BIG PICARD THEOREM AND ALGEBRAIC HYPERBOLICITY FOR VARIETIES ADMITTING A VARIATION OF HODGE STRUCTURES

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Abstract. In this paper we study various notions of hyperbolicity for varieties admitting complex polarized variation of Hodge structures ($\mathbb{C}$-PVHS for short). In the first part we prove that if a quasi-projective manifold U admits a $\mathbb{C}$-PVHS whose period map is quasi-finite, then U is algebraically hyperbolic in the sense of Demailly, and that the generalized big Picard theorem holds for U: any holomorphic map from the punctured unit disk to U extends to a holomorphic map of the unit disk Δ into any projective compactification of U. This result generalizes a recent work by Bakker-Brunebarbe-Tsimerman. In the second part, we prove the strong hyperbolicity for varieties admitting $\mathbb{C}$-PVHS, which is analogous to previous works by Nadel, Rousseau, Brunebarbe and Cadorel on arithmetic locally symmetric varieties. In the last part, we show how the techniques developed in this paper yield some new perspectives for hyperbolicity of arithmetic locally symmetric varieties.

Contents

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0. Introduction

0.1. **Background.** The classical big Picard theorem says that any holomorphic map from the punctured disk Δ^* into $\mathbb{P}^1$ which omits three points can be extended to a holomorphic map $\Delta \to \mathbb{P}^1$, where Δ denotes the unit disk. Therefore, we introduce a new notation of hyperbolicity which generalizes the big Picard theorem.

Definition 0.1 (Picard hyperbolicity)**.** Let U be a Zariski open set of a compact Kähler manifold Y. U is called *Picard hyperbolic* if any holomorphic map $f : \Delta^* \to U$ extends to a holomorphic map $\bar{f} : \Delta \to Y$.

Picard hyperbolic varieties first attracted the author's interests because of the recent interesting work [JK18b] by Javanpeykar-Kucharczyk on the *algebraicity of analytic maps*. In [JK18b, Definition 1.1], they introduce a new notion of hyperbolicity: a quasi-projective variety U is *Borel hyperbolic* if any holomorphic map from a quasi-projective variety to U is necessarily *algebraic*. In [JK18b, Corollary 3.11] they prove that a Picard hyperbolic variety is Borel hyperbolic. We refer the readers to [JK18b, §1] for their motivation on the Borel hyperbolicity. Picard hyperbolic varieties fascinate us further when we realize in Proposition 3.4 that a more general extension theorem is also valid for them: any holomorphic map from $\Delta^p \times (\Delta^*)^q$ to the manifold U in Definition 0.1 extends to a *meromorphic map* from Δ^{p+q} to Y.

By A. Borel [Bor72] and Kobayashi-Ochiai [KO71], it has long been known to us that the quotients of bounded symmetric domains by torsion free arithmetic lattice are hyperbolically embedded into their Baily-Borel compactification, and thus they are Picard hyperbolic (see [Kob98, Theorem 6.1.3]). A *transcendental* analogue of bounded symmetric domains is the rich theory of period domain, which was first introduced by Griffiths [Gri68a] and was later systematically studied by him in the seminal work [Gri68b, Gri70a, Gri70b]. Griffiths further conjectured that the image of a 'period map' is algebraic and that the period map is algebraic. In [JK18b, §1.1] Javanpeykar-Kucharczyk formulated an inspiring variant of Griffiths' conjecture as follows.

Conjecture 0.2 (Griffiths, Javanpeykar-Kucharczyk)**.** An algebraic variety U which admits a quasi-finite period map $U \to \mathcal{D}/\Gamma$ is Borel hyperbolic.

Unlike Hermitian symmetric spaces, except the classical cases (abelian varieties, and K3 type), the quotient of period domain $\mathcal{D}/\Gamma$ in Conjecture 0.2 is never an algebraic variety, and the global monodromy groups Γ is not arithmetic in general. However, it is still expected and conjectured by Griffiths that there is a 'partial compactification' for $\mathcal{D}/\Gamma$ analogous to the Baily-Borel-Satake compactification in the sense of [Gri70b, Conjecture 9.2] or [GGLR17, Conjecture 1.2.2]. For a period map $p : U \to \mathcal{D}/\Gamma$, in [GGLR17] Green-Griffiths-Lazza-Robles constructed Hodge theoretic completion for the image $p(U)$ when $\dim p(U) = 1, 2$.

In a recent remarkable work [BBT18], Bakker-Brunebarbe-Tsimerman proved (among others) that a variety (or more generally Deligne-Mumford stacks) admitting a quasi-finite $\mathbb{R}_{\mathrm{an,exp}}$-period map is Borel hyperbolic. Since they applied the tools from o-minimal structures, they have to assume that the monodromy group of variation of Hodge structures they studied are *arithmetic*. In this paper, we extend their theorem to the Picard hyperbolicity, and we also remove their arithmeticity condition for monodromy groups.

0.2. **Big Picard theorem and algebraic hyperbolicity.** The first result is the following.

Theorem A. *Let Y be a complex projective manifold and let D be a simple normal crossing divisor on Y. Assume that there is a complex polarized variation of Hodge structures ($\mathbb{C}$-PVHS for short) over $U := Y - D$ with discrete monodromy group and local unipotent monodromies around D whose period map is quasi-finite* (i.e. *every fiber is a finite set). Then U is both algebraically hyperbolic, and Picard hyperbolic. In particular, U is Borel hyperbolic.*

We refer the reader to § 1.1 for complex polarized variation of Hodge structures ($\mathbb{C}$-PVHS for short), and to Definition 3.1 for the definition of algebraic hyperbolicity. Let us mention that this result yields a new proof for Borel's extension theorem (see Theorem 5.2), since there is a canonical $\mathbb{C}$-PVHS on the quotient of bounded symmetric domain by torsion free arithmetic lattice whose period map is immersive (see Theorem 5.1). As a consequence of Theorem A, we obtain the following result for varieties admitting an integral variation of Hodge structures, which in particular proves Conjecture 0.2.

Theorem B. *Let U be a quasi-projective manifold and let $(V, \nabla, F^\bullet, Q)$ be an integral polarized variation of Hodge structures over U, whose period map is quasi-finite. Then U is both algebraically hyperbolic and Picard hyperbolic. In particular, U is Borel hyperbolic.*

Let us mention that when the monodromy group of polarized variation of Hodge structures $(V, \nabla, F^\bullet, Q)$ in Theorem B is assumed to be *arithmetic*, Borel hyperbolicity of the quasi-projective manifold U in Theorem B has been proven in [BBT18, Corollary 7.1], and algebraic hyperbolicity of U is implicitly shown by Javanpeykar-Litt in [JL19, Theorem 4.2] if local monodromies $(V, \nabla, F^\bullet, Q)$ at infinity are unipotent (see Remark 3.3). Our proofs of Theorems A and B are based on complex analytic and Hodge theoretic methods, and it does not use the delicate o-minimal geometry in [PS08, PS09, BKT18, BBT18].

We can even generalize Theorems A and B to higher dimensional domain spaces.

Corollary C (=Theorems A and B+Proposition 3.4)**.** *Let U be the quasi-projective manifold in Theorem A or Theorem B, and let Y be a smooth projective compactification of U. Then any holomorphic map $f : \Delta^p \times (\Delta^*)^q \to U$ extends to a meromorphic map $\overline{f} : \Delta^{p+q} \dashrightarrow Y$. In particular, if W is a Zariski open set of a compact complex manifold X, then any holomorphic map $g : W \to U$ extends to a meromorphic map $\overline{g} : X \dashrightarrow Y$.*

0.3. **Lang conjecture and strong hyperbolicity.** Let us introduce several definitions of hyperbolicity. We refer the readers to the recent survey by Javenpeykar [Jav20b, §8] for more details and the relations among them.

Definition 0.3 (Notions of hyperbolicity)**.** Let X be a complex projective manifold and let $Z \subsetneq X$ be a closed subset of X.

(1) The variety X is called *Kobayashi hyperbolic modulo Z* if the Kobayashi pseudo-distance $d_X(x, y) > 0$ for distinct points $x, y \in X$ not both contained in Z. If Z can be chosen to be a proper Zariski closed subset, then we say X is *pseudo Kobayashi hyperbolic.*
(2) The variety X is called *Picard hyperbolic modulo Z* if any holomorphic map $\gamma : \Delta^* \to X$ not contained in Z extends across the origin. If Z can be chosen to be a proper Zariski closed subset, then we say X is *pseudo Picard hyperbolic.*
(3) The variety X is called *Brody hyperbolic modulo Z* if any entire curve $\gamma : \mathbb{C} \to X$ is contained in Z.

It is easy to show that if X is pseudo Kobayashi hyperbolic or pseudo Picard hyperbolic modulo a proper closed subvariety Z, then X is Brody hyperbolic moduli Z.

A tantalizing conjecture by Lang [Lan91, Chapter VIII. Conjecture 1.4] predicts the connection between positivity in algebraic geometry and hyperbolicity as follows.

Conjecture 0.4 (Lang)**.** *A projective manifold is of general type if and only if it is pseudo Kobayashi hyperbolic.*

In a similar vein, it is quite natural to propose the following conjecture, as suggested in [Jav20b, Remark 8.10].

Conjecture 0.5. *A projective manifold is of general type if and only if it is pseudo Picard hyperbolic.*

In the second part of the paper we prove *strong hyperbolicity* (notion first introduced in [Bru16]) for varieties admitting $\mathbb{C}$-PVHS, motivated by previous works of [Nad89, Rou16, Bru16, Cad16, Cad18] for *arithmetic locally symmetric varieties.*

Theorem D. *Let Y be a complex projective manifold and let $D = \sum_{i=1}^{c} D_i$ be a simple normal crossing divisor on Y. Assume that there is a $\mathbb{C}$-PVHS $(V, \nabla, F^\bullet, Q)$ over $U := Y - D$ whose period map is generically immersive. Assume moreover that the local monodromies of $(V, \nabla, F^\bullet, Q)$ around D is unipotent, and that $(V, \nabla, F^\bullet, Q)$ has* injective local monodromy representation around *D (see Definition 4.2). Then there is a finite étale cover $\tilde{U} \to U$ and a projective compactification X of $\tilde{U}$ so that*

(i) *the variety X is of general type;*
(ii) *the variety X is pseudo Kobayashi hyperbolic;*
(iii) *the variety X is pseudo Picard hyperbolic.*

Based on Theorem D, one can prove a more refined result, which is also conjectured by Javanpeykar [Jav20a].

Corollary E. *Let (Y, D) be a log pair. Assume that there is a $\mathbb{C}$-PVHS $(V, \nabla, F^\bullet, Q)$ over $U := Y - D$ with discrete monodromy group whose period map is quasi-finite. Assume moreover that the local monodromies of $(V, \nabla, F^\bullet, Q)$ around D is unipotent, and that $(V, \nabla, F^\bullet, Q)$ has* injective local monodromy representation around *D (see Definition 4.2). Then there is a finite étale cover $\tilde{U} \to U$ and a projective compactification X of $\tilde{U}$ so that*

(i) *any Zariski closed subvariety of X is of general type if it is not contained in $\tilde{D} := X - \tilde{U}$.*
(ii) *The variety X is Picard hyperbolic modulo $\tilde{D}$;*
(iii) *the variety X is Brody hyperbolic modulo $\tilde{D}$.*

Theorem D and Corollary E provide a new class of examples verifying Conjectures 0.4 and 0.5. Moreover, we can apply Theorem D to prove a strong hyperbolicity for arithmetic locally symmetric varieties.

Theorem F (=Theorem 5.4)**.** *Let $U := \mathcal{D}/\Gamma$ be the quotient of be a bounded symmetric domain $\mathcal{D}$ by a torsion free arithmetic lattice $\Gamma \in \mathrm{Aut}(\mathcal{D})$. Then there is a finite étale cover $\tilde{U} \to U$ and a projective compactification X of $\tilde{U}$ so that X is Picard hyperbolic modulo $X - \tilde{U}$.*

Let us stress here that, previously, Rousseau [Rou16] proved that the variety X in Theorem F is Kobayashi hyperbolic modulo $X - \tilde{U}$, and Brunebarbe [Bru16] proved that any Zariski closed subvariety not contained in $X - \tilde{U}$ is of general type.

0.4. **Main strategy.**

0.4.1. *Why not Hodge metric?* Let Y be a projective manifold and let D be a simple normal crossing divisor on Y. Assume that there is a $\mathbb{C}$-PVHS $(V,\nabla,F^\bullet,Q)$ on $U=Y-D$. Then there is a natural holomorphic map, so-called *period map*, $p:U\to \mathscr{D}/\Gamma$ where $\mathscr{D}$ is the *period domain* associated to $(V,\nabla,F^\bullet,Q)$ (see [CMSP17] or [KKM11, §4.3] for the definition) and Γ is the monodromy group. The period domain $\mathscr{D}$ admits a canonical (Γ-invariant) hermitian metric $h_{\mathscr{D}}$, and by Griffiths-Schmid [GS69] its holomorphic sectional curvatures along horizontal directions are bounded from above by a negative constant. One can thus easily show the Kobayashi hyperbolicity of U if p is immersive everywhere. Indeed, since p is tangent to the horizontal subbundle of $T_{\mathscr{D}}$ by the Griffiths transversality, one can pull back the metric $h_{\mathscr{D}}$ to U by p and by the curvature decreasing property, the holomorphic sectional curvature of the hermitian (moreover Kähler) metric $h_U:=p^*h_{\mathscr{D}}$ on U is also bounded from above by a negative constant. This Kähler metric h_U is quite useful in proving that the log cotangent bundle $\Omega_Y(\log D)$ is big and that (Y,D) is of log general type in the work [Zuo00,Bru18,BC17]. However, such metric h_U is not sufficient to prove the Picard hyperbolicity of U since h_U might *degenerate* in a bad way near the boundary D and thus its curvature behavior near D is unclear to us. To the best of our knowledge, it should be quite difficult to prove that U is Picard hyperbolic or algebraically hyperbolic without knowing the precise information of h_U near D.

0.4.2. *A Finsler metric on the compactification.* The recent work [DLSZ19] on the Picard hyperbolicity of moduli of polarized manifolds by Lu, Sun, Zuo and the author motivated us to prove Theorem A. An important tool (amongs others) in this work, is a particular Higgs bundle constructed by Viehweg-Zuo [VZ02,VZ03] (later developed by Popa el al. [PS17,PTW19] using mixed Hodge modules), which contains a *globally* positive line bundle over the compactification Y rather than U. This positive line bundle originates from Kawamata's deep work [Kaw85] on the Iitaka conjecture: for an algebraic fiber space $f:X\to Y$ between projective manifolds whose geometric generic fiber admits a good minimal model, $\det f_*(mK_{X/Y})$ is big for $m\gg 0$ if f has maximal variation. In an ingenious way, Viehweg-Zuo [VZ02, VZ03] applied Viehweg's fiber product and cyclic cover tricks to transfer Kawamata's positivity $\det f_*(mK_{X/Y})$ to their Higgs bundles.

We first note that in the case that there is a $\mathbb{C}$-PVHS $(V,\nabla,F^\bullet,Q)$ over $Y-D$ where (Y,D) is a log pair, one also has a strictly positive line bundle on U if the period map is generically immersive, which was constructed by Griffiths in [Gri70a] half century ago! Based on the work [CKS86,Kas85] on the asymptotic estimate for Hodge metrics at infinity, Bakker-Brunebarbe-Tsimerman [BBT18] showed that this Griffiths line bundle extends to a big line bundle $L_{\rm Gri}$ over Y if the monodromies of $(V,\nabla,F^\bullet,Q)$ around D are unipotent (see Lemma 1.4). As we will see later, the Griffiths line bundle plays a similar role as the Kawamata positivity described above. Indeed, based on the above $\mathbb{C}$-PVHS $(V,\nabla,F^\bullet,Q)$ we construct a Higgs bundle $(E,\theta)=(\oplus_{p+q=m}E^{p,q},\oplus_{p+q=m}\theta_{p,q})$ on the log pair (Y,D) so that the Griffiths line bundle $L_{\rm Gri}$ is contained in some higher stage E^{p_0,q_0} of E. This Higgs bundle shares some similarities with the Viehweg-Zuo Higgs bundle in [VZ02, VZ03] (see Remark 1.7). Inspired by our previous work [Den18] on the proof of Viehweg-Zuo's conjecture on Brody hyperbolicity of moduli of polarized manifolds, in Theorem 1.9 we show that (E,θ) still enjoys a 'partially' infinitesimal Torelli property. This enables us construct a negatively curved, and generically positively definite Finsler metric on U, in a similar vein as [Den18].

Theorem 0.6 (=Theorem 1.6+Theorem 2.6)**.** *Let Y be a projective manifold and let D be a simple normal crossing divisor on Y. Assume that there is a $\mathbb{C}$-PVHS over $Y-D$ with local unipotent monodromies around D, whose period map is generically immersive.*

Then there are a Finsler metric h *(see Definition 2.1) on* $T_Y(-\log D)$ *which is positively definite on a dense Zariski open set* U° *of* $Y - D$*, and a smooth Kähler form* ω *on* Y *such that for any holomorphic map* $\gamma : C \to U$ *from an open set* $C \subset \mathbb{C}$ *to* U*, one has*

$$\sqrt{-1}\partial\bar{\partial}\log|\gamma'(t)|_h^2 \geq \gamma^*\omega. \tag{0.4.1}$$

Let us mention that, though we only construct (possibly degenerate) Finsler metric over $T_Y(-\log D)$, it follows from (0.4.1) that we know exactly the behavior of its curvature near the boundary D since ω is a smooth Kähler form over Y. The proof of Theorem A is then based on Theorem 0.6 and the following criteria for big Picard theorem established in [DLSZ19] whose proof is Nevanlinna theoretic.

Theorem 0.7 ([DLSZ19, Theorem A]). *Let* Y *be a projective manifold and let* D *be a simple normal crossing divisor on* Y*. Let* $f : \Delta^* \to Y - D$ *be a holomorphic map. Assume that there is a (possibly degenerate) Finsler metric* h *of* $T_Y(-\log D)$ *such that* $|f'(t)|_h^2 \not\equiv 0$*, and*

$$\frac{1}{\pi}\sqrt{-1}\partial\bar{\partial}\log|f'(t)|_h^2 \geq f^*\omega \tag{0.4.2}$$

for some smooth Kähler metric ω *on* Y*. Then* f *extends to a holomorphic map* $\overline{f} : \Delta \to Y$*.*

Let us also mention that the Finsler metric constructed in Theorem 0.6 is also crucially used in the proof of Theorem D.

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Notations and Conventions

- A *log pair* (Y, D) consists of a smooth projective manifold and a simple normal crossing divisor D, and such log pair (Y, D) is called a *log-compactification* of the quasi-projective manifold $Y - D$.
- A *log morphism* $f : (X, E) \to (Y, D)$ between log pairs is a morphism $f : X \to Y$ with $E \subset f^{-1}(D)$.
- For line bundles L_1 and L_2 on a projective manifold, we write $L_1 \geq L_2$ if $L_1 \otimes L_2^{-1}$ is effective.
- For a big line bundle L on a projective manifold, $\mathbf{B}_+(L)$ denotes its *augmented base locus* (see [Laz04, Definition 10.3.2]).
- $\mathbb{C}$-PVHS stands for *complex polarized variation of Hodge structures.*
- An *arithmetic locally symmetric variety* is the quotient $\mathcal{D}/\Gamma$ of a bounded symmetric domain $\mathcal{D}$ by a torsion-free arithmetic lattice $\Gamma \in \mathrm{Aut}(\mathcal{D})$.

1. Construction of special Higgs bundles

1.1. Preliminary on $\mathbb{C}$-PVHS.

Definition 1.1. A Higgs bundle on a log pair (Y, D) is a pair (E, θ) consisting of a holomorphic vector bundle E on Y and an $\mathcal{O}_Y$-linear map

$$\theta : E \to E \otimes \Omega_Y(\log D)$$

so that $\theta \wedge \theta = 0$. Such θ is called *Higgs field.*

Following Simpson [Sim88], a complex polarized variation of Hodge structures of weight m over $U = Y - D$ is a C^∞-vector bundle $V = \oplus_{p+q=m} V^{p,q}$ and a flat connection ∇ satisfying Griffiths' transversality condition.

$$\nabla : V^{p,q} \to A^{0,1}(V^{p+1,q-1}) \oplus A^1(V^{p,q}) \oplus A^{1,0}(V^{p-1,q+1}) \tag{1.1.1}$$

and such that a polarization exists; this is a sesquilinear form $Q(\bullet,\bullet)$ over V, hermitian symmetric or antisymmetric as m is even or odd, invariant under ∇, such that the Hodge decomposition $V = \oplus_{p+q=m} V^{p,q}$ is orthogonal and such that

$$h := (\sqrt{-1})^{p-q} Q(\bullet,\bullet) > 0$$

on $V^{p,q}$.

Let us decompose ∇ into operators of $(1,0)$ and $(0,1)$

$$\nabla = \nabla' + \nabla''$$

and thus ∇'' induces a complex structure on V. We define a filtration

$$F^p V := V^{p,q} \oplus V^{p+1,q-1} \oplus \cdots \oplus V^{m,0}$$

and by (1.1.1) F^pV is invariant under ∇''. Hence F^pV can be equipped with the complex structure inherited from (V, ∇''), and the filtration

$$F^\bullet : V = F^0V \supset F^1V \supset \cdots \supset F^mV \supset F^{m+1}V = \{0\}$$

is called the *Hodge filtration*. Such data $(V, \nabla, F^\bullet, Q)$ is called a complex polarized variation of Hodge structures ($\mathbb{C}$-PVHS for short) on U.

Note that the flat connection ∇ in (1.1.1) induces an $\mathcal{O}_U$-linear map

$$\eta_{p,q} : F^pV/F^{p+1}V \to (F^{p-1}V/F^pV) \otimes \Omega_U.$$

Let us denote by $F := \oplus_p (F^pV/F^{p+1}V)$ and $\eta = \oplus_p \eta_{p,q}$. Then (F, η) is a Higgs bundle on U.

We say the $\mathbb{C}$-PVHS $(V, \nabla, F^\bullet, Q)$ on U has *unipotent monodromies* around D if local monodromies around D of the local system on U induced by the flat bundle (V, ∇) are all *unipotent.*

For two $\mathbb{C}$-PVHS $(V_1, \nabla_1, F^\bullet V_1, Q_1)$ and $(V_2, \nabla_2, F^\bullet V_2, Q_2)$ of weight m_1 and m_2 over $Y - D$, one can define their tensor product, which is still $\mathbb{C}$-PVHS with weight $m_1 + m_2$. Moreover, if they both have unipotent monodromies around D, so is their tensor product.

Remark 1.2. It is well-known that $\mathbb{C}$-PVHS are quite close to real variation of Hodge structures ($\mathbb{R}$-PVHS for short, see [CKS86] for a previse definition). Indeed, one can obtain a $\mathbb{R}$-PVHS by adding the $\mathbb{C}$-PVHS with its conjugate. In particular, the estimate of Hodge metric at infinity of a $\mathbb{R}$-PVHS in [CKS86] also holds true for $\mathbb{C}$-PVHS.

For a $\mathbb{C}$-PVHS $(V, \nabla, F^\bullet, Q)$ defined over $U = Y - D$ with unipotent monodromies around D, there is a *canonical* way to extend it to a Higgs bundle over the log pair (Y, D). By Deligne, V has a locally free extension $\overline{V}$ to Y such that ∇ extends to a logarithmic connection

$$\overline{\nabla} : \overline{V} \to \overline{V} \otimes \Omega_Y(\log D)$$

with nilpotent residues. For each p we set

$$\overline{F}^p V := \iota_* F^p V \cap \overline{V}$$

where $\iota : U \hookrightarrow Y$ is the inclusive map. By Schmid's nilpotent orbit theorem [Sch73], both $\overline{F}^p V$ and the graded term $\overline{F}^{p,q} = \overline{F}^p V/\overline{F}^{p+1} V$ are locally free, and $\overline{\nabla}$ induces an $\mathcal{O}_Y$-linear map

$$\overline{\eta}_{p,q} : \overline{F}^{p,q} \to \overline{F}^{p-1,q+1} \otimes \Omega_Y(\log D).$$

Hence the pair

$$(\overline{F},\overline{\eta}) := (\oplus_{p+q=m}\overline{F}^{p,q}, \oplus_{p+q=m}\overline{\eta}_{p,q}) \tag{1.1.2}$$

is a Higgs bundle on the log pair (Y,D), which extends (F,η) defined over U.

Definition 1.3. We say that Higgs bundle $(\overline{F},\overline{\eta})$ over (Y,D) in (1.1.2) is *canonically* induced by the $\mathbb{C}$-PVHS $(V,\nabla,F^{\bullet},Q)$.

1.2. **Griffiths line bundle.** For the $\mathbb{C}$-PVHS $(V,\nabla,F^{\bullet},Q)$ defined over U as above, in [Gri70a], Griffiths constructed a line bundle $L_{\rm Gri}$ on U, which he called the *canonical bundle* of $(V,\nabla,F^{\bullet},Q)$. When the local monodromies of $(V,\nabla,F^{\bullet},Q)$ around D are unipotent and the period map of $(V,\nabla,F^{\bullet},Q)$ is generically immersive, it seems well-known to the experts (see *e.g.* [Zuo00, p. 280] or [BBT18, Lemma 6.4]) that the Griffiths bundle extends to a big and nef line bundle on Y.

Lemma 1.4. *Let (Y,D) be a log pair. Let $(V,\nabla,F^{\bullet},Q)$ be a $\mathbb{C}$-PVHS of weight m over $Y-D$ with unipotent monodromies around D, whose period map is generically immersive. Then the* Griffiths line bundle

$$L_{\rm Gri} := (\det\overline{F}^{m,0})^{\otimes m}\otimes(\det\overline{F}^{m-1,1})^{\otimes(m-1)}\otimes\cdots\otimes\det\overline{F}^{1,m-1}$$

is a big and nef line bundle on Y. Here $(\oplus_{p+q=m}\overline{F}^{p,q}, \oplus_{p+q=m}\overline{\eta}_{p,q})$ is the Higgs bundle on (Y,D) canonically induced by $(V,\nabla,F^{\bullet},Q)$ defined in Definition 1.3.

We need a refined result of Lemma 1.4, which shall be used in the proof of Theorem D.

Lemma 1.5. *Let (Y,D) be a log pair. Let $(V,\nabla,F^{\bullet},Q)$ be a $\mathbb{C}$-PVHS over $Y-D$ with discrete monodromy group and local unipotent monodromies around D, whose period map is quasi-finite. Then the Griffiths line bundle L is a big and nef line bundle on Y, which is* ample *over $Y-D$; namely its augmented base locus $\mathbf{B}_+(L)\subset D$.*

Proof. By Lemma 1.4 we know that L is big and nef. By a theorem of Nakamaye [Laz04, Theorem 10.3.5], we know that the null locus $\mathrm{Null}(L)=\mathbf{B}_+(L)$. Recall that the null locus is defined by

$$\mathrm{Null}(L) := \bigcup_{Z\subsetneq Y;\ L^{\dim Z}\cdot Z=0} Z$$

where the union is taken over all irreducible positive dimensional closed subvarieties Z of Y. It then suffices to prove that $L^{\dim\tilde{Z}}\cdot\tilde{Z}>0$ for any irreducible closed subvariety $\tilde{Z}$ not contained in D. We take a desingularization $\nu: Z\to\tilde{Z}$ so that $D_Z:=\nu^{-1}(D)$ is simple normal crossing. Then $\nu:(Z,D_Z)\to(Y,D)$ is a log morphism, and one has a natural morphism $\nu^*\Omega_Y(\log D)\to\Omega_Z(\log D_Z)$.

Let $(\oplus_{p+q=m}\overline{F}^{p,q}, \oplus_{p+q=m}\overline{\eta}_{p,q})$ be the Higgs bundle on (Y,D) canonically induced by $(V,\nabla,F^{\bullet},Q)$ defined in Definition 1.3. Set $E^{p,q}:=\nu^*\overline{F}^{p,q}$, and define $\theta_{p,q}$ to be the composition map

$$\nu^*\overline{F}^{p,q}\to\nu^*\overline{F}^{p-1,q+1}\otimes\nu^*\Omega_Y(\log D)\to\nu^*\overline{F}^{p-1,q+1}\otimes\Omega_Z(\log D_Z).$$

Then $(\oplus_{p+q=m}E^{p,q},\oplus_{p+q=m}\theta_{p,q})$ is a Higgs bundle on (Z,D_Z) canonically induced by the $\mathbb{C}$-PVHS $\mu^*(V,\nabla,F^{\bullet},Q)$, where $\mu: Z-D_Z\to Y-D$ is the restriction of ν to $Z-D_Z$. Note that $\mu^*(V,\nabla,F^{\bullet},Q)$ also has local unipotent monodromies around D_Z. Since μ is generically immersive, the period map of $\mu^*(V,\nabla,F^{\bullet},Q)$ is also generically immersive. By Lemma 1.4, the Griffiths line bundle of $(\oplus_{p+q=m}E^{p,q},\oplus_{p+q=m}\theta_{p,q})$ is also big and nef. Namely,

$$L' := (\det E^{m,0})^{\otimes m}\otimes(\det E^{m-1,1})^{\otimes(m-1)}\otimes\cdots\otimes\det E^{1,m-1}$$

is a big and nef line bundle. Note that $L := (\det \overline{F}^{m,0})^{\otimes m} \otimes (\det \overline{F}^{m-1,1})^{\otimes(m-1)} \otimes \cdots \otimes \det \overline{F}^{1,m-1}$. Hence $\nu^*L = L'$ is big and nef. Hence

$$L^{\dim \tilde{Z}} \cdot \tilde{Z} = (\nu^* L)^{\dim Z} > 0.$$

It follows that $\mathbf{B}_+(L) \subset D$. The lemma is proved. ◻

1.3. **Special Higgs bundles induced by $\mathbb{C}$-PVHS.** Let (Y, D) be a log pair. Let $(V, \nabla, F^\bullet, Q)$ be a $\mathbb{C}$-PVHS of weight m over $Y - D$ with unipotent monodromies around D, whose period map is generically immersive. Let $(\overline{F}, \overline{\eta})$ be the Higgs bundle over the log pair (Y, D) canonically induced by $(V, \nabla, F^\bullet, Q)$ defined in Definition 1.3. Let us denote by $r_p := \operatorname{rank} F^{p,q}$, and $r := mr_m + (m-1)r_{m-1} + \cdots + r_1$.

We define a new Higgs bundle (E, θ) on (Y, D) by setting $(E, \theta) := (\overline{F}, \bar{\eta})^{\otimes r}$. Precisely, $E := \overline{F}^{\otimes r}$, and

$$\theta := \bar{\eta} \otimes \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{(r-1)-\text{tuple}} + \mathbb{1} \otimes \bar{\eta} \otimes \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{(r-2)-\text{tuple}} + \cdots + \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{(r-1)-\text{tuple}} \otimes \bar{\eta}.$$

We have the (Hodge) decomposition

$$E = \oplus_{P+Q=rm} E^{P,Q}$$

with

$$E^{P,Q} := \oplus_{p_1+\cdots+p_r=P; q_1+\cdots+q_r=Q} \overline{F}^{p_1,q_1} \otimes \cdots \otimes \overline{F}^{p_r,q_r} \tag{1.3.1}$$

Hence

$$\theta : E^{P,Q} \to E^{P-1,Q+1} \otimes \Omega_Y(\log D).$$

One can easily show that (E, θ) is *canonically induced* by the $\mathbb{C}$-PVHS $(V, \nabla, F^\bullet, Q)^{\otimes r}$ in the sense of Definition 1.3. Note that the tensor product $(V, \nabla, F^\bullet, Q)^{\otimes r}$ has weight $m \cdot r$, and also has unipotent monodromies around D.

Note that $\det \overline{F}^{p,q} = \wedge^{r_p} \overline{F}^{p,q} \subset (\overline{F}^{p,q})^{\otimes r_p} \subset \overline{F}^{\otimes r_p}$. Hence

$$L_{\rm Gri} := (\det \overline{F}^{m,0})^{\otimes m} \otimes (\det \overline{F}^{m-1,1})^{\otimes(m-1)} \otimes \cdots \otimes \det \overline{F}^{1,m-1} \subset (\overline{F}^{m,0})^{\otimes m r_m} \otimes \cdots \otimes (\overline{F}^{1,m-1})^{\otimes r_1} \subset E$$

Moreover, by (1.3.1), one has

$$L_{\rm Gri} \subset E^{P_0,Q_0}$$

with $P_0 = r_m m^2 + r_{m-1}(m-1)^2 + \cdots + r_1$, and $P_0 + Q_0 = rm$.

In summary, we construct a special Higgs bundle on the log pair (Y, D) as follows.

Theorem 1.6. *Let (Y, D) be a log pair. Let $(V, \nabla, F^\bullet, Q)$ be a $\mathbb{C}$-PVHS over $Y - D$ with unipotent monodromies around D, whose period map is generically immersive. Then there is a Higgs bundle $(E, \theta) = (\oplus_{p+q=\ell} E^{p,q}, \oplus_{p+q=\ell} \theta_{p,q})$ on the log pair (Y, D) satisfying the following conditions.*

(i) *The Higgs field θ satisfies*

$$\theta : E^{p,q} \to E^{p-1,q+1} \otimes \Omega_Y(\log D)$$

(ii) *(E, θ) is canonically induced (in the sense of Definition 1.3) by some $\mathbb{C}$-PVHS over $Y - D$ of weight ℓ with unipotent monodromies around D.*

(iii) *There is a big and nef line bundle L over Y such that $L \subset E^{p_0,q_0}$ for some $p_0 + q_0 = \ell$.* ◻

Remark 1.7. The interested readers can compare the Higgs bundle in Theorem 1.6 with the Viehweg-Zuo Higgs bundle in [VZ02, VZ03] (see also [PTW19]). Loosely speaking, a *Viehweg-Zuo Higgs bundle* for a log pair (Y, D) is a Higgs bundle $(E = \oplus_{p+q=m} E^{p,q}, \theta)$ over $(Y, D + S)$ induced by some (geometric) $\mathbb{Z}$-PVHS defined over a Zariski open set of $Y-(D\cup S)$, where S is another divisor on Y so that $D+S$ is simple normal crossing. The extra data is that there is a sub-Higgs sheaf $(F=\oplus_{p+q=m}F^{p,q},\eta)\subset(E,\theta)$ such that the *first stage* $F^{n,0}$ is a big line bundle, and

$$\eta: F^{p,q}\longrightarrow F^{p-1,q+1}\otimes\Omega_Y(\log D).$$

As we explained in § 0.4.2, the positivity $F^{n,0}$ comes in a sophisticated way from the Kawamata's big line bundle $\det f_*(mK_{X/Y})$ where $f:X\to Y$ is some algebraic fiber space between projective manifolds. For our Higgs bundle $(E=\oplus_{p+q=m}E^{p,q},\theta)$ over the log pair (Y,D) in Theorem 1.6, the global positivity is the Griffiths line bundle which is contained in some *intermediate stage* E^{p_0,q_0} of $(E=\oplus_{p+q=m}E^{p,q},\theta)$.

1.4. **Iterating Higgs fields.** Let $(E=\oplus_{p+q=\ell}E^{p,q},\theta)$ be a Higgs bundle on a log pair (Y,D) satisfying the three conditions in Theorem 1.6. We apply ideas by Viehweg-Zuo [VZ02, VZ03] to iterate Higgs fields.

Since $\theta: E^{p,q}\to E^{p-1,q+1}\otimes\Omega_Y(\log D)$, one can iterate θ by k-times to obtain

$$E^{p_0,q_0}\longrightarrow E^{p_0-1,q_0+1}\otimes\Omega_Y(\log D)\to\cdots\to E^{p_0-k,q_0+k}\otimes\otimes^k\Omega_Y(\log D)$$

Since $\theta\wedge\theta=0$, the above morphism factors through

$$E^{p_0,q_0}\longrightarrow E^{p_0-k,q_0+k}\otimes\mathrm{Sym}^k\Omega_Y(\log D)\tag{1.4.1}$$

Since L is a subsheaf of E^{p_0,q_0}, it induces

$$L\longrightarrow E^{p_0-k,q_0+k}\otimes\mathrm{Sym}^k\Omega_Y(\log D)$$

which is equivalent to a morphism

$$\tau_k:\mathrm{Sym}^kT_Y(-\log D)\longrightarrow L^{-1}\otimes E^{p_0-k,q_0+k}\tag{1.4.2}$$

The readers might be worried that all τ_k might be trivial so that the above construction will be meaningless. In the next subsection, we will show that this indeed cannot happen.

1.5. **An infinitesimal Torelli-type theorem.** We first follow ideas in [VZ03, §7] to give some "proper" metric on the special Higgs bundle (E,θ) constructed in Theorem 1.6. A more general result for $\mathbb{Z}$-PVHS with quasi-unipotent monodromies are obtained by Popa-Taji-Wu [PTW19].

Let $(E=\oplus_{p+q=\ell}E^{p,q},\theta)$ be a Higgs bundle on a log pair (Y,D) satisfying the three conditions in Theorem 1.6. Write the simple normal crossing divisor $D=D_1+\cdots+D_k$. Let $f_{D_i}\in H^0\big(Y,\mathcal{O}_Y(D_i)\big)$ be the canonical section defining D_i. We fix a smooth hermitian metrics g_{D_i} on $\mathcal{O}_Y(D_i)$. After rescaling g_{D_i}, we assume that $|f_{D_i}|_{g_{D_i}}<1$ for $i=1,\ldots,k$. Set

$$r_D:=\prod_{i=1}^k(-\log|f_{D_i}|^2_{g_{D_i}}).$$

Let g be a singular hermitian metric with analytic singularities of the big and nef line bundle L such that g is smooth on $Y\setminus\mathbf{B}_+(L)$ where $\mathbf{B}_+(L)$ is the augmented base locus of L, and the curvature current $\sqrt{-1}\Theta_g(L)\geqslant\omega$ for some smooth Kähler form ω on Y. For $\alpha\in\mathbb{N}$, define

$$h_L:=g\cdot(r_D)^\alpha$$

The following proposition is a variant of [VZ03, §7] (see also [PTW19, §3] for a more general statement).

Proposition 1.8. *When $\alpha\gg0$, after rescaling f_{D_i}, there exists a continuous, positively definite hermitian form ω_α on $T_Y(-\log D)$ such that*

(i) *the curvature form*

$$\sqrt{-1}\Theta_{h_L}(L)_{\upharpoonright U_0} \geqslant r_D^{-2}\cdot\omega_{\alpha\upharpoonright U_0}, \qquad \sqrt{-1}\Theta_{h_L}(L)\geq\omega$$

where ω *is a smooth Kähler metric on* Y*, and* $U_0 := Y\setminus\big(D\cup\mathbf{B}_+(L)\big)$.

(ii) *The singular hermitian metric* $h := h_L^{-1}\otimes h_{\rm hod}$ *on* $L^{-1}\otimes E$ *is locally bounded on* Y*, and smooth outside* $D\cup\mathbf{B}_+(L)$*, where* $h_{\rm hod}$ *is the Hodge metric for the Higgs bundle* $(E,\theta)|_U$*. Moreover,* h *vanishes on* $D\cup\mathbf{B}_+(L)$.

(iii) *The singular hermitian metric* r_D^2h *on* $L^{-1}\otimes E$ *is also locally bounded on* Y *and vanishes on* D. □

Let us explain the idea of the proof for Proposition 1.8. Proposition 1.8.(i) follows from an easy computation. Recall that local monodromies around D of the local system induced by $\mathbb{C}$-PVHS $(E,\theta)|_U$ are assumed to be unipotent. By the deep work by Cattani-Kaplan-Schmid [CKS86] (see also [VZ03, Claim 7.8]) on the estimate of Hodge metrics, we know that the Hodge norms for local sections of E have at most logarithmic growth near D, which can be controlled by $r_D^{-\alpha}$ if $\alpha\gg 0$.

Now let us prove the following result which is a variant of [Den18, Theorem D]. It in particular answers the question in last subsection, and this result is crucial in constructing negatively curved Finsler metric over $T_Y(-\log D)$ in Theorem 0.6.

Theorem 1.9 (Infinitesimal Torelli-type property)**.** *The morphism* $\tau_1 : T_Y(-\log D)\to L^{-1}\otimes E^{p_0-1,q_0+1}$ *defined in* (1.4.2) *is always generically injective.*

The proof is almost the same at that of [Den18, Theorem D]. We provide it here for completeness sake.

Proof of Theorem 1.9. By Theorem 1.6.(iii), the inclusion $L\subset E^{p_0,q_0}$ induces a global section $s\in H^0(Y,L^{-1}\otimes E^{p_0,q_0})$, which is *generically* non-vanishing over $U=Y-D$. Set

$$U_1 := \{y\in Y-(D\cup\mathbf{B}_+(L))\mid s(y)\neq 0\} \tag{1.5.1}$$

which is a non-empty Zariski open set of U. Since the Hodge metric $h_{\rm hod}$ is a direct sum of metrics h_p on $E^{p,q}$, the metric h for $L^{-1}\otimes E$ is a direct sum of metrics $h_L^{-1}\cdot h_p$ on $L^{-1}\otimes E^{p,q}$, which is smooth over $U_0 := Y-(D\cup\mathbf{B}_+(L))$. Let us denote D' to be the $(1,0)$-part of its Chern connection over U_1, and Θ to be its curvature form. Then by the Griffiths curvature formula of Hodge bundles (see [CMSP17, p. 363]), over U_0 we have

$$\begin{aligned}\Theta &= -\Theta_{L,h_L}\otimes\mathbb{1}+\mathbb{1}\otimes\Theta_{h_{p_0}}(E^{p_0,q_0})\\ &= -\Theta_{L,h_L}\otimes\mathbb{1}-\mathbb{1}\otimes(\theta^*_{p_0,q_0}\wedge\theta_{p_0,q_0})-\mathbb{1}\otimes(\theta_{p_0+1,q_0-1}\wedge\theta^*_{p_0+1,q_0-1})\\ &= -\Theta_{L,h_L}\otimes\mathbb{1}-\tilde{\theta}^*_{p_0,q_0}\wedge\tilde{\theta}_{p_0,q_0}-\tilde{\theta}_{p_0+1,q_0-1}\wedge\tilde{\theta}^*_{p_0+1,q_0-1}\end{aligned} \tag{1.5.2}$$

where we set

$$\theta_{p,q}=\theta|_{E^{p,q}} : E^{p,q}\to E^{p-1,q+1}\otimes\Omega_Y(\log D)$$

and

$$\tilde{\theta}_{p,q}=\mathbb{1}\otimes\theta_{p,q} : L^{-1}\otimes E^{p,q}\to L^{-1}\otimes E^{p-1,q+1}\otimes\Omega_Y(\log D)$$

and define $\tilde{\theta}^*_{p,q}$ to be the adjoint of $\tilde{\theta}_{p,q}$ with respect to the metric $h_L^{-1}\cdot h$. Hence over U_1 one has

$$\begin{aligned}-\sqrt{-1}\partial\bar{\partial}\log|s|_h^2 &= \frac{\{\sqrt{-1}\Theta(s),s\}_h}{|s|_h^2}+\frac{\sqrt{-1}\{D's,s\}_h\wedge\{s,D's\}_h}{|s|_h^4}-\frac{\sqrt{-1}\{D's,D's\}_h}{|s|_h^2}\\ &\leqslant \frac{\{\sqrt{-1}\Theta(s),s\}_h}{|s|_h^2}\end{aligned} \tag{1.5.3}$$

thanks to Cauchy-Schwarz inequality

$$\sqrt{-1}|s|_h^2 \cdot \{D's, D's\}_h \geqslant \sqrt{-1}\{D's, s\}_h \wedge \{s, D's\}_h.$$

Putting (1.5.2) to (1.5.3), over U_1 one has

$$\begin{aligned}\sqrt{-1}\Theta_{L,h_L} - \sqrt{-1}\partial\bar\partial \log |s|_h^2 &\leqslant -\frac{\left\{\sqrt{-1}\tilde\theta^*_{p_0,q_0} \wedge \tilde\theta_{p_0,q_0}(s), s\right\}_h}{|s|_h^2} \\ &\quad - \frac{\left\{\sqrt{-1}\tilde\theta_{p_0+1,q_0-1} \wedge \tilde\theta^*_{p_0+1,q_0-1}(s), s\right\}_h}{|s|_h^2} \\ &= \frac{\sqrt{-1}\left\{\tilde\theta_{p_0,q_0}(s), \tilde\theta_{p_0,q_0}(s)\right\}_h}{|s|_h^2} + \frac{\left\{\tilde\theta^*_{p_0+1,q_0-1}(s), \tilde\theta^*_{p_0+1,q_0-1}(s)\right\}_h}{|s|_h^2} \\ &\leq \frac{\sqrt{-1}\left\{\tilde\theta_{p_0,q_0}(s), \tilde\theta_{p_0,q_0}(s)\right\}_h}{|s|_h^2}\end{aligned} \tag{1.5.4}$$

where $\tilde\theta_{p_0,q_0}(s) \in H^0\big(Y, L^{-1} \otimes E^{p_0-1,q_0+1} \otimes \Omega_Y(\log D)\big)$. By Proposition 1.8.(ii), one has $|s|_h^2(y) = 0$ for any $y \in D \cup \mathbf{B}_+(L)$. Therefore, there exists $y_0 \in U_0$ so that $|s|_h^2(y_0) \geqslant |s|_h^2(y)$ for any $y \in U_0$. Hence $|s|_h^2(y_0) > 0$, and by (1.5.1), $y_0 \in U_1$. Since $|s|_h^2$ is smooth over U_0, $\sqrt{-1}\partial\bar\partial \log |s|_h^2$ is semi-negative at y_0 by the maximal principle. By Proposition 1.8.(i), $\sqrt{-1}\Theta_{L,h_L}$ is strictly positive at y_0. By (1.5.4) and $|s|_h^2(y_0) > 0$, we conclude that $\sqrt{-1}\left\{\tilde\theta_{p_0,q_0}(s), \tilde\theta_{p_0,q_0}(s)\right\}_h$ is strictly positive at y_0. In particular, for any non-zero $\xi \in T_{Y,y_0}$, $\tilde\theta_{p_0,q_0}(s)(\xi) \neq 0$. For

$$\tau_1 : T_Y(-\log D) \to L^{-1} \otimes E^{p_0-1,q_0+1}$$

in (1.4.2), over U it is defined by $\tau_1(\xi) := \tilde\theta_{p_0,q_0}(s)(\xi)$, which is thus *injective at* $y_0 \in U_1$. Hence τ_1 is *generically injective*. The theorem is thus proved. □

2. Construction of negatively curved Finsler metric

We first introduce the definition of Finsler metric.

Definition 2.1 (Finsler metric)**.** Let E be a holomorphic vector bundle on a complex manifold X. A *Finsler metric* on E is a real non-negative *continuous* function $h : E \to [0, +\infty[$ such that

$$h(av) = |a|h(v)$$

for any $a \in \mathbb{C}$ and $v \in E$. The metric h is *positively definite* at a subset $U \subset X$ if $h(v) > 0$ for any nonzero $v \in E_x$ and any $x \in U$.

We shall mention that our definition is a bit different from that in [Kob98, Chapter 2, §3], which requires *convexity*, and the Finsler metric therein can be upper-semi continuous.

Let $(E = \oplus_{p+q=\ell} E^{p,q}, \theta)$ be a Higgs bundle on a log pair (Y, D) satisfying the three conditions in Theorem 1.6. We adopt the same notations as those in Theorem 1.6 and § 1.5 throughout this section. Let us denote by n the largest non-negative number for k so that τ_k in (1.4.2) is not trivial. By Theorem 1.9, $n > 0$. Following [Den18, §2.3] we construct Finsler metrics $F_1, \ldots, F_n$ on $T_Y(-\log D)$ as follows. By (1.4.2), for each $k = 1, \ldots, n$, there exists

$$\tau_k : \mathrm{Sym}^k T_Y(-\log D) \to L^{-1} \otimes E^{p_0-k,q_0+k}.$$

Then it follows from Proposition 1.8.(ii) that the (Finsler) metric h on $L^{-1} \otimes E^{p_0-k,q_0+k}$ induces a Finsler metric F_k on $T_Y(-\log D)$ defined as follows: for any $e \in T_Y(-\log D)_y$,

$$F_k(e) := h\big(\tau_k(e^{\otimes k})\big)^{\frac{1}{k}} \tag{2.0.1}$$

Let $C \subset \mathbb{C}$ be any open set of $\mathbb{C}$. For any holomorphic map $\gamma : C \to U := Y - D$, one has

$$d\gamma : T_C \to \gamma^* T_U \hookrightarrow \gamma^* T_Y(-\log D). \tag{2.0.2}$$

We denote by $\partial_t := \frac{\partial}{\partial t}$ the canonical vector fields in $C \subset \mathbb{C}$, $\bar{\partial}_t := \frac{\partial}{\partial \bar{t}}$ its conjugate. The Finsler metric F_k induces a continuous Hermitian pseudo-metric on C, defined by

$$\gamma^* F_k^2 = \sqrt{-1} G_k(t) dt \wedge d\bar{t}. \tag{2.0.3}$$

Hence $G_k(t) = |\tau_k\big(d\gamma(\partial_t)^{\otimes k}\big)|_h^{\frac{2}{k}}$, where τ_k is defined in (1.4.2).

By Theorem 1.9, there is a Zariski open set U° of U such that $U^\circ \cap \mathbf{B}_+(L) = \varnothing$, and τ_1 is injective at any point of U°. We now fix any holomorphic map $\gamma : C \to U$ with $\gamma(C) \cap U^\circ \neq \varnothing$. By Proposition 1.8.(ii), the metric h for $L^{-1} \otimes E$ is smooth and positively definite over $U - \mathbf{B}_+(L)$. Hence $G_1(t) \not\equiv 0$. Let C° be an (non-empty) open set of C whose complement $C \setminus C^\circ$ is a *discrete set* so that

- The image $\gamma(C^\circ) \subset U^\circ$.
- For every $k = 1, \ldots, n$, either $G_k(t) \equiv 0$ on C° or $G_k(t) > 0$ for any $t \in C^\circ$.
- $\gamma'(t) \neq 0$ for any $t \in C^\circ$, namely $\gamma|_{C^\circ} : C^\circ \to U^0$ is immersive everywhere.

By the definition of $G_k(t)$, if $G_k(t) \equiv 0$ for some $k > 1$, then $\tau_k(\partial_t^{\otimes k}) \equiv 0$ where τ_k is defined in (1.4.2). Note that one has $\tau_{k+1}(\partial_t^{\otimes (k+1)}) = \tilde{\theta}\big(\tau_k(\partial_t^{\otimes k})\big)(\partial_t)$, where

$$\tilde{\theta} = \mathbb{1}_{L^{-1}} \otimes \theta : L^{-1} \otimes E \to L^{-1} \otimes E \otimes \Omega_Y(\log D)$$

We thus conclude that $G_{k+1}(t) \equiv 0$. Hence it exists $1 \leq m \leq n$ so that the set $\{k \mid G_k(t) > 0 \text{ over } C^\circ\} = \{1, \ldots, m\}$, and $G_\ell(t) \equiv 0$ for all $\ell = m+1, \ldots, n$. From now on, *all the computations* are made over C° if not specified.

Using the same computations in the proof of [Den18, Proposition 2.10], we have following curvature formula.

Theorem 2.2. *For $k = 1, \ldots, m$, over C° one has*

$$\frac{\partial^2 \log G_1}{\partial t \partial \bar{t}} \geq \Theta_{L,h_L}(\partial_t, \bar{\partial}_t) - \frac{G_2^2}{G_1} \qquad \text{if } k = 1, \tag{2.0.4}$$

$$\frac{\partial^2 \log G_k}{\partial t \partial \bar{t}} \geq \frac{1}{k}\Big(\Theta_{L,h_L}(\partial_t, \bar{\partial}_t) + \frac{G_k^k}{G_{k-1}^{k-1}} - \frac{G_{k+1}^{k+1}}{G_k^k}\Big) \qquad \text{if } k > 1. \tag{2.0.5}$$

Here we make the convention that $G_{m+1} \equiv 0$ and $\frac{0}{0} = 0$. We also write ∂_t (resp. $\bar{\partial}_t$) for $d\gamma(\partial_t)$ (resp. $d\gamma(\bar{\partial}_t)$) abusively, where $d\gamma$ is defined in (2.0.2). □

Let us mention that in [Den18, eq. (2.2.11)] we drop the term $\Theta_{L,h_L}(\partial_t, \bar{\partial}_t)$ in (2.0.5), though it can be easily seen from the proof of [Den18, Lemma 2.7].

We will follows ideas in [Den18, §2.3] (inspired by [TY15, BPW17, Sch17]) to introduce a new Finsler metric F on $T_Y(-\log D)$ by taking convex sum in the following form

$$F := \sqrt{\sum_{k=1}^{n} k \alpha_k F_k^2}. \tag{2.0.6}$$

where $\alpha_1, \ldots, \alpha_n \in \mathbb{R}^+$ are some constants which will be fixed later.

For the above $\gamma : C \to U$ with $\gamma(C) \cap U^\circ \neq \varnothing$, we write

$$\gamma^* F^2 = \sqrt{-1} H(t) dt \wedge d\bar{t}.$$

Then

$$H(t) = \sum_{k=1}^{n} k\alpha_k G_k(t), \tag{2.0.7}$$

where G_k is defined in (2.0.3). Recall that for $k = 1, \ldots, m$, $G_k(t) > 0$ for any $t \in C^\circ$.

We first recall a computational lemma by Schumacher.

Lemma 2.3 ([Sch17, Lemma 17]). *Let $\alpha_j > 0$ and G_j be positive real numbers for $j = 1, \ldots, n$. Then*

$$\begin{aligned} &\sum_{j=2}^{n} \Big(\alpha_j \frac{G_j^{j+1}}{G_{j-1}^{j-1}} - \alpha_{j-1} \frac{G_j^{j}}{G_{j-1}^{j-2}} \Big) \\ &\geqslant \frac{1}{2} \Bigg(-\frac{\alpha_1^3}{\alpha_2^2} G_1^2 + \frac{\alpha_{n-1}^{n-1}}{\alpha_n^{n-2}} G_n^2 + \sum_{j=2}^{n-1} \Big(\frac{\alpha_{j-1}^{j-1}}{\alpha_j^{j-2}} - \frac{\alpha_j^{j+2}}{\alpha_{j+1}^{j+1}} \Big) G_j^2 \Bigg) \end{aligned} \tag{2.0.8}$$

□

Now we are ready to compute the curvature of the Finsler metric F based on Theorem 2.2.

Theorem 2.4. *Fix a smooth Kähler metric ω on Y. There exist* universal *constants $0 < \alpha_1 < \ldots < \alpha_n$ and $\delta > 0$, such that for any $\gamma : C \to U = Y - D$ with C an open set of $\mathbb{C}$ and $\gamma(C) \cap U^\circ \neq \varnothing$, one has*

$$\sqrt{-1}\partial\bar{\partial} \log |\gamma'(t)|_F^2 \geq \delta \gamma^* \omega \tag{2.0.9}$$

Proof. By Theorem 1.9 and the assumption that $\gamma(C) \cap U^\circ \neq \varnothing$, $G_1(t) \not\equiv 0$. We first recall a result in [Den18, Lemma 2.9], and we write its proof here for it is crucial in what follows.

Claim 2.5. *There is a universal constant $c_0 > 0$ (*i.e.* it does not depend on γ) so that $\Theta_{L,h_L}(\partial_t, \bar{\partial}_t) \geq c_0 G_1(t)$ for all $t \in C$.*

Proof of Claim 2.5. Indeed, by Proposition 1.8.(i), it suffices to prove that

$$\frac{|\partial_t|^2_{\gamma^*(r_D^{-2} \cdot \omega_\alpha)}}{|\tau_1(d\gamma(\partial_t))|_h^2} \geqslant c_0 \tag{2.0.10}$$

for some $c_0 > 0$, where ω_α is a positively definite Hermitian metric on $T_Y(-\log D)$. Note that

$$\frac{|\partial_t|^2_{\gamma^*(r_D^{-2} \cdot \omega_\alpha)}}{|\tau_1(d\gamma(\partial_t))|_h^2} = \frac{|\partial_t|^2_{\gamma^*(r_D^{-2} \cdot \omega_\alpha)}}{|\partial_t|^2_{\gamma^*\tau_1^* h}} = \frac{|\partial_t|^2_{\gamma^*(\omega_\alpha)}}{|\partial_t|^2_{\gamma^*\tau_1^*(r_D^2 \cdot h)}},$$

where $\tau_1^*(r_D^2 \cdot h)$ is a Finsler metric (indeed continuous pseudo hermitian metric) on $T_Y(-\log D)$ by Proposition 1.8.(iii). Since Y is compact, there exists a constant $c_0 > 0$ such that

$$\omega_\alpha \geqslant c_0 \tau_1^*(r_D^2 \cdot h).$$

Hence (2.0.10) holds for any $\gamma : C \to U$ with $\gamma(C) \cap U^\circ \neq \varnothing$. The claim is proved. □

By [Sch12, Lemma 8],

$$\sqrt{-1}\partial\bar{\partial}\log(\sum_{j=1}^{n} j\alpha_j G_j) \geqslant \frac{\sum_{j=1}^{n} j\alpha_j G_j \sqrt{-1}\partial\bar{\partial}\log G_j}{\sum_{i=1}^{n} j\alpha_j G_i} \tag{2.0.11}$$

Putting (2.0.4) and (2.0.5) to (2.0.11), and making the convention that $\frac{0}{0}=0$, we obtain

$$\begin{aligned}
\frac{\partial^2 \log H(t)}{\partial t\partial\bar{t}} &\geq \frac{1}{H}\Big(-\alpha_1 G_2^2 + \sum_{k=2}^{n}\alpha_k\big(\frac{G_k^{k+1}}{G_{k-1}^{k-1}} - \frac{G_{k+1}^{k+1}}{G_k^{k-1}}\big)\Big) + \frac{\sum_{k=1}^{n}\alpha_k G_k}{H}\Theta_{L,h_L}(\partial_t,\bar{\partial}_t)\\
&= \frac{1}{H}\Big(\sum_{j=2}^{n}\Big(\alpha_j\frac{G_j^{j+1}}{G_{j-1}^{j-1}} - \alpha_{j-1}\frac{G_j^{j}}{G_{j-1}^{j-2}}\Big)\Big) + \frac{\sum_{k=1}^{n}\alpha_k G_k}{H}\Theta_{L,h_L}(\partial_t,\bar{\partial}_t)\\
&\overset{(2.0.8)}{\geq} \frac{1}{H}\Big(-\frac{1}{2}\frac{\alpha_1^3}{\alpha_2^2}G_1^2 + \frac{1}{2}\sum_{j=2}^{n-1}\Big(\frac{\alpha_{j-1}^{j-1}}{\alpha_j^{j-2}} - \frac{\alpha_j^{j+2}}{\alpha_{j+1}^{j+1}}\Big)G_j^2 + \frac{1}{2}\frac{\alpha_{n-1}^{n-1}}{\alpha_n^{n-2}}G_n^2\Big)\\
&+ \frac{\sum_{k=1}^{n}\alpha_k G_k}{H}\Theta_{L,h_L}(\partial_t,\bar{\partial}_t)\\
&\overset{\text{Claim } 2.5}{\geq} \frac{1}{H}\Big(\frac{\alpha_1}{2}(c_0 - \frac{\alpha_1^2}{\alpha_2^2})G_1^2 + \frac{1}{2}\sum_{j=2}^{n-1}\Big(\frac{\alpha_{j-1}^{j-1}}{\alpha_j^{j-2}} - \frac{\alpha_j^{j+2}}{\alpha_{j+1}^{j+1}}\Big)G_j^2 + \frac{1}{2}\frac{\alpha_{n-1}^{n-1}}{\alpha_n^{n-2}}G_n^2\Big)\\
&+ \frac{1}{H}(\frac{1}{2}\alpha_1 G_1 + \sum_{k=2}^{n}\alpha_k G_k)\Theta_{L,h_L}(\partial_t,\bar{\partial}_t)
\end{aligned}$$

One can take $\alpha_1 = 1$, and choose the further $\alpha_j > \alpha_{j-1}$ inductively so that

$$c_0 - \frac{\alpha_1^2}{\alpha_2^2} > 0, \quad \frac{\alpha_{j-1}^{j-1}}{\alpha_j^{j-2}} - \frac{\alpha_j^{j+2}}{\alpha_{j+1}^{j+1}} > 0 \quad \forall\, j = 2,\ldots,n-1. \tag{2.0.12}$$

Hence

$$\frac{\partial^2 \log H(t)}{\partial t\partial\bar{t}} \geq \frac{1}{H}(\frac{1}{2}\alpha_1 G_1 + \sum_{k=2}^{n}\alpha_k G_k)\Theta_{L,h_L}(\partial_t,\bar{\partial}_t) \overset{(2.0.7)}{\geq} \frac{1}{n}\Theta_{L,h_L}(\partial_t,\bar{\partial}_t)$$

over C°. By Proposition 1.8.(i), this implies that

$$\sqrt{-1}\partial\bar{\partial}\log|\gamma'|_F^2 = \sqrt{-1}\partial\bar{\partial}\log H(t) \geq \frac{1}{n}\gamma^*\sqrt{-1}\Theta_{L,h_L} \geq \delta\gamma^*\omega \tag{2.0.13}$$

over C° for some positive constant δ, which does not depend on γ. Since $|\gamma'(t)|_F^2$ is continuous and locally bounded from above over C, by the extension theorem of subharmonic function, (2.0.13) holds over the whole C. Since $c_0 > 0$ is a constant which does not depend on γ, so are $\alpha_1,\ldots,\alpha_n$ by (2.0.12). The theorem is thus proved. □

In summary of results in this subsection, we obtain the following theorem.

Theorem 2.6. *Let $(E = \oplus_{p+q=\ell}E^{p,q},\theta)$ be a Higgs bundle on a log pair (Y,D) satisfying the three conditions in Theorem 1.6. Then there are a Finsler metric h on $T_Y(-\log D)$ which is* positively definite *on a dense Zariski open set U° of $U := Y - D$, and a smooth Kähler form ω on Y such that for any holomorphic map $\gamma : C \to U$ from any open subset C of $\mathbb{C}$ with $\gamma(C)\cap U^\circ \neq \varnothing$, one has*

$$\sqrt{-1}\partial\bar{\partial}\log|\gamma'|_h^2 \geq \gamma^*\omega. \tag{2.0.14}$$

Let U' be the open set τ_1 in (1.4.2) is injective. Then $U^\circ = U' - \mathbf{B}_+(L)$, where L is the big line bundle in Theorem 1.6.

3. Big Picard theorem and Algebraic hyperbolicity

3.1. **Definition of algebraic hyperbolicity.** *Algebraic hyperbolicity* for a compact complex manifold X was introduced by Demailly in [Dem97, Definition 2.2], and he proved in [Dem97, Theorem 2.1] that X is algebraically hyperbolic if it is Kobayashi hyperbolicity. The notion of algebraic hyperbolicity was generalized to log pairs by Chen [Che04].

Definition 3.1 (Algebraic hyperbolicity)**.** Let (X, D) be a log pair. For any reduced irreducible curve $C \subset X$ such that $C \not\subset X$, we denote by $i_X(C, D)$ the number of distinct points in the set $\nu^{-1}(D)$, where $\nu : \tilde{C} \to C$ is the normalization of C. The log pair (X, D) is *algebraically hyperbolic* if there is a smooth Kähler metric ω on X such that

$$2g(\tilde{C}) - 2 + i(C, D) \geq \deg_\omega C := \int_C \omega$$

for all curves $C \subset X$ as above.

Note that $2g(\tilde{C})-2+i(C, D)$ depends only on the complement $X-D$. Hence the above notion of hyperbolicity also makes sense for quasi-projective manifolds: we say that a quasi-projective manifold U is algebraically hyperbolic if it has a log compactification (X, D) which is algebraically hyperbolic.

However, unlike Demailly's theorem, it is unclear to us that Kobayashi hyperbolicity or Picard hyperbolicity of $X - D$ will imply the algebraic hyperbolicity of (X, D). In [PR07] Pacienza-Rousseau proved that if $X - D$ is hyperbolically embedded into X, the log pair (X, D) (and thus $X - D$) is algebraically hyperbolic.

3.2. **Proofs of main results.** In this subsection, we will combine Theorem 0.7 with Theorem 0.6 to prove main results in this paper.

Proof of Theorem A. By Theorem 0.6, there exist finite log pairs $\{(X_i, D_i)\}_{i=0,\ldots,N}$ so that

(1) There are morphisms $\mu_i : X_i \to Y$ with $\mu_i^{-1}(D) = D_i$, so that each $\mu_i : X_i \to \mu_i(X_i)$ is a birational morphism, and $X_0 = Y$ with $\mu_0 = \mathbb{1}$.
(2) There are smooth Finsler metrics h_i for $T_{X_i}(-\log D_i)$ which is positively definite over a Zariski open set U_i° of $U_i := X_i - D_i$.
(3) $\mu_i|_{U_i^\circ} : U_i^\circ \to \mu_i(U_i^\circ)$ is an isomorphism.
(4) There are smooth Kähler metrics ω_i on X_i such that for any curve $\gamma : C \to U_i$ with C an open set of $\mathbb{C}$ and $\gamma(C) \cap U_i^\circ \neq 0$, one has

$$\sqrt{-1}\partial\bar{\partial} \log |\gamma'|^2_{h_i} \geq \gamma^* \omega_i. \tag{3.2.1}$$

(5) For any $i \in \{0, \ldots, N\}$, either $\mu_i(U_i) - \mu_i(U_i^\circ)$ is zero dimensional, or there exists $I \subset \{0, \ldots, N\}$ so that

$$\mu_i(U_i) - \mu_i(U_i^\circ) \subset \cup_{j\in I}\mu_j(X_j)$$

Let us explain how to construct these log pairs. By the assumption, there is a $\mathbb{C}$-PVHS $(V, \nabla, F^\bullet, Q)$ on $Y - D$ with the period map quasi-finite, which is thus generically immersive. We then apply Theorem 0.6 to construct a Finsler metric on $T_Y(-\log D)$ which is positively definite over some Zariski open set U° of $U = Y - D$ with the desired curvature property (2.0.14). Set $X_0 = Y$, $\mu_0 = \mathbb{1}$ and $U_0^\circ = U^\circ$. Let $Z_1, \ldots, Z_m$ be all irreducible varieties of $Y - U^\circ$ which are not components of D. Then $Z_1 \cup \ldots \cup Z_m \supset U \setminus U^\circ$. For each i, we take a desingularization $\mu_i : X_i \to Z_i$ so that $D_i := \mu_i^{-1}(D)$ is a simple normal crossing divisor in X_i. For the $\mathbb{C}$-PVHS $\mu_i^*(V, \nabla, F^\bullet, Q)$ on $U_i = X_i - D_i$ by pulling-back $(V, \nabla, F^\bullet, Q)$ via μ_i, its period map is generically immersive, and it also has unipotent monodromies around D_i. We then apply Theorem 0.6 to construct the desired Finsler metrics in Item 4 for $T_{X_i}(-\log D_i)$. We iterate this construction, and since each step the dimension of X_i is strictly decreased, this algorithm stops after finite steps.

(i) We will first prove that U is Picard hyperbolic. Fix any holomorphic map $f : \Delta^* \to U$. If $f(\Delta^*) \cap U_0^\circ \neq \varnothing$, then by Theorem 0.7 and Item 4, we conclude that f extends to a holomorphic map $\overline{f} : \Delta \to X_0 = Y$.

Assume now $f(\Delta^*) \cap \mu_0(U_0^\circ) = \varnothing$. By Item 5, there exists $I_0 \subset \{0, \ldots, N\}$ so that

$$f(\Delta^*) \subset \mu_0(U_0) - \mu_0(U_0^\circ) \subset \cup_{j \in I_0} \mu_j(X_j)$$

Since $\mu_j(X_j)$ are all irreducible, there exists $k \in I_0$ so that $f(\Delta^*) \subset \mu_k(X_k)$. Note that $U_k := \mu_k^{-1}(U)$. Hence $f(\Delta^*) \subset \mu_k(U_k)$. If $f(\Delta^*) \cap \mu_k(U_k^\circ) \neq \varnothing$, by Item 3 $f(\Delta^*)$ is not contained in the exceptional set of μ_k. Hence f can be lift to $f_k : \Delta^* \to U_k$ so that $\mu_k \circ f_k = f$ and $f_k(\Delta^*) \cap U_k^\circ \neq \varnothing$. By Theorem 0.7 and Item 4 again we conclude that f_k extends to a holomorphic map $\overline{f}_k : \Delta \to X_k$. Hence $\mu_k \circ \overline{f}_k$ extends f. If $f(\Delta^*) \cap \mu_k(U_k^\circ) = \varnothing$, we apply Item 5 to iterate the above arguments and after finite steps there exists X_i so that $f(\Delta^*) \subset \mu_i(U_i)$ and $f(\Delta^*) \cap \mu_i(U_i^\circ) \neq \varnothing$. By Item 3, f can be lifted to $f_i : \Delta^* \to U_i$ so that $\mu_i \circ f_i = f$ and $f_i(\Delta^*) \cap U_i^\circ \neq \varnothing$. By Theorem 0.7 and Item 4 again, f_i extends to the origin, and so is f. We prove the Picard hyperbolicity of $U = Y - D$.

(ii) Let us prove the algebraic hyperbolicity of U. Fix any reduced and irreducible curve $C \subset Y$ with $C \not\subset D$. By the above arguments, there exists $i \in \{0, \ldots, N\}$ so that $C \subset \mu_i(X_i)$ and $C \cap \mu_i(U_i^\circ) \neq \varnothing$. Let $C_i \subset X_i$ be the strict transform of C under μ_i. By Item 3 $h_i|_{C_i}$ is not identically equal to zero.

Denote by $\nu_i : \tilde{C}_i \to C_i \subset X_i$ the normalization of C_i, and set $P_i := (\mu_i \circ \nu_i)^{-1}(D) = \nu_i^{-1}(D_i)$. One has

$$d\nu_i : T_{\tilde{C}_i}(-\log P_i) \to \nu_i^* T_{X_i}(-\log D_i)$$

which induces a (non-trivial) pseudo hermitian metric $\tilde{h}_i := \nu_i^* h_i$ over $T_{\tilde{C}_i}(-\log P_i)$. By (3.2.1), the *curvature current*

$$\frac{\sqrt{-1}}{2\pi}\Theta_{\tilde{h}_i^{-1}}(K_{\tilde{C}_i}(\log P_i)) \geq \nu_i^* \omega_i$$

Hence

$$2g(\tilde{C}_i) - 2 + i(C, D) = \int_{\tilde{C}_i} \frac{\sqrt{-1}}{2\pi}\Theta_{\tilde{h}_i^{-1}}(K_{\tilde{C}_i}(\log P_i)) \geq \int_{\tilde{C}_i} \nu_i^* \omega_i$$

Fix a Kähler metric ω_Y on Y. Then there is a constant $\varepsilon_i > 0$ so that $\omega_i \geq \varepsilon_i \mu_i^* \omega_Y$. We thus have

$$2g(\tilde{C}_i) - 2 + i(C, D) \geq \varepsilon_i \int_{\tilde{C}_i} (\mu_i \circ \nu_i)^* \omega_Y = \varepsilon_i \deg_{\omega_Y} C,$$

for $\mu_i \circ \nu_i : \tilde{C}_i \to C$ is the normalization of C. Set $\varepsilon := \inf_{i=0,\ldots,N} \varepsilon_i$. Then we conclude that for any reduced and irreducible curve $C \subset Y$ with $C \not\subset D$, one has

$$2g(\tilde{C}) - 2 + i(C, D) \geq \varepsilon \deg_{\omega_Y} C$$

where $\tilde{C} \to C$ is its normalization. This shows the algebraic hyperbolicity of U.

The proof of the theorem is accomplished. □

To prove Theorem B, we need the following fact on Picard and algebraic hyperbolicity.

Lemma 3.2. *Let U be a quasi-projective manifold and let $p : \tilde{U} \to U$ be a finite étale cover. If $\tilde{U}$ is Picard hyperbolic (resp. algebraically hyperbolic), then U is also Picard hyperbolic (resp. algebraically hyperbolic).*

Proof. Let us take log-compactifications (X, D) and (Y, E) for $\tilde{U}$ and U respectively, so that p extends to a morphism $\overline{p} : X \to Y$ with $\overline{p}^{-1}(E) = D$.

(i) Assume now $\tilde{U}$ is Picard hyperbolic. For any holomorphic map $f : \Delta^* \to U$, we claim that there is a finite covering

$$\begin{aligned} \pi : \Delta^* &\to \Delta^* \\ z &\mapsto z^n \end{aligned}$$

so that there is a holomorphic map $\tilde{f} : \Delta^* \to \tilde{U}$ with

$$\begin{tikzcd} \Delta^* \arrow[r, "\tilde{f}"] \arrow[d, "\pi"] & \tilde{U} \arrow[d, "p"] \\ \Delta^* \arrow[r, "f"] & U \end{tikzcd}$$

Indeed, fix any based point $z_0 \in \Delta^*$ with $x_0 := f(z_0)$. Pick any $y_0 \in p^{-1}(x_0)$. Then either $f_*\big(\pi_1(\Delta^*, z_0)\big)$ is a finite group or $f_*\big(\pi_1(\Delta^*, z_0)\big) \cap p_*\big(\pi_1(\tilde{U}, y_0)\big) \supsetneq \{0\}$ since $p_*\big(\pi_1(\tilde{U}, y_0)\big)$ is a subgroup of $\pi_1(U, x_0)$ with finite index. Let $\gamma \in \pi_1(\Delta^*, z_0) \simeq \mathbb{Z}$ be a generator. Then $f_*(\gamma^n) \subset p_*\big(\pi_1(\tilde{U}, y_0)\big)$ for some $n \in \mathbb{Z}_{>0}$. Therefore, $(f \circ \pi)_*\big(\pi_1(\Delta^*, z_0)\big) \subset p_*\big(\pi_1(\tilde{U}, y_0)\big)$, which implies that the lift $\tilde{f}$ of $f \circ \pi$ for the covering map p exists.

Since $\tilde{U}$ is Picard hyperbolic, $\tilde{f}$ extends to a holomorphic map $\overline{\tilde{f}} : \Delta \to X$. The composition $\overline{p} \circ \overline{\tilde{f}}$ extends $f \circ \pi$. Since π extends to a map $\overline{\pi} : \Delta \to \Delta$, we thus has

$$\lim_{z \to 0} f(z) = \overline{p} \circ \overline{\tilde{f}}(0).$$

By the *Riemann extension theorem*, f extends to the origin holomorphically.

(ii) Assume that (X, D) is algebraically hyperbolic. Fix smooth Kähler metrics ω_X and ω_Y on X and Y so that $\overline{p}^*\omega_Y \leq \omega_X$. Then there is a constant $\varepsilon > 0$ such that for any reduced and irreducible curve $C \subset X$ with $C \not\subset D$, one has

$$2g(\tilde{C}) - 2 + i(C, D) \geq \varepsilon \deg_{\omega_X} C$$

where $\tilde{C} \to C$ is its normalization.

Take any reduced and irreducible curve $C \subset Y$ with $C \not\subset E$. Then there is a reduced and irreducible curve C' of X so that $\overline{p}(C') = C$. Let $\nu : \tilde{C} \to C$ and $\nu' : \tilde{C}' \to C'$ be their normalization respectively, which induces a (possibly ramified) covering map $\pi : \tilde{C}' \to \tilde{C}$ so that

$$\begin{tikzcd} \tilde{C}' \arrow[r, "\nu'"] \arrow[d, "\pi"] & C' \arrow[d, "\overline{p}|_{C'}"] \\ \tilde{C} \arrow[r, "\nu"] & C \end{tikzcd}$$

Set $P := \nu^{-1}(E)$ and $Q := (\nu')^{-1}(D)$. Then $\pi^\circ : \tilde{C}' - Q \to \tilde{C} - P$ is an unramified covering map. By Riemann–Hurwitz formula one has

$$\begin{aligned} 2g(\tilde{C}) - 2 + i(C, E) &= \frac{1}{\deg \pi}\big(2g(\tilde{C}') - 2 + i(C', D)\big) \\ &\geq \frac{\varepsilon}{\deg \pi} \deg_{\omega_X} C' \geq \frac{\varepsilon}{\deg \pi} \deg_{\overline{p}^*\omega_Y} C' = \varepsilon \deg_{\omega_Y} C \end{aligned}$$

Hence (Y, E) is also algebraically hyperbolic, and so is U.

The lemma is proved. □

Note that in [JK18a, Proposition 5.2.(1)], Javanpeykar-Kamenova proved that if $X \to Y$ is a finite étale morphism of projective varieties over an algebraically closed field of characteristic zero, then Y is algebraically hyperbolic provided that X is algebraically hyperbolic.

We now show how to reduce Theorem B to Theorem A by applying Lemma 3.2.

Proof of Theorem B. Let (Y, D) be a log-compactification of U. Since there is a $\mathbb{Z}$-PVHS $(V, \nabla, F^\bullet, Q)$ on U, by a theorem of A. Borel, its local monodromies around D is quasi-unipotent. By [Bru18, §3.2], there is a finite étale cover $p : \tilde{U} \to U$ and a log-compactification (X, E) of $\tilde{U}$ so that $p^*(V, \nabla, F^\bullet, Q)$ has unipotent monodromies around E. Since the period map of $(V, \nabla, F^\bullet, Q)$ is assumed to be quasi-finite, so is that of $p^*(V, \nabla, F^\bullet, Q)$. By Theorem A, we know that $\tilde{U}$ is both Picard hyperbolic and algebraically hyperbolic, and it follows immediately from Lemma 3.2 that the same holds for U. □

Remark 3.3. Let U be a quasi-projective manifold admitting an integral variation of Hodge structures $(V, \nabla, F^\bullet, Q)$ with *arithmetic* monodromy group whose period map is quasi-finite. In [JL19, Theorem 4.2] Javanpeykar-Litt proved that U is *weakly bounded* in the sense of Kovács-Lieblich [KL10, Definition 2.4] (which is weaker than algebraic hyperbolicity). Though not mentioned explicitly, their proof of [JL19, Theorem 4.2] implicitly shows that such U is also algebraically hyperbolic when local monodromies of $(V, \nabla, F^\bullet, Q)$ at infinity are unipotent. Their proof is different from that of Theorem B, and it uses the work [BBT18] on the ampleness of Griffiths line bundle of $(V, \nabla, F^\bullet, Q)$ as well as the Arakelov-type inequality for Hodge bundles by Peters [Pet00].

Corollary C immediately follows from the following proposition, which is a consequence of the deep extension theorem of meromorphic maps by Siu [Siu75]. The *meromorphic map* in this paper is defined in the sense of Remmert, and we refer the reader to [FG02, p. 243] for the precise definition.

Proposition 3.4. *Let Y° be a Zariski open set of a compact Kähler manifold Y. Assume that Y° is Picard hyperbolic. Then any holomorphic map $f : \Delta^p \times (\Delta^*)^q \to Y^\circ$ extends to a meromorphic map $\overline{f} : \Delta^{p+q} \dashrightarrow Y$. In particular, any holomorphic map g from a Zariski open set X° of a compact complex manifold X to Y° extends to a meromorphic map from X to Y.*

Proof. By [Siu75, Theorem 1], any meromorphic map from a Zariski open set Z° of a complex manifold Z to a compact Kähler manifold Y extends to a meromorphic map from Z to Y provided that the codimension of $Z - Z^\circ$ is at least 2. It then suffices to prove the extension theorem for any holomorphic map $f : \Delta^r \times \Delta^* \to Y^\circ$. By the assumption that Y° is Picard hyperbolic, for any $z \in \Delta^r$, the holomorphic map $f|_{\{z\}\times\Delta^*} : \{z\} \times \Delta^* \to Y^\circ$ can be extended to a holomorphic map from $\{z\} \times \Delta$ to Y. It then follows from [Siu75, p.442, $(*)$] that f extends to a meromorphic map $\overline{f} : \Delta^{r+1} \dashrightarrow Y$. This proves the first part of the proposition. To prove the second part, we first apply the Hironaka theorem on resolution of singularities to assume that $X - X^\circ$ is a simple normal crossing divisor on X. Then for any point $x \in X - X^\circ$ it has an open neighborhood Ω_x which is isomorphic to Δ^{p+q} so that $X^\circ \simeq \Delta^p \times (\Delta^*)^q$ under this isomorphism. The above arguments show that $g|_{\Omega_x \cap X^\circ}$ extends to a meromorphic map from Ω_x to Y, and thus g can be extended to a meromorphic map from X to Y. The proposition is proved. □

By the Chow theorem, this extension theorem in particular gives an alternative proof of the fact that Picard hyperbolic variety is moreover Borel hyperbolic, proven in [JK18b, Corollary 3.11].

We end this section with the following remark.

Remark 3.5. Let (E,θ) be the Higgs bundle on a log pair (Y,D) as that in Theorem 2.6. One can also use the idea by Viehweg-Zuo [VZ02] in constructing their *Viehweg-Zuo* sheaf (based on the negativity of kernels of Higgs fields by Zuo [Zuo00]) to prove a weaker result than Theorem 2.6: for any holomorphic map $\gamma : C \to U$ from any open subset C of $\mathbb{C}$ with $\gamma(C)\cap U^\circ \neq \varnothing$, there exists a Finsler metric h_C of $T_Y(-\log D)$ (depending on C) and a Kähler metric ω_C for Y (also depending on C) so that $|\gamma'(t)|_h^2 \not\equiv 0$ and

$$\sqrt{-1}\partial\bar{\partial}\log|\gamma'|_{h_C}^2 \geq \gamma^*\omega_C.$$

It follows from our proof of Theorem A that one can also combine Theorem 0.7 with this weaker result to prove Theorem A. The more general result Theorem 0.6 will be used to prove Theorem D.(ii) in next section.

4. Strong hyperbolicity of varieties admitting $\mathbb{C}$-PVHS

We begin this section with some definitions.

Definition 4.1. Let (Y,D) be a log pair. Let y be any point of D, and let $D_{i_1},\ldots,D_{i_\ell}$ be components of D containing y. An *admissible coordinate* around y is the tuple $(\mathcal{U},\varphi)$:

- $\mathcal{U}$ is an open subset of Y containing y.
- φ is an isomorphism $\varphi : \mathcal{U} \to \Delta^d = \{(z_1,\ldots,z_d) \mid |z_i| < 1\}$ such that $\varphi(y) = (0,\ldots,0)$, and $\varphi(D_{i_j}) = \{z_j = 0\}$ for any $j = 1,\ldots,\ell$.

Definition 4.2. Let $(Y, D = \sum_{i=1}^c D_i)$ be a log pair and let $(V,\nabla,F^\bullet,Q)$ be a $\mathbb{C}$-PVHS over $U := Y - D$. Let γ_i be the generator of the local monodromy of $(V,\nabla,F^\bullet,Q)$ around D_i. We say $(V,\nabla,F^\bullet,Q)$ has *injective local monodromy representation around* D if for any non-empty set $\{i_1,\ldots,i_k\} \subset \{1,\ldots,c\}$ so that $D_{i_1}\cap\cdots\cap D_{i_k} \neq \varnothing$, the subgroup of the monodromy group generated by $\{\gamma_{i_1},\ldots,\gamma_{i_k}\}$ is a free abelian group of rank k.

Note that the generator of the local monodromy of $(V,\nabla,F^\bullet,Q)$ around D_i does not depend on the choice of admissible coordinate, and thus the subgroup generated by $\{\gamma_{i_1},\ldots,\gamma_{i_k}\}$ is always an abelian group (but might not have rank k and might have torsion). Based on this fact, on can easily obtain the following lemma.

Lemma 4.3. *Let (Y,D) be a log pair and let $(V,\nabla,F^\bullet,Q)$ be a $\mathbb{C}$-PVHS over $U := Y - D$. $(V,\nabla,F^\bullet,Q)$ has injective local monodromy representation around D if and only if for any point $y \in D$, the monodromy representation of the $\mathbb{C}$-PVHS $(V,\nabla,F^\bullet,Q)|_{\mathcal{U}-D}$ is injective, where $(\mathcal{U},\varphi)$ is an admissible coordinate around y.*

Example 4.4. Let (Y,D) be a log pair. When D is a *smooth* divisor, a $\mathbb{C}$-PVHS on $Y-D$ has injective local monodromy representation around D if and only if it has infinite local monodromies on each irreducible component of D.

Let us briefly explain the idea of the proof for Theorem D and some related results. The starting point is the special Higgs bundle $(E,\theta) := (\oplus_{p+q=\ell}E^{p,q}, \oplus_{p+q=\ell}\theta_{p,q})$ on (Y,D) constructed in Theorem 1.6. We divide the proof into four steps.

(1) The first step is to construct a generically finite surjective log morphism $\mu : (X,\tilde{D}) \to (Y,D)$ which is étale over U and has sufficiently high ramification over D (depending on the above (E,θ)). To find this highly ramified morphism μ, we apply the well-known result that monodromy group of a $\mathbb{C}$-PVHS is residually finite, and use the Cauchy argument principle to show the high ramification around D. Let us mention that this step is also the crucial ingredient in [Nad89,Rou16,Bru16,Cad18] for the strong hyperbolicity of hermitian symmetric spaces, whereas they all apply

Mumford's work [Mum77] to add level structures to find such ramified cover for toroidal compactifications of quotient of bounded symmetric domains.

(2) The second step is the pull-back of the above Higgs bundle (E,θ) to $(X,\tilde{D})$ via μ, which we denote by $(\tilde{E},\tilde{\theta}) := (\oplus_{p+q=\ell}\tilde{E}^{p,q},\oplus_{p+q=\ell}\tilde{\theta}_{p,q})$. This Higgs bundle $(\tilde{E},\tilde{\theta})$ on $(X,\tilde{D})$ satisfies the three conditions in Theorem 1.6. Moreover, some stage $\tilde{E}^{p_0,\ell-p_0}$ contains a big line bundle L' which admits enough local positivity on $\tilde{D}$.

(3) The third step is to prove Theorem D.(i). We start with $\tilde{E}^{p_0,\ell-p_0}$ and iterate the Higgs field $\tilde{\theta}$, which stops at finite steps. By the negativity of the kernel of $\tilde{\theta}$ due to Zuo [Zuo00], and the strong positivity of L', we construct an ample sheaf contained in some symmetric differential $\mathrm{Sym}^{\beta}\Omega_X$ (rather than on $\mathrm{Sym}^{\beta}\Omega_X(\log\tilde{D})$!). It follows from a celebrated theorem of Campana-Păun [CP19] that, X is of general type. Let us mention that this idea of iterating Higgs fields to their kernels, originally due to Viehweg-Zuo [VZ02], has been used by Brunebarbe in [Bru16] in which he proved similar result for hermitian symmetric space. There are also some similar results for *quotients of bounded domains* by Boucksom-Diverio [BD18] and Cadorel-Diverio-Guenancia [CDG19].

(4) The last step is to prove Theorems D.(ii) and D.(iii). We use the above Higgs bundle $(\tilde{E},\tilde{\theta})$ and ideas in § 2 to construct a Finsler metric F on T_X instead of $T_X(-\log\tilde{D})$ due to the enough local positivity of L' around $\tilde{D}$. Such a metric F is generically positive, and has holomorphic sectional curvature bounded from above by a negative constant by Theorem 2.4. By Ahlfors-Schwarz lemma, we conclude that X is pseudo Kobayashi hyperbolic; and by Theorem 0.7 the pseudo Picard hyperbolicity of X follows. Let us mention that Rousseau [Rou16] has proved similar result for hermitian symmetric spaces, which was later refined by Cadorel [Cad18]. Their methods use Bergman metrics for bounded symmetric domains instead of Hodge theory.

Now we start the detailed proof of Theorem D.

Proof of Theorem D. By Theorem 1.6, there is a Higgs bundle $(E,\theta) = (\oplus_{p+q=\ell}E^{p,q},\oplus_{p+q=\ell}\theta_{p,q})$ over (Y,D) satisfying the three conditions therein. In particular, there is a big line bundle L on Y and an inclusion $L\subset E^{p_0,\ell-p_0}$ for some $p_0>0$. Pick $m\gg 0$ so that $L-\frac{\ell}{m}D$ is a big $\mathbb{Q}$-line bundle.

Step 1. Let us denote by $\rho:\pi_1(U)\to GL(r,\mathbb{C})$ the monodromy representation of $(V,\nabla,F^{\bullet},Q)$, and denote by $\Gamma:=\rho(\pi_1(U))$ its monodromy group, which is a finitely generated linear group hence residually finite. For any irreducible component D_i of D, let us denote by $\gamma_i\in\Gamma$ the generator of the local monodromy of $(V,\nabla,F^{\bullet},Q)$ around D_i. We set

$$\mathfrak{S}:=\{\{i_1,\ldots,i_k\}\subset\{1,\ldots,c\}\mid D_{i_1}\cap\cdots\cap D_{i_k}\neq\varnothing\}.$$

Then for any $\{i_1,\ldots,i_k\}\in\mathfrak{S}$, $\gamma_{i_1},\ldots,\gamma_{i_k}$ commute pairwise. Indeed, pick any point $y\in D_{i_1}\cap\cdots\cap D_{i_k}-\cup_{p\neq i_j}D_p$, and take an admissible coordinate $(\mathcal{U},\varphi)$ around y, the fundamental group $\pi_1(\mathcal{U})\simeq\pi_1((\Delta^*)^k\times\Delta^{d-k})\simeq\mathbb{Z}^k$ which is abelian. Let $e_1,\ldots,e_k$ be the generators of $\pi_1((\Delta^*)^k\times\Delta^{d-k})$, namely e_i is a clockwise loop around the origin in the i-th factor Δ^*. Then $\gamma_i:=\rho(e_i)$. Clearly, $\gamma_{i_1},\ldots,\gamma_{i_k}$ commute pairwise. By the assumption that $(V,\nabla,F^{\bullet},Q)$ has injective local monodromy representation around D and by Lemma 4.3, $\gamma_{i_1},\ldots,\gamma_{i_k}$ generates a free abelian group of rank k.

Let m be the integer chosen at the beginning. It follows from the definition of residually finite group that there is a normal subgroup $\tilde{\Gamma}$ of Γ with finite index so that

$$\gamma_{i_1}^{\ell_1}\cdots\gamma_{i_k}^{\ell_k}\notin\tilde{\Gamma}\tag{4.0.1}$$

for each $\{i_1,\ldots,i_k\}\in\mathfrak{S}$, and each $0<|\ell_i|<m$. Then $\rho^{-1}(\tilde{\Gamma})$ is a normal subgroup of $\pi_1(U)$ with finite index. Let $\nu:\tilde{U}\to U$ be the finite étale cover of U whose fundamental group is $\rho^{-1}(\tilde{\Gamma})$. Then the monodromy group of $\nu^*(V,\nabla,F^\bullet,Q)$ is $\tilde{\Gamma}$.

Let us take a smooth projective compactification X of $\tilde{U}$ with $\tilde{D}:=X-\tilde{U}$ simple normal crossing so that $\nu:\tilde{U}\to U$ extends to a log morphism $\mu:(X,\tilde{D})\to(Y,D)$. Write $\tilde{D}=\sum_{j=1}^n\tilde{D}_j$ where $\tilde{D}_j$'s are irreducible components of $\tilde{D}$. Then there is $\{m_{ij}\in\mathbb{Z}_{\geq 0}\}_{i=1,\ldots,c;j=1,\ldots,n}$ so that

$$\mu^*(D_i)=\sum_{j=1}^n m_{ij}\tilde{D}_j.$$

Let us endow $\mathfrak{S}$ a partial order $\leq$ so that $I\leq J$ if and only if $I\subset J$. For any $\tilde{D}_j$, let $\{i_1,\ldots,i_k\}\in\mathfrak{S}$ be the largest element so that $\mu(\tilde{D}_j)\subset D_{i_1}\cap\cdots\cap D_{i_k}$. Hence $m_{ij}=0$ if and only if $i\notin\{i_1,\ldots,i_k\}$. Then there is a point $x\in\tilde{D}_j-\cup_{p\neq j}\tilde{D}_p$ so that $y:=\mu(x)\in D_{i_1}\cap\cdots\cap D_{i_k}-\cup_{p\neq i_j}D_p$. Take admissible coordinates $(\mathcal{W},\phi;z_1,\ldots,z_d)$ and $(\mathcal{U},\varphi;w_1,\ldots,w_d)$ around x and y respectively so that $\mu(\mathcal{W})\subset\mathcal{U}$. Within these coordinates, $\tilde{D}_j\cap\mathcal{W}=\{z_1=0\}$, and $D_{i_\ell}\cap\mathcal{U}=\{w_\ell=0\}$. Denote by $(\mu_1(z),\ldots,\mu_d(z))$ the expression of μ within these coordinates. Then

$$(\mu_1(z),\ldots,\mu_d(z))=(z_1^{m_{i_1,j}}\nu_1(z),\ldots,z_1^{m_{i_k,j}}\nu_k(z),\mu_{k+1}(z),\ldots,\mu_d(z))$$

where $\nu_1(z),\ldots,\nu_k(z)$ are holomorphic functions defined on $\mathcal{W}$ so that neither of them is identically equal to zero on $\{z_1=0\}$, and $m_{i_p,j}\geq 1$ for $p=1,\ldots,k$. We thus can choose a slice $S:=\{(z_1,\ldots,z_d)\mid\{|z_1|\leq\varepsilon,z_2=\zeta_2,\ldots,z_d=\zeta_d\}\subset\mathcal{W}$ so that $\nu_i(z)\neq 0$ for any $z\in S$ and any $i=1,\ldots,k$. Let us define a loop $e(\theta):[0,1]\to\mathcal{W}-\tilde{D}$ by $e(\theta):=(\varepsilon e^{2\pi i\theta},\zeta_2,\ldots,\zeta_d)$ which is the generator of $\pi_1(\mathcal{W}-\tilde{D})$. By *Cauchy's argument principle*, the winding number of $\mu_p\circ e(\theta)$ around 0 is $m_{i_p,j}$ for $p=1,\ldots,k$. Hence by the following diagram

$$\begin{tikzcd}
\tilde{\Gamma} \arrow[r, hook] & \Gamma \\
\pi_1(\mathcal{W}-\tilde{D}) \arrow[u] \arrow[r, "\mu_*"] \arrow[d, "\simeq"] & \pi_1(\mathcal{U}-D) \arrow[u] \arrow[d] \\
\mathbb{Z} \arrow[r] & \mathbb{Z}^k
\end{tikzcd}$$

one has $\mu_*(1)=(m_{i_1,j},\ldots,m_{i_k,j})$. In other words, $\gamma_{i_1}^{m_{i_1,j}}\cdots\gamma_{i_k}^{m_{i_k,j}}\in\tilde{\Gamma}$. By (4.0.1), there is some $n\in\{1,\ldots,k\}$ so that $m_{i_n,j}\geq m$. Note that

$$\mathrm{ord}_{\tilde{D}_j}(\mu^*D)=\sum_{i=1}^c m_{ij}=\sum_{p=1}^k m_{i_p,j}\geq m_{i_n,j}\geq m.$$

This implies that the divisor $\mu^*D-m\tilde{D}$ is effective.

Step 2. Let us introduce a Higgs bundle $(\tilde{E},\tilde{\theta})=(\oplus_{p+q=\ell}\tilde{E}^{p,q},\oplus_{p+q=\ell}\tilde{\theta}_{p,q})$ on $(X,\tilde{D})$ by pulling-back (E,θ) via μ. More precisely, $\tilde{E}^{p,q}:=\mu^*E^{p,q}$ and $\tilde{\theta}_{p,q}$ is defined to be the composition of the following maps

$$\mu^*E^{p,q}\to\mu^*E^{p-1,q+1}\otimes\mu^*\Omega_Y(\log D)\to\mu^*E^{p-1,q+1}\otimes\Omega_X(\log\tilde{D}).$$

Note that $(\tilde{E},\tilde{\theta})$ is a Higgs bundle on $(X,\tilde{D})$ canonically induced (in the sense of Definition 1.3) by the $\mathbb{C}$-PVHS $\nu^*(V,\nabla,F^\bullet,Q)$ defined on $\tilde{U}$. Since the period map of $\nu^*(V,\nabla,F^\bullet,Q)$ is still generically immersive, and $\nu^*(V,\nabla,F^\bullet,Q)$ has local unipotent monodromies around $\tilde{D}$, the Higgs bundle $(\tilde{E},\tilde{\theta})$ verifies the first two conditions in Theorem 1.6. The pull-back μ^*L is still a big line bundle for μ is generically finite. Hence $(\tilde{E},\tilde{\theta})$ satisfies the third conditions in Theorem 1.6 for $\mu^*L\subset \tilde{E}^{p_0,\ell-p_0}$.

Step 3. Now we iterate $\tilde{\theta}$ by k-times as in § 1.4 to obtain a morphism

$$\tilde{E}^{p_0,\ell-p_0}\longrightarrow \tilde{E}^{p_0-k,\ell-p_0+k}\otimes \mathrm{Sym}^k\Omega_X(\log \tilde{D}). \tag{4.0.2}$$

Since μ^*L is a subsheaf of $\tilde{E}^{p_0,\ell-p_0}$, it induces a morphism

$$\eta_k:\mu^*L\longrightarrow \tilde{E}^{p_0-k,\ell-p_0+k}\otimes \mathrm{Sym}^k\Omega_X(\log \tilde{D}).$$

Write k_0 for the largest k so that η_k is non-trivial. Then $0\le k_0\le p_0\le \ell$. Let us denote by N_p the kernel of $\theta_{p,\ell-p}$. Hence η_{k_0} admits a factorization

$$\eta_{k_0}:\mu^*L\longrightarrow N_{p_0-k_0}\otimes \mathrm{Sym}^{k_0}\Omega_X(\log \tilde{D}).$$

This implies that $k_0>0$; or else, there is a morphism from the big line bundle μ^*L to N_{p_0}, whose dual $N_{p_0}^*$ is weakly positive in the sense of Viehweg by [Zuo00]. Write $\tilde{L}:=\mu^*L-\ell\tilde{D}$, which is a big line bundle since $\mu^*D-m\tilde{D}$ is an effective divisor and $L-\frac{\ell}{m}D$ is a big $\mathbb{Q}$-line bundle. Hence η_{k_0} induces

$$\tilde{L}\longrightarrow N_{p_0-k_0}\otimes \mathrm{Sym}^{k_0}\Omega_X(\log \tilde{D})\otimes \mathcal{O}_X(-\ell\tilde{D})\subset N_{p_0-k_0}\otimes \mathrm{Sym}^{k_0}\Omega_X.$$

In other words, there exists a non-trivial morphism

$$\tilde{L}\otimes N_{p_0-k_0}^*\longrightarrow \mathrm{Sym}^{k_0}\Omega_X.$$

The torsion free coherent sheaf $\tilde{L}\otimes N_{p_0-k_0}^*$ is big in the sense of Viehweg. Hence there is $\alpha>0$ so that

$$\mathrm{Sym}^{\alpha}(\tilde{L}\otimes N_{p_0-k_0}^*)\otimes \mathcal{O}_X(-A)$$

is generically globally generated for some ample divisor A. Hence there is a non-trivial morphism

$$\mathcal{O}_X(A)\longrightarrow \mathrm{Sym}^{\alpha k_0}\Omega_X.$$

By a theorem of Campana-Păun [CP19, Corollary 8.7], X is of log general type.

Step 4. Let us prove that X is pseudo Kobayashi hyperbolic. Note that η_k induces a morphism

$$\tau_k:\mathrm{Sym}^kT_X(-\log \tilde{D})\longrightarrow \tilde{E}^{p_0-k,\ell-p_0+k}\otimes \mu^*L^{-1}$$

By Theorem 1.9 we know that τ_1 is injective on a Zariski open set $\tilde{U}'\subset\tilde{U}$. Since $\tilde{L}=\mu^*L-\ell\tilde{D}$, τ_k induces a morphism

$$\tilde{\tau}_k:\mathrm{Sym}^kT_X\longrightarrow \mathrm{Sym}^kT_X(-\log \tilde{D})\otimes \mathcal{O}_X(\ell\tilde{D})\longrightarrow \tilde{E}^{p_0-k,\ell-p_0+k}\otimes \tilde{L}^{-1}$$

which coincides with τ_k over $\tilde{U}$. Hence $\tilde{\tau}_1$ is also injective over $\tilde{U}'$. By Proposition 1.8.(ii), we can take a singular hermitian metric $h_{\tilde{L}}$ for $\tilde{L}$ so that $h:=h_{\tilde{L}}^{-1}\otimes h_{\mathrm{hod}}$ on $\tilde{L}^{-1}\otimes\tilde{E}$ is locally bounded on Y, and smooth outside $\tilde{D}\cup\mathbf{B}_+(\tilde{L})$, where h_{hod} is the Hodge metric for the Higgs bundle $(\tilde{E},\tilde{\theta})|_{\tilde{U}}$. Moreover, h vanishes on $\tilde{D}\cup\mathbf{B}_+(\tilde{L})$. This metric h on $\tilde{L}^{-1}\otimes\tilde{E}$ induces a Finsler metric F_k on T_X defined as follows: for any $e\in T_{X,x}$,

$$F_k(e):=h\big(\tilde{\tau}_k(e^{\otimes k})\big)^{\frac{1}{k}}$$

We apply the same method in § 2 to construct a new Finsler metric F on T_X by taking convex sum in the following form

$$F:=\sqrt{\sum_{i=1}^{k_0}\alpha_iF_i^2}.$$

where $\alpha_1, \ldots, \alpha_{k_0} \in \mathbb{R}^+$ are certain constants. This Finsler metric F on T_X is positively definite over $\tilde{U}^\circ := \tilde{U}' - \mathbf{B}_+(\tilde{L})$ for $\tilde{\tau}_1$ is injective over $\tilde{U}'$ and h is smooth on $\tilde{U} - \mathbf{B}_+(\tilde{L})$. By Theorem 2.4 one can choose $\alpha_1, \ldots, \alpha_{k_0} \in \mathbb{R}^+$ properly so that for any $\gamma : C \to X$ with C an open set of $\mathbb{C}$ and $\gamma(C) \cap \tilde{U}^\circ \neq \varnothing$, one has

$$\sqrt{-1}\partial\bar{\partial}\log|\gamma'(t)|_F^2 \geq \gamma^*\omega \tag{4.0.3}$$

for some fixed smooth Kähler form ω on X. Indeed, it follows from the proof of Theorem 2.4 that there is an open subset C° of C whose complement is a discrete set such that (4.0.3) holds over C°. By Definition 2.1, $|\gamma'(t)|_F^2$ is continuous and locally bounded from above over C, and by the extension theorem of subharmonic function, (4.0.3) holds over the whole unit disk C. By Theorem 0.7 and (4.0.3), X is Picard hyperbolic modulo $X - \tilde{U}^\circ$. Hence Theorem D.(iii) follows.

By Definition 2.1 again, there is $\varepsilon > 0$ so that $\omega \geq \varepsilon F^2$. Hence (4.0.3) implies that

$$\frac{\partial^2 \log|\gamma'(t)|_F^2}{\partial t\partial\bar{t}} \geq \varepsilon|\gamma'(t)|_F^2$$

for any $\gamma : \Delta \to X$ with $\gamma(\Delta)\cap\tilde{U}^\circ \neq \varnothing$. In other words, the holomorphic sectional curvature of F is bounded from above by the negative constant $-\varepsilon$ (see [Kob98, Theorem 2.3.5]). By the Ahlfors-Schwarz lemma, we conclude that X is Kobayashi hyperbolic modulo $X - \tilde{U}^\circ$ (see [Den18, Lemma 2.4]). In particular, X is pseudo Kobayashi hyperbolic. We complete the proof of the theorem. □

As a byproduct of the above proof, we obtain the following result.

Theorem 4.5. *Let (Y, D) be a log pair and let $(E, \theta) = (\oplus_{p+q=\ell}E^{p,q}, \oplus_{p+q=\ell}\theta_{p,q})$ be a Higgs bundle on (Y, D) satisfying the following conditions.*

(1) The Higgs field θ satisfies

$$\theta : E^{p,q} \to E^{p-1,q+1} \otimes \Omega_Y(\log D)$$

(2) (E, θ) is canonically induced (in the sense of Definition 1.3) by some $\mathbb{C}$-PVHS over $Y - D$ of weight ℓ with unipotent monodromies around D.

(3) There is a big line bundle L over Y such that $L \subset E^{p_0,q_0}$ for some $p_0 + q_0 = \ell$.

(4) The line bundle $L - p_0D$ is still big.

Then the projective manifold Y

(i) *is of general type;*

(ii) *is pseudo Kobayashi hyperbolic;*

(iii) *is pseudo Brody hyperbolic.*

Now we are able to prove Corollary E.

Proof of Corollary E. By Theorem 1.6, there is a Higgs bundle $(E, \theta) = (\oplus_{p+q=\ell}E^{p,q}, \oplus_{p+q=\ell}\theta_{p,q})$ over (Y, D) satisfying the three conditions therein. In particular, there is a big and nef line bundle L on Y and an inclusion $L \subset E^{p_0,\ell-p_0}$ for some $p_0 > 0$. Moreover, the augmented base locus $\mathbf{B}_+(L) \subset D$ by Lemma 1.5. Pick $m \gg 0$ so that $L - \frac{\ell}{m}D$ is a big $\mathbb{Q}$-line bundle, and $\mathbf{B}_+(L - \frac{\ell}{m}D) \subset D$.

Now we perform the same argument in Step 1 of the proof of Theorem D to construct a log pair (X, D) and a log morphism $\mu : (X, \tilde{D}) \to (Y, D)$ which is étale over U. Moreover, $\mu^*D - m\tilde{D}$ is effective. Hence, one has $\mathbf{B}_+\big(\mu^*(L - \frac{\ell}{m}D)\big) \subset \tilde{D}$ and $\mathbf{B}_+(\mu^*L) \subset \tilde{D}$.

We construct a Higgs bundle $(\tilde{E}, \tilde{\theta}) = (\oplus_{p+q=\ell}\tilde{E}^{p,q}, \oplus_{p+q=\ell}\theta_{p,q})$ on $(X, \tilde{D})$ by pullingback (E, θ) via μ as in Step 2 of the proof of Theorem D. It satisfies the three conditions in Theorem 1.6, and $\mu^*L \subset \tilde{E}^{p_0,\ell-p_0}$ for some p_0.

Let $\tilde{Z}$ be any Zariski closed subvariety of X which is not contained in $\tilde{D}$. Take a resolution of singularities $g : Z \to \tilde{Z}$ so that $D_Z := \nu^{-1}(\tilde{D})$ is simple normal crossing. Then $g : (Z, D_Z) \to (X, \tilde{D})$ is a log morphism which is generically finite. Following Step 2 of the proof of Theorem D again, we construct a Higgs bundle $(F, \eta) = (\oplus_{p+q=\ell} F^{p,q}, \oplus_{p+q=\ell} \eta_{p,q})$ on (Z, D_Z) by pulling-back $(\tilde{E}, \tilde{\theta})$ via g, which satisfies the first two conditions in Theorem 4.5. Moreover, one has $g^*\mu^*L \subset F^{p_0, \ell-q_0}$. Note that $g|_{Z-D_Z} : Z - D_Z \to X - \tilde{D}$ is generically immersive. Recall that $\mathbf{B}_+(\mu^*L) \subset \tilde{D}$ and $\mathbf{B}_+\big(\mu^*(L - \frac{\ell}{m}D)\big) \subset \tilde{D}$. We conclude that both $g^*\mu^*L$ and $g^*\mu^*(L - \frac{\ell}{m}D)$ are big. Moreover, one has

$$g^*\mu^*L - \ell D_Z \geq g^*(\mu^*L - \ell\tilde{D}) \geq g^*\mu^*(L - \frac{\ell}{m}D).$$

The Higgs bundle (F, η) on (Z, D_Z) thus satisfies the last two conditions in Theorem 4.5. In summary, we construct a Higgs bundle on (Z, D_Z) satisfying all the conditions in Theorem 4.5. By Theorem 4.5, Z is of general type. We proved Corollary E.(i).

Let us prove Corollary E.(ii). For any $\tilde{\gamma} : \Delta^* \to X$ whose image is not contained in $\tilde{D}$, let $\tilde{Z}$ be its Zariski closure. Take a desingularization $\nu : Z \to \tilde{Z}$ as above, and let $\gamma : \Delta^* \to Z$ be the lift of γ. By the above argument and Theorem 4.5, Z is pseudo Picard hyperbolic. Hence γ extends to a holomorphic map $\overline{\gamma} : \Delta \to Z$. Hence $\nu \circ \overline{\gamma}$ extends $\tilde{\gamma}$. We proved Corollary E.(ii). It is an easy to see that Corollary E.(ii) implies Corollary E.(iii). □

Remark 4.6. Let us mention that a compact complex manifold is Kobayashi hyperbolic if and only if it is Picard hyperbolic (see [Kob98]). However, we do not know the relation between pseudo Kobayshi hyperbolicity and pseudo Picard hyperbolicity. We refer the reader to [Jav20b, §8] for a conjectural picture.

5. Applications to arithmetic locally symmetric varieties

In this section we will give some applications of techniques developed in this paper to hyperbolicity of arithmetic locally symmetric varieties. A complex manifold $U = \mathcal{D}/\Gamma$ is called an *arithmetic locally symmetric variety* if it is the quotient of a bounded symmetric domain $\mathcal{D}$ by a torsion free arithmetic lattice $\Gamma \in \mathrm{Aut}(\mathcal{D})$. We know that U is quasi-projective and admits a *toroidal compactification*, which is indeed a projective normal crossing compactification. The purpose of this section is as follows.

(1) We apply techniques in the proof of Theorem A to reprove the Borel extension theorem Theorem 5.2.
(2) We apply Theorem D to reprove a strong hyperbolicity result by Rousseau [Rou16] for arithmetic locally symmetric varieties.
(3) We reprove the algebraic hyperbolicity for arithmetic locally symmetric varieties, which is a consequence of [Bor72] and [PR07].
(4) We prove the Picard hyperbolicity modulo the boundary for compactifications of arithmetic locally symmetric varieties after passing to a finite étale cover, in the same vein as Corollary E.(ii).

This subsection is strongly inspired by [Rou16, Bru16].

By the work of Gross [Gro94], Sheng-Zuo [SZ10], and Friedman-Laza [FL13], we know that there is a $\mathbb{C}$*-PVHS of Calabi-Yau type* on any irreducible bounded symmetric domain. Instead of introducing the precise definition of $\mathbb{C}$-PVHS of Calabi-Yau type and its relation with arithmetic locally symmetric varieties, we will make a summary of results in these references following [Bru16, §5] which shall be used in this subsection.

Theorem 5.1. *Let U be an arithmetic locally symmetric variety. Then there are a toroidal compactification of Y of U, and a Higgs bundle $(E,\theta) = (\oplus_{p+q=\ell}E^{p,q}, \oplus_{p+q=\ell}\theta_{p,q})$ over the log pair $(Y, D := Y - U)$ satisfying the following conditions.*

(1) The Higgs field θ satisfies

$$\theta : E^{p,q} \longrightarrow E^{p-1,q+1} \otimes \Omega_Y(\log D)$$

(2) (E,θ) is canonically induced by some $\mathbb{C}$-PVHS over $Y - D$ of weight ℓ with unipotent monodromies around D.

(3) $E^{\ell,0}$ is a big and nef line bundle on Y such that $\mathbf{B}_+(E^{\ell,0}) \subset D$.

(4) The map $T_Y(-\log D) \to \mathrm{Hom}(E^{\ell,0}, E^{\ell-1,1})$ induced by $\theta_{\ell,0} : E^{\ell,0} \to E^{\ell-1,1} \otimes \Omega_Y(\log D)$ is injective when restricted to U.

Let us first apply our techniques to give a new proof for Borel's extension theorem.

Theorem 5.2 ([Bor72, Theorem A]). *Let U be an arithmetic locally symmetric variety and let Y be a toroidal compactification of U. Then any holomorphic map from $f : (\Delta^*)^p \times \Delta^q \to U$ extends to a meromorphic map $\overline{f} : \Delta^{p+q} \dashrightarrow Y$.*

Proof. By Theorem 2.6, the Higgs bundle on U in Theorem 5.1 induces a Finsler metric F over $T_Y(-\log D)$, which is positively definite over U, and satisfies the curvature estimate (2.0.14). Then by Theorem 0.7, U is Picard hyperbolic. The desired meromorphic map $\overline{f}$ then follows from Proposition 3.4 immediately. □

We can also give a proof for algebraic hyperbolicity of arithmetic locally symmetric varieties.

Theorem 5.3. *Arithmetic locally symmetric varieties are algebraically hyperbolic.*

Proof. Pick a toroidal compactification Y of U and set $D := Y - U$. It follows from our proof of Theorem 5.2 that we can construct a Finsler metric F on $T_X(-\log D)$ which is positively definite over U, and satisfies the curvature estimate (2.0.14). Using the same argument in the proof of algebraic hyperbolicity in Theorem A, one can prove that U is algebraically hyperbolic. The theorem is proved. □

Theorem 5.3 was already known. Indeed, Pacienza-Rousseau [PR07] proved that if a quasi-projective manifold U is hyperbolically embedded into some projective compactification, then U is algebraically hyperbolic. Arithmetic locally symmetric varieties are hyperbolically embedded into their Baily-Borel compactifications by [Bor72]. Theorem 5.3 thus follows from a combination of [PR07, Bor72].

Let us now use previous techniques to prove a new result for hyperbolicity of arithmetic locally symmetric varieties.

Theorem 5.4. *Let U be an arithmetic locally symmetric variety. Then there is a finite étale cover $\tilde{U} \to U$ and a projective compactification X of $\tilde{U}$ so that X is Picard hyperbolic modulo $X - \tilde{U}$.*

Recall that Rousseau [Rou16] has proved that the variety X in Theorem 5.4 is Kobayashi hyperbolic modulo $X - \tilde{U}$, and Brunebarbe [Bru16] has proved that any Zariski closed subvariety not contained in $X - \tilde{U}$ is of general type. As we will see below, our methods can give a new proof of Rousseau's theorem as a byproduct.

Proof of Theorem 5.4. By Theorem 5.1 there is a Higgs bundle $(E,\theta) = (\oplus_{p+q=\ell}E^{p,q}, \oplus_{p+q=\ell}\theta_{p,q})$ over the log pair (Y, D) which satisfies the conditions therein. Take $m \gg 0$ so that $\mathbf{B}_+(E^{\ell,0} - \frac{\ell}{m}D) \subset D$. By [Mum77], one can add the level structure so that there is an étale cover $\tilde{U} \to U$ and a log-compactification X of $\tilde{U}$ with $\tilde{D} := X - \tilde{U}$ so that for the induced log morphism $\mu : (X, \tilde{D}) \to (Y, D)$, one has $\mu^*D \geq m\tilde{D}$. By pulling back (E,θ) to $(X,\tilde{D})$ as Step 2 in the proof of Theorem D, we construct a Higgs bundle $(\tilde{E},\tilde{\theta}) = (\oplus_{p+q=\ell}\tilde{E}^{p,q},\oplus_{p+q=\ell}\tilde{\theta}_{p,q})$ on $(X,\tilde{D})$ which satisfies the four conditions in Theorem 5.1. Moreover, $\tilde{E}^{\ell,0} - \ell\tilde{D}$ is big and $\mathbf{B}_+(\tilde{E}^{\ell,0} - \ell\tilde{D}) \subset \tilde{D}$. By Step 4 in the proof of Theorem D, we can construct a Finsler metric on T_X which is positively definite on $\tilde{U}$ so that for any $\gamma : C \to X$ with C an open set of $\mathbb{C}$ and $\gamma(C) \cap \tilde{U} \neq \varnothing$, one has

$$\sqrt{-1}\partial\bar{\partial}\log|\gamma'(t)|_F^2 \geq \gamma^*\omega.$$

By the same argument in Step 4 again, X is both Picard hyperbolic and Kobayashi hyperbolic modulo $\tilde{D}$. □

Remark 5.5. Let us mention that one can also give a direct proof of Theorem 5.4 by combining [Rou16] with Theorem 0.7, as communicated to us by Erwan Rousseau. Indeed, in [Rou16, Proposition 2.4], Rousseau constructed a singular hermitian metric h for T_X which is positively definite over $\tilde{U}$ and vanishes on the boundary $D := X - \tilde{U}$ so that for any $\gamma : C \to X$ with C an open set of $\mathbb{C}$ and $\gamma(C) \cap \tilde{U} \neq \varnothing$, one has

$$\sqrt{-1}\partial\bar{\partial}\log|\gamma'(t)|_h^2 \geq \varepsilon\gamma^* g$$

over $\gamma^{-1}(\tilde{U})$ for some constant $\varepsilon > 0$. Here g is the Bergman metric for $T_{\tilde{U}}$. Note that the metric g^{-1} for $\Omega_{\tilde{U}}$ is 'good' in the sense of [Mum77] with respect to the extension $\Omega_X(\log D)$ of $\Omega_{\tilde{U}}$ (see [Mum77, Main Theorem 3.1 & Proposition 3.4.a)]). In other words, for any local holomorphic section e of $\Omega_X(\log D)$, the norm $\|e\|^2_{g^{-1}}$ has at most logarithmic growth near the boundary. In particular, one can easily verify that $g \geq \omega|_{\tilde{U}}$ for some smooth Kähler form of X. Hence

$$\sqrt{-1}\partial\bar{\partial}\log|\gamma'(t)|_h^2 \geq \varepsilon\gamma^*\omega$$

for any $\gamma : \Delta^* \to X$ with $\gamma(\Delta^*) \cap \tilde{U} \neq \varnothing$. It follows from Theorem 0.7 that γ extends across the origin.

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