_{\mathrm{Deb},n} = (2n)^{n+3}.$$

It is worth to mention that the techniques in [4] are more intrinsic and the ideas of their proof brought new geometric insights in the understanding of the positivity of cotangent bundles. Later, Brotbek [3] extended these techniques from the setting of symmetric differentials to that of higher order jet differentials, so that he was able to prove a long-standing conjecture of Kobayashi in [19].

THEOREM 0.2 (Brotbek). – *Let $X$ be a projective manifold of dimension $n$. For any very ample line bundle $\mathscr{A}$ on $X$, there exists $d_{\mathrm{Kob},n} \in \mathbb{N}$ depending only on the dimension $n$ such that for any $d \geqslant d_{\mathrm{Kob},n}$, a general smooth hypersurface $H \in |\mathscr{A}^d|$ is Kobayashi hyperbolic.*

The proof of Theorem 0.2 in [3] is also "almost" effective on $d_{\mathrm{Kob},n}$ because of two noetherianity arguments: the first concerns the increasing sequences of Wronskians ideal sheaves; the second concerns a constant arising in Nakamaye's theorem as that of [4], which can be made effective by Theorem 2.2. Our second goal of the present paper is to give an intrinsic interpretation of Brotbek's Wronskians (see § 1.2), and as a byproduct, we can render the above-mentioned first noetherianity argument effective. This in turn provides effective lower bounds for the Kobayashi conjecture in combination with the explicit formula of $d_{\mathrm{Kob},n}$ in [3].

THEOREM B. – *In the same setting as Theorem 0.2, one can take*

$$d_{\mathrm{Kob},n} = n^{2n+3}(n+1).$$

Let us mention that in [3] Brotbek obtained a much stronger result than Theorem 0.2. Indeed, he proved that for the hypersurface $H$ in Theorem 0.2, the tautological line bundle $\mathscr{O}_{H_k}(a_k, \ldots, a_1)$ on the *Demailly-Semple $k$-jet tower $H_k$* of the direct manifold $(H, T_H)$ is "almost ample" for some $(a_1, \ldots, a_k) \in \mathbb{N}^k$ when $k \geqslant n-1 = \dim H$. In view of the following vanishing theorem by Diverio in [13], the above-mentioned lower bound for $k$ in [3] is optimal.

THEOREM 0.3 (Diverio). – *Let $Z \subset \mathbb{P}^n$ be a smooth complete intersection of hypersurfaces of any degree in $\mathbb{P}^n$. Then*

$$H^0(Z, E_{k,m}^{\mathrm{GG}} T_Z^*) = 0$$

*for all $m \geqslant 1$ and $1 \leqslant k < \dim(Z)/\mathrm{codim}(Z)$. Here $E_{k,m}^{\mathrm{GG}} T_Z^*$ denotes the* Green-Griffiths jet bundle *of order $k$ and weighted degree $m$.*

Motivated by the above vanishing theorem, in the same vein as the Debarre conjecture, Diverio-Trapani proposed the following generalized conjecture in [16].

CONJECTURE 0.4 (Diverio-Trapani). – *Let $Z \subset \mathbb{P}^n$ be the complete intersection of $c$-general hypersurfaces of sufficiently high degree. Then the* invariant jet bundle $E_{k,m}T_Z^*$ *is ample provided that $k \geqslant \frac{n}{c} - 1$ and $m \gg 0$.*

The last aim of the present paper is to study Conjecture 0.4 using geometric methods in [4, 3].

THEOREM C. – *Let $X$ be an $n$-dimensional projective manifold equipped with a very ample line bundle $\mathscr{A}$, and let $Z \subset X$ be the complete intersection of $c$-general hypersurfaces $H_1 \ldots, H_c \in |\mathscr{A}^d|$. Then $Z$ is* almost $k$-jet ample *(see Definition 1.2) if $k \geqslant \frac{n}{c} - 1$, and $d \geqslant 2cn^{c\lceil \frac{n}{c}\rceil+1} \cdot \lceil \frac{n}{c}\rceil^{c\lceil \frac{n}{c}\rceil+3}$. In particular, $Z$ is Kobayashi hyperbolic.*

Let us mention that we apply the results in the first part of the present paper to obtain the effective lower degree bounds in Theorem C.

In view of the correspondence between tautological line bundles on the Demailly-Semple jet towers and invariant jet bundles studied in [9, Proposition 6.16], the following result on Conjecture 0.4 is a consequence of Theorem C.

COROLLARY D. – *In the same setting as Theorem C, for any $k \geqslant \frac{n}{c} - 1$, there exists a subbundle $\mathscr{F} \subset E_{k,m}T_Z^*$ for some $m \gg 0$ such that*

(i) *$\mathscr{F}$ is ample.*
(ii) *For any regular germ of curve $f : (\mathbb{C}, 0) \to (Z, z)$, there is a global section $P \in H^0(Z, \mathscr{F} \otimes \mathscr{A}^{-1})$ so that $P([f]_k)(0) \neq 0$.*

In other words, one can find a subbundle $\mathscr{F}$ of the invariant jet bundle $E_{k,m}T_Z^*$, which is ample, and the *Demailly-Semple locus* (see [15, §2.1] for the definition) induced by $\mathscr{F}$ is empty.

Lastly, let us mention that the techniques in [4, 3] were extended by Brotbek and the author to prove a logarithmic analogue of the Debarre conjecture in [5], and to prove the logarithmic (orbifold) Kobayashi conjecture in [6]. To achieve the effective lower degree bounds, both the articles [5, 6] rely on the methods in the present paper.

This paper is organized as follows. In § 1.1 we recall the fundamental tools of jet differentials by Demailly, Green-Griffiths and Siu, which can be seen as higher order analogues of symmetric differential forms and provide obstructions to the existence of entire curves. § 1.2 is devoted to the study of new techniques of Wronskians introduced by Brotbek in his proof of the Kobayashi conjecture [3]. We bring a new perspective of Brotbek's Wronskians, which we interpret as a certain *morphism of $\mathscr{O}$-modules* from the jet bundles of a line bundle to the invariant jet bundles. In view of this result one can immediately make the first noetherianity argument in [3] effective. In § 2, by means of an explicit construction of global sections with a "negative twist", we obtain a *slightly weaker but effective* Nakamaye's theorem for the universal families of zero-dimensional subschemes introduced in [4, 3]. This in turn renders the second noetherianity argument in [3] as well as that in [4] effective, and in combination with the formulas for lower degree bounds in [4, 3], we prove Theorems A and B. The aim of § 3 is to study Conjecture 0.4. In § 3.1 we briefly recall the essential results in [3], and we show in § 3.2 and § 3.3 how to deduce Theorem C from Brotbek's techniques.

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## 1. Jet differentials and Brotbek's Wronskians

By the work of Nadel [21] and Demailly-El Goul [12], the *Wronskians* induced by meromorphic connections provide an abundant supply of invariant jet differentials. In [3] Brotbek introduced an alternative approach to construct Wronskian jet differentials associated to sections of a given line bundle. In § 1.2 we give an intrinsic definition of Brotbek's Wronskians via the jet bundles of line bundles.

### 1.1. Jet spaces and jet differentials

In this subsection, we collect the main techniques of jet differentials in [9]. A *direct manifold* is a pair $(X, V)$ where $X$ is a complex manifold and $V \subset T_X$ is a holomorphic subbundle of the tangent bundle. Denote by $p_k : J_k V \to X$ the bundle of $k$-jets of germs of parametrized curves in $(X, V)$, that is, the set of equivalent classes of holomorphic maps $f : (\mathbb{C}, 0) \to (X, x)$ which are tangent to $V$, with the equivalence relation $f \sim g$ if and only if all derivatives $f^{(j)}(0) = g^{(j)}(0)$ coincide for $0 \leqslant j \leqslant k$, when computed in some local coordinate system of $X$ near $x$. The class $f$ in $J_k V$ is denoted by $[f]_k$. The projection map $p_k : J_k V \to X$ is simply $[f]_k \mapsto f(0)$. When $V = T_X$, we simply write $J_k X$ in place of $J_k V$. Note that $J_k X \to X$ is a local trivial fibration with fibers $\mathbb{C}^{nk}$. Indeed, local coordinates $(z_1, \ldots, z_n)$ for an open set $U \subset X$ induce coordinates

$$(z_1, \ldots, z_n, z_1', \ldots, z_n', \ldots, z_1^{(k)}, \ldots, z_n^{(k)})$$

on $p_k^{-1}(U)$, and any $k$-jet $[f]_k \in p_k^{-1}(U)$ has coordinates

$$\big(f_1(0), \ldots, f_n(0), \ldots, f_1^{(k)}(0), \ldots, f_n^{(k)}(0)\big).$$

Let $\mathbb{G}_k$ be the group of germs of $k$-jets of biholomorphisms of $(\mathbb{C}, 0)$, that is, the group of germs of biholomorphic maps

$$t \mapsto \varphi(t) = a_1 t + a_2 t^2 + \cdots + a_k t^k, \quad a_1 \in \mathbb{C}^*, a_j \in \mathbb{C}, \forall j \geqslant 2,$$

in which the composition law is taken modulo terms $t^j$ of degree $j > k$. Then $\mathbb{G}_k$ admits a natural fiberwise right action on $J_k X$ defined by $\varphi \cdot [f]_k := [f \circ \varphi]_k$. Note that $\mathbb{C}^*$ can be seen as a subgroup of $\mathbb{G}_k$ defined by $(a_2 = \cdots = a_k = 0)$.

In [18], Green-Griffiths introduced the vector bundle $E_{k,m}^{\mathrm{GG}} T_X^* \to X$ whose fibers are complex valued polynomials $Q([f]_k)$ on the fibers of $J_k X$, of weighted degree $m$ with respect to the $\mathbb{C}^*$-action, that is, $Q(\lambda \cdot [f]_k) = \lambda^m Q([f]_k)$, for all $\lambda \in \mathbb{C}^*$ and $[f]_k \in$

$J_k X$. Let $U \subset X$ be an open set with local coordinates $(z_1, \ldots, z_n)$. Then any local section $Q \in E_{k,m}^{\mathrm{GG}} T_X^*(U)$ can be written as

$$Q = \sum_{|\alpha_1|+2|\alpha_2|+\cdots+k|\alpha_k|=m} c_\alpha(z)(d^1 z)^{\alpha_1}(d^2 z)^{\alpha_2} \cdots (d^k z)^{\alpha_k},$$

where $c_\alpha(z) \in \mathscr{O}(U)$ for any $\alpha := (\alpha_1, \ldots, \alpha_k) \in (\mathbb{N}^n)^k$, such that for any holomorphic map $\gamma : \Omega \to U$ from an open set $\Omega \subset \mathbb{C}$, one has

$$Q\big([\gamma]_k\big)(t) = \sum_{|\alpha_1|+2|\alpha_2|+\cdots+k|\alpha_k|=m} c_\alpha\big(\gamma(t)\big)\big(\gamma'(t)\big)^{\alpha_1}\big(\gamma''(t)\big)^{\alpha_2} \cdots \big(\gamma^{(k)}(t)\big)^{\alpha_k} \in \mathscr{O}(\Omega),$$

where $[\gamma]_k(t) : \Omega \to J_k X_{\restriction U}$ is the lifted holomorphic curve on $J_k X$ induced by $\gamma$.

The bundle $E_{k,\bullet}^{\mathrm{GG}} T_X^* := \bigoplus_{m \geqslant 0} E_{k,m}^{\mathrm{GG}} T_X^*$ is in a natural way a bundle of graded algebras (the product is obtained simply by taking the product of polynomials). There are natural inclusions $E_{k,\bullet}^{\mathrm{GG}} T_X^* \subset E_{k+1,\bullet}^{\mathrm{GG}} T_X^*$ of algebras, hence $E_{\infty,\bullet}^{\mathrm{GG}} T_X^* := \bigcup_{k \geqslant 0} E_{k,\bullet}^{\mathrm{GG}} T_X^*$ is also an algebra. It follows from [9, §6] that the sheaf of holomorphic sections $\mathscr{O}(E_{\infty,\bullet}^{\mathrm{GG}} T_X^*)$ admits a canonical derivation $D$ given by a collection of $\mathbb{C}$-linear maps

$$D : \mathscr{O}(E_{k,m}^{\mathrm{GG}} T_X^*) \to \mathscr{O}(E_{k+1,m+1}^{\mathrm{GG}} T_X^*) \tag{1.1.1}$$

constructed as follows. For any germ of curve $f : (\mathbb{C}, 0) \to X$, and any $Q \in \mathscr{O}(E_{k,m}^{\mathrm{GG}} T_X^*)$,

$$(DQ)([f]_{k+1})(t) := \frac{d}{dt} Q([f]_k)(t).$$

We can also inductively define $D^k := D \circ D^{k-1}$. In particular, for any holomorphic function $s \in \mathscr{O}(U)$, $D^k(s) \in E_{k,k}^{\mathrm{GG}} T_X^*(U)$.

In this present paper, we are interested in the more geometric context introduced by Demailly in [9]: the subbundle $E_{k,m} T_X^* \subset E_{k,m}^{\mathrm{GG}} T_X^*$ which consists of polynomial differential operators $Q$ which are invariant under arbitrary changes of parametrization, that is, for any $\varphi \in \mathbb{G}_k$ and any $[f]_k \in J_k X$, one has

$$Q\big(\varphi \cdot [f]_k\big) = \varphi'(0)^m Q([f]_k).$$

The bundle $E_{k,m} T_X^*$ is called the *invariant jet bundle of order $k$ and weighted degree $m$*. It is noticeable that *Wronskians* provide a very natural construction for invariant jet differentials.

For any direct manifold $(X, V)$ with rank $V = r$, Demailly [9] introduced a *fonctorial construction* of a sequence of direct manifolds

$$\cdots \to (\mathrm{P}_k V, V_k) \xrightarrow{\pi_k} (\mathrm{P}_{k-1}, V_{k-1}) \xrightarrow{\pi_{k-1}} \cdots \xrightarrow{\pi_2} (\mathrm{P}_1 V, V_1) \xrightarrow{\pi_1} (\mathrm{P}_0 V, V_0) = (X, V) \tag{1.1.2}$$

so that $\mathrm{P}_k V := \mathrm{P}(V_{k-1})$ is a $\mathbb{P}^{r-1}$-bundle over $\mathrm{P}_{k-1} V$ for each $k \geqslant 1$, and we say $\mathrm{P}_k V$ the *Demailly-Semple $k$-jet tower* of $(X, V)$. In the absolute case $(X, T_X)$, we simply write $X_k := \mathrm{P}_k V$. In the case of smooth family of compact complex manifolds $\mathscr{X} \to T$, $\mathscr{X}_k^{\mathrm{rel}}$ denotes to be the Demailly-Semple $k$-jet tower of the direct manifold $(\mathscr{X}, T_{\mathscr{X}/T})$, where $T_{\mathscr{X}/T}$ denotes the relative tangent bundle. It follows from [9, §6] that the Demailly-Semple jet tower has the following geometric properties.

1. Any germ of curve $f : (\mathbb{C}, 0) \to X$ tangent to $V$ can be lifted to $f_{[k]} : (\mathbb{C}, 0) \to \mathrm{P}_k V$.

2. Denote by $J_k^{\mathrm{reg}}V := \{[f]_k \mid f'(0) \neq 0\}$ the set of *regular $k$-jets* tangent to $V$. Then there exists a morphism $J_k^{\mathrm{reg}}V \to \mathrm{P}_k V$, which sends $[f]_k$ to $f_{[k]}(0)$, whose image is a Zariski open subset $\mathrm{P}_k V^{\mathrm{reg}} \subset \mathrm{P}_k V$ which can be identified with the quotient $J_k^{\mathrm{reg}}V/\mathbb{G}_k$. Moreover, the complement $\mathrm{P}_k V^{\mathrm{sing}} := \mathrm{P}_k V \setminus \mathrm{P}_k V^{\mathrm{reg}}$ is a divisor in $\mathrm{P}_k V$.
3. For any $k, m \geqslant 0$ one has

$$(\pi_{0,k})_* \mathscr{O}_{\mathrm{P}_k V}(m) = E_{k,m} V^*, \tag{1.1.3}$$

where we write $\pi_{j,k} = \pi_{j+1} \circ \cdots \circ \pi_k : \mathrm{P}_k V \to \mathrm{P}_j V$ for any $0 \leqslant j \leqslant k$, and $\mathscr{O}_{\mathrm{P}_k V}(1)$ denotes the tautological line bundle over $\mathrm{P}_k V = \mathrm{P}(V_{k-1})$.

More generally, for a $k$-tuple $(a_1, \ldots, a_k) \in \mathbb{N}^k$, we write

$$\mathscr{O}_{\mathrm{P}_k V}(a_k, \ldots, a_1) := \mathscr{O}_{\mathrm{P}_k V}(a_k) \otimes \pi_{k-1,k}^* \mathscr{O}_{\mathrm{P}_{k-1} V}(a_{k-1}) \otimes \cdots \otimes \pi_{1,k}^* \mathscr{O}_{\mathrm{P}_1 V}(a_1).$$

The *fundamental vanishing theorem* shows that the jet differentials vanishing along any ample divisor give rise to obstructions to the existence of entire curves.

THEOREM 1.1 (Demailly, Green-Griffiths, Siu-Yeung). – *Let $(X, V)$ be any direct manifold equipped with an ample line bundle $\mathscr{A}$. For any non-constant entire curve $f : \mathbb{C} \to X$ tangent to $V$, and any $\omega \in H^0\big(P_k V, \mathscr{O}_{P_k V}(a_k, \ldots, a_1) \otimes \pi_{0,k}^* \mathscr{A}^{-1}\big)$ with $(a_1, \ldots, a_k) \in \mathbb{N}^k$, one has $f_{[k]}(\mathbb{C}) \subset (\omega = 0)$.*

Observe that for any non-constant entire curve $f : \mathbb{C} \to X$ tangent to $V$, the image of its lift $f_{[k]} : \mathbb{C} \to \mathrm{P}_k V$ is not entirely contained in $\mathrm{P}_k V^{\mathrm{sing}}$. In view of Theorem 1.1, we introduce the following definition.

DEFINITION 1.2. – *Let $X$ be a projective manifold. We say that $X$* is almost $k$-jet ample *if there exists some $(a_1, \ldots, a_k) \in \mathbb{N}^k$ so that $\mathscr{O}_{X_k}(a_k, \ldots, a_1)$ is big and its augmented base locus $\mathbf{B}_+\big(\mathscr{O}_{X_k}(a_k, \ldots, a_1)\big) \subset X_k^{\mathrm{sing}}$. In particular, $X$ is Kobayashi hyperbolic.*

Note that almost 1-jet ampleness is equivalent to the ampleness of cotangent bundle.

### 1.2. Brotbek's Wronskians

This subsection is devoted to the study of the Wronskians defined by Brotbek in [3, §2.2]. Let $X$ be an $n$-dimensional compact complex manifold. Recall that for any holomorphic line bundle $L$ on $X$, one can define the bundle $J^k L$ of *$k$-jet sections* of $L$ by $J^k L_x = \mathscr{O}(L)_x / \big(\mathfrak{m}_x^{k+1} \cdot \mathscr{O}(L)_x\big)$ for every $x \in X$, where $\mathfrak{m}_x$ is the maximal ideal of $\mathscr{O}_x$. Pick an open set $U \subset X$ with coordinates $(z_1, \ldots, z_n)$ so that $L_{\upharpoonright U}$ can be trivialized by a nowhere vanishing section $e_U \in L(U)$. The fiber $J^k L_x$ can be identified with the set of Taylor developments of order $k$

$$\sum_{|\gamma| \leqslant k} c_\gamma (z - x)^\gamma \cdot e_U,$$

and the coefficients $\{c_\gamma\}_{\gamma \in \mathbb{N}^n, |\gamma| \leqslant k}$ define coordinates along the fibers of $J^k L$. This in turn gives rise to a natural local trivialization of $J^k L$ defined by

$$\begin{aligned} \Psi_U : U \times \mathbb{C}^{I_{n,k}} &\xrightarrow{\simeq} J^k L_{\upharpoonright U}, \\ (x, c_\gamma) &\mapsto \sum_{\gamma \in I_{n,k}} c_\gamma (z - x)^\gamma \cdot e_U, \end{aligned}$$

where $I_{n,k} := \{\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{N}^n \mid |\gamma| \leqslant k\}$. Observe that there exists a $\mathbb{C}$-linear morphism

$$j_L^k : L \to J^k L,$$

which is not a morphism of $\mathscr{O}_X$-modules, defined as follows. For any $s \in L(U)$, define

$$j_L^k(s)(x) := \sum_{|\gamma| \leqslant k} \frac{1}{\gamma!} \frac{\partial^{|\gamma|} s_U}{\partial z^\gamma}(x)(z-x)^\gamma \cdot e_U, \tag{1.2.1}$$

where $s_U \in \mathscr{O}(U)$ so that $s = s_U \cdot e_U$. When $L = \mathscr{O}_X$, we simply write $j^k := j^k_{\mathscr{O}_X}$. The jet bundle $J^k L$ will be used to interpret the canonical derivative $D : \mathscr{O}(E^{\mathrm{GG}}_{k,m} T^*_X) \to \mathscr{O}(E^{\mathrm{GG}}_{k+1,m+1} T^*_X)$ defined in (1.1.1) in an alternative way. Let us first give a more precise expression of $D$.

Lemma 1.3. – *Take any open set $U \subset X$ with coordinates $(z_1, \dots, z_n)$. For any $k \geqslant 1$, and any holomorphic function $s \in \mathscr{O}(U)$, one has*

$$D^k(s)(z) = \sum_{|\alpha_1|+2|\alpha_2|+\cdots+k|\alpha_k|=k} c_{k,\alpha}(z)(d^1 z)^{\alpha_1}(d^2 z)^{\alpha_2} \cdots (d^k z)^{\alpha_k} \in E^{\mathrm{GG}}_{k,k} T^*_X(U) \tag{1.2.2}$$

*such that for each $\alpha := (\alpha_1, \dots, \alpha_k) \in (\mathbb{N}^n)^k$, $c_{k,\alpha}(z) \in \mathscr{O}(U)$ is a $\mathbb{Z}$-linear combination of $\frac{\partial^{|\gamma|} s}{\partial z^\gamma}(z)$ with $|\gamma| = \gamma_1 + \cdots + \gamma_n \leqslant k$.*

*Proof.* – We will prove the lemma by induction on $k$. For $k = 1$, we simply have

$$D(s) = ds = \sum_{i=1}^n \frac{\partial s}{\partial z_i}(z) dz_i \in T^*_X(U),$$

and thus (1.2.2) remains valid for $k = 1$.

Now we assume that $D^k(s)$ has the form (1.2.2). By (1.1.1), one has

$$D^{k+1}(s) = \sum_{|\alpha_1|+2|\alpha_2|+\cdots+k|\alpha_k|=k} \Bigg( \sum_{i=1}^{k-1} \sum_{\substack{j=1,\dots n \\ \alpha_i - \mathbf{e}_j \in \mathbb{N}^n}} c_{k,\alpha}(z)(d^1 z)^{\alpha_1} \cdots (d^i z)^{\alpha_i - \mathbf{e}_j}(d^{i+1} z)^{\alpha_{i+1}+\mathbf{e}_j} \cdots (d^k z)^{\alpha_k}$$
$$+ \sum_{j=1}^n \frac{\partial c_{k,\alpha}(z)}{\partial z_j}(d^1 z)^{\alpha_1+\mathbf{e}_j} \cdots (d^k z)^{\alpha_k} + \sum_{\substack{j=1,\dots n \\ \alpha_k - \mathbf{e}_j \in \mathbb{N}^n}} c_{k,\alpha}(z)(d^1 z)^{\alpha_1} \cdots (d^k z)^{\alpha_k - \mathbf{e}_j}(d^{k+1} z)^{\mathbf{e}_j} \Bigg),$$

where $\mathbf{e}_j$ denotes the vector in $\mathbb{N}^n$ with a 1 in the $j$th coordinate and 0's elsewhere. By the assumption, for every $j = 1, \dots, n$ and every $\alpha$, $\frac{\partial c_{k,\alpha}(z)}{\partial z_j} \in \mathscr{O}(U)$ is a $\mathbb{Z}$-linear combination of $\frac{\partial^{|\gamma|} s}{\partial z^\gamma}(z)$ with $|\gamma| = \gamma_1 + \cdots + \gamma_n \leqslant k+1$. From the above expression we conclude that (1.2.2) also holds true for $D^{k+1}(s)$. The lemma follows. □

It follows from (1.2.1) and Lemma 1.3 that there exists a morphism of $\mathscr{O}_X$-modules, denoted by $j^k D : J^k \mathscr{O}_X \to E^{\mathrm{GG}}_{k,k} T^*_X$, so that $D^k : \mathscr{O}_X \to E^{\mathrm{GG}}_{k,k} T^*_X$ factors through $j^k D$, that is, $D^k = j^k D \circ j^k$.

Following [3], given $k+1$ holomorphic functions $g_0, \dots, g_k \in \mathscr{O}(U)$, one can associate them to a jet differentials of order $k$ and weighted degree $k' := \frac{k(k+1)}{2}$, say *Wronskians*, in the following way

$$W_U(g_0, \dots, g_k) := \begin{vmatrix} g_0 & g_1 & \cdots & g_k \\ D(g_0) & D(g_1) & \cdots & D(g_k) \\ \vdots & \vdots & \ddots & \vdots \\ D^k(g_0) & D^k(g_1) & \cdots & D^k(g_k) \end{vmatrix} \in E^{\mathrm{GG}}_{k,k'} T^*_X(U). \tag{1.2.3}$$

It follows from [3, Proposition 2.2] that Wronskians are indeed *invariant jet differentials*. From its alternating property, $W_U$ induces a $\mathbb{C}$-linear map, which we still denoted by $W_U : \Lambda^{k+1}\mathscr{O}(U) \to E_{k,k'}T^*_X(U)$ abusively. By the factorization property of $D^k$, $W_U$ gives rise to a morphism of $\mathscr{O}_U$-module

$$W_{J^k\mathscr{O}_U} : \Lambda^{k+1} J^k \mathscr{O}_U \to E_{k,k'} T^*_U$$

so that one has

$$W_U(g_0, \dots, g_k) = W_{J^k\mathscr{O}_U}\big(j^k(g_0) \wedge \cdots \wedge j^k(g_k)\big).$$

In other words, Brotbek's Wronskians $W_U$ can be factorized as follows.

(1.2.4)

$$W_U : \Lambda^{k+1}\mathscr{O}(U) \xrightarrow{\Lambda^{k+1} j^k} \Lambda^{k+1}\big(J^k\mathscr{O}_U(U)\big) \to \big(\Lambda^{k+1}J^k\mathscr{O}_U\big)(U) \xrightarrow{W_{J^k\mathscr{O}_U}} E_{k,k'}T^*_U(U).$$

Now we consider the Demailly-Semple $k$-jet tower $X_k$ of $(X, T_X)$. For the open set $U_k := \pi_{0,k}^{-1}(U)$ of $X_k$, the coordinate system $(z_1, \dots, z_n)$ on $U$ induces a trivialization $U_k \simeq U \times \mathbb{R}_{n,k}$, where $\mathbb{R}_{n,k}$ is some smooth rational variety introduced in [9, Theorem 9.1]. Hence

$$\mathscr{O}_{X_k}(1)_{\restriction U_k} \simeq \mathrm{pr}_2^*(\mathscr{O}_{\mathbb{R}_{n,k}}(1)), \tag{1.2.5}$$

where $\mathrm{pr}_2 : U_k \xrightarrow{\simeq} U \times \mathbb{R}_{n,k} \to \mathbb{R}_{n,k}$ is the composition of the isomorphism with the projection map. By (1.1.3), we conclude that, under the above trivialization, the direct image $(\pi_{0,k})_*$ induces a local trivialization of the vector bundle $E_{k,k'}T^*_U$

$$\varphi_U : U \times H^0\big(\mathbb{R}_{n,k}, \mathscr{O}_{\mathbb{R}_{n,k}}(k')\big) \xrightarrow{\simeq} E_{k,k'}T^*_U. \tag{1.2.6}$$

Write $F_{n,k} := H^0\big(\mathbb{R}_{n,k}, \mathscr{O}_{\mathbb{R}_{n,k}}(k')\big)$. Therefore, under the trivializations $\varphi_U$ and $\Psi_U$, the morphism of $\mathscr{O}_U$-module $W_{J^k\mathscr{O}_U}$ is indeed *constant*, i.e., there is a $\mathbb{C}$-linear map $\nu_{n,k} : \Lambda^{k+1}\mathbb{C}^{I_{n,k}} \to F_{n,k}$ such that one has the following diagram.

$$\begin{array}{ccc} U \times \Lambda^{k+1}\mathbb{C}^{I_{n,k}} & \xrightarrow{1_U \times \nu_{n,k}} & U \times F_{n,k} \\ \Psi_U \big\downarrow \wr & & \wr \big\downarrow \varphi_U \\ \Lambda^{k+1} J^k \mathscr{O}_U & \xrightarrow{W_{J^k\mathscr{O}_U}} & E_{k,k'}T^*_U. \end{array}$$

Denote by $\mathfrak{I}_{n,k} \subset \mathscr{O}_{\mathbb{R}_{n,k}}$ the base ideal of the linear system $|\mathrm{Im}(\nu_{n,k})| \subset |\mathscr{O}_{\mathbb{R}_{n,k}}(k')|$, and set $\mathfrak{w}_{k,U}$ to be the ideal sheaf $\mathrm{pr}_2^*(\mathfrak{I}_{n,k})$ on $U_k$.

By [3], Wronskians can also be associated to global sections of any line bundle $L$. Take an open set $U \subset X$ with coordinates $(z_1, \dots, z_n)$ so that $L_{\restriction U}$ can be trivialized by a nowhere vanishing section $e_U \in L(U)$. Consider any $s_0, \dots, s_k \in H^0(X, L)$. There exists unique $s_{i,U} \in \mathscr{O}(U)$ so that $s_i = s_{i,U} \cdot e_U$ for every $i = 0, \dots, k$. It was proved in [3, Proposition 2.3] that the section

$$W_U(s_{0,U}, \dots, s_{k,U}) \cdot e_U^{k+1} \in (E_{k,k'} T_X^* \otimes L^{k+1})(U) \tag{1.2.7}$$

is *intrinsically* defined, *i.e.,* it does not depend on the choice of $e_U$. Hence they can be glued together into a global section, denoted to be $W(s_0, \dots, s_k) \in H^0(X, E_{k,k'} T_X^* \otimes L^{k+1})$. Set

$$\omega(s_0, \dots, s_k) := (\pi_{0,k})_*^{-1} W(s_0, \dots, s_k) \in H^0\big(X_k, \mathscr{O}_{X_k}(k') \otimes \pi_{0,k}^* L^{k+1}\big) \tag{1.2.8}$$

to be the inverse image of the Wronskian $W(s_0, \dots, s_k)$ under (1.1.3).

Following [3, §2.3], define

$$\begin{aligned} \mathbb{W}(X_k, L) : = \operatorname{Span}\{\omega(s_0, \dots, s_n) \mid s_0, \dots, s_n \in H^0(X, L)\} \\ \subset H^0\big(X_k, \mathscr{O}_{X_k}(k') \otimes \pi_{0,k}^* L^{k+1}\big) \end{aligned}$$

and define the *k-th Wronskian ideal sheaf* of $L$, denoted by $\mathfrak{w}(X_k, L)$, to be the base ideal of $\mathbb{W}(X_k, L)$. It was also shown in [3, §2.3] that if $L$ is very ample, one has a chain of inclusions

$$\mathfrak{w}(X_k, L) \subset \mathfrak{w}(X_k, L^2) \subset \cdots \subset \mathfrak{w}(X_k, L^m) \subset \cdots .$$

By noetherianity, this increasing sequence stabilizes after some $m_\infty(X_k, L) \in \mathbb{N}$, and the obtained asymptotic ideal sheaf is denoted by $\mathfrak{w}_\infty(X_k, L)$. Let us mention that $m_\infty(X_k, L)$ concerns the first noetherianity argument in [3], and in the rest of this subsection we will apply our new interpretation of Brotbek's Wronskians in (1.2.4) to render $m_\infty(X_k, L)$ effective. The strategy is to compare the globally defined Wronskian ideal sheaves $\{\mathfrak{w}(X_k, L^m)\}_{m \in \mathbb{N}}$ to the intrinsic ideal sheaf $\mathfrak{w}_{k,U}$.

One direction is easy to see from the very definition of $\mathfrak{w}(X_k, L)$. By (1.2.7), for any $s_0, \dots, s_k \in H^0(X, L)$, the Wronskian can be localized by

$$W(s_0, \dots, s_k)_{\upharpoonright U} = W_U(s_{0,U}, \dots, s_{k,U}) \cdot e_U^{k+1} \in (E_{k,k'} T_X^* \otimes L^{k+1})(U).$$

We denote by $\omega_U(s_{0,U}, \dots, s_{k,U}) \in \mathscr{O}_{X_k}(k')(U_k)$ the corresponding element of $W_U(s_{0,U}, \dots, s_{k,U})$ under the isomorphism (1.1.3), where $U_k := \pi_{0,k}^{-1}(U)$. In view of (1.2.5), one has $\mathscr{O}_{X_k}(k')(U_k) \simeq H^0(U, U \times F_{n,k})$, or more precisely,

$$\begin{array}{ccc} H^0(U, \Lambda^{k+1} J^k \mathscr{O}_U) & \xrightarrow{W_{J^k \mathscr{O}_U}} & H^0(U, E_{k,k'} T_U^*) \\ \wr\| \downarrow \Psi_U^{-1} & & \wr\| \downarrow \varphi_U^{-1} \\ H^0(U, U \times \Lambda^{k+1} \mathbb{C}^{I_{n,k}}) & \xrightarrow{1_U \times \nu_{n,k}} & H^0(U, U \times F_{n,k}). \end{array}$$

By (1.2.4), $W_U(s_{0,U}, \dots, s_{k,U}) = W_{J^k \mathscr{O}_U}(j^k s_{0,U} \wedge \cdots \wedge j^k s_{k,U})$. Hence

$$\omega_U(s_{0,U}, \dots, s_{k,U}) \simeq (1_U \times \nu_{n,k}) \circ \Psi_U^{-1}(j^k s_{0,U} \wedge \cdots \wedge j^k s_{k,U}). \tag{1.2.9}$$

Recall that $\mathfrak{I}_{n,k} \subset \mathscr{O}_{\mathbb{R}_{n,k}}$ is the base ideal of the linear system $|\operatorname{Im}(\nu_{n,k})|$, and $\mathfrak{w}_{k,U}$ is defined to be the ideal sheaf $\operatorname{pr}_2^*(\mathfrak{I}_{n,k})$ on $U_k \simeq U \times \mathbb{R}_{n,k}$. By (1.2.9), the base ideal of $\omega_U(s_{0,U}, \ldots, s_{k,U})$ is contained in $\mathfrak{w}_{k,U}$. As $s_0, \ldots, s_k \in H^0(X, L)$ are arbitrary, this leads to

$$\mathfrak{w}(X_k, L)_{\upharpoonright U_k} \subset \mathfrak{w}_{k,U}. \tag{1.2.10}$$

Now we further assume that the line bundle $L$ separates $k$-jets everywhere, *i.e.,* the $\mathbb{C}$-linear map

$$H^0(X, L) \xrightarrow{j_L^k} H^0(X, J^k L) \to J^k L_x$$

is surjective for any $x \in X$. Then

$$\Lambda^{k+1} H^0(X, L) \to \Lambda^{k+1} \mathscr{O}(U) \xrightarrow{j^k} \Lambda^{k+1} J^k \mathscr{O}_U(U) \to \Lambda^{k+1} J^k \mathscr{O}_x \simeq \Lambda^{k+1} \mathbb{C}^{I_{n,k}}$$

is also surjective for any $x \in U$. By (1.2.9) again,

$$\mathrm{Im}(\nu_{n,k}) = \mathrm{Span}\{\omega_U(s_{0,U}, \ldots, s_{k,U})_{\upharpoonright \{x\} \times \mathbb{R}_{n,k}} \mid s_0, \ldots, s_k \in H^0(X, L)\},$$

where we identify $\{x\} \times \mathbb{R}_{n,k}$ with the fiber $\pi_{0,k}^{-1}(x)$. Write $\iota_x$ to be the composition $\mathbb{R}_{n,k} \to \{x\} \times \mathbb{R}_{n,k} \hookrightarrow U \times \mathbb{R}_{n,k} \xrightarrow{\simeq} U_k \hookrightarrow X_k$. This in turn implies that

$$\iota_x^* \mathfrak{w}(X_k, L) := \iota_x^{-1} \mathfrak{w}(X_k, L) \otimes_{\iota_x^{-1} \mathscr{O}_{X_k}} \mathscr{O}_{\mathbb{R}_{n,k}} = \mathfrak{I}_{n,k}.$$

It follows from $\mathfrak{w}_{k,U} := \mathrm{pr}_2^* \mathfrak{I}_{n,k}$ that $\mathfrak{w}(X_k, L)_{\upharpoonright U_k} = \mathfrak{w}_{k,U}$. By the inclusive relation (1.2.10), one has

$$\mathfrak{w}_{k,U} = \mathfrak{w}(X_k, L)_{\upharpoonright U_k} = \mathfrak{w}(X_k, L^2)_{\upharpoonright U_k} = \cdots = \mathfrak{w}(X_k, L^k)_{\upharpoonright U_k} = \cdots. \tag{1.2.11}$$

As is well-known, $A^k$ separates $k$-jets everywhere once $A$ is very ample. By (1.2.11), we conclude the following result.

THEOREM 1.4. – *Let $X$ be a projective manifold, and let $A$ be a very ample line bundle on $X$. Then $\mathfrak{w}(X_k, A^k) = \mathfrak{w}_\infty(X_k, A)$ and $m_\infty(X_k, A) = k$.*

Moreover, it follows from the relation (1.2.11) that the asymptotic Wronskian ideal sheaf is intrinsically defined, *i.e.,* $\mathfrak{w}_\infty(X_k, L)$ does not depend on the very ample line bundle $L$. This reproves [3, Lemma 2.8]. It also allows us to denote by $\mathfrak{w}_\infty(X_k)$ the asymptotic Wronskian ideal sheaf.

REMARK 1.5. – *In a joint work with Brotbek* [6]*, we generalize the alternative interpretation of Wronskians by jets of sections of line bundles in this subsection to the logarithmic settings.*

### 1.3. Blow-up of the Wronskian ideal sheaf

This subsection is mainly borrowed from [3]. We will state some important results without proof, and the readers who are interested in further details are encouraged to refer to [3, §2.4]. Let us first recall the following crucial property of the Wronskian ideal sheaf in [3].

LEMMA 1.6 ([3, Lemma 2.4]). – *Let $X$ be a projective manifold equipped with a very ample line bundle $L$. Then*

$$\mathrm{Supp}\big(\mathscr{O}_{X_k} / \mathfrak{w}(X_k, L^k)\big) \subset X_k^{\mathrm{sing}},$$

*where $X_k^{\mathrm{sing}}$ is the set of singular $k$-jets of $X_k$.*

Based on the above lemma, as was shown in [3], Brotbek introduced a *fonctorial birational morphism* of the Demailly-Semple $k$-jet tower $\nu_k : \hat{X}_k \to X_k$ by blowing-up the asymptotic Wronskian ideal sheaf $\mathfrak{w}_\infty(X_k)$, so that he was able to establish a *strong Zariski open property* for hyperbolicity. Indeed, Brotbek even built the strong Zariski open property for almost $k$-jet ampleness. We require the following results in [3] to proceed further.

THEOREM 1.7 ([3, Propositions 2.10, 2.11 and 2.13]). – *Let $X$ be a projective manifold.*

(i) *For any smooth closed submanifold $Y \subset X$, the inclusion $Y_k \subset X_k$ induces an inclusion $\hat{Y}_k \subset \hat{X}_k$. Moreover, $\hat{Y}_k$ is the strict transform of $Y_k$ in $\hat{X}_k$.*

(ii) *If*

$$(*) \qquad \exists a_0, \ldots, a_k \in \mathbb{N} \quad \text{s.t.} \quad \nu_k^* \mathscr{O}_{X_k}(a_k, \ldots, a_1) \otimes \mathscr{O}_{\hat{X}_k}(-a_0 F) \ \text{ is ample,}$$

*then $X$ is* almost $k$-jet ample. *Here $F$ is an effective divisor on $\hat{X}_k$ defined by $\mathscr{O}_{\hat{X}_k}(-F) = \nu_k^* \mathfrak{w}_\infty(X_k)$.*

(iii) *Let $\mathscr{X} \overset{\rho}{\to} T$ be a smooth and projective morphism between non-singular varieties. We denote by $\mathscr{X}_k^{\mathrm{rel}}$ the Demailly-Semple $k$-jet tower of the relative directed variety $(\mathscr{X}, T_{\mathscr{X}/T})$. Take $\nu_k : \hat{\mathscr{X}}_k^{\mathrm{rel}} \to \mathscr{X}_k^{\mathrm{rel}}$ to be the blow-up of the asymptotic Wronskian ideal sheaf $\mathfrak{w}_\infty(\mathscr{X}_k^{\mathrm{rel}})$. Then for any $t_0 \in T$ writing $X_{t_0} := \rho^{-1}(t_0)$, we have $\nu_k^{-1}(X_{t_0,k}) = \hat{X}_{t_0,k}$.*

(iv) *Property $(*)$ is a Zariski open property. Precisely speaking, in the same setting as above, if there exists $t \in T$ such that $X_t$ satisfies $(*)$, then there exists a non-empty Zariski open subset $T_0 \subset T$ such that for any $s \in T_0$, $X_s$ satisfies $(*)$ as well. In particular, $X_s$ is almost $k$-jet ample for all $s \in T_0$.*

## 2. An effective Nakamaye's theorem

As mentioned in § 0, both [4] and [3] apply Nakamaye's Theorem on the augmented base locus [22] for families of zero-dimensional subschemes to provide a geometric control on base locus. In this section we render their noetherianity arguments effective.

We start by setting notations as in [3, §3]. Consider $V := H^0\big(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(\delta)\big)$, which can be identified with the space of homogeneous polynomials of degree $\delta$ in $\mathbb{C}[z_0, \ldots, z_N]$. For any $J \subset \{0, \ldots, N\}$ we set

$$\mathbb{P}_J := \{[z_0, \ldots, z_N] \in \mathbb{P}^N \mid z_j = 0 \text{ if } j \in J\}.$$

Given any $\Delta \in \mathrm{Gr}_{k+1}(V)$ and $[z] \in \mathbb{P}^N$, we denote by $\Delta([z]) = 0$ once $P([z]) = 0$ for any $P \in \Delta \subset V$. Define the *universal family of complete intersections* to be

$$\mathscr{Y} := \{(\Delta, [z]) \in \mathrm{Gr}_{k+1}(V) \times \mathbb{P}^N \mid \Delta([z]) = 0\}. \tag{2.1}$$

For any $J \subset \{0, \ldots, N\}$, set

$$\mathscr{Y}_J := \mathscr{Y} \cap (\mathrm{Gr}_{k+1}(V) \times \mathbb{P}_J). \tag{2.2}$$

Let us denote by $p : \mathscr{Y} \to \mathrm{Gr}_{k+1}(V)$ and $q : \mathscr{Y} \to \mathbb{P}^N$ the projection maps. The next lemma is our starting point.

Lemma 2.1. – *For any $J \subset \{0, \ldots, N\}$, $\mathscr{Y}_J \to \mathbb{P}_J$ is a locally trivial holomorphic fibration with fibers isomorphic to the Grassmannian* $\mathrm{Gr}\big(k+1, \dim(V)-1\big)$. *In particular, $\mathscr{Y}_J$ is a smooth projective manifold.*

*Proof.* – Any linear transformation $g \in \mathrm{GL}_{N+1}(\mathbb{C})$ induces a natural action $\tilde{g} \in \mathrm{GL}(V)$, hence also induces a biholomorphism $\hat{g}$ of $\mathrm{Gr}_{k+1}(V)$. Observe that for any $[z] \in \mathbb{P}^N$, $\hat{g}$ maps the fiber $q^{-1}([z])$ to $q^{-1}([g \cdot z])$ bijectively. Since $\mathrm{GL}_{N+1}(\mathbb{C})$ acts transitively on $\mathbb{P}^N$, the fibration $q : \mathscr{Y} \to \mathbb{P}^N$ can thus be trivialized locally.

Take a special point $[\mathbf{e}_0] := [1, 0, \ldots, 0] \in \mathbb{P}^N$. For any $P = \sum_{|I|=\delta} a_I z^I \in V$, $P([\mathbf{e}_0]) = 0$ if and only if the coefficient of $z_0^\delta$ in $P$ is zero. If we denote by $V_0$ the subspace of $V$ spanned by $\{z^I \mid |I| = \delta, z^I \neq z_0^\delta\}$, then $q^{-1}([\mathbf{e}_0]) = \mathrm{Gr}_{k+1}(V_0) \simeq \mathrm{Gr}\big(k+1, \dim(V)-1\big)$. The lemma is thus proved. □

Observe that when $k+1 \geqslant N$, $p : \mathscr{Y} \to \mathrm{Gr}_{k+1}(V)$ is a *generically finite to one* morphism. Let us denote by $\mathscr{L}$ the very ample line bundle on $\mathrm{Gr}_{k+1}(V)$ which is the pull back of $\mathscr{O}(1)$ on $\mathrm{P}(\Lambda^{k+1} V)$ under the Plücker embedding $\mathrm{Gr}_{k+1}(V) \hookrightarrow \mathrm{P}(\Lambda^{k+1} V)$. Then $p^*\mathscr{L}_{\upharpoonright \mathscr{Y}_J}$ is a big and nef line bundle on $\mathscr{Y}_J$ for any $J \subset \{0, \ldots, N\}$. Write $p_J : \mathscr{Y}_J \to \mathrm{Gr}_{k+1}(V)$ and $q_J : \mathscr{Y}_J \to \mathbb{P}_J$ for the natural projections, and define

$$E_J := \{y \in \mathscr{Y} \mid \dim_y\big(p_J^{-1}(p_J(y))\big) > 0\}$$
$$G_J^\infty := p_J(E_J) \subset \mathrm{Gr}_{k+1}(V).$$

When $J = \emptyset$ we simply write $E := E_\emptyset$ and $G^\infty := G_\emptyset^\infty$. By the definition of *null locus* [20, Definition 10.3.4], $E_J = \mathrm{Null}(p_J^*\mathscr{L})$. It then follows from Nakamaye's theorem [22] that

$$\mathbf{B}_+(p_J^*\mathscr{L}) = \mathrm{Null}(p_J^*\mathscr{L}) = E_J.$$

Observe that the line bundle $\mathscr{L} \boxtimes \mathscr{O}_{\mathbb{P}^N}(1)$ on $\mathrm{Gr}_{k+1}(V) \times \mathbb{P}^N$ is ample, and so is its restriction to $\mathscr{Y}_J$. Hence by the definition of augmented base locus and noetherianity, there exists $m_J \in \mathbb{N}$ such that

$$\mathrm{Bs}\big(\mathscr{L}^m \boxtimes \mathscr{O}_{\mathbb{P}_J}(-1)_{\upharpoonright \mathscr{Y}_J}\big) = \mathbf{B}_+(p_J^*\mathscr{L}) = E_J \subset p_J^{-1}(G^\infty), \quad \forall m \geqslant m_J. \tag{2.3}$$

We emphasize that the value $M := \max\{m_J \mid J \subset \{0, \ldots, N\}\}$ concerns the second noetherianity argument in [3] resulting in the loss of effective lower degree bounds $d_{\mathrm{Kob},n}$ in Theorem 0.2.

Instead of requiring (2.3), we will provide a slightly weaker base control but with an effective estimate on $M$, which still remains valid in Brotbek's proof (see [3, Remark 3.13]).

Theorem 2.2. – *When $m \geqslant \delta^k$, for any $J \subset \{0, \ldots, N\}$, one has*

$$\mathrm{Bs}\big(\mathscr{L}^m \boxtimes \mathscr{O}_{\mathbb{P}_J}(-1)_{\upharpoonright \mathscr{Y}_J}\big) \subset p_J^{-1}(G_J^\infty). \tag{2.4}$$

To prove Theorem 2.2, we construct *sufficiently many* global sections of $\mathscr{L}^m \boxtimes \mathscr{O}_{\mathbb{P}_J}(-1)_{\upharpoonright \mathscr{Y}_J}$ in an explicit manner to control their base locus. Precisely speaking, for any $\Delta \notin G_J^\infty$, by definition $p_J^{-1}(\Delta)$ is a finite set. We will show that for each $m \geqslant \delta^k$ there exists an effective divisor $D_\Delta \in |\mathscr{L}^m \boxtimes \mathscr{O}_{\mathbb{P}_J}(-1)_{\upharpoonright \mathscr{Y}_J}|$ so that $D_\Delta \cap p_J^{-1}(\Delta) = \emptyset$.

Let us first recall a version of *projection formula* in *intersection theory*, which is indeed a direct consequence of [17, Example 8.1.7].

THEOREM 2.3 (Projection formula). – *Let $f : X \to Y$ be a generically finite to one and surjective morphism between non-singular irreducible varieties, and $x$ (resp. $y$) be cycle on $X$ (resp. $Y$) of dimension $k$ (resp.* $\dim(X) - k$*). Then*

$$\deg\Big(f_*\big(f^*(y)\cdot x\big)\Big) = \deg\big(y \cdot f_*(x)\big),$$

*where $f^*$ and $f_*$ are defined in the Chow group. When the scheme-theoretic inverse image $f^{-1}(y)$ is of pure dimension* $\dim(X) - k$*, one has $f^*(y) = [f^{-1}(y)]$.*

*Proof of Theorem 2.2.* – We first deal with the case $k+1 = N$, and then reduce the general setting $k+1 \geqslant N$ to this case.

CLAIM 2.4. – *When $k+1 = N$, $H^0\big(\mathscr{Y}, \mathscr{L}^m \boxtimes \mathscr{O}_{\mathbb{P}^N}(-1)_{\restriction \mathscr{Y}}\big) \neq \emptyset$ for all $m \geqslant \delta^{N-1}$.*

*Proof.* – Let us pick a smooth curve $C$ in $\mathrm{Gr}_N(V)$ of degree 1 with respect to $\mathscr{L}$, given by

$$\Delta([t_0, t_1]) := \mathrm{Span}(z_1^\delta, z_2^\delta, \dots z_{N-1}^\delta, t_0 z_N^\delta + t_1 z_0^\delta),$$

where $[t_0, t_1] \in \mathbb{P}^1$. Indeed, the curve $C$ is the line in the Plücker embedding $\mathrm{P}(\Lambda^N V)$ defined by two vectors $z_1^\delta \wedge \cdots \wedge z_{N-1}^\delta \wedge z_0^\delta$ and $z_1^\delta \wedge \cdots \wedge z_N^\delta$ in $\Lambda^N V$. Hence $\mathscr{L}\cdot C = 1$.

Consider a hyperplane $D$ in $\mathbb{P}^N$ given by $\{[z_0, \dots, z_N] \mid z_0 + z_N = 0\}$. Since $p : \mathscr{Y} \to \mathrm{Gr}_N(V)$ is a generically finite to one and surjective morphism, $p_*q^*D$ is an effective divisor in $\mathrm{Gr}_N(V)$.

Since $p^{-1}(C)$ has pure dimension 1, then $p^*C$ is a 1-cycle in $\mathscr{Y}$. An easy computation shows that $p^*C$ and $q^*D$ intersect only at the point

$$\mathrm{Span}(z_1^\delta, z_2^\delta, \dots z_{N-1}^\delta, z_N^\delta + (-1)^{\delta+1} z_0^\delta) \times [1, 0, \dots, 0, -1] \in \mathscr{Y}$$

with multiplicity $\delta^{N-1}$. Hence $p^*C \cdot q^*D = \delta^{N-1}$. By Theorem 2.3, one has

$$C \cdot p_*(q^*D) = p_*(p^*C \cdot q^*D) = \delta^{N-1}.$$

Note that the Picard group $\mathrm{Pic}\big(\mathrm{Gr}_N(V)\big) \simeq \mathbb{Z}$ is generated by $\mathscr{L}$, which in turn implies

$$p_*q^*D \in |\mathscr{L}^{\delta^{N-1}}| \tag{2.5}$$

by the fact that $\mathscr{L}\cdot C = 1$. It follows from Lemma 2.1 that $q^*D$ is a smooth hypersurface in $\mathscr{Y}$. Since $\mathrm{Supp}(q^*D) \subset \mathrm{Supp}(p^*p_*q^*D)$, $p^*p_*q^*D - q^*D$ is thus an effective divisor of $\mathscr{Y}$, and by (2.5)

$$p^*p_*q^*D - q^*D \in |\mathscr{L}^{\delta^{N-1}} \boxtimes \mathscr{O}_{\mathbb{P}^N}(-1)_{\restriction \mathscr{Y}}|. \tag{2.6}$$

The claim follows from the fact that $\mathscr{L}$ is very ample. □

The base locus of $|\mathscr{L}^{\delta^{N-1}} \boxtimes \mathscr{O}_{\mathbb{P}^N}(-1)_{\restriction \mathscr{Y}}|$ can be well understood.

CLAIM 2.5. – *For any $m \geqslant \delta^{N-1}$, the base locus*

$$\mathrm{Bs}\big(\mathscr{L}^m \boxtimes \mathscr{O}_{\mathbb{P}^N}(-1)_{\restriction \mathscr{Y}}\big) \subset p^{-1}(G^\infty). \tag{2.7}$$

*Proof*. – For given any $\Delta_0 \notin G^\infty$, $p^{-1}(\Delta_0)$ is a finite set by the definition of $G^\infty$. Then one can take a general hyperplane $D \in |\mathscr{O}_{\mathbb{P}^N}(1)|$ such that $D \cap q\big(p^{-1}(\Delta_0)\big) = \emptyset$. By (2.6), $D$ gives rise to an effective divisor

$$p^* p_* q^* D - q^* D \in |\mathscr{L}^{\delta^{N-1}} \boxtimes \mathscr{O}_{\mathbb{P}^N}(-1)_{\restriction \mathscr{Y}}|.$$

For any $\Delta \in \mathrm{Gr}_N(V)$, if we denote by

$$\mathrm{Int}(\Delta) := \{[z] \in \mathbb{P}^N \mid \Delta([z]) = 0\},$$

then $q\big(p^{-1}(\Delta)\big) = \mathrm{Int}(\Delta)$. Hence the condition $D \cap q\big(p^{-1}(\Delta_0)\big) = \emptyset$ is equivalent to that $\mathrm{Int}(\Delta_0) \cap D = \emptyset$. On the other hand, for any $\Delta \in \mathrm{Supp}(p_* q^* D)$, one has $\mathrm{Int}(\Delta) \cap D \neq \emptyset$, and thus we conclude that $\Delta_0 \notin \mathrm{Supp}(p_* q^* D)$. In particular,

$$p^{-1}(\Delta_0) \cap \mathrm{Supp}(p^* p_* q^* D - q^* D) = \emptyset.$$

As $\Delta_0$ is an arbitrary point outside $G^\infty$, we conclude that

$$\mathrm{Bs}\big(\mathscr{L}^{\delta^{N-1}} \boxtimes \mathscr{O}_{\mathbb{P}^N}(-1)_{\restriction \mathscr{Y}}\big) \subset p^{-1}(G^\infty).$$

Since $\mathscr{L}$ is very ample, we have

$$\mathrm{Bs}\big(\mathscr{L}^m \boxtimes \mathscr{O}_{\mathbb{P}^N}(-1)_{\restriction \mathscr{Y}}\big) \subset \mathrm{Bs}\big(\mathscr{L}^{\delta^{N-1}} \boxtimes \mathscr{O}_{\mathbb{P}^N}(-1)_{\restriction \mathscr{Y}}\big) \subset p^{-1}(G^\infty)$$

for any $m \geqslant \delta^{N-1}$. The claim is thus proved. □

Let us deal with the general case $J \supsetneq \emptyset$. Without loss of generality we can assume that $J = \{n+1, \ldots, N\}$. For any $\Delta_0 \in p_J(\mathscr{Y}_J) \setminus G_J^\infty$, the set $p_J^{-1}(\Delta_0) = \mathrm{Int}(\Delta_0) \cap \mathbb{P}_J$ is finite. We can also take a general hyperplane $D \in |\mathscr{O}_{\mathbb{P}^N}(1)|$ such that $\mathrm{Int}(\Delta_0) \cap D \cap \mathbb{P}_J = \emptyset$. One can further choose a proper coordinate for $\mathbb{P}^N$ such that $D = (z_n = 0)$.

By Lemma 2.1, $q_J^*(D \cap \mathbb{P}_J)$ is a smooth hypersurface in $\mathscr{Y}_J$. Set $F := p_J\big(q_J^{-1}(D \cap \mathbb{P}_J)\big)$ set-theoretically. Then for any effective divisor $\tilde{H} \in |\mathscr{L}^m|$ on $\mathrm{Gr}_N(V)$ such that $F \subset \mathrm{Supp}(\tilde{H})$ and $p_J(\mathscr{Y}_J) \not\subset \mathrm{Supp}(\tilde{H})$,

$$p_J^*(\tilde{H}) - q_J^*(D \cap \mathbb{P}_J) \in |\mathscr{L}^m \boxtimes \mathscr{O}_{\mathbb{P}^N}(-1)_{\restriction \mathscr{Y}_J}| \tag{2.8}$$

is an effective divisor of $\mathscr{Y}_J$. However, it may happen that for any hyperplane $\tilde{D} \in |\mathscr{O}_{\mathbb{P}^N}(1)|$, all constructed divisors of the form $p_* q^*(\tilde{D})$ will always contain $\Delta_0$.

Choose a decomposition of $V = V_1 \oplus V_2$ such that $V_1$ is spanned by the vectors $\{z^\alpha \in V \mid \alpha_n = \cdots = \alpha_N = 0\}$ and $V_2$ is spanned by other $z^\alpha$'s. Let us denote $G$ to be the subgroup of the general linear group $GL(V)$ which is the lower triangle matrix with respect to the decomposition of $V = V_1 \oplus V_2$ as follows:

$$G := \left\{ g \in GL(V) \mid g = \begin{bmatrix} I & 0 \\ A & B \end{bmatrix}, B \in GL(V_2), A \in \mathrm{Hom}(V_1, V_2) \right\}. \tag{2.9}$$

The subgroup $G$ also induces a natural group action on the Grassmannian $\mathrm{Gr}_N(V)$, and we have the following

Claim 2.6. – *Set $H := p_*(q^* D)$. Then for any $g \in G$, $F \subset g(H)$ and there exists a $g_0 \in G$ such that $\Delta_0 \notin g_0(H)$.*

*Proof.* – For any $\Delta \in \mathrm{Gr}_N(V)$, choose $\{s_1, \ldots, s_N\} \subset V$ which spans $\Delta$. Let $s_i = u_i + v_i$ be the unique decomposition of $s_i$ under $V = V_1 \oplus V_2$. Recall that $F := p_J\big(q_J^{-1}(D \cap \mathbb{P}_J)\big)$. Then

$$\Delta \in F \iff \bigcap_{i=1}^{N}(u_i = 0) \cap \mathbb{P}^{n-1} \neq \emptyset, \tag{2.10}$$

where $\mathbb{P}^{n-1} := \{[z_0, \ldots, z_N] \in \mathbb{P}^N \mid z_j = 0 \text{ for } j \geqslant n\} = D \cap \mathbb{P}_J$, and we can identify $V_1$ with $H^0\big(\mathbb{P}^{n-1}, \mathscr{O}_{\mathbb{P}^{n-1}}(\delta)\big)$.

For any $g \in GL(V)$, $g(\Delta)$ is spanned by $\{g(s_1), \ldots, g(s_N)\}$. By the definition of $G$, for any $g \in G$, we have the decomposition $g(s_i) = u_i + v_i'$ with respect to $V = V_1 \oplus V_2$ which keeps the $V_1$ factors invariant. Then $g(F) = F$ for any $g \in G$ by (2.10). The first statement follows from the fact $F \subset H$.

Now we take $\{t_1, \ldots, t_N\} \subset V$ which spans $\Delta_0$. Denote $t_i = u_i + v_i$ to be the decomposition of $t_i$ under $V = V_1 \oplus V_2$. By our choice of $D$, $\mathrm{Int}(\Delta_0) \cap \mathbb{P}^{n-1} = \emptyset$, which is equivalent to $\bigcap_{i=1}^N (u_i = 0) \cap \mathbb{P}^{n-1} = \emptyset$ by (2.10). We can then choose the proper basis $\{t_1, \ldots, t_N\}$ spanning $\Delta_0$, so that

(i) $\bigcap_{i=1}^n (u_i = 0) \cap \mathbb{P}^{n-1} = \emptyset$;
(ii) for some $m \geqslant n$, $\{u_1, \ldots, u_m\}$ is a set of vectors in $V_1$ which is linearly independent;
(iii) $u_{m+1} = \cdots = u_N = 0$.

Then $\bigcap_{i=1}^n (u_i = 0) \cap \{z_n = 0\} = \mathbb{P}^{N-n-1} := \{[z_0, \ldots, z_N] \in \mathbb{P}^N \mid z_j = 0 \text{ for } j \leqslant n\}$, and $\{v_{m+1}, \ldots, v_N\}$ is a set of linearly independent vectors in $V_2$.

Take a point $\Delta' \in \mathrm{Gr}_N(V)$ spanned by

$$\begin{cases} \tilde{t}_1 := u_1 \\ \vdots \\ \tilde{t}_n := u_n \\ \tilde{t}_{n+1} := u_{n+1} + z_{n+1}^\delta \\ \vdots \\ \tilde{t}_m := u_m + z_m^\delta \\ \tilde{t}_{m+1} := u_{m+1} + z_{m+1}^\delta = z_{m+1}^\delta \\ \vdots \\ \tilde{t}_N := u_N + z_N^\delta = z_N^\delta \end{cases}.$$

Then one can easily observe that $\mathrm{Int}(\Delta') \cap (z_n = 0) = \emptyset$, which is equivalent to that $\Delta' \notin H = p_* q^*(D)$. We will find a $g_0 \in G$ such that $g_0(\Delta') = \Delta_0$.

Indeed, since $\{v_{m+1}, \ldots, v_N\} \subset V_2$ and $\{u_1, \ldots, u_m\} \subset V_1$ are both linearly independent, we can find a $B \in GL(V_2)$ such that $B(z_i^\delta) = v_i$ for all $i \geqslant m+1$, and $A \in \mathrm{Hom}(V_1, V_2)$ satisfying that

$$\begin{cases} A(u_i) = v_i & \text{for} \quad 1 \leqslant i \leqslant n, \\ A(u_j) = v_j - B(z_j^\delta) & \text{for} \quad n+1 \leqslant j \leqslant m. \end{cases}$$

Set $g_0 := \begin{bmatrix} I & 0 \\ A & B \end{bmatrix}$ which is of the type (2.9). We have

$$g_0(\Delta') = \mathrm{Span}\{g_0(\tilde{t}_1), \dots, g_0(\tilde{t}_N)\} = \mathrm{Span}\{t_1, \dots, t_N\} = \Delta_0.$$

Recall that $\Delta' \notin H$. Then $\Delta_0 \notin g_0(H)$ and we finish the proof of the claim. □

Since $H \in |\mathscr{L}^{\delta^{N-1}}|$ by (2.5), we claim that $g_0(H) \in |\mathscr{L}^{\delta^{N-1}}|$. Indeed, since the complex general linear group $GL(V)$ is connected, the biholomorphism of $\mathrm{Gr}_N(V)$ induced by $g_0 \in GL(V)$ is homotopic to the identity map, and thus $H$ and $g_0(H)$ lie on the same linear system. By Claim 2.6, $F \subset g_0(H)$ and $\Delta_0 \notin g_0(H)$. By (2.8), the divisor

$$p_J^*\big(g_0(H)\big) - q_J^*(D \cap \mathbb{P}_J) \in |\mathscr{L}^{\delta^{N-1}} \boxtimes \mathscr{O}_{\mathbb{P}^N}(-1)_{\upharpoonright \mathscr{Y}_J}|$$

is effective and avoids the finite set $p_J^{-1}(\Delta_0)$.

Note that $\Delta_0 \in \mathrm{Gr}_N(V)$ is an arbitrary point in $p_J(\mathscr{Y}_J) \setminus G_J^\infty$. This in turn proves Theorem 2.2 for the case $k+1 = N$.

Let us show how to deal with the general cases $k+1 > N$.

For any $J \subset \{0, \dots, N\}$, one can see $\mathbb{P}_J \subset \mathbb{P}^N$ as subspaces of $\mathbb{P}^{k+1}$ defined by

$$\begin{aligned} \mathbb{P}^N &:= \{[z_0, \dots, z_{k+1}] \in \mathbb{P}^{k+1} \mid z_{N+1} = \cdots = z_{k+1} = 0\}, \\ \mathbb{P}_J &:= \big\{[z_0, \dots, z_{k+1}] \in \mathbb{P}^{k+1} \mid z_j = 0 \text{ if } j \in J \cup \{N+1, \dots, k+1\}\big\}. \end{aligned}$$

Set $V_k := H^0\big(\mathbb{P}^{k+1}, \mathscr{O}_{\mathbb{P}^{k+1}}(\delta)\big)$, and

$$\tilde{\mathscr{Y}}_J := \{(\Delta, [z]) \in \mathrm{Gr}_{k+1}(V_k) \times \mathbb{P}_J \mid \Delta([z]) = 0\}.$$

There is a natural inclusion $\mathrm{Gr}_{k+1}(V) \subset \mathrm{Gr}_{k+1}(V_k)$. Define $\tilde{p}_J : \tilde{\mathscr{Y}}_J \to \mathrm{Gr}_{k+1}(V_k)$ and $\tilde{q}_J : \tilde{\mathscr{Y}}_J \to \mathbb{P}_J$ to be the natural projections. Set

$$\tilde{G}_J^\infty := \{\Delta \in \mathrm{Gr}_{k+1}(V_k) \mid \tilde{p}_J^{-1}(\Delta) \text{ is not finite set}\}.$$

Hence by the above arguments, for $m \geqslant \delta^k$, we have

$$\mathrm{Bs}(\mathscr{L}_k^m \boxtimes \mathscr{O}_{\mathbb{P}^{k+1}}(-1)_{\upharpoonright \tilde{\mathscr{Y}}_J}) \subset \tilde{p}_J^{-1}(\tilde{G}_J^\infty), \tag{2.11}$$

where $\mathscr{L}_k$ is the tautological line bundle on $\mathrm{Gr}_{k+1}(V_k)$.

Recall that $\mathscr{Y} \subset \mathrm{Gr}_{k+1}(V) \times \mathbb{P}^N$ and $\mathscr{Y}_J \subset \mathrm{Gr}_{k+1}(V) \times \mathbb{P}_J$ are the universal families of complete intersections defined in (2.1) and (2.2). The inclusion $\iota_k : \mathrm{Gr}_{k+1}(V) \hookrightarrow \mathrm{Gr}_{k+1}(V_k)$ induces the following inclusions

$$\begin{array}{ccc} \mathscr{Y}_J & \hookrightarrow & \mathrm{Gr}_{k+1}(V) \times \mathbb{P}_J \\ \big\downarrow & & \big\downarrow {\scriptstyle \iota_k \times 1} \\ \tilde{\mathscr{Y}}_J & \hookrightarrow & \mathrm{Gr}_{k+1}(V_k) \times \mathbb{P}_J. \end{array}$$

Observe that $G_J^\infty = \tilde{G}_J^\infty \cap \mathrm{Gr}_{k+1}(V)$. Note that $\iota_k^* \mathscr{L}_k := \mathscr{L}$, which is still the tautological line bundle on $\mathrm{Gr}_{k+1}(V)$. Hence by the above arguments, for $m \geqslant \delta^k$, we have

$$\begin{aligned}\mathrm{Bs}(\mathscr{L}^m \boxtimes \mathscr{O}_{\mathbb{P}^N}(-1)_{\restriction \mathscr{Y}_J}) &= \mathrm{Bs}(\mathscr{L}_k^m \boxtimes \mathscr{O}_{\mathbb{P}^{k+1}}(-1)_{\restriction \mathscr{Y}_J}) \\ &\subset \mathrm{Bs}(\mathscr{L}_k^m \boxtimes \mathscr{O}_{\mathbb{P}^{k+1}}(-1)_{\restriction \tilde{\mathscr{Y}}_J}) \cap \mathscr{Y}_J \\ &\subset \tilde{p}_J^{-1}(\tilde{G}_J^\infty) \cap \mathscr{Y}_J \qquad \big(\text{by } (2.11)\big) \\ &= p_J^{-1}\big(G_J^\infty\big),\end{aligned}$$

where $p_J : \mathscr{Y}_J \to \mathrm{Gr}_{k+1}(V)$ and $q_J : \mathscr{Y}_J \to \mathbb{P}_J$ are the projection maps. This in turn proves Theorem 2.2 for the general cases $k+1 \geqslant N$. □

REMARK 2.7. – *Let us mention that the proof of Theorem 2.2 is indeed constructive, and we do not rely on the general results by Nakamaye.*

Now we are able to apply Theorems 1.4 and 2.2 to prove Theorem B using the explicit formula of $d_{\mathrm{Kob},n}$ in [3].

*Proof of Theorem B.* – In [3, p. 18], Brotbek obtained the following formula

$$d_{\mathrm{Kob},n} = m_\infty(X_k, \mathscr{A}) + \delta + (R+k)\delta,$$

where $R := M(k+1)\big(m_\infty(X_k, \mathscr{A}) + \delta - 1 + k\delta\big) + 1$ with $M \in \mathbb{N}$ the lower bound of $m$ so that (2.4) remains valid, and one can take $k = n-1$, $\delta = n^2$ by [3]. By Theorems 1.4 and 2.2, we can take $m_\infty(X_k, \mathscr{A}) = k = n-1$, and $M = \delta^k = \delta^{n-1}$. Hence

$$\begin{aligned}d_{\mathrm{Kob},n} &\leqslant m_\infty(X_k, \mathscr{A}) + \delta + (R+k)\delta \\ &= k + \delta + \delta\Big(\delta^k(k+1)\big(k+\delta-1+k\delta\big) + 1 + k\Big) \\ &= n^{2n+1}(n^3+n-2) + n^3 + n^2 + n - 1 \\ &\leqslant n^{2n+3}(n+1),\end{aligned}$$

and the theorem follows. □

REMARK 2.8. – *Along Siu's line of slanted vector fields on higher jet spaces outlined in* [24]*, Diverio-Merker-Rousseau* [14] *first proved the* weak hyperbolicity *(say that a projective variety $X$ is weakly hyperbolic if all entire curves lie in a proper subvariety $Y \subsetneq X$) of general hypersurfaces in $\mathbb{P}^n$ of degree $d \geqslant 2^{(n-1)^5}$. This lower bound was improved by Demailly* [10] *to $d \geqslant \Big\lfloor \frac{n^4}{3}\Big(n\log\big(n\log(24n)\big)\Big)^n \Big\rfloor$, and the latest best known bound $d \geqslant (5n)^2 n^n$ was obtained by Darondeau* [7]*. Very recently, Demailly* [11] *gave a simple proof of the Kobayashi conjecture as well as an effective lower bound $d_{\mathrm{Kob},n} = \frac{1}{5}\big(e(n-1)\big)^{2n}$ for the degrees.*

Now we will generalize Theorem 2.2 to the cases of products of Grassmannians. Let us fix $c, k, n \in \mathbb{N}$ with $c(k+1) \geqslant n$. Write $V_{\delta_i} := H^0\big(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(\delta_i)\big)$ and $\mathbf{G} := \prod_{i=1}^c \mathrm{Gr}_{k+1}(V_{\delta_i})$ for any $(\delta_1, \ldots, \delta_c) \in \mathbb{N}^c$. Set $\mathscr{Y}$ to be the *universal family of complete intersections* defined by

$$\mathscr{Y} := \{(\Delta_1, \ldots, \Delta_c, [z]) \in \mathbf{G} \times \mathbb{P}^n \mid \Delta_i([z]) = 0, \forall\, i = 1, \ldots, c\}. \tag{2.12}$$

Denote by $p : \mathscr{Y} \to \mathbf{G}$ and $q : \mathscr{Y} \to \mathbb{P}^n$ the projection maps. Then $p$ is a generically finite to one morphism. Define a group homeomorphism

$$\begin{aligned} \mathscr{L} : \mathbb{Z}^c &\to \mathrm{Pic}(\mathbf{G}) \\ \mathbf{a} = (a_1, \ldots, a_c) &\mapsto \mathscr{O}_{\mathrm{Gr}_{k+1}(V_{\delta_1})}(a_1) \boxtimes \cdots \boxtimes \mathscr{O}_{\mathrm{Gr}_{k+1}(V_{\delta_c})}(a_c), \end{aligned} \tag{2.13}$$

which is indeed an isomorphism.

Let us introduce $c$-smooth curves $C_1, \ldots, C_c$ on $\mathbf{G}$, defined by

$$\begin{aligned} \Delta_i([t_0, t_1]) := \mathrm{Span}(z_1^{\delta_1}, z_{c+1}^{\delta_1}, \ldots, z_{kc+1}^{\delta_1}) \times \mathrm{Span}(z_2^{\delta_2}, z_{c+2}^{\delta_2}, \ldots, z_{kc+2}^{\delta_2}) \times \cdots \\ \times \mathrm{Span}(t_0 z_i^{\delta_i} + t_1 z_0^{\delta_i}, z_{c+i}^{\delta_i}, \ldots, z_{kc+i}^{\delta_i}) \times \cdots \times \mathrm{Span}(z_c^{\delta_c}, z_{2c}^{\delta_c}, \ldots, z_{(k+1)c}^{\delta_c}), \end{aligned}$$

for $[t_0, t_1] \in \mathbb{P}^1$. It is easy to verify that $\mathscr{L}(\mathbf{a}) \cdot C_i = a_i$ for each $i$. Consider the hyperplane $D_i \in |\mathscr{O}_{\mathbb{P}^n}(1)|$ given by $\{[z_0, \ldots, z_n] \mid z_i + z_0 = 0\}$. Then we have the similar result as Claim 2.4.

LEMMA 2.9. – *Suppose that $n = k(c+1)$. For any hyperplane $D \in |\mathscr{O}_{\mathbb{P}^n}(1)|$, $p_* q^* D \in |\mathscr{L}(\mathbf{b})|$, where $\mathbf{b} := (b_1, \ldots, b_c) \in \mathbb{N}^c$ with $b_i := \frac{\prod_{j=1}^c \delta_j^{k+1}}{\delta_i}$.*

*Proof.* – It is easy to show that $p^* C_i$ and $q^* D_i$ intersect only at one point with multiplicity $b_i$ for each $i = 1, \ldots, c$. By the projection formula in Theorem 2.3, one has

$$(p_* q^* D_i) \cdot C_i = p_*(q^* D_i \cdot p^* C_i) = b_i. \tag{2.14}$$

Recall that $\mathscr{L}(\mathbf{a}) \cdot C_i = a_i$ for any $\mathbf{a} \in \mathbb{Z}^c$. Then $p_* q^* D \in |\mathscr{L}(\mathbf{b})|$ by (2.14). □

By similar arguments as Claim 2.5, $\mathscr{L}(\mathbf{b}) \boxtimes \mathscr{O}_{\mathbb{P}^n}(-1)_{\restriction \mathscr{Y}}$ is effective, and its base locus

$$\mathrm{Bs}\big(\mathscr{L}(\mathbf{b}) \boxtimes \mathscr{O}_{\mathbb{P}^n}(-1)_{\restriction \mathscr{Y}}\big) \subset p^{-1}(G^\infty), \tag{2.15}$$

where $G^\infty$ is the set of points in $\mathbf{G}$ at which the fiber in $\mathscr{Y}$ is positive dimensional. We can apply the same methods in proving Theorem 2.2 to obtain a more general result.

THEOREM 2.10. – *Let $\mathscr{Y}$ be the universal complete intersection defined by*

$$\mathscr{Y} := \Big\{(\Delta_1, \ldots, \Delta_c, [z]) \in \prod_{i=1}^c \mathrm{Gr}_{k+1}(V_{\delta_i}) \times \mathbb{P}^n \mid \Delta_i([z]) = 0, \forall\, i = 1, \ldots, c\Big\},$$

*where $V_{\delta_i} := H^0\big(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(\delta_i)\big)$, and $(k+1)c \geqslant n$. For any $J \subset \{0, \ldots, n\}$, define $\mathscr{Y}_J := \mathscr{Y} \cap \prod_{i=1}^c \mathrm{Gr}_{k+1}(V_{\delta_i}) \times \mathbb{P}_J$. Then for any $\mathbf{a} = (a_1, \ldots, a_c) \in \mathbb{N}^c$ with $a_i \geqslant \frac{\prod_{j=1}^c \delta_j^{k+1}}{\delta_i}$ for $i = 1, \ldots, c$, the base locus*

$$\mathrm{Bs}\big(\mathscr{L}(\mathbf{a}) \boxtimes \mathscr{O}_{\mathbb{P}^n}(-1)_{\restriction \mathscr{Y}_J}\big) \subset p_J^{-1}(G_J^\infty),$$

*where $G_J^\infty$ is the set of points in $\prod_{i=1}^c \mathrm{Gr}_{k+1}(V_{\delta_i})$ at which the fiber in $\mathscr{Y}_J$ is positive dimensional.*

REMARK 2.11. – *Very recently, Brotbek and the author* [5, 6] *extended the techniques in* [4, 3] *to the logarithmic settings using* meromorphic connections*, and we proved*

(i) *the logarithmic analogue of the Debarre conjecture: for general hypersurfaces $H_1, \ldots, H_n \in |\mathscr{O}_{\mathbb{P}^n}(d)|$ with $d \geqslant (4n)^n$ and $D := \sum_{i=1}^n H_i$ simple normal crossing, the logarithmic cotangent bundle $\Omega_{\mathbb{P}^n}(\log D)$ is* almost ample*;*

(ii) *a result towards the* orbifold Kobayashi conjecture *by Rousseau* [23]*: for general hypersurfaces* $H \in |\mathscr{O}_{\mathbb{P}^n}(d)|$ *with* $d \geqslant (n+1)^{n+3} \cdot (n+2)^{n+3}$*, the* Campana orbifold $\big(\mathbb{P}^n, (1-\frac{1}{d})H\big)$ *is* orbifold hyperbolic.

*Let us mention that we have to apply Theorem 2.10 to obtain the effective lower bounds of degrees in* [5, 6].

## 3. On the Diverio-Trapani Conjecture

In this section, we apply the techniques in [4, 3] to prove Theorem C. Let us mention that § 3.1 is not self-contained, and we strongly recommend the readers who are interested in further details to refer to the paper [3].

### 3.1. Families of Fermat-type Hypersurfaces

In [3], Brotbek introduced the families of *Fermat-type Hypersurfaces* as a candidate for the examples satisfying a strong Zariski open property for hyperbolicity. In this subsection, we briefly recall his constructions and the essential techniques in [3] which will be used in the proof of Theorem C.

Let $X$ be an $n$-dimensional projective manifold endowed with a very ample line bundle $A$. We fix $n+1$ sections in general position $\tau_0, \ldots, \tau_n \in H^0(X, A)$. Let us fix a positive integer $r$ and $k$. For any $\varepsilon, \delta \in \mathbb{N}$, set $V_\delta := H^0\big(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(\delta)\big)$, and $\mathbb{A}_{\varepsilon,\delta} := H^0(X, A^\varepsilon) \otimes V_\delta$. Consider for any $\mathbf{a} := \big(a_I \in H^0(X, A^\varepsilon)\big)_{|I|=\delta} \in \mathbb{A}_{\varepsilon,\delta}$, the hypersurface $H_{\mathbf{a}}$ in $X$ defined by the zero locus of the section

$$\sigma(\mathbf{a}) := \sum_{|I|=\delta} a_I \tau^{(r+k)I} \in H^0(X, A^m), \tag{3.1.1}$$

where $m = \varepsilon + (r+k)\delta$ and $\tau^{(r+k)I} := (\tau_0^{i_0} \cdots \tau_n^{i_n})^{r+k}$ for $I = (i_0, \ldots, i_n)$. Consider the universal family

$$\mathscr{H}_{\varepsilon,\delta} := \{(\mathbf{a}, x) \in \mathbb{A}_{\varepsilon,\delta} \times X \mid \sigma(\mathbf{a})(x) = 0\}.$$

There exists a Zariski open set of $\mathbb{A}^{\mathrm{sm}}_{\varepsilon,\delta} \subset \mathbb{A}_{\varepsilon,\delta}$ so that over $\mathbb{A}^{\mathrm{sm}}_{\varepsilon,\delta}$, $\mathscr{H}_{\varepsilon,\delta}$ is a smooth family. Let us also denote by $\mathscr{H}_{\varepsilon,\delta} \to \mathbb{A}^{\mathrm{sm}}_{\varepsilon,\delta}$ the restrict family, $\mathscr{H}^{\mathrm{rel}}_{\varepsilon,\delta,k}$ the (relative) Demailly-Semple $k$-jet tower of $(\mathscr{H}_{\varepsilon,\delta}, T_{\mathscr{H}_{\varepsilon,\delta}/\mathbb{A}^{\mathrm{sm}}_{\varepsilon,\delta}})$, and $\hat{\mathscr{H}}^{\mathrm{rel}}_{\varepsilon,\delta,k}$ the blow-up of $\mathscr{H}^{\mathrm{rel}}_{\varepsilon,\delta,k}$ defined in Theorem 1.7.

Let us define a finite set $\Sigma := \bigcup_{\{j_1,\ldots,j_n\}\subset\{0,\ldots,n\}} (\tau_{j_1} = \cdots = \tau_{j_n} = 0)$ of $X$, and write $X^\circ := X \setminus \Sigma$. Denote by $\hat{X}^\circ_k := (\pi_{0,k} \circ \nu_k)^{-1}(X^\circ)$. We can shrink $\mathbb{A}^{\mathrm{sm}}_{\varepsilon,\delta}$ to a Zariski open set so that $\mathscr{H}_{\varepsilon,\delta} \subset \mathbb{A}^{\mathrm{sm}}_{\varepsilon,\delta} \times X^\circ$ and, *a fortiori*, $\hat{\mathscr{H}}^{\mathrm{rel}}_{\varepsilon,\delta,k} \subset \mathbb{A}^{\mathrm{sm}}_{\varepsilon,\delta} \times \hat{X}^\circ_k$.

We need to cover $X$ by a natural stratification induced by the vanishing of the $\tau_j$'s. For any $J \subset \{0, \ldots, n\}$, define

$$X_J := \{x \in X \mid \tau_j(x) = 0 \Leftrightarrow j \in J\},$$
$$\mathbb{P}_J := \{[z] \in \mathbb{P}^n \mid z_j = 0 \text{ if } j \in J\},$$
$$V_{\delta,J} := H^0\big(\mathbb{P}_J, \mathscr{O}_{\mathbb{P}_J}(\delta)\big),$$

$\hat{X}_{k,J} := (\pi_{0,k} \circ \nu_k)^{-1}(X_J)$ and $\hat{X}^\circ_{k,J} := \hat{X}_{k,J} \cap \hat{X}^\circ_k$.

We are now in position to recall the main results in [3], which will be applied in § 3.2.

THEOREM 3.1 (Brotbek). – *Fix any $r \in \mathbb{N}$. For each $\varepsilon, \delta \in \mathbb{N}$, there exists a rational map*

$$\Phi_{\varepsilon,\delta} : \mathbb{A}_{\varepsilon,\delta} \times \hat{X}_k \dashrightarrow \mathrm{Gr}_{k+1}(V_\delta) \tag{3.1.2}$$

*induced by Brotbek's Wronskians. Suppose that $\varepsilon \geqslant m_\infty(X_k, A)$ and $\delta \geqslant n(k+1)$.*

(i) *There exists a non-empty Zariski open subset $\mathbb{A}^\circ_{\varepsilon,\delta} \subset \mathbb{A}^{\mathrm{sm}}_{\varepsilon,\delta}$ so that the restriction of $\Phi_{\varepsilon,\delta}$ to $\mathbb{A}^\circ_{\varepsilon,\delta} \times \hat{X}^\circ_k$ is a regular morphism.*
(ii) *Set $\mathscr{L}$ to be the tautological line bundle on $\mathrm{Gr}_{k+1}(V_\delta)$, and $F$ to be the effective divisor in $\hat{X}_k$ defined by $\mathscr{O}_{\hat{X}_k}(-F) := \nu_k^* \mathfrak{w}_\infty(X_k)$. One has*

$$\Phi^*_{\varepsilon,\delta}\mathscr{L} = \nu_k^*\big(\mathscr{O}_{X_k}(k') \otimes \pi^*_{0,k} A^{(k+1)(\varepsilon+k\delta)}\big) \otimes \mathscr{O}_{\hat{X}_k}(-F). \tag{3.1.3}$$

(iii) *Define a rational map*

$$\begin{aligned}\Psi_{\varepsilon,\delta} : \mathbb{A}_{\varepsilon,\delta} \times \hat{X}_k &\dashrightarrow \mathrm{Gr}_{k+1}(V_\delta) \times \mathbb{P}^n\\ (\mathbf{a}, w) &\mapsto \big(\Phi_{\varepsilon,\delta}(\mathbf{a}, w), [\tau^r(w)]\big),\end{aligned}$$

*where $[\tau^r(w)] := \big[\tau_0^r\big(\pi_{0,k} \circ \nu_k(w)\big), \ldots, \tau_n^r\big(\pi_{0,k} \circ \nu_k(w)\big)\big]$. The restriction of $\Psi_{\varepsilon,\delta}$ to $\hat{\mathscr{H}}^{\mathrm{rel}}_{\varepsilon,\delta,k}$ factors through $\mathscr{Y}$, where $\mathscr{Y} \subset \mathrm{Gr}_{k+1}(V_\delta) \times \mathbb{P}^n$ is the universal family of complete intersections defined in* (2.1). *In other words, for any $\mathbf{a} \in \mathbb{A}^\circ_{\varepsilon,\delta}$, $\hat{H}_{\mathbf{a},k} \subset \hat{X}^\circ_k$ and $\Psi_{\varepsilon,\delta}(\hat{H}_{\mathbf{a},k}) \subset \mathscr{Y}$.*
(iv) *For any $w \in \hat{X}^\circ_k$, there exists a $\mathbb{C}$-linear map*

$$\varphi_{\varepsilon,\delta,w} : \mathbb{A}_{\varepsilon,\delta} \to V_\delta^{k+1} \tag{3.1.4}$$

*such that $\Phi_{\varepsilon,\delta}$ is defined at $(\mathbf{a}, w) \in \mathbb{A}_{\varepsilon,\delta} \times \hat{X}^\circ_k$ if and only if $\dim[\varphi_{\varepsilon,\delta,w}(\mathbf{a})] = k+1$. Here $[\varphi_{\varepsilon,\delta,w}(\mathbf{a})]$ denotes to be the subspace in $V_\delta$ spanned by $(k+1)$-vectors $\varphi_{\varepsilon,\delta,w}(\mathbf{a})$. Moreover, for any $\mathbf{a} \in \mathbb{A}^\circ_{\varepsilon,\delta}$, $\Phi_{\varepsilon,\delta}(\mathbf{a}, w) = [\varphi_{\varepsilon,\delta,w}(\mathbf{a})] \in \mathrm{Gr}_{k+1}(V_\delta)$.*
(v) *Same setting as above. For the (unique) $J \subset \{0, \ldots, n\}$ so that $w \in \hat{X}^\circ_{k,J}$, the composition of $\mathbb{C}$-linear maps*

$$\phi_{\varepsilon,\delta,w} : \mathbb{A}_{\varepsilon,\delta} \xrightarrow{\varphi_{\varepsilon,\delta,w}} V_\delta^{k+1} \xrightarrow{\rho_w} V_{\delta,J}^{k+1}$$

*is surjective. Here $\rho_w : V_\delta^{k+1} \to V_{\delta,J}^{k+1}$ is the projection map.*

### 3.2. Families of complete intersections of Fermat-type hypersurfaces

Let us construct families of complete intersection varieties in $X$ cut out by Fermat-type hypersurfaces defined in § 3.1. As we will see in Theorem 3.4, these examples satisfy the strong Zariski open property $(*)$ for almost $k$-jet ampleness defined in Definition 1.2.

We fix $1 \leqslant c \leqslant n-1$, $r \in \mathbb{N}$, $k \geqslant \frac{n}{c} - 1$, and two $c$-tuples of positive integers $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_c), \delta = (\delta_1, \ldots, \delta_c) \in \mathbb{N}^c$. Consider the family $\mathscr{X} \subset \mathbb{A}_{\varepsilon_1,\delta_1} \times \cdots \times \mathbb{A}_{\varepsilon_c,\delta_c} \times X$ of *complete intersection varieties* in $X$ defined by

$$\mathscr{X} := \{(\mathbf{a}_1, \ldots, \mathbf{a}_c, x) \in \mathbb{A}_{\varepsilon_1,\delta_1} \times \cdots \times \mathbb{A}_{\varepsilon_c,\delta_c} \times X \mid \sigma(\mathbf{a}_1)(x) = \cdots = \sigma(\mathbf{a}_c)(x) = 0\}, \tag{3.2.1}$$

where $\sigma(\mathbf{a}_i)$ is the section defined in (3.1.1). Let us denote by $\rho : \mathscr{X} \to \mathbb{A}_{\varepsilon_1,\delta_1} \times \cdots \times \mathbb{A}_{\varepsilon_c,\delta_c}$ the natural projection, and for any $\mathbf{a} := (\mathbf{a}_1, \ldots, \mathbf{a}_c)$, set $Z_{\mathbf{a}} := \rho^{-1}(\mathbf{a})$. One can show that there is a non-empty Zariski open set $\mathbb{A}_{\mathrm{sm}} \subset \mathbb{A} := \mathbb{A}_{\varepsilon_1,\delta_1} \times \cdots \times \mathbb{A}_{\varepsilon_c,\delta_c}$ so that $Z_{\mathbf{a}}$ is smooth for any $\mathbf{a} \in \mathbb{A}_{\mathrm{sm}}$. In other words, for any $\mathbf{a} \in \mathbb{A}_{\mathrm{sm}}$, the $c$-hypersurfaces $H_{\mathbf{a}_1}, \ldots, H_{\mathbf{a}_c}$ are smooth and intersect transversely so that $Z_{\mathbf{a}} := H_{\mathbf{a}_1} \cap \cdots \cap H_{\mathbf{a}_c}$ is a smooth subvariety in $X$ of codimension $c$. Let us also denote by $\mathscr{Z} \to \mathbb{A}_{\mathrm{sm}}$ the restricted (smooth) family. Denote by $\mathscr{Z}_k^{\mathrm{rel}}$ the relative Demailly-Semple $k$-jet tower of $(\mathscr{Z}, T_{\mathscr{Z}/\mathbb{A}_{\mathrm{sm}}})$, and $\hat{\mathscr{Z}}_k^{\mathrm{rel}}$ its blow-up defined in Theorem 1.7. Observe that $Z_{\mathbf{a},k} = H_{\mathbf{a}_1,k} \cap \cdots \cap H_{\mathbf{a}_c,k}$ for any $\mathbf{a} \in \mathbb{A}_{\mathrm{sm}}$, and by Theorem 1.7, one has

$$\hat{Z}_{\mathbf{a},k} \subset \hat{H}_{\mathbf{a}_1,k} \cap \cdots \cap \hat{H}_{\mathbf{a}_c,k}. \tag{3.2.2}$$

Consider a rational map $\Phi : \mathbb{A} \times \hat{X}_k \dashrightarrow \mathrm{Gr}_{k+1}(V_{\delta_1}) \times \cdots \times \mathrm{Gr}_{k+1}(V_{\delta_c})$ by taking the products of (3.1.2). Precisely speaking, $\Phi$ is defined by

$$\begin{aligned} \Phi : \mathbb{A} \times \hat{X}_k &\dashrightarrow \mathrm{Gr}_{k+1}(V_{\delta_1}) \times \cdots \times \mathrm{Gr}_{k+1}(V_{\delta_c}) \\ (\mathbf{a}_1, \ldots, \mathbf{a}_c, w) &\mapsto \big(\Phi_{\varepsilon_1,\delta_1}(\mathbf{a}_1, w), \ldots, \Phi_{\varepsilon_c,\delta_c}(\mathbf{a}_c, w)\big) \end{aligned}$$

Write $\mathbf{G} := \mathrm{Gr}_{k+1}(V_{\delta_1}) \times \cdots \times \mathrm{Gr}_{k+1}(V_{\delta_c})$ for short. As a direct consequence of Theorems 1.4 and 3.1, we have the following result.

THEOREM 3.2. – *Assume that $\varepsilon_i \geqslant k, \delta_i \geqslant n(k+1)$ for every $i = 1, \ldots, c$. Then*

(i) *the restriction of $\Phi$ to $\mathbb{A}^{\circ}_{\varepsilon_1,\delta_1} \times \cdots \times \mathbb{A}^{\circ}_{\varepsilon_c,\delta_c} \times \hat{X}^{\circ}_k$ is regular.*

(ii) *Set $\mathbb{A}^{\circ} := \mathbb{A}^{\circ}_{\varepsilon_1,\delta_1} \times \cdots \times \mathbb{A}^{\circ}_{\varepsilon_c,\delta_c} \cap \mathbb{A}_{\mathrm{sm}}$. We also denote by $\hat{\mathscr{Z}}_k^{\mathrm{rel}} \to \mathbb{A}^{\circ}$ the restricted family. Then $\hat{\mathscr{Z}}_k^{\mathrm{rel}} \subset \mathbb{A}^{\circ} \times \hat{X}^{\circ}_k$.*

(iii) *For any $(b_1, \ldots, b_c) \in \mathbb{N}^c$, one has*

$$\Phi^*\mathscr{L}(b_1, \ldots, b_c) = \nu_k^*\Big(\mathscr{O}_{X_k}\big(\sum_{i=1}^{c} b_i k'\big) \otimes \pi_{0,k}^* A^{\sum_{i=1}^{c} b_i (k+1)(\varepsilon_i + k\delta_i)}\Big) \otimes \mathscr{O}_{\hat{X}_k}\Big(-\big(\sum_{i=1}^{c} b_i\big)F\Big), \tag{3.2.3}$$

*where $\mathscr{L}(b_1, \ldots, b_c)$ is the tautological line bundle defined in* (2.13).

(iv) *Define a rational map*

$$\begin{aligned} \Psi : \mathbb{A} \times \hat{X}_k &\dashrightarrow \mathrm{Gr}_{k+1}(V_{\delta_1}) \times \cdots \times \mathrm{Gr}_{k+1}(V_{\delta_c}) \times \mathbb{P}^n \\ (\mathbf{a}, w) &\mapsto \big(\Phi(\mathbf{a}, w), [\tau^r(w)]\big), \end{aligned}$$

*where $[\tau^r(w)] := \big[\tau_0^r\big(\pi_{0,k} \circ \nu_k(w)\big), \ldots, \tau_n^r\big(\pi_{0,k} \circ \nu_k(w)\big)\big]$. The restriction of $\Psi$ to $\hat{\mathscr{Z}}_k^{\mathrm{rel}}$ factors through $\mathscr{Y}$, where $\mathscr{Y} \subset \mathrm{Gr}_{k+1}(V_{\delta_1}) \times \ldots \times \mathrm{Gr}_{k+1}(V_{\delta_c}) \times \mathbb{P}^n$ is the universal family of complete intersections defined in* (2.12). *In other words, for any $\mathbf{a} \in \mathbb{A}^{\circ}$, $\hat{Z}_{\mathbf{a},k} \subset \hat{X}^{\circ}_k$ and $\Psi(\hat{Z}_{\mathbf{a},k}) \subset \mathscr{Y}$.*

*Proof.* – We apply Theorem 1.4 to take $m_{\infty}(X_k, A) = k$. (i), (ii) and (iii) can be easily derived from Theorem 3.1. To prove (iv), it is enough to show that for any $\mathbf{a} \in \mathbb{A}^{\circ}$, $\Psi(\hat{Z}_{\mathbf{a},k}) \subset \mathscr{Y}$. By (3.2.2), for any $w \in \hat{Z}_{\mathbf{a},k}$, $i = 1, \ldots, c$ and $P \in \Phi_{\varepsilon_i,\delta_i}(\mathbf{a}_i, w)$, one has

$$P\big([\tau^r(w)]\big) = 0.$$

This proves (iv) by the definition of $\mathscr{Y}$. □

Set $\mathscr{Y}_J := \mathscr{Y} \cap \big(\mathbf{G} \times \mathbb{P}_J\big) \subset \mathbf{G} \times \mathbb{P}^n$, and denote by $G_J^{\infty}$ the set of points in $\mathbf{G}$ at which the fiber in $\mathscr{Y}_J$ is positive dimensional.

Now we are ready to prove the following lemma, which is a variant of [3, Lemma 3.11].

LEMMA 3.3 (Avoiding positive dimensional fibers). – *Assume that* $\varepsilon_i \geqslant k$, $\delta_i \geqslant \dim \hat{X}_k = (n-1)(k+1)+1$ *for* $i = 1, \ldots, c$. *Then for any* $J \subset \{0, \ldots, n\}$, *there exists a non-empty Zariski open subset* $\mathbb{A}_J \subset \mathbb{A}^\circ$ *such that*

$$\Phi^{-1}(G_J^\infty) \cap (\mathbb{A}_J \times \hat{X}_{k,J}^\circ) = \emptyset.$$

*Proof.* – We introduce the following analogues of $\mathscr{Y}_J$ parametrized by affine spaces

$$\widetilde{\mathscr{Y}_{1,J}} := \big\{(\alpha_{10}, \ldots, \alpha_{ck}, [z]) \in \prod_{i=1}^{c} V_{\delta_i}^{k+1} \times \mathbb{P}_J \mid \forall 1 \leqslant i \leqslant c, 0 \leqslant p \leqslant k, \alpha_{ip}([z]) = 0\big\},$$

$$\widetilde{\mathscr{Y}_{2,J}} := \{(\alpha_{10}, \ldots, \alpha_{ck}, [z]) \in \prod_{i=1}^{c} V_{\delta_i,J}^{k+1} \times \mathbb{P}_J \mid \forall 1 \leqslant i \leqslant c, 0 \leqslant p \leqslant k, \alpha_{ip}([z]) = 0\}.$$

By analogy with $G_J^\infty$, we denote by $\mathbb{V}_{1,J}^\infty$ (resp. $\mathbb{V}_{2,J}^\infty$) the set of points in $\prod_{i=1}^{c} V_{\delta_i}^{k+1}$ (resp. $\prod_{i=1}^{c} V_{\delta_i,J}^{k+1}$) at which the fiber in $\widetilde{\mathscr{Y}_{1,J}}$ (resp. $\widetilde{\mathscr{Y}_{2,J}}$) is positive dimensional.

Fix any $w \in \hat{X}_{k,J}^\circ$. By Theorem 3.1.(iv), for any $\mathbf{a} = (\mathbf{a}_1, \ldots, \mathbf{a}_c) \in \mathbb{A}^\circ$ we have

$$\Phi(\mathbf{a}, w) = \big([\varphi_{\varepsilon_1,\delta_1,w}(\mathbf{a}_1)], \ldots, [\varphi_{\varepsilon_c,\delta_c,w}(\mathbf{a}_c)]\big),$$

where $\varphi_{\varepsilon_i,\delta_i,w} : \mathbb{A}_{\varepsilon_i,\delta_i} \to V_{\delta_i}^{k+1}$ is the linear map defined in Theorem 3.1.(iv). Let us define a $\mathbb{C}$-linear map

$$\begin{aligned} \varphi_w : \mathbb{A} &\to \prod_{i=1}^{c} V_{\delta_i}^{k+1} \\ \mathbf{a} &\mapsto \big(\varphi_{\varepsilon_1,\delta_1,w}(\mathbf{a}_1), \ldots, \varphi_{\varepsilon_c,\delta_c,w}(\mathbf{a}_c)\big). \end{aligned}$$

Then we have

$$\Phi^{-1}(G_J^\infty) \cap (\mathbb{A}^\circ \times \{w\}) = \varphi_w^{-1}(\mathbb{V}_{1,J}^\infty) \cap \mathbb{A}^\circ = (\rho_w \circ \varphi_w)^{-1}(\mathbb{V}_{2,J}^\infty) \cap \mathbb{A}^\circ,$$

where

$$\rho_w : \prod_{i=1}^{c} V_{\delta_i}^{k+1} \to \prod_{i=1}^{c} V_{\delta_i,J}^{k+1}$$

is the projection map. Since the linear map $\rho_w \circ \varphi_w$ is diagonal by blocks, by Theorem 3.1.(v) we have

$$\operatorname{rank} \rho_w \circ \varphi_w = \sum_{i=1}^{c} (k+1) \dim V_{\delta_i,J}.$$

Therefore

$$\begin{aligned} \dim\big(\Phi^{-1}(G_J^\infty) \cap (\mathbb{A}^\circ \times \{w\})\big) &\leqslant \dim\big((\rho_w \circ \varphi_w)^{-1}(\mathbb{V}_{2,J}^\infty)\big) \\ &\leqslant \dim(\mathbb{V}_{2,J}^\infty) + \dim \ker(\rho_w \circ \varphi_w) \\ &\leqslant \dim(\mathbb{V}_{2,J}^\infty) + \dim \mathbb{A} - \operatorname{rank}(\rho_w \circ \varphi_w) \\ &= \dim(\mathbb{V}_{2,J}^\infty) + \dim \mathbb{A} - \sum_{i=1}^{c} (k+1) \dim V_{\delta_i,J} \\ &= \dim \mathbb{A} - \operatorname{codim}(\mathbb{V}_{2,J}^\infty, \prod_{i=1}^{c} V_{\delta_i,J}^{k+1}), \end{aligned}$$

which in turn implies that

$$\dim\big(\Phi^{-1}(G_J^{\infty}) \cap \mathbb{A}^{\circ} \times \hat{X}_{k,J}^{\circ}\big) \leqslant \dim \mathbb{A} - \operatorname{codim}(\mathbb{V}_{2,J}^{\infty}, \prod_{i=1}^{c} V_{\delta_i,J}^{k+1}) + \dim \hat{X}_k.$$

By a result due to Benoist [1] and Brotbek-Darondeau (see [4, Corollary 3.2]), we have

$$\operatorname{codim}(\mathbb{V}_{2,J}^{\infty}, \prod_{i=1}^{c} V_{\delta_i,J}^{k+1}) \geqslant \min_{i=1,\ldots,c} \delta_i + 1.$$

Therefore, if

$$\dim \hat{X}_k < \min_{i=1,\ldots,c} \delta_i + 1, \tag{3.2.4}$$

$\Phi^{-1}(G_J^{\infty})$ doesn't dominate $\mathbb{A}^{\circ}$ via the projection $\mathbb{A}^{\circ} \times \hat{X}_{k,J}^{\circ} \to \mathbb{A}^{\circ}$, and we can thus find a non-empty Zariski open subset $\mathbb{A}_J \subset \mathbb{A}^{\circ}$ such that

$$\Phi^{-1}(G_J^{\infty}) \cap (\mathbb{A}_J \times \hat{X}_{k,J}^{\circ}) = \emptyset. \qquad \square$$

### 3.3. Proof of Theorem C

We are now in position to prove Theorem C. Indeed, we establish the following more refined result than Theorem C.

THEOREM 3.4. – *Let $X$ be an $n$-dimensional projective manifold equipped with a very ample line bundle $A$. Let $c$ be any integer satisfying $1 \leqslant c \leqslant n-1$, and set $k := \lceil \frac{n}{c} \rceil - 1$. Assume that the multi-degrees $(d_1, \ldots, d_c) \in (\mathbb{N})^c$ satisfy the following condition:*

$$\begin{aligned}
&\exists \delta := (\delta_1, \ldots, \delta_c) \in \mathbb{N}^c \text{ with } \delta_i \geqslant \delta_0 := n(k+1) \text{ for } i = 1, \ldots, c. \\
&\exists \varepsilon := (\varepsilon_1, \ldots, \varepsilon_c) \in \mathbb{N}^c \text{ with } \varepsilon_i \geqslant k \text{ for } i = 1, \ldots, c. \\
&\exists r > \sum_{i=1}^{c} b_i (k+1)(\varepsilon_i + k\delta_i), \text{ where } b_i := \frac{\prod_{j=1}^{c} \delta_j^{k+1}}{\delta_i} \\
&\text{s.t. } d_i = \varepsilon_i + (r+k)\delta_i \text{ for } i = 1, \ldots, c.
\end{aligned}$$

*Then for general hypersurfaces $H_1 \in |A^{d_1}|, \ldots, H_c \in |A^{d_c}|$, their complete intersection (smooth) variety $Z := H_1 \cap \cdots \cap H_c$ is* almost $\tilde{k}$-jet ample *for any $\tilde{k} \geqslant k$.*

*Proof.* – Observe that, the choice for $(\varepsilon, \delta)$ and $k$ in the theorem fits all the requirements in Theorem 3.2 and Lemma 3.3. In the same vein as [4, 3], let us first prove the nefness.

CLAIM 3.5. – *Set $\mathbb{A}_{\mathrm{nef}} := \bigcap_J \mathbb{A}_J$. For any $\mathbf{a} \in \mathbb{A}_{\mathrm{nef}}$, the line bundle*

$$\nu_k^*\big(\mathscr{O}_{X_k}(\sum_{i=1}^{c} b_i k') \otimes \pi_{0,k}^* A^{-q(\varepsilon,\delta,r)}\big) \otimes \mathscr{O}_{\hat{X}_k}(-\sum_{i=1}^{c} b_i F)_{\restriction \hat{Z}_{\mathbf{a},k}}$$

*on $\hat{Z}_{\mathbf{a},k}$ is nef. Here we write $q(\varepsilon, \delta, r) := r - \sum_{i=1}^{c} b_i (k+1)(\varepsilon_i + k\delta_i) > 0$.*

*Proof.* – To prove that a line bundle on a projective variety is nef, it suffices to show that for any irreducible curve, its intersection with this line bundle is non-negative. For any fixed $\mathbf{a} \in \mathbb{A}_{\mathrm{nef}}$, and any irreducible curve $C \subset \hat{Z}_{\mathbf{a},k}$, there is a unique $J \subset \{0,\ldots,n\}$ such that $C^\circ := \hat{X}^\circ_{k,J} \cap C$ is a non-empty Zariski open subset of $C$, and thus $C^\circ \subset \hat{\mathscr{Z}}_{k,J}$. It follows from Theorem 3.2.(iv) that $\Psi$ factors through $\mathscr{Y}_J$ when restricted to $\hat{\mathscr{Z}}_{k,J}$. Hence $\Psi_{\restriction C^\circ}$ also factors through $\mathscr{Y}_J$, and by the properness of $\mathscr{Y}_J$, $\Psi(C) \subset \mathscr{Y}_J$. By Lemma 3.3 and the definition of $\mathbb{A}_{\mathrm{nef}}$, we have

$$\Phi(C^\circ) \cap G_J^\infty = \emptyset,$$

and thus

$$\Psi(C) \not\subset p_J^{-1}(G_J^\infty).$$

By Theorem 2.10, one has

$$\mathrm{Bs}\big(\mathscr{L}(b_1,\ldots,b_c) \boxtimes \mathscr{O}_{\mathbb{P}^n}(-1)_{\restriction \mathscr{Y}_J}\big) \subset p_J^{-1}(G_J^\infty),$$

which yields

$$\Psi(C) \cdot \big(\mathscr{L}(b_1,\ldots,b_c) \boxtimes \mathscr{O}_{\mathbb{P}^n}(-1)_{\restriction \mathscr{Y}}\big) = \Psi(C) \cdot \big(\mathscr{L}(b_1,\ldots,b_c) \boxtimes \mathscr{O}_{\mathbb{P}^n}(-1)_{\restriction \mathscr{Y}_J}\big) \geqslant 0.$$

Write $\Psi_{\mathbf{a}} : \hat{Z}_{\mathbf{a},k} \to \mathscr{Y}$ the restriction of $\Psi$ to $\hat{Z}_{\mathbf{a},k}$. By (3.2.3), we have

$$\Psi_{\mathbf{a}}^*\big(\mathscr{L}(b_1,\ldots,b_c) \boxtimes \mathscr{O}_{\mathbb{P}^n}(-1)_{\restriction \mathscr{Y}}\big) = \nu_k^*\big(\mathscr{O}_{X_k}(\sum_{i=1}^c b_i k') \otimes \pi_{0,k}^* A^{-q(\varepsilon,\delta,r)}\big) \otimes \mathscr{O}_{\hat{X}_k}(-\sum_{i=1}^c b_i F)_{\restriction \hat{Z}_{\mathbf{a},k}},$$

and thus

$$C \cdot \Big(\nu_k^*\big(\mathscr{O}_{X_k}(\sum_{i=1}^c b_i k') \otimes \pi_{0,k}^* A^{-q(\varepsilon,\delta,r)}\big) \otimes \mathscr{O}_{\hat{X}_k}(-\sum_{i=1}^c b_i F)_{\restriction \hat{Z}_{\mathbf{a},k}}\Big) \geqslant 0,$$

which proves the claim. □

By [9, Proposition 6.16], we can find an ample line bundle

$$\nu_k^*\big(\mathscr{O}_{X_k}(a_k,\ldots,a_1) \otimes \pi_{0,k}^* A^{a_0}\big) \otimes \mathscr{O}_{\hat{X}_k}(-F)$$

on $\hat{X}_k$ for some $a_0,\ldots,a_k \in \mathbb{N}$. Denote by $\nu_{\mathbf{a},k} : \hat{Z}_{\mathbf{a},k} \to Z_{\mathbf{a},k}$ the blow-up of the asymptotic Wronskian ideal sheaf $\mathfrak{w}_\infty(Z_{\mathbf{a},k})$ of $Z_{\mathbf{a},k}$. Write $A_{\mathbf{a}} := A_{\restriction Z_{\mathbf{a}}}$ and $F_{\mathbf{a}} := F \cap \hat{Z}_{\mathbf{a},k}$. Therefore, for any $\ell > a_0$, by Claim 3.5 the line bundle

$$\nu_{\mathbf{a},k}^*\big(\mathscr{O}_{Z_{\mathbf{a},k}}(a_k + \sum_{i=1}^c \ell b_i k', a_{k-1},\ldots,a_1) \otimes \pi_{0,k}^* A_{\mathbf{a}}^{a_0-\ell q(\varepsilon,\delta,r)}\big) \otimes \mathscr{O}_{\hat{Z}_{\mathbf{a},k}}\big(-(\sum_{i=1}^c \ell b_i + 1)F_{\mathbf{a}}\big)$$

$$= \nu_k^*\big(\mathscr{O}_{X_k}(a_k + \sum_{i=1}^c \ell b_i k', a_{k-1},\ldots,a_1) \otimes \pi_{0,k}^* A^{a_0-\ell q(\varepsilon,\delta,r)}\big) \otimes \mathscr{O}_{\hat{X}_k}\big(-(\sum_{i=1}^c \ell b_i + 1)F\big)_{\restriction \hat{Z}_{\mathbf{a},k}}$$

is ample for $\mathbf{a} \in \mathbb{A}_{\mathrm{nef}}$, which verifies the condition $(*)$. By the Zariski open property $(*)$ in Theorem 1.7.(ii), we conclude that there exists a non-empty Zariski open subset $S_{\mathrm{ample}} \subset \prod_{i=1}^c |A^{d_i}|$ such that for any $(H_1,\ldots,H_c) \in S_{\mathrm{ample}}$, their complete intersection $Z := H_1 \cap \cdots \cap H_c$ is a reduced smooth variety of codimension $c$ in $X$, and $Z$ is almost $k$-jet ample. By [9, Lemma 7.6], if a complex manifold $Y$ is almost $k$-jet ample, then it is also almost $\tilde{k}$-jet ample for any $\tilde{k} \geqslant k$. This finishes the proof of the theorem. □

Let us deduce Theorem C from Theorem 3.4.

*Proof of Theorem C.* – Let us keep the same notations in Theorem 3.4. We will fix $\varepsilon_1 = \cdots = \varepsilon_c \geqslant k := \lceil \frac{n}{c} \rceil - 1$ and $\delta := (\delta_0, \ldots, \delta_0)$ with $\delta_0 = n(k+1)$. Then $b_1 = \cdots = b_c = \delta_0^{c(k+1)-1}$. If we take

$$d_0 := \delta_0\big(c(k+1)(k+\delta_0+k\delta_0-1)\delta_0^{c(k+1)-1}+1+k\big)+k,$$

then any $d \geqslant d_0$ has a decomposition

$$d = \delta_0(r+k)+\varepsilon$$

with $k \leqslant \varepsilon < k+\delta_0$, and

$$r \geqslant c\delta_0^{c(k+1)-1}(k+1)(k+\delta_0-1+k\delta_0)+1 > \sum_{i=1}^{c} b_i(k+1)(\varepsilon+k\delta_0),$$

satisfying the conditions in Theorem 3.4. Observe that

$$\begin{aligned} d_0 &= \delta_0\big(c(k+1)(k+\delta_0+k\delta_0-1)\delta_0^{c(k+1)-1}+1+k\big)+k \\ &\leqslant \delta_0^{c(k+1)}c(k+1)^2(\delta_0+1) \\ &\leqslant 2cn^{c\lceil \frac{n}{c}\rceil+1}\cdot \lceil \frac{n}{c}\rceil^{c\lceil \frac{n}{c}\rceil+3}. \end{aligned} \tag{3.3.1}$$

In conclusion, the complete intersection $H_1 \cap \cdots \cap H_c$ of $c$-general hypersurfaces $H_1, \ldots, H_c \in |\mathscr{A}^d|$ with $d \geqslant 2cn^{c\lceil \frac{n}{c}\rceil+1}\cdot \lceil \frac{n}{c}\rceil^{c\lceil \frac{n}{c}\rceil+3}$ is almost $\tilde{k}$-jet ample for any $\tilde{k} \geqslant \frac{n}{c}-1$. □

Let us mention that when $\frac{n}{2} \leqslant c \leqslant n-1$, by [4, Corollary 2.9], one can take $\delta_0 := 2n-1$, which is slightly better than that in Theorem 3.4. Now we apply the estimate in [4] to provide a slight better bound in the case $\frac{n}{2} \leqslant c \leqslant n-1$.

*Proof of Theorem A.* – Note that if $X$ is a smooth projective variety whose cotangent bundle $\Omega_X$ is ample, then for any smooth closed subvariety $Y \subset X$, $\Omega_Y$ is also ample. Hence it suffices to prove the theorem for $c = \lceil \frac{n}{2} \rceil$, $k = 1$. By (3.3.1) and $\delta_0 = 2n-1$, one can take

$$\begin{aligned} d_{\mathrm{Deb},n} &= \delta_0\big(c(k+1)(k+\delta_0+k\delta_0-1)\delta_0^{c(k+1)-1}+1+k\big)+k \\ &= 4(2n-1)^{2\lceil \frac{n}{2}\rceil+1}\cdot \lceil \frac{n}{2}\rceil + 2(2n-1)+1 \\ &\leqslant 2(2n-1)^{n+2}\cdot(n+1)+4n-1 \\ &\leqslant (2n)^{n+3}. \end{aligned}$$

□

### 3.4. Proof of Corollary D

This subsection is devoted to prove Corollary D.

*Proof of Corollary D.* – Recall that the Demailly-Semple $k$-jet tower $Z_k$ of $(Z, T_Z)$ is a locally trivial product as well as its blow-up $\nu_k : \hat{Z}_k \to Z_k$ along the Wronskian ideal sheaf $\mathfrak{w}_\infty(Z_k)$. Indeed, by § 1.2 for any $z \in Z$ there exists an open set $U$ containing $z$ so that $U_k := \pi_{0,k}^{-1}(U) \simeq U \times \mathbb{R}_{n-c,k}$ and $\mathfrak{w}_\infty(Z_k)_{\restriction U_k} \simeq \mathrm{pr}_2^* \mathfrak{I}_{n-c,k}$, where $\mathrm{pr}_2 : U \times \mathbb{R}_{n-c,k} \to \mathbb{R}_{n-c,k}$ is the projection map. Let us denote by $\mu_k : \hat{\mathbb{R}}_{n-c,k} \to \mathbb{R}_{n-c,k}$ the blow-up of $\mathbb{R}_{n-c,k}$ along $\mathfrak{I}_{n-c,k}$. Write $\hat{U}_k := \nu_k^{-1}(U_k)$. Then

$$
\begin{tikzcd}
\hat{U}_k \arrow[r, "\simeq"] \arrow[d, "\nu_k"'] & U \times \hat{\mathbb{R}}_{n-c,k} \arrow[d, "1\times\mu_k"] \\
U_k \arrow[r, "\simeq"] & U \times \mathbb{R}_{n-c,k}.
\end{tikzcd}
\tag{3.4.1}
$$

It follows from the proof of Theorem 3.4 that there exists $a_1, \ldots, a_k, q \in \mathbb{N}$ such that $\nu_k^* \mathscr{O}_{Z_k}(a_k, \ldots, a_1) \otimes \mathscr{O}_{\hat{Z}_k}(-qF)$ is ample. Write $\pi_k = \pi_{0,k} \circ \nu_k : \hat{Z}_k \to Z$. One thus can take $a_1, \ldots, a_k, q \gg 0$ so that all higher direct images

$$
R^i(\pi_k)_*\big(\nu_k^* \mathscr{O}_{Z_k}(a_k, \ldots, a_1) \otimes \mathscr{O}_{\hat{Z}_k}(-qF)\big) = 0 \quad \forall\, i > 0, \tag{3.4.2}
$$

and $\mathscr{L} := \nu_k^* \mathscr{O}_{Z_k}(a_k, \ldots, a_1) \otimes \mathscr{O}_{\hat{Z}_k}(-qF) \otimes \pi_k^* \mathscr{A}^{-1}$ is ample for some very ample line bundle $\mathscr{A}$ on $Z$.

Claim 3.6. – *$(\pi_k)_*\big(\nu_k^* \mathscr{O}_{Z_k}(ma_k, \ldots, ma_1) \otimes \mathscr{O}_{\hat{Z}_k}(-mqF)\big)$ is an ample vector bundle for each $m \gg 0$.*

*Proof of Claim 3.6.* – Denote by $\mathscr{E}_m := (\pi_k)_*(\mathscr{L}^m)$. From the local trivial product structure of $\hat{Z}_k$ as in (3.4.1), $\mathscr{E}_m$ is locally free for each $m > 0$.

By (3.4.2) and the degeneration of Leray spectral sequences, one has

$$
H^i(Z, \mathscr{E}_m \otimes \mathscr{F}) = H^i(\hat{Z}_k, \mathscr{L}^m \otimes \pi_k^* \mathscr{F}) \quad \forall\, i > 0,\ m > 0
$$

for any coherent sheaf $\mathscr{F}$ on $Z$. Fix any point $y \in Z$, with the maximal ideal of $\mathscr{O}_{Z,y}$ denoted by $\mathfrak{m}_y$. As $\mathscr{L}$ is ample, there is a positive integer $m_y \gg 0$ such that

$$
H^1(Z, \mathscr{E}_m \otimes \mathfrak{m}_y) = H^1(\hat{Z}_k, \mathscr{L}^m \otimes \pi_k^* \mathfrak{m}_y) = 0 \quad \forall\, m \geqslant m_y,
$$

which in turn implies that $\mathscr{E}_m$ is globally generated at $y$ for all $m \geqslant m_y$. As $Z$ is compact, we can find an integer $m_0 \gg 0$ such that $\mathscr{E}_m$ is globally generated when $m \geqslant m_0$. Observe that

$$
\mathscr{E}_m = (\pi_k)_*\big(\nu_k^* \mathscr{O}_{Z_k}(ma_k, \ldots, ma_1) \otimes \mathscr{O}_{\hat{Z}_k}(-mqF)\big) \otimes \mathscr{A}^{-m},
$$

where $\mathscr{A}$ is a very ample line bundle on $Z$. Hence $(\pi_k)_*\big(\nu_k^* \mathscr{O}_{Z_k}(ma_k, \ldots, ma_1) \otimes \mathscr{O}_{\hat{Z}_k}(-mqF)\big)$ is a quotient of a direct sum of copies of the very ample line bundle $\mathscr{O}_Z(\mathscr{A}^m)$. By the *cohomological characterization of ample vector bundles* in [20, Theorem 6.1.10], $(\pi_k)_*\big(\nu_k^* \mathscr{O}_{Z_k}(ma_k, \ldots, ma_1) \otimes \mathscr{O}_{\hat{Z}_k}(-mqF)\big)$ is ample for $m \geqslant m_0$. □

By the projection formula

$$\mathscr{F} := (\pi_k)_*\big(\nu_k^*\mathscr{O}_{Z_k}(a_k,\dots,a_1)\otimes\mathscr{O}_{\hat{Z}_k}(-qF)\big) = (\pi_{0,k})_*\big(\mathscr{O}_{Z_k}(a_k,\dots,a_1)\otimes\mathfrak{J}_q\big), \tag{3.4.3}$$

where $\mathfrak{J}_q := (\nu_k)_*\mathscr{O}_{\hat{Z}_k}(-qF)$ is the ideal sheaf of $Z_k$ with the subscheme $\mathscr{O}_{Z_k}/\mathfrak{J}_q$ supported on $Z_k^{\mathrm{sing}}$. By Claim 3.6, for proper $a_1,\dots,a_k,q \gg 0$, $\nu_k^*\mathscr{O}_{Z_k}(a_k,\dots,a_1)\otimes\mathscr{O}_{\hat{Z}_k}(-qF)\otimes\pi_k^*\mathscr{A}^{-1}$ is very ample. For any regular germ of curve $f:(\mathbb{C},0)\to(Z,z)$, its $k$-th lift $f_{[k]}\in Z_k^{\mathrm{reg}}$. Hence there exists a global section $\sigma\in H^0\big(Z_k,\mathscr{O}_{Z_k}(a_k,\dots,a_1)\otimes\pi_{0,k}^*\mathscr{A}^{-1}\otimes\mathfrak{J}_q\big)$ so that $\sigma(f_{[k]})\neq 0$. Let $P_\sigma\in H^0(Z,\mathscr{F}\otimes\mathscr{A}^{-1})$ be the corresponding element of $\sigma$ under the isomorphism (3.4.3). Hence $P_\sigma([f]_k)\neq 0$. It follows from [9, Proposition 6.16.i] that $\mathscr{F}\subset E_{k,m}T_Z^*$ for $m := a_1+\cdots+a_k$. The corollary is thus proved. □

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